JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **131**, 106–112 (1988)

A Note on Disfocality

STEEN MARKVORSEN

Mathematical Institute, The Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark

Submitted by Ky Fan

Received April 14, 1986

1. INTRODUCTION

Consider the solution $\psi(s)$ to the initial value problem

$$\begin{aligned} (g(s)\,\psi'(s))' + f(s)\,\psi(s) = 0, \qquad & s \in [0, \infty[\\ \psi(0) = 1, \qquad \psi'(0) = 0, & \end{aligned} \tag{$*$}$$

where we have assumed as usual, that f and g are given continuous functions of s and that $g(s) > 0$ for all s.

If the solution ψ has a zero (a focal point), then it has a smallest zero, which we denote by $\xi(*)$.

In the literature there are several lower bounds for the distance between consecutive zeros of ψ and/or ψ'—cf. [2, 3]. These lower bounds are usually given implicitly as in the following result.

PROPOSITION A [2, 5]. *If* $g(s) \equiv 1$ *and if*

$$\int_0^c (c - t) \max \{f(t), 0\}\, dt < 1, \tag{1.1}$$

then $\xi(*) > c$.

An inspection of the Sturm comparison theorems (cf. [3, pp. 334 ff]) reveals that—intuitively speaking—the occurrence of negative values of f and large values of g should help make $\xi(*)$ large. We prove, that this is true in the following sense.

0022-247X/88 $3.00

PROPOSITION 1. *Suppose that* $f(s)g(s) \leqq \kappa^2$ *for some* $\kappa \in R_+$ *and all* $s \in [0, c]$. *If*

$$-\frac{\pi}{2} \leqq \int_0^s ((\kappa^2 + f(t)\, g(t))/\kappa g(t))\, dt \leqq \frac{\pi}{2} \qquad \textit{for all} \quad s \in [0, c], \tag{1.2}$$

then $\xi(*) > c$.

Remark. In particular, it is now easy to construct examples of problems $(*)$ which for any given c and Ω in R_+ have $g \equiv 1$ and

$$\int_0^c (c - t) \max\{f(t), 0\}\, dt > \Omega,$$

but still $\xi(*) > c$.

As one should expect, the lower bound $(-\pi/2)$ in condition (1.2) is not essential. This will be proved in a technical refinement of Proposition 1 below (see Theorem 1).

Our proofs of these results use a modification of the well-known Prüfer transformation [3, p. 332]. In the last section we use the same transformation to obtain a corresponding upper bound on $\xi(*)$; In effect we show, that if g is small and if f is large in the average (again compared with $\pi/2$), then $\xi(*)$ is small (Theorem 2).

2. The Modified Prüfer Transformation

Let $\psi(s)$ be the solution to $(*)$ and let κ be a positive constant. Then

$$z(s) = \operatorname{Arc\,tan}\left(\frac{-g(s)\, \psi'(s)}{\kappa \psi(s)}\right) \tag{2.1}$$

defines a continuously differentiable function of s, which is the "upside down" of the ordinary Prüfer transformation. A straightforward computation gives (compare with [1, p. 554])

$$\begin{aligned} z'(s) &= \kappa^{-1} f(s) \cos^2(z(s)) + \kappa g(s)^{-1} \sin^2(z(s)) \\ &= \tfrac{1}{2}(\kappa g(s)^{-1} + \kappa^{-1} f(s)) \\ &\quad + \tfrac{1}{2}(\kappa^{-1} f(s) - \kappa g(s)^{-1}) \cos(2z(s)). \end{aligned} \tag{2.2}$$

Now assume that $f(s)g(s) \leqq \kappa^2$ for all s in $[0, c]$. Then we get the following inequalities

$$z'(s) \geqq \tfrac{1}{2}(\kappa g(s)^{-1} + \kappa^{-1} f(s)) + \tfrac{1}{2}(\kappa^{-1} f(s) - \kappa g(s)^{-1}) = \kappa^{-1} f(s) \quad (2.3)$$

and

$$z'(s) \leqq \begin{cases} \kappa g(s)^{-1} & \text{if} \quad \cos(2z(s)) \leqq 0 \\ \tfrac{1}{2}(\kappa g(s)^{-1} + \kappa^{-1} f(s)) & \text{if} \quad \cos(2z(s)) \geqq 0. \end{cases} \quad (2.4)$$

We can now prove Proposition 1 as stated in the Introduction. We use the inequality (2.4) and have by assumption

$$z(s) \leqq \int_0^s \frac{1}{2}(\kappa g(t)^{-1} + \kappa^{-1} f(t))\, dt \leqq \frac{\pi}{4} \quad (2.5)$$

for all s in $[0, c]$ as long as $z(s) \geqq -\pi/4$. But since the integral is assumed to be not less than $-\pi/4$ we see, that both inequalities in (2.5) hold true for all s in the interval $[0, c]$ (the details of the argument are given below). Therefore $\xi(*)$ cannot be in $[0, c]$, since $z(\xi(*)) = \pi/2$.

In order to weaken the lower bound on the integral in (1.2) we introduce the notation

$$\omega(s) = \frac{1}{2}(\kappa g(s)^{-1} + \kappa^{-1} f(s)); \qquad s \in [0, c] \quad (2.6)$$

$$P(a, b) = \int_a^b \omega(t)\, dt; \qquad a, b \in [0, c]. \quad (2.7)$$

We construct a continuous function $Q(s)$ on $[0, c]$ as follows:

Let α be the largest element in $[0, c]$ for which $P(0, s) \geqq -\pi/4$ for all $s \in [0, \alpha]$ and define $Q(s) = P(0, s)$ for $s \in [0, \alpha]$. In the interval $I = [\alpha, c]$ we consider the continuous function $h(s) = P(\alpha, s)$ and its nonincreasing part $\bar{h}(s)$ which we define as follows: Let $J = \{t \in [\alpha, c] \mid h(t) \leqq h(s) \text{ for all } s < t\}$ and define

$$\bar{h}(s) = h(\sup\{t \in J \mid t \leqq s\}) \qquad \text{for} \quad s \in I.$$

Then $\bar{h}$ is nonincreasing and continuous on I. We define $Q(s) = h(s) - \bar{h}(s) - \pi/4$ for $s \in I$. In total Q is a well-defined continuous function on $[0, c]$ with no value less than $-\pi/4$.

We can now prove the following

LEMMA. *Let $\psi(s)$ be the solution to* $(*)$ *and let $z(s)$ be defined by* (2.1). *If* $z(s) \leqq \pi/4$ *for all* $s \in [0, c]$, *then*

$$z(s) \leqq Q(s) \qquad \textit{for all} \quad s \in [0, c]. \tag{2.8}$$

Proof. We must prove the inequality (2.8) in the two intervals $[0, \alpha]$ and $[\alpha, c]$:

(i) $z(s) \leqq P(0, s)$ for all $s \in [0, \alpha]$. Indeed, if we let $\beta \leqq \alpha$ be the smallest solution to $z(\beta) = -\pi/4$, then by (2.4) we have $z(s) \leqq P(0, s)$ for all $s \in [0, \beta]$. Now assume for contradiction that $z(\delta) > P(0, \delta)$ for some $\delta \in [\beta, \alpha]$. Then $z(\delta) > -\pi/4$ and there is by continuity an element $\eta \in [\beta, \delta[$ closest to δ such that $z(\eta) = -\pi/4$ and $z(s) > -\pi/4$ for all $s \in]\eta, \delta]$. From (2.4) we get $z(s) \leqq -\pi/4 + P(\eta, s)$ for all $s \in [\eta, \delta]$. In particular, $z(\delta) \leqq -\pi/4 + P(\eta, \delta) \leqq P(0, \eta) + P(\eta, \delta) = P(0, \delta)$, which is the desired contradiction.

(ii) $z(s) \leqq Q(s) = h(s) - \bar{h}(s) - \pi/4$ for all $s \in I = [\alpha, c]$. In fact, suppose, for contradiction, that $z(\delta) > Q(\delta)$ for some $\delta \in]\alpha, c]$. Then $z(\delta) > -\pi/4$ and again by continuity there is an element $\eta \in [\alpha, \delta[$ closest to δ such that $z(\eta) = -\pi/4$ and $z(s) > -\pi/4$ for all $s \in]\eta, \delta]$. From (2.4) we have $z(s) \leqq -\pi/4 + P(\eta, s)$ and hence $0 < P(\eta, s) = h(s) - h(\eta)$ for all $s \in]\eta, \delta]$.

We conclude that $\bar{h}(s)$ is constant on this interval: $\bar{h}(s) \equiv \bar{h}_0 \leqq h(\eta)$ for all $s \in [\eta, \delta]$. Therefore

$$\begin{aligned} P(\eta, s) &= h(s) - \bar{h}(s) + (\bar{h}(s) - h(\eta)) \\ &\leqq h(s) - \bar{h}(s), \end{aligned}$$

and finally,

$$\begin{aligned} z(\delta) &\leqq -\frac{\pi}{4} + P(\eta, \delta) \\ &\leqq -\frac{\pi}{4} + h(\delta) - \bar{h}(\delta) \\ &= Q(\delta), \end{aligned}$$

which is the desired contradiction. ∎

Our main result can now be stated as follows.

THEOREM 1. *Suppose that* $f(s)g(s) \leqq \kappa^2$ *for some* $\kappa \in R_+$ *and all*

$s \in [0, c]$. *If* $Q(s) \leqq \pi/4$ *for all* $s \in [0, c]$, *where* Q *is defined in terms of* f, g, *and* κ *as above, then the initial value problem* $(*)$ *has*

$$\xi(*) > c.$$

Proof. Assume for contradiction, that $\xi(*) \leqq c$. Then $z(\xi(*)) = \pi/2$, which is clearly impossible by the lemma, which gives

$$z(s) \leqq Q(s) \leqq \frac{\pi}{4} \qquad \text{for all} \quad s \in [0, c]. \quad \blacksquare$$

3. Some Examples and Discussion

In order to illustrate the use of Theorem 1 we consider the following class of Mathieu equations

$$\psi''(s) + (a \cdot \cos(bs) - 1)\, \psi(s) = 0, \tag{3.1}$$

where a and b are constants. It is well known that (3.1) has no focal point when $a \leqq 1$. As a consequence of Theorem 1 we have

Corollary. *For any given number* a *in the interval* $]1, 2]$ *there is a positive number* $b_0(a)$ *such that if* $b \geqq b_0(a)$, *then Equation* (3.1) *has no focal point in* R_+.

Proof. Since $g(s) \equiv 1$ and $f(s) = a\cos(bs) - 1$ we get $f(s)\, g(s) \leqq a - 1 = \kappa^2$ for all $s \in R_+$. Thus

$$\omega(s) = \frac{1}{2}\left(\frac{a-2}{\sqrt{a-1}} + \frac{a}{\sqrt{a-1}} \cos(bs)\right),$$

and $P(0, s) = \frac{1}{2}(\lambda s + \mu \sin(bs))$, where $\lambda = (a-2)/\sqrt{a-1} \leqq 0$ and $\mu = a/b\sqrt{a-1}$. We see that $P(0, s)$ is the sum of a linear function and a periodic function which can be uniformly scaled by b. Hence there is a $\tilde{b}_0(a)$ such that $Q(s) = P(0, s) \leqq \pi/4$ for all $s \leqq \alpha$ when $b \geqq \tilde{b}_0(a)$. Furthermore, when subtracting the nonincreasing part of $P(0, s)$ from $P(0, s)$ in the interval $[\alpha, \infty[$ what remains is again a periodic function whose maximum is uniquely determined by b when a is given. In total, the function $Q(s)$ for this particular problem has a b-controlled maximum value which can be made arbitrarily small so that Theorem 1 applies and the corollary follows. $\blacksquare$

We note here that the otherwise strong disconjugacy results of Hille and Nehari (see [1] for most general statements) do not apply to the setting of the corollary, since they all require the weight function f to be positive. One of the main features of Theorem 1 is precisely that one can obtain disfocality by balancing the effect of the positive part of f with the effect of the negative part of f.

In fact, if we only consider nonnegative weights f, our result cannot even compete with the original Liapunov estimate in Proposition A. As an example, we consider the problem $(*)$ with $g(s) \equiv 1$ and $f(s) = (1+\delta^2 - s)^{-1}$ on the interval $[0, 1+\frac{1}{2}\delta^2]$, where δ is any small number in R_+. Then Proposition A gives $\xi(*) > 1$ independent of $\delta \to 0$, whereas Proposition 1 only gives $\xi(*) > 0.61$ as $\delta \to 0$.

We also note that any construction of a uniform approximation to the actual function $z(s)$ defined in (2.1) automatically will give an estimate of $\xi(*)$. This approach has been discussed by W. Leighton in [6], where he uses step-function approximations of f and g to approximate $\xi(*)$. Such a procedure will, of course, produce much more information than what is needed to bound $\xi(*)$ and hence it will, in general, take much more (numerical) work than what is required to construct the function $Q(s)$ for Theorem 1.

4. An Implicit Upper Bound

As mentioned in the introduction we finally show the following partial complement to the previous results.

Theorem 2. *If $f(s)g(s) \leqq \kappa^2$ for some $\kappa \in R_+$ and all $s \in [0, c]$, and if*

$$\int_0^c \kappa^{-1} f(t)\, dt \geqq \frac{\pi}{2}, \tag{4.1}$$

then

$$\xi(*) \leqq c.$$

Proof. Assume, for contradiction, that $\xi(*) > c$. Then $z(s) < \pi/2$ for all $s \in [0, c]$. But from (2.3) and the assumption (4.1) we also have

$$z(c) = \int_0^c z'(t)\, dt \geqq \int_0^c \kappa^{-1} f(t)\, dt \geqq \frac{\pi}{2},$$

which is the desired contradiction. ∎

References

1. J. H. Barrett, Disconjugacy of second order linear differential equations with nonnegative coefficients, *Proc. Amer. Math. Soc.* **10**, II (1959), 552–561.
2. J. H. E. Cohn, Consecutive zeros of solutions of ordinary second order differential equations, *J. London Math. Soc.* **5** (1972), 465–468.
3. P. Hartman, "Ordinary Differential Equations," 2nd ed., Birkhäuser, Basel 1982.
4. P. Hartman and A. Wintner, On an oscillation criterion of Liapunov, *Amer. J. Math.* **73** (1951), 885–890.
5. M. K. Kwong, On Lyapunov's inequality for disfocality, *J. Math. Anal. Appl.* **83** (1981), 486–494.
6. W. Leighton, Computing bounds for focal points and for σ-points for second-order linear differential equations, *in* "Ordinary Differential Equations, 1971 NRL–MRC Conference," Academic Press, New York 1972, (L. Weiss, Ed.), pp. 497–503.