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Universal L^p improving for averages along polynomial curves in low dimensions

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Abstract

We prove sharp $L^p \to L^q$ estimates for averaging operators along general polynomial curves in two and three dimensions. These operators are translation-invariant, given by convolution with the so-called affine arclength measure of the curve and we obtain universal bounds over the class of curves given by polynomials of bounded degree. Our method relies on a geometric inequality for general vector polynomials together with a combinatorial argument due to M. Christ. Almost sharp Lorentz space estimates are obtained as well.

Keywords: Averaging operators; Polynomial curves; Universal bounds

1. Introduction and statement of results

Recently there has been considerable attention given to certain euclidean harmonic analysis problems associated to a curve or surface where the underlying euclidean arclength or surface measure (which typically defines the classical problem) is replaced by the so-called affine arclength or surface measure. This has the effect of making the problem affine invariant as well as invariant under reparameterisations of the underlying variety. For this reason there have been

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many attempts to obtain universal results, establishing uniform bounds over a large class of curves or surfaces. The affine arclength or surface measure also has the mitigating effect of dampening any curvature degeneracies of the curve or surface and therefore the expectation is that the universal bounds one seeks will be the same as those arising from the most non-degenerate situation.

This line of research has been actively pursued for the problem of Fourier restriction, a central problem in euclidean harmonic analysis; see for example [1,2,5,6,14–17,21,23,28]. Drury initiated an investigation along these lines for the problem of achieving precise regularity results for averages along curves or surfaces, in particular determining sharp $L^p \to L^q$ estimates, and this has been followed up by several authors; see for example [7,8,15,19,20,22,24–27].

In this paper we continue an investigation by Oberlin to establish such a result for averaging operators along general polynomial curves in $\mathbb{R}^d$ when $d = 2$ or $d = 3$ (in [22], the $d = 2$ case was fully resolved and partially resolved for $d = 3$). More specifically, if $\gamma : I \to \mathbb{R}^d$ parametrises a smooth curve in $\mathbb{R}^d$ on an interval I, set

$$L_\gamma(t) = \det\bigl(\gamma'(t) \cdots \gamma^{(d)}(t)\bigr);$$

this is the determinant of a $d \times d$ matrix whose jth column is given by the jth derivative of γ, $\gamma^{(j)}(t)$. The affine arclength measure $\nu = \nu_\gamma$ on γ is defined on a test function ϕ by

$$\nu(\phi) = \int_I \phi\bigl(\gamma(t)\bigr)\bigl|L_\gamma(t)\bigr|^{\frac{2}{d(d+1)}}\,dt;$$

one easily checks that this measure is invariant under reparameterisations of γ. A basic problem in the theory of averaging operators along curves (or more generally, for generalised Radon transforms) is to determine the exponents p and q so that the a priori estimate

$$\|Tf\|_{L^q(\mathbb{R}^d)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

holds uniformly for a large class of curves γ where

$$Tf(x) = f * \nu(x) = \int_I f\bigl(x - \gamma(t)\bigr)\bigl|L_\gamma(t)\bigr|^{\frac{2}{d(d+1)}}\,dt.$$

The use of the affine arclength measure allows us to think about global estimates, not only establishing (1) with a constant C uniform over a large class of curves but also possibly obtaining such a constant independent of the parametrising interval I. As discussed above, the exponents p and q in (1) that we expect, should come from the most non-degenerate situation which in this case is the curve $\gamma(t) = (t, \ldots, t^d)$ in $\mathbb{R}^d$ where $L_\gamma \equiv \mathit{constant}$. A simple scaling argument shows that necessarily we must have $1/q = 1/p - 2/d(d+1)$ if (1) is to hold as a global estimate. Furthermore by testing (1) on $f = \chi_{B_\delta}$ where B_δ is the ball of radius δ with centre 0, we obtain the added necessary condition $(d^2 + d)/(d^2 - d + 2) \leqslant p \leqslant (d + 1)/2$. It is a remarkable result of Christ [9] that (up to the endpoints) these restrictions on p and q are in fact sufficient for (1) to hold in this non-degenerate situation. Stovall [30], building on an argument of Christ [10], has converted Christ's restricted weak-type estimates at the endpoints into strong type estimates.

To date, progress that has been made to establish universal bounds in (1) for curves γ where $L_\gamma \not\equiv$ *constant* has not been as substantial as for the corresponding problem of Fourier restriction. The case for curves $\gamma(t) = (t, \phi(t))$ given as the graph of a convex function ϕ has been considered by Choi, Drury, Oberlin and Pan and the best result here is due to Oberlin [20] where the additional hypothesis that ϕ'' is monotone increasing is imposed and then only a weak-type estimate is obtained at the endpoint $(2/3, 1/3)$ (in [7] Choi obtained strong type estimates at $(2/3, 1/3)$ but these estimates are not universal – the constant C in (1) depends on ϕ – and in fact the author needs to impose much more stringent conditions on ϕ).

Compare this with the situation for the corresponding Fourier restriction problem in two dimensions where Sjölin [29] obtained uniform bounds over the class of *all* convex curves – see also [21]. The class of convex curves is a natural class to examine in light of simple counterexamples to (1) where L_γ changes sign too often (of course if γ is convex, L_γ does not change sign). By the above discussion on necessary conditions, we see that the endpoint estimate to aim for in (1) is $(2/3, 1/3)$ in two dimensions. Consider the curve γ given by $\gamma(t) = (t, t^k \sin(1/t))$. By testing (1) on $f = \chi_{D_\delta}$ where $D_\delta = \{(x, y)\colon |x| \leqslant \delta,\ |y| \leqslant \delta^k\}$ one easily shows that if (1) were to hold for this example, then $1/q \geqslant 1/p - (k-1)/3(k+1)$. Therefore if L_γ changes sign too often then (1) may not hold uniformly for all curves in the expected L^p range.

In [22] Oberlin established (1) in two dimensions for the family of polynomial curves $\gamma(t) = \mathbf{P}(t) = (P_1(t), P_2(t))$ where each P_1 and P_2 is a general real polynomial of bounded degree. Specifically he established (1) with a constant C only depending on the degrees of the polynomials defining $\mathbf{P}$. This is a natural class of curves to consider as the number of sign changes of $L_{\mathbf{P}}$ is controlled by the degree of the polynomials P_j. Furthermore Oberlin established (1) in three dimensions for polynomial curves of the form $\mathbf{P}(t) = (t, P_2(t), P_3(t))$ but the estimates are not universal in the sense that the constant C can be taken to depend only on the degrees of the polynomials. For the corresponding Fourier restriction problem in the setting of polynomial curves, see [2] and [14].

In this paper we give an alternative approach to the results in [22] and strengthen the three-dimensional result to general polynomial curves $\mathbf{P}(t) = (P_1(t), P_2(t), P_3(t))$; furthermore all estimates will be uniform over the class of polynomials of bounded degree. Our hope is that this approach will generalise to general polynomials curves in all dimensions.

From now on we shall focus on the operator

$$\mathcal{A}f(x) = \int\limits_I f\big(x - \mathbf{P}(t)\big)\big|L_{\mathbf{P}}(t)\big|^{\frac{2}{d(d+1)}}\,dt. \tag{2}$$

We are now ready to state our main result which is a global estimate.

Theorem 1. *Let $d = 2, 3$. Then for every $\epsilon > 0$,*

$$\|\mathcal{A}f\|_{L^{\frac{d^2+d}{2d-2}, \frac{d+1}{2}+\epsilon}(\mathbb{R}^d)} \leqslant C\|f\|_{L^{\frac{d+1}{2}}(\mathbb{R}^d)}$$

and

$$\|\mathcal{A}f\|_{L^{\frac{d+1}{d-1}, \frac{d^2+d}{d^2-d+2}+\epsilon}(\mathbb{R}^d)} \leqslant C\|f\|_{L^{\frac{d^2+d}{d^2-d+2}}(\mathbb{R}^d)},$$

where the constant C depends only on $\epsilon > 0$, the degrees of the polynomials defining the curve $\mathbf{P}$ *and in particular not on the parametrising interval I.*

When $d = 2$ there is just a single endpoint and the above two estimates agree. Here $L^{p,r}(\mathbb{R}^d)$ denote the familiar Lorentz spaces. As discussed above, it follows from Christ [9] that the p, q exponents in the estimates $\mathcal{A} : L^{p,r} \to L^{q,s}$ in Theorem 1 are best possible. Also, up to the ϵ factor, the r, s exponents are also best possible in general. In fact by considering the local operator $\mathcal{A}_{loc}$ (defined by restricting the integration to the unit interval) in the non-degenerate case $\mathbf{P}(t) = (t, t^2, \dots, t^d)$ an $L^{p,r}$ to $L^{q,s}$ estimate would give rise to an $L^{p,r} \to L^{p,s}$ estimate for $\mathcal{A}_{loc}$ and a result of A. Blozinski [4] states that there are no nontrivial positive[2] bounded linear translation invariant operators from $L^{p,r}$ to $L^{p,s}$ whenever $s < r$.[3]

Since C can be taken to be independent of I and $\mathcal{A}$ is a positive operator, Theorem 1 is equivalent to establishing the concluding estimates for the global analogue of $\mathcal{A}$ where the integration in (2) is replaced by the entire real line.

The proof of Theorem 1 combines an elegant combinatorial argument of Christ in [9], together with a recent geometric inequality for vector polynomials which was established in [14]. Christ's method is elementary but powerful and has seen applications outside the model curve case $(t, \dots, t^d)$ (see [3,11,18]) as well as substantial generalisations (see [12] and [31]). We mention again that Christ has developed a method that may be used to deduce strong-type estimates (even Lorentz type estimates) from restricted weak-type estimates (see [10]) and we will follow this method to deduce the Lorentz bounds in Theorem 1.

Finally, we wish to emphasise the fact that the result of Theorem 1 is obtained by using slightly different ingredients in different dimensions; whilst the basic techniques employed do not change, the relevant arguments need to be suitably adjusted. This is reflected in the structure of the paper: in the next section we recall the rudiments of Christ's argument in [9] followed by a description in Section 3 of the *key* geometric inequality for polynomial curves established in [14], an essential fact in our arguments. In Section 4 we deal with the restricted weak-type estimates in three dimensions, and in Section 5 we show how these can be turned into strong-type and indeed Lorentz-space estimates, again in three dimensions. In the last section we produce the necessary arguments needed to deal with the two-dimensional case.

Notation. Throughout this paper, whenever we write $A \lesssim B$ or $A = O(B)$ for any two nonnegative quantities A and B, we mean that there exists a strictly positive constant c, possibly depending on the degree of the map $\mathbf{P}$, so that $A \leqslant cB$; this constant is subject to change from line to line and even from step to step. We also write $A \sim B$ if $A \lesssim B \lesssim A$.

2. Rudiments of Christ's argument

For a nonnegative finite measure μ supported on an interval I and a curve parametrised by $\gamma : I \to \mathbb{R}^d$, consider the averaging operator

$$Af(x) = \int f\big(x - \gamma(t)\big)\, d\mu(t).$$

In this section we recall the basics of the combinatorial argument of Christ in [9] to prove a restricted weak-type estimate $A : L^{p,1}(\mathbb{R}^d) \to L^{q,\infty}(\mathbb{R}^d)$. This is equivalent to proving

[2] The assumption of positivity was removed by Cowling and Fournier in [13].

[3] We thank the referee for pointing out that in fact there is a general principle that if a bounded operator is translation *and* dilation invariant, then no $L^{p,r} \to L^{q,s}$ are possible when $s < r$.

$$\langle A\chi_E, \chi_F\rangle \lesssim |E|^{1/p}|F|^{1/q'} \tag{3}$$

for any two measurable sets $E, F \subset \mathbb{R}^d$ where $|\cdot|$ denotes the Lebesgue measure. Without loss of generality we may assume that $|E|$, $|F|$ and $\langle A\chi_E, \chi_F\rangle$ are all positive quantities. Define two positive parameters α and β by the relations

$$\alpha := \frac{1}{|F|}\langle A\chi_E, \chi_F\rangle, \qquad \beta := \frac{1}{|E|}\big\langle A^*\chi_F, \chi_E\big\rangle \quad \text{so that} \quad \alpha|F| = \beta|E|$$

where $A^*f(y) = \int f(y+\gamma(t))\,d\mu(t)$. Thus α is the average value of $A\chi_E$ on F and β is the average of $A^*\chi_F$ on E.

By passing to refinements of the sets E and F, without changing significantly the basic quantity $K := \langle A\chi_E, \chi_F\rangle = \langle \chi_E, A^*\chi_F\rangle$ to be estimated in (3), we will be able to bound pointwise $A\chi_E$ by α on F and bound pointwise $A^*\chi_F$ by β on E. Precisely one defines the following refinements of E and F:

$$F_1 = \big\{x \in F\colon\ A\chi_E(x) \geqslant \alpha/2\big\}, \qquad E_1 = \big\{y \in E\colon\ A^*\chi_{F_1}(y) \geqslant \beta/4\big\},$$
$$F_2 = \big\{x \in F_1\colon\ A\chi_{E_1}(x) \geqslant \alpha/8\big\}, \quad \ldots, \quad E_n = \big\{y \in E_{n-1}\colon\ A^*\chi_{F_n}(y) \geqslant \beta/2^{2n}\big\},$$

etc. It is a simple matter to check that $\langle A\chi_{E_n}, \chi_{F_n}\rangle \geqslant K/2^{2n}$ and $\langle \chi_{E_n}, A^*\chi_{F_{n+1}}\rangle \geqslant K/2^{2n+1}$ for each n and so $E_n, F_n \neq \emptyset$.

If $d = 3$, we fix an $x_0 \in F_2$, set $S = \{s \in I\colon\ x_0 - \gamma(s) \in E_1\}$ and note

$$\mu(S) = A\chi_{E_1}(x_0) \geqslant \alpha/8. \tag{4}$$

Next observe that for every $s \in S$, if $T_s = \{t \in I\colon\ x_0 - \gamma(s) + \gamma(t) \in F_1\}$, then

$$\mu(T_s) = A^*\chi_{F_1}\big(x_0 - \gamma(s)\big) \geqslant \beta/4. \tag{5}$$

Finally we see that for every $s \in S$ and $t \in T_s$, if $U_{s,t} = \{u \in I\colon\ x_0 - \gamma(s) + \gamma(t) - \gamma(u) \in E\}$, then

$$\mu(U_{s,t}) = A\chi_E\big(x_0 - \gamma(s) + \gamma(t)\big) \geqslant \alpha/2. \tag{6}$$

Hence we end up with a structured parameter domain $\mathcal{P} = \{(s,t,u) \in I^3\colon\ s \in S,\ t \in T_s,\ u \in U_{s,t}\}$ so that if $\Phi_\gamma(s,t,u) := x_0 - \gamma(s) + \gamma(t) - \gamma(u)$, $\Phi_\gamma(\mathcal{P}) \subset E$. Therefore if Φ_γ is injective we have

$$|E| \geqslant \iiint_{\mathcal{P}} \big|J_{\Phi_\gamma}(s,t,u)\big|\,ds\,dt\,du = \int_S \int_{T_s} \int_{U_{s,t}} \big|J_{\Phi_\gamma}(s,t,u)\big|\,ds\,dt\,du$$

where $J_{\Phi_\gamma}(s,t,u) = \det(\gamma'(s)\gamma'(t)\gamma'(u))$ is the determinant of the Jacobian matrix for the mapping Φ_γ, reducing matters to understanding the smallness of J_{Φ_γ} (for instance, sublevel sets of J_{Φ_γ}) in order to bound from below the above integral over the structured set $\mathcal{P}$. If $\gamma(t) = (t, t^2, t^3)$ (the non-degenerate example in three dimensions) and $\mu = |\cdot|$ is the Lebesgue measure, then $J_{\Phi_\gamma}(s,t,u) = 6(s-t)(t-u)(s-u)$ and so (4), (5) and (6) quickly imply $|E| \geqslant \beta^2\alpha^4$ which gives (3) with $p=2$ and $q=3$, the desired endpoint estimate in this case.

If $d=2$, we fix a $y_0 \in E_1$, set $S = \{s \in I\colon\ y_0 + \gamma(s) \in F_1\}$ and note

$$\mu(S) = A^*\chi_{E_1}(y_0) \geqslant \beta/4. \tag{7}$$

Next observe that for every $s \in S$, if $T_s = \{t \in I\colon\ y_0 + \gamma(s) - \gamma(t) \in E\}$, then

$$\mu(T_s) = A\chi_E\big(y_0 + \gamma(s)\big) \geqslant \alpha/2. \tag{8}$$

Hence we end up with a structured parameter domain $\mathcal{P} = \{(s,t) \in I^2\colon\ s \in S,\ t \in T_s\}$ so that if $\Phi_\gamma(s,t) := y_0 + \gamma(s) - \gamma(t)$, $\Phi_\gamma(\mathcal{P}) \subset E$. Therefore if Φ_γ is injective we have

$$|E| \geqslant \iint\limits_{\mathcal{P}} \big|J_{\Phi_\gamma}(s,t)\big|\,ds\,dt = \int\limits_S \int\limits_{T_s} \big|J_{\Phi_\gamma}(s,t)\big|\,ds\,dt$$

where $J_{\Phi_\gamma}(s,t) = -\det(\gamma'(s)\gamma'(t))$. If $\gamma(t) = (t,t^2)$ (the non-degenerate example in two dimensions) and $\mu = |\cdot|$ is the Lebesgue measure, then $J_{\Phi_\gamma}(s,t) = 2(s-t)$ and so (7), (8) imply $|E| \geqslant \beta\alpha^2$ which gives (3) with $p = 3/2$ and $q=3$, the desired endpoint estimate in this case.

When we consider a general polynomial curve $\gamma(t) = \mathbf{P}(t) = (P_1(t), P_2(t))$ in two dimensions with μ the affine arclength measure on $\mathbf{P}$, we will only be able to prove

$$\iint\limits_{\mathcal{P}} \big|J_{\Phi_\gamma}(s,t)\big|\,ds\,dt = \int\limits_S \int\limits_{T_s} \big|J_{\Phi_\gamma}(s,t)\big|\,ds\,dt \geqslant \beta\alpha^2 \tag{9}$$

in the range $\alpha \leqslant \beta$. In fact, without further refinements in the argument (see, for example, [9]), this integral bound can be false in the range $\beta \leqslant \alpha$. Nevertheless, due to the fact that the sharp endpoint estimate lies on the line of duality $L^p \to L^{p'}$, it will be the case that $|E| \geqslant \beta\alpha^2$ for *all* α, β. Knowing only (9) in the range $\beta \leqslant \alpha$ leads to some further difficulties when establishing the Lorentz bounds and these difficulties do not present themselves in the three-dimensional case. This is why we choose to address the three-dimensional case first.

3. A geometric inequality

As we have seen in the previous section, Christ's argument in [9] is based in part on analysis of the map

$$\Phi_{\mathbf{P}}(t_1,\ldots,t_d) = (-1)^d\mathbf{P}(t_1) + (-1)^{d+1}\mathbf{P}(t_2) + \cdots - \mathbf{P}(t_d).$$

In particular it would be desirable to have the following properties about $\Phi_{\mathbf{P}}$:

Key properties

(a) $\Phi_{\mathbf{P}}$ is 1–1;

(b) $|J_{\Phi_{\mathbf{P}}}(t_1,\ldots,t_d)| \geqslant C\prod_{j=1}^d |L_{\mathbf{P}}(t_j)|^{\frac{1}{d}} \prod_{j<k} |t_j - t_k|$

where $J_{\Phi_{\mathbf{P}}}(t_1,\ldots,t_d) = \pm\det(\mathbf{P}'(t_1)\cdots\mathbf{P}'(t_d))$ *is the determinant of the Jacobian matrix for the mapping* $\Phi_{\mathbf{P}}$ *and* $L_{\mathbf{P}}(t) = \det(\mathbf{P}'(t)\cdots\mathbf{P}^{(d)}(t))$ *was introduced in the introduction as part of the definition of the affine arclength measure along* $\mathbf{P}$.

As we have seen the injectivity of $\Phi_{\mathbf{P}}$ allows us to reduce matters to examining integrals of $J_{\Phi_{\mathbf{P}}}$ over various structured sets of $(t_1, \ldots, t_d)$. And then the geometric inequality, property (b), will make the examination of these integrals feasible. Even in the non-degenerate case $\mathbf{P}(t) = (t, t^2, \ldots, t^d)$, $\Phi_{\mathbf{P}}$ is not quite 1–1 but it is $d!$ to 1 off a set of measure zero. Furthermore in this case, the geometric inequality (b) is an equality.

For polynomial curves both (a) and (b) are false in general. However in [14], a collection of $O(1)$ disjoint open intervals $\{I\}$ was found which decomposes $\mathbb{R} = \bigcup \overline{I}$ so that on each I^d, $\Phi_{\mathbf{P}}$ is $d!$ to 1 off a set of measure zero and the geometric inequality (b) holds. With this decomposition we will restrict our original operator $\mathcal{A}$ to each interval I and apply Christ's argument. The decomposition is valid only under the assumption that $L_{\mathbf{P}} \not\equiv 0$. Of course if $L_{\mathbf{P}} \equiv 0$, then the estimates in (1) are trivial and so, without loss of generality, the non-degeneracy assumption $L_{\mathbf{P}} \not\equiv 0$ will be in force for the remainder of the paper.

The decomposition is produced in two stages. The first stage produces an elementary decomposition of $\mathbb{R} = \bigcup \overline{J}$ so that on each open interval J, various polynomial quantities (more precisely, certain determinants of minors of the $d \times d$ matrix $(\mathbf{P}'(t) \cdots \mathbf{P}^{(d)})$, including $L_{\mathbf{P}}$) are single-signed. This allows us to write down a formula relating $J_{\Phi_{\mathbf{P}}}$ and $L_{\mathbf{P}}$. When $d = 2$ this formula is particularly simple; namely,

$$J_{\Phi_{\mathbf{P}}}(s,t) = P_1'(s)P_1'(t) \int_s^t \frac{L_{\mathbf{P}}(w)}{P_1'(w)^2}\, dw$$

for any $s, t \in J$ (here $\mathbf{P} = (P_1, P_2)$). From this, one can establish the injectivity of $\Phi_{\mathbf{P}}$ on $\{(t_1, \ldots, t_d) \in J^d\colon\ t_1 < \cdots < t_d\}$. Next we decompose each $\overline{J} = \bigcup \overline{I}$ further so that on each open interval I, (b) holds. More precisely, we have inequality (b) for all $(t_1, \ldots, t_d) \in I^d$ where C depends only on d and the degrees of the polynomials defining $\mathbf{P}$.

This second stage decomposition $\overline{J} = \bigcup \overline{I}$ is more technical and derived from a certain algorithm which uses two further decomposition procedures generated by individual polynomials. These further decomposition procedures are used in tandem and have the effect of reducing (2) to open intervals I on which various polynomials, including $L_{\mathbf{P}}$, behave like a centred monomial. Furthermore the algorithm exploits in a crucial way the affine invariance of the inequality (b); that is, the inequality is invariant under replacement of $\mathbf{P}$ by $A\mathbf{P}$ for any invertible $d \times d$ matrix A.

To recapitulate, in [14] a decomposition $\mathbb{R} = \bigcup \overline{I}$ where $\{I\}$ is an $O(1)$ collection of open disjoint intervals was produced so that the following three properties hold for each I:

(P1) the map $\Phi_{\mathbf{P}}$ is 1–1 on the region $D = \{(t_1, \ldots, t_d) \in I^d\colon\ t_1 < \cdots < t_d\}$;

(P2) for $t \in I$, $|L_{\mathbf{P}}(t)| \sim A_I |t - b_I|^{k_I}$ for some $A_I > 0$, $b_I \notin I$ and a nonnegative integer k_I which is bounded above by a constant only depending on the degrees of the polynomials defining $\mathbf{P}$;

(P3) for $(t_1, \ldots, t_d) \in I^d$,

$$\left|J_{\Phi_\Gamma}(t_1, \ldots, t_d)\right| \geqslant C \prod_{j=1}^{d} \left|L_\Gamma(t_j)\right|^{\frac{1}{d}} \prod_{j<k} |t_j - t_k|$$

where C depends only on d and the degrees of the polynomials defining $\mathbf{P}$.

In the following we will assume that the constant A_I, which appears in (P2) is equal to 1. This assumption is justified because, after performing the above decomposition, one can make a reparameterisation $s = ct$, where $c = A_I^{2/[2k_I+d(d+1)]}$, on each I. As a consequence of this reparameterisation, we then have to consider a polynomial $Q(s) = P(s/c)$ defined on a scaled interval cI. This choice of c implies that for $s \in cI$, $|L_{\mathbf{Q}}(s)| \sim |s - cb_I|^{k_I}$. The geometric inequality in (P3) still holds for Q on the scaled interval and, since our estimates will only depend on the degree of Q, we can carry out our arguments with Q in the place of P.

4. Restricted weak-type estimates

As mentioned above it suffices to carry out our analysis for the globally defined operator

$$\mathcal{A}_{\mathbb{R}} f(x) = \int_{\mathbb{R}} f\big(x - \mathbf{P}(t)\big)\big|L_{\mathbf{P}}(t)\big|^{\frac{2}{d(d+1)}}\,dt, \tag{10}$$

and we begin by proving the desired restricted weak-type estimates. We have the following.

Theorem 2. *Let $d = 3$; the operator* (10) *satisfies*

$$\mathcal{A}_{\mathbb{R}} : L^{2,1}\big(\mathbb{R}^3\big) \to L^{3,\infty}\big(\mathbb{R}^3\big), \tag{11}$$

$$\mathcal{A}_{\mathbb{R}} : L^{3/2,1}\big(\mathbb{R}^3\big) \to L^{2,\infty}\big(\mathbb{R}^3\big), \tag{12}$$

where the bounds depend only on the degree of $\mathbf{P}$.

Proof. By duality it suffices to establish just one of these estimates, say (11), and as we have seen in Section 2, this in turn is equivalent to proving

$$\langle \mathcal{A}_{\mathbb{R}} \chi_E, \chi_F\rangle \lesssim |E|^{1/2}|F|^{2/3} \tag{13}$$

for all pairs of measurable sets $E, F \subset \mathbb{R}^3$. We now apply the decomposition procedure described in Section 3 to the vector polynomial $\mathbf{P}(t) = (P_1(t), P_2(t), P_3(t))$, decomposing $\mathbb{R} = \bigcup \bar{I}$ into $O(1)$ disjoint open intervals $\{I\}$ so that for each I, properties (P1), (P2) and (P3) hold.

For each I, we define the measure μ_I by

$$\mu_I(J) = \int_J |t - b_I|^{k_I/6}\,dt.$$

We need only consider the operator

$$\mathcal{A}_I f(x) = \int_I f\big(x - \mathbf{P}(t)\big)|t - b|^{k/6}\,dt = \int_I f\big(x - \mathbf{P}(t)\big)\,d\mu(t),$$

and prove (13) for $\mathcal{A}_I$, uniformly in I. Here $b = b_I \notin I$, $k = k_I$ is some nonnegative integer and $\mu = \mu_I$ is a measure supported in I. Introducing the positive parameters $\alpha = \alpha_I$ and $\beta = \beta_I$ as in Section 2, we see that

$$\left|\langle \mathcal{A}_I \chi_E, \chi_F\rangle\right| \lesssim |E|^{1/2}|F|^{2/3} \quad \Leftrightarrow \quad |E| \gtrsim \alpha^4\beta^2, \tag{14}$$

uniformly in I. From Section 2, we see that there is a point $x_0 \in F$ and

$S \subset I$ so that $\mu(S) \gtrsim \alpha$;

for each $s \in S$ there is a $T_s \subset I$ so that $\mu(T_s) \gtrsim \beta$;

for each $t \in T_s$ there is a $U_{s,t} \subset I$ so that $\mu(U_{s,t}) \gtrsim \alpha$;

if $\mathcal{P} = \{(s,t,u) \in I^3\colon\ s \in S,\ t \in T_s,\ u \in U_{s,t}\}$ then $x_0 + \Phi_{\mathbf{P}}(\mathcal{P}) \subset E$.

Thanks to these properties, as well as (P1), (P2) and (P3), we have the bound

$$\begin{aligned} |E| &\gtrsim \iiint\limits_{\mathcal{P}} \left|J_{\Phi_{\mathbf{P}}}(s,t,u)\right| ds\,dt\,du \\ &\gtrsim \int\limits_S |s-b|^{k/3} \int\limits_{T_s} |t-b|^{k/3}|s-t| \int\limits_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds. \end{aligned} \tag{15}$$

To estimate the last integral from below, we will split our argument into a number of cases, depending on the relative sizes of the factors $|s-b|$, $|t-b|$, $|u-b|$, $|s-t|$, $|s-u|$ and $|t-u|$ appearing the integrand. By a simple pigeonhole argument (and restricting to a subset of S with μ-mass still at least α if necessary) we may assume that either

$$\begin{aligned} &\mu\big(T_s \cap \{t \in I\colon\ |t-b| \leqslant (1/8)|s-b|\}\big) \gtrsim \beta \quad \text{for all } s \in S, \quad \text{or} \\ &\mu\big(T_s \cap \{t \in I\colon\ (1/8)|s-b| < |t-b| \leqslant 2|s-b|\}\big) \gtrsim \beta \quad \text{for all } s \in S, \quad \text{or} \\ &\mu\big(T_s \cap \{t \in I\colon\ |t-b| \geqslant 2|s-b|\}\big) \gtrsim \beta \quad \text{for all } s \in S. \end{aligned}$$

Therefore, without loss of generality (by restricting further each T_s to one of the above subsets), we may assume either

- $|t-b| \leqslant (1/8)|s-b|$ holds on T_s for each $s \in S$, or
- $(1/8)|s-b| < |t-b| \leqslant 2|s-b|$ holds on T_s for each $s \in S$, or
- $|t-b| \geqslant 2|s-b|$ holds on T_s for each $s \in S$.

These will make up our three basic cases; each case will be split further into three subcases. In each case above, by a similar pigeonhole argument, we may assume (again restricting to subsets of $U_{s,t}$ if necessary) either

- $|u-b| \leqslant (1/4)|t-b|$ holds on $U_{s,t}$ for every $s \in S$ and $t \in T_s$, or
- $(1/4)|t-b| < |u-b| \leqslant 4|t-b|$ holds on $U_{s,t}$ for every $s \in S$ and $t \in T_s$, or
- $|u-b| \geqslant 4|t-b|$ holds on $U_{s,t}$ for every $s \in S$ and $t \in T_s$.

Our goal is to establish the uniform bound

$$\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \gtrsim \alpha^4\beta^2, \tag{16}$$

in each of the 3×3 cases above.

To establish (16) we will need to excise various intervals from subsets of S, T_s and $U_{s,t}$ without changing their μ measures significantly. For this purpose we introduce the following dynamic notation.

- For $\delta > 0$, let $B_\alpha = \{u \in I\colon\ |u-b| \leqslant \delta\alpha^{6/(k+6)}\}$ so that $\mu(B_\alpha) \leqslant c_k\delta^{(k+6)/6}\alpha$. We will choose $\delta > 0$ to be sufficiently small in each instance so that the following holds: if $W \subset I$ is a set satisfying $\mu(W) > c_0\alpha$ for some $c_0 > 0$, then $\mu(W \setminus B_\alpha) \geqslant (c_0/2)\alpha$.
- For $\delta > 0$ and t, set $B_{t,\alpha} = \{u \in I\colon\ |u-t| \leqslant \delta\alpha|t-b|^{-k/6}\}$.
  - If for all $u \in W \subset I$, $|u-b| \leqslant C_0|t-b|$, then $\mu(W \cap B_{t,\alpha}) \leqslant 2C_0^{k/6}\delta\alpha$ (in fact $\int_{W\cap B_{t,\alpha}} |u-b|^{k/6}\,du \leqslant C_0^{k/6}|t-b|^{k/6}\mu(B_{t,\alpha}) \leqslant 2C_0^{k/6}\delta\alpha$) and therefore if $\mu(W) \geqslant c_0\alpha$, we have $\mu(W \setminus B_{t,\alpha}) \geqslant (c_0/2)\alpha$ if $\delta > 0$ is chosen sufficiently small.
  - On the other hand, if we do not know a priori that $|u-b| \leqslant C_0|t-b|$ on W but we happen to know $|t-b| \geqslant C_0\alpha^{6/(k+6)}$, then automatically we have the control $|u-b| \lesssim |t-b|$ on $B_{t,\alpha}$ since $|t-b| \geqslant C_0\alpha^{6/(k+6)}$ implies $\alpha|t-b|^{-k/6} \lesssim |t-b|$ and thus $|u-t| \lesssim |t-b|$ on $B_{t,\alpha}$.

Case 1. On T_s, $|t-b| \leqslant (1/8)|s-b|$ holds; note then that $|s-t| \sim |s-b|$.

Case (1a). On $U_{s,t}$, $|u-b| \leqslant (1/4)|t-b|$ holds; note then that $|u-t| \sim |t-b|$ and $|u-s| \sim |s-b|$. Thus

$$\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \\
&\quad\sim \int_S |s-b|^{k/3+2} \int_{T_s} |t-b|^{k/3+1} \int_{U_{s,t}} |u-b|^{k/3}\,du\,dt\,ds \\
&\quad\geqslant \int_{S\setminus B_\alpha} |s-b|^{k/6+k/6+2} \int_{T_s\setminus B_\beta} |t-b|^{k/6+k/6+1} \int_{U_{s,t}\setminus B_\alpha} |u-b|^{k/6+k/6}\,du\,dt\,ds.
\end{aligned}$$

Now choosing $\delta > 0$ in each B_α, B_β to ensure that the μ measures of the above sets have not been altered significantly, and using the fact that on $U_{s,t} \setminus B_\alpha$ we have $|u-b| \gtrsim \alpha^{\frac{6}{k+6}}$ (as well as analogous estimates on $T_s \setminus B_\beta$ and $S \setminus B_\alpha$), we see that the last iterated integral is bounded below by a constant multiple of

$$\alpha^{\frac{6}{k+6}(k/6+2)} \times \alpha \times \beta^{\frac{6}{k+6}(k/6+1)} \times \beta \times \alpha^{\frac{k}{k+6}} \times \alpha = \alpha^4\beta^2.$$

Case (1b). On $U_{s,t}$, $(1/4)|t-b| \leqslant |u-b| \leqslant 4|t-b|$ holds; here then $|u-s| \sim |s-b|$ but now $|u-t|$ may vanish. Then

$$\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3} |s-t| \int_{U_{s,t}} |u-b|^{k/3} |u-s||u-t| \, du\, dt\, ds \\
&\quad \sim \int_S |s-b|^{k/3+2} \int_{T_s} |t-b|^{k/3} \int_{U_{s,t}} |u-b|^{k/3} |u-t| \, du\, dt\, ds \\
&\quad \geqslant \int_S |s-b|^{k/3+2} \int_{T_s} |t-b|^{k/3} \int_{U_{s,t} \setminus B_{t,\alpha}} |u-b|^{k/3} |u-t| \, du\, dt\, ds,
\end{aligned}$$

and using that on $U_{s,t} \setminus B_{t,\alpha}$ one has $|u-t| \gtrsim \alpha |t-b|^{-k/6}$ (together with the fact that $|s-b| \gtrsim |t-b|$ and $|u-b| \sim |t-b|$ in this case) this last quantity is bounded below by

$$\begin{aligned}
&\alpha \int_S |s-b|^{k/3+2} \int_{T_s} |t-b|^{k/6} \int_{U_{s,t} \setminus B_{t,\alpha}} |u-b|^{k/3} \, du\, dt\, ds \\
&\quad \gtrsim \alpha \int_{S \setminus B_\alpha} |s-b|^{k/3+1} \int_{T_s \setminus B_\beta} |t-b|^{k/3+1} \int_{(U_{s,t} \setminus B_{t,\alpha}) \setminus B_\alpha} |u-b|^{k/6} \, du\, dt\, ds.
\end{aligned}$$

Since $|u-b| \leqslant 2|t-b|$ on $U_{s,t}$, we see that we can choose $\delta > 0$ in each B_α, B_β and $B_{t,\alpha}$ so as not to change the μ measures much when we excise these intervals from S, T_s and $U_{s,t}$. Therefore the last iterated integral above is bounded below by a constant times $\alpha \times \alpha^2 \times \beta^2 \times \alpha = \alpha^4 \beta^2$ (here we have used the fact, and will continue to do so, that for any set $E \subset \mathbb{R}$, $\int_E |t-b|^{(k/3)+1} \, dt \gtrsim \mu(E)^2$).

Case (1c). On $U_{s,t}$, $|u-b| \geqslant 4|t-b|$; here $|u-t| \sim |u-b|$ but now $|u-s|$ may vanish. Then

$$\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3} |s-t| \int_{U_{s,t}} |u-b|^{k/3} |u-s||u-t| \, du\, dt\, ds \\
&\quad \gtrsim \int_{S \setminus B_\alpha} |s-b|^{k/3+1} \int_{T_s} |t-b|^{k/3} \int_{U_{s,t} \setminus B_{s,\alpha}} |u-b|^{k/3+1} |u-s| \, du\, dt\, ds \\
&\quad \gtrsim \alpha \int_{S \setminus B_\alpha} |s-b|^{k/6+1} \int_{T_s} |t-b|^{k/3} \int_{U_{s,t} \setminus B_{s,\alpha}} |u-b|^{k/3+1} \, du\, dt\, ds \\
&\quad \gtrsim \alpha \int_{S \setminus B_\alpha} |s-b|^{k/6} \int_{T_s \setminus B_\beta} |t-b|^{k/3+1} \int_{(U_{s,t} \setminus B_{s,\alpha}) \setminus B_\alpha} |u-b|^{k/3+1} \, du\, dt\, ds.
\end{aligned}$$

Since we do have the control $|u-b| \lesssim |s-b|$ on $B_{s,\alpha}$ (since for $s \in S \setminus B_\alpha$, $|s-b| \gtrsim \alpha^{6/(k+6)}$), we see that by appropriate choices of $\delta > 0$ in B_α, B_β and $B_{s,\alpha}$, the above sets do not change in μ measure. Thus the final iterated integral is at least a constant multiple of $\alpha \times \alpha \times \beta^2 \times \alpha^2 = \alpha^4 \beta^2$.

Case 2. On T_s, $(1/8)|s-b| \leqslant |t-b| \leqslant 2|s-b|$ holds.

Case (2a). On $U_{s,t}$, $|u-b| \leqslant (1/4)|t-b|$ holds; here then $|u-t| \sim |t-b|$, and we may also deduce $|u-s| \sim |s-b|$. Since $|t-b| \sim |s-b|$,

$$\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \\
&\quad \gtrsim \int_S |s-b|^{k/3+1} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/3+1}|s-t| \int_{U_{s,t}} |u-b|^{k/3}\,du\,dt\,ds \\
&\quad \gtrsim \beta \int_S |s-b|^{k/6+1} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/3+1} \int_{U_{s,t}} |u-b|^{k/3}\,du\,dt\,ds \\
&\quad \gtrsim \beta \int_{S\setminus B_\alpha} |s-b|^{k/3+2} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/6} \int_{U_{s,t}\setminus B_\alpha} |u-b|^{k/3}\,du\,dt\,ds.
\end{aligned}$$

Again since $|t-b| \lesssim |s-b|$, appropriate choices of $\delta > 0$ can be made so as not to change the μ measures of S, T_s and $U_{s,t}$ when we excise from them the above intervals. Hence the last iterated integral is bounded below by a constant multiple of

$$\beta \times \alpha \times \alpha^{\frac{6}{k+6}(k/6+2)} \times \beta \times \alpha \times \alpha^{\frac{6}{k+6}(k/6)} = \alpha^4\beta^2.$$

Case (2b). On $U_{s,t}$, $(1/4)|t-b| \leqslant |u-b| \leqslant 4|t-b|$ holds. Here we may compare all quantities containing b; namely $|s-b| \sim |t-b| \sim |u-b|$. Hence

$$\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \\
&\quad \gtrsim \int_S |s-b|^{k/2} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/3}|s-t| \int_{U_{s,t}\setminus(B_{t,\alpha}\cup B_{s,\alpha})} |u-b|^{k/6}|u-t||u-s|\,du\,dt\,ds \\
&\quad \gtrsim \beta\alpha^2 \int_{S\setminus B_\alpha} |s-b|^{k/6} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/6} \int_{U_{s,t}\setminus(B_{t,\alpha}\cup B_{s,\alpha})} |u-b|^{k/6}\,du\,dt\,ds.
\end{aligned}$$

Again we see that the sets we are integrating over have not changed in μ measure much when we remove intervals and so the last iterated integral is at least a constant times $\beta\alpha^2 \times \alpha \times \beta \times \alpha = \alpha^4\beta^2$.

Case (2c). On $U_{s,t}$, $|u-b| \geqslant 4|t-b|$; here $|u-t| \sim |u-b|$ but $|u-s|$ and $|t-s|$ may vanish. Since $|u-b| \gtrsim |s-b| \sim |t-b|$,

$$\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds$$

$$\gtrsim \int_{S\setminus B_\alpha} |s-b|^{k/3} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/3}|s-t| \int_{U_{s,t}\setminus B_{s,\alpha}} |u-b|^{k/3+1}|u-s|\,du\,dt\,ds$$

$$\gtrsim \alpha\beta \int_{S\setminus B_\alpha} |s-b|^{k/3+1} \int_{T_s\setminus B_{s,\beta}} |t-b|^{k/6} \int_{U_{s,t}\setminus B_{s,\alpha}} |u-b|^{k/6}\,du\,dt\,ds.$$

One checks that removing B_α, $B_{s,\beta}$ and $B_{s,\alpha}$ has not changed the μ measures of our sets very much and so this last iterated integral is at least a constant times $\alpha\beta \times \alpha^2 \times \beta \times \alpha = \alpha^4\beta^2$.

Case 3. On T_s, $|t-b| \geqslant 2|s-b|$ holds; in this case $|t-s| \sim |t-b|$.

Case (3a). On $U_{s,t}$, $|u-b| \leqslant (1/4)|t-b|$ holds; here $|t-u| \sim |t-b|$ but $|u-s|$ may vanish. Since $|t-b| \gtrsim |s-b|$,

$$\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds$$

$$\gtrsim \int_{S\setminus B_\alpha} |s-b|^{k/3} \int_{T_s\setminus B_\beta} |t-b|^{k/3+2} \int_{U_{s,t}\setminus B_{s,\alpha}} |u-b|^{k/3}|u-s|\,du\,dt\,ds$$

$$\gtrsim \alpha \int_{S\setminus B_\alpha} |s-b|^{k/6+1} \int_{T_s\setminus B_\beta} |t-b|^{k/3+1} \int_{(U_{s,t}\setminus B_{s,\alpha})\setminus B_\alpha} |u-b|^{k/3}\,du\,dt\,ds.$$

Again the removal of intervals has not changed significantly the μ measures and so the last iterated integral is at least a constant multiple of $\alpha \times \alpha^{k/(k+6)+1} \times \beta^2 \times \alpha^{6/(k+6)+1} = \alpha^4\beta^2$.

Case (3b). On $U_{s,t}$, $(1/4)|t-b| \leqslant |u-b| \leqslant 4|t-b|$ holds; here $|s-u| \sim |u-b|$ but $|u-t|$ can vanish. Since $|u-b| \sim |t-b|$,

$$\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds$$

$$\gtrsim \int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3+1} \int_{U_{s,t}\setminus B_{t,\alpha}} |u-b|^{k/3+1}|u-t|\,du\,dt\,ds$$

$$\gtrsim \alpha \int_{S\setminus B_\alpha} |s-b|^{k/3} \int_{T_s\setminus B_\beta} |t-b|^{k/3+1} \int_{U_{s,t}\setminus B_{t,\alpha}} |u-b|^{k/6+1}\,du\,dt\,ds,$$

and as before we see that the last iterated integral is bounded below by a constant times $\alpha \times \alpha^{k/(k+6)+1} \times \beta^2 \times \alpha^{6/(k+6)+1} = \alpha^4\beta^2$.

Case (3c). On $U_{s,t}$, $|u-b| \geqslant 4|t-b|$; here we may deduce that $|u-t| \sim |u-b|$ and $|u-s| \sim |u-b|$. Thus

$$\int\limits_S |s-b|^{k/3} \int\limits_{T_s} |t-b|^{k/3}|s-t| \int\limits_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds$$
$$\gtrsim \int\limits_{S\setminus B_\alpha} |s-b|^{k/3} \int\limits_{T_s\setminus B_\beta} |t-b|^{k/3+1} \int\limits_{U_{s,t}\setminus B_\alpha} |u-b|^{k/3+2}\,du\,dt\,ds,$$

and as before this last iterated integral is at least a constant multiple of $\alpha \times \alpha^{\frac{6}{k+6}(k/6)} \times \beta^2 \times \alpha \times \alpha^{\frac{6}{k+6}(k/6+2)} = \alpha^4\beta^2$.

This completes the bound for (15) and thus proves (14), completing the proof of Theorem 2. □

5. Strong-type inequalities

We now wish to complete the proof of Theorem 1 when $d=3$. We shall suitably modify the arguments in [10] in order to achieve this goal. We will concentrate only on the first estimate stated in Theorem 1 and thanks to our geometric inequality and previous arguments, we just have to show that the operator $\mathcal{A}_I : L^2(\mathbb{R}^3) \to L^{3,2+\epsilon}(\mathbb{R}^3)$, uniformly in I. This is equivalent to showing

$$\bigl|\langle \mathcal{A}_I f, g\rangle\bigr| \leqslant C_\epsilon \|f\|_2 \|g\|_{3/2,2-\epsilon} \quad \text{for any } f \in L^2\bigl(\mathbb{R}^3\bigr),\ g \in L^{3/2,2-\epsilon}\bigl(\mathbb{R}^3\bigr). \tag{17}$$

Following [10], it suffices to select f, g of the form

$$f = \sum_{\ell\in\mathbb{Z}} 2^\ell \chi_{E_\ell}, \qquad g = \sum_{m\in\mathbb{Z}} 2^m \chi_{F_m},$$

where the sets E_ℓ are pairwise disjoint and so are the sets F_m. However, we shall specialise further, and pick the function $g = g_0$ to be simply the characteristic function of a measurable set, $g_0 := \chi_F$. If we prove estimate (17) with g replaced by g_0, we then have an $L^2 \to L^{3,\infty}$ bound; one can then use Christ's arguments to turn this into the claimed Lorentz space bound. We may normalise the L^2 norm of f, so that $\sum_\ell 2^{2\ell}|E_\ell| = 1$, and then the desired $L^2 \to L^{3,\infty}$ bound becomes

$$\sum_\ell 2^\ell \langle \mathcal{A}_I \chi_{E_\ell}, \chi_F\rangle \lesssim |F|^{2/3}. \tag{18}$$

We decompose the ℓ sum above in order to stabilise certain quantities. For dyadic numbers $\epsilon, \eta \in (0, 1/2]$ we define $L_{\epsilon,\eta}$ to be those ℓ where

$$|E_\ell| \sim \eta 2^{-2\ell} \quad \text{and} \quad \langle \mathcal{A}_I \chi_{E_\ell}, \chi_F\rangle \sim \epsilon |E_\ell|^{1/2}|F|^{2/3}.$$

The number M of indices ℓ in $L_{\epsilon,\eta}$ is therefore finite and satisfies $M\eta \lesssim 1$. Our aim is then to prove

$$\sum_{\ell\in L_{\epsilon,\eta}} 2^{\ell}\langle \mathcal{A}_I\chi_{E_\ell},\chi_F\rangle \lesssim \min\bigl(\epsilon^a,\eta^b\bigr)|F|^{2/3} \tag{19}$$

for some positive exponents a, b. By summing over the dyadic ϵ and η, we see that (19) implies (18).

Next we may assume that $|i-j| \geqslant C\log(1/\epsilon)$ for any two distinct indices appearing in the sum over $L_{\epsilon,\eta}$ where $C>0$ will be an absolute constant.[4] One now defines sets

$$G_\ell = \bigl\{x\in F\colon \mathcal{A}_I\chi_{E_\ell} \geqslant c_0|E_\ell|^{1/2}|F|^{2/3}|F|^{-1}\bigr\},$$

for a certain $c_0>0$. If c_0 is chosen sufficiently small, then $\langle \mathcal{A}_I\chi_{E_\ell},\chi_{F\setminus G_\ell}\rangle \leqslant 1/2\langle \mathcal{A}_I\chi_{E_\ell},\chi_F\rangle$ and so $\langle \mathcal{A}_I\chi_{E_\ell},\chi_{G_\ell}\rangle \sim \langle \mathcal{A}_I\chi_{E_\ell},\chi_F\rangle$. By Theorem 2 we have $\langle \mathcal{A}_I\chi_{E_\ell},\chi_{G_\ell}\rangle \lesssim |E_\ell|^{1/2}|G_\ell|^{2/3}$ and so

$$|G_\ell| \gtrsim \epsilon^{3/2}|F|. \tag{20}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\Bigl(|F|^{-1}\sum_{\ell\in L_{\epsilon,\eta}}|G_\ell|\Bigr)^2 &\leqslant |F|^{-1}\int_F\Bigl(\sum_{\ell\in L_{\epsilon,\eta}}\chi_{G_\ell}\Bigr)^2\\ &\leqslant |F|^{-1}\sum_{\ell\in L_{\epsilon,\eta}}|G_\ell| + |F|^{-1}\sum_{k\neq\ell}|G_k\cap G_\ell|\end{aligned}$$

and therefore either $(|F|^{-1}\sum_{\ell\in L_{\epsilon,\eta}}|G_\ell|)^2 \lesssim |F|^{-1}\sum_{k\neq\ell}|G_k\cap G_\ell|$ holds or we have $\sum_{\ell\in L_{\epsilon,\eta}}|G_\ell| \lesssim |F|$. If the former holds, then by (20)

$$\bigl(M\epsilon^{3/2}\bigr)^2 \lesssim \Bigl(\sum_{\ell\in L_{\epsilon,\eta}}|G_\ell|\Bigr)^2 \lesssim M^2|F|^{-1}\max_{k\neq\ell}|G_k\cap G_\ell|$$

and the above dichotomy becomes

$$\text{either} \sum_{\ell\in L_{\epsilon,\eta}}|G_\ell| \lesssim |F| \tag{21}$$

$$\text{or there exist } i\neq j \text{ so that } |G_i\cap G_j| \gtrsim \epsilon^3|F|. \tag{22}$$

[4] By splitting the sum over $L_{\epsilon,\eta}$ into $O(C\log(1/\epsilon))$ sums, this assumption will cost us only a factor of $O(C\log(1/\epsilon))$ in the estimate (19).

The key is now to show that (22) leads to a contradiction; this implies that (21) holds, and therefore

$$\sum_{\ell\in L_{\epsilon,\eta}} 2^{\ell}\langle \mathcal{A}_I\chi_{E_\ell},\chi_F\rangle \sim \sum_{\ell\in L_{\epsilon,\eta}} 2^{\ell}\langle \mathcal{A}_I\chi_{E_\ell},\chi_{G_\ell}\rangle \lesssim \Big(\sum_{\ell\in L_{\epsilon,\eta}} 2^{3\ell}|E_\ell|^{3/2}\Big)^{1/3}\Big(\sum_{\ell\in L_{\epsilon,\eta}} |G_\ell|\Big)^{2/3} \lesssim \eta^{1/6}|F|^{2/3}.$$

On the other hand,

$$\sum_{\ell\in L_{\epsilon,\eta}} 2^{\ell}\langle \mathcal{A}_I\chi_{E_\ell},\chi_F\rangle \sim \sum_{\ell\in L_{\epsilon,\eta}} 2^{\ell}\epsilon|E_\ell|^{1/2}|F|^{2/3} \lesssim \epsilon M\eta^{1/2}|F|^{2/3} \lesssim \epsilon\eta^{1/2}|F|^{2/3}$$

and these two estimates together imply (19).

To disprove (22) we need the following result.

Lemma 1. *There exists a* finite *set of pairs (A,B) satisfying $1\leqslant A<2$, $2<B\leqslant 3$ and $A+B=4$ so that whenever $E,E',G\subset\mathbb{R}^3$ are measurable sets of finite measure satisfying*

$$\mathcal{A}_I\chi_E(x)\geqslant\beta \quad\text{and}\quad \mathcal{A}_I\chi_{E'}(x)\geqslant\theta \quad\text{for all } x\in G,$$

there exists a pair (A,B) from our collection so that

$$|E'|\gtrsim \beta^A\beta'^2\theta^B$$

where $\beta'=\beta\frac{|G|}{|E|}$.

Proof. Set $\Phi_{\mathbf{P}}(s,t,u)=-\mathbf{P}(s)+\mathbf{P}(t)-\mathbf{P}(u)$ and define refinements

$$E^1=\{y\in E:\ \mathcal{A}_I^*\chi_G(y)\geqslant\beta'/2\},$$
$$G^1=\{x\in G:\ \mathcal{A}_I\chi_{E^1}(x)\geqslant\beta/4\}.$$

We have

$$\langle\mathcal{A}_I^*\chi_{G^1},\chi_{E^1}\rangle=\langle\mathcal{A}_I\chi_{E^1},\chi_G\rangle-\langle\mathcal{A}_I\chi_{E^1},\chi_{G\setminus G^1}\rangle\geqslant\langle\mathcal{A}_I\chi_{E^1},\chi_G\rangle-\frac{\beta|G|}{4}$$
$$=\langle\mathcal{A}_I^*\chi_G,\chi_E\rangle-\langle\mathcal{A}_I^*\chi_G,\chi_{E\setminus E^1}\rangle-\frac{\beta|G|}{4}\geqslant\langle\mathcal{A}_I\chi_E,\chi_G\rangle-\frac{3\beta|G|}{4}\geqslant\frac{\beta|G|}{4}.$$

Hence $G^1\neq\emptyset$. Now, pick $x_0\in G^1$ and set

$$S=\{s\in I:\ x_0-\mathbf{P}(s)\in E^1\}\quad\Rightarrow\quad\mu(S)=\mathcal{A}_I\chi_{E^1}(x_0)\geqslant\beta/4.$$

For $s \in S$, set

$$T_s = \{t \in I\colon\ x_0 - \mathbf{P}(s) + \mathbf{P}(t) \in G\} \quad \Rightarrow \quad \mu(T_s) = \mathcal{A}_I^* \chi_G\big(x_0 - \mathbf{P}(s)\big) \geqslant \frac{\beta'}{2}.$$

Finally for $s \in S$ and $t \in T_s$, set

$$U_{s,t} = \{u \in I\colon\ x_0 + \Phi_{\mathbf{P}}(s,t,u) \in E'\} \quad \Rightarrow \quad \mu(U_{s,t}) = \mathcal{A}_I \chi_{E'}\big(x_0 - \mathbf{P}(s) + \mathbf{P}(t)\big) \geqslant \theta.$$

The idea is to estimate the measure of E' by observing that if

$$\mathcal{P} = \{(s,t,u) \in I^3\colon\ s \in S,\ t \in T_s,\ u \in U_{s,t}\} \quad \text{then } x_0 + \Phi_{\mathbf{P}}(\mathcal{P}) \subset E'.$$

Hence the arguments of Section 4 apply and we have

$$|E'| \gtrsim \int\limits_S |s-b|^{k/3} \int\limits_{T_s} |t-b|^{k/3}|s-t| \int\limits_{U_{s,t}} |u-b|^{k/3}|t-u||s-u|\,du\,dt\,ds.$$

To estimate the last iterated integral we can proceed in exactly the same manner as in the proof of Theorem 2 and split the analysis into $9 = 3 \times 3$ cases (1a)–(3c). In all cases, we obtain a bound from below equal to a constant multiple of $\beta^A \beta'^2 \theta^B$, for A and B belonging to a fixed finite set and always satisfying $1 \leqslant A < 2$, $2 < B \leqslant 3$ and $A + B = 4$. We explicitly present here a couple of cases to show that A and B can take different values, and leave the remaining cases to the interested reader.

Let us suppose that for all $t \in T_s$, $|t-b| \leqslant (1/8)|s-b|$ and for all $u \in U_{s,t}$, $|u-b| \leqslant (1/4)|t-b|$. Then $|s-t| \sim |s-b|$, $|u-t| \sim |t-b|$ and $|u-s| \sim |s-b|$. Hence

$$\begin{aligned}
&\int\limits_S |s-b|^{k/3} \int\limits_{T_s} |t-b|^{k/3}|s-t| \int\limits_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \\
&\quad\sim \int\limits_S |s-b|^{k/3+2} \int\limits_{T_s} |t-b|^{k/3+1} \int\limits_{U_{s,t}} |u-b|^{k/3}\,du\,dt\,ds \\
&\quad\geqslant \int\limits_{S\setminus B_\beta} |s-b|^{k/6+k/6+2} \int\limits_{T_s\setminus B_{\beta'}} |t-b|^{k/6+k/6+1} \int\limits_{U_{s,t}\setminus B_\theta} |u-b|^{k/6+k/6}\,du\,dt\,ds,
\end{aligned}$$

and the last iterated integral is bounded below by a constant times

$$\beta^{\frac{6}{k+6}(k/6+2)} \times \beta \times \beta'^{\frac{6}{k+6}(k/6+1)} \times \beta' \times \theta^{\frac{k}{k+6}} \times \theta = \beta^{\frac{2k+18}{k+6}} \beta'^2 \theta^{\frac{2k+6}{k+6}}.$$

Another case is one where, for all $t \in T_s$, $(1/8)|s-b| \leqslant |t-b| \leqslant 2|s-b|$, and for all $U_{s,t}$, $(1/4)|t-b| \leqslant |u-b| \leqslant 4|t-b|$. Here $|s-b| \sim |t-b| \sim |u-b|$. Hence

$$
\begin{aligned}
&\int_S |s-b|^{k/3} \int_{T_s} |t-b|^{k/3}|s-t| \int_{U_{s,t}} |u-b|^{k/3}|u-s||u-t|\,du\,dt\,ds \\
&\quad \gtrsim \int_S |s-b|^{k/2} \int_{T_s\setminus B_{s,\beta'}} |t-b|^{k/3}|s-t| \int_{U_{s,t}\setminus(B_{t,\theta}\cup B_{s,\theta})} |u-b|^{k/6}|u-t||u-s|\,du\,dt\,ds \\
&\quad \gtrsim \beta'\theta^2 \int_{S\setminus B_\beta} |s-b|^{k/6} \int_{T_s\setminus B_{s,\beta'}} |t-b|^{k/6} \int_{U_{s,t}\setminus(B_{t,\theta}\cup B_{s,\theta})} |u-b|^{k/6}\,du\,dt\,ds.
\end{aligned}
$$

Again we see that the sets we are integrating over have not changed in μ measure much when we remove intervals and so the last iterated integral is at least a constant times $\beta'\theta^2 \times \beta \times \beta' \times \theta = \beta\beta'^2\theta^3$.

The remaining seven cases can be treated in a similar way. □

We can now conclude our argument; pick $E = E_i$, $E' = E_j, G = G_i \cap G_j$, and $\beta = \epsilon|E|^{1/2}|F|^{-1/3}, \theta = |E'|^{1/2}|F|^{-1/3}, \beta' = \beta|G|/|E|$. By Lemma 1 we have

$$
\begin{aligned}
|E'| &\gtrsim \epsilon^{A+B}|F|^{(A+B)/3}|E|^{A/2}|F|^{B/2}\beta^2|G|^2|E|^{-2} \\
&\gtrsim \epsilon^4|F|^{-4/3}|E|^{A/2}|E'|^{B/2}\epsilon^2|E||F|^{-2/3}|G|^2|E|^{-2} \gtrsim \epsilon^{12}|E|^{A/2-1}|E'|^{B/2},
\end{aligned}
$$

where we have used the fact that $|G| \gtrsim \epsilon^3|F|$. Using the relation $A+B=4$ we deduce

$$
|E'|^{1-A/2} \lesssim \epsilon^{-12}|E|^{1-A/2},
$$

which is equivalent to

$$
2^{-jp} \lesssim \epsilon^{24/(2-A)}2^{-ip},
$$

implying $j \geqslant i - C'\log(1/\epsilon)$; since the roles of i, j can be exchanged one has $|i-j| \leqslant C'\log(1/\epsilon)$, which contradicts our assumptions and therefore (22) cannot hold. This gives us the weak-type bound (18). As we have already mentioned, the arguments in [10] can now be reproduced verbatim to obtain the Lorentz bound (17), completing the proof of Theorem 1 for $d=3$.

6. Two-dimensional estimates

In this section we present the arguments necessary to prove Theorem 1 in the case $d=2$, starting with the restricted weak type estimates.

Theorem 3. *Let $d=2$. The operator* (10) *satisfies*

$$
\mathcal{A}_{\mathbb{R}} : L^{3/2,1}(\mathbb{R}^2) \to L^{3,\infty}(\mathbb{R}^2). \tag{23}
$$

Proof. The preparatory statements of Sections 3 and 4 can obviously be applied also in this setting and we quickly reduce our analysis to the operators

$$\mathcal{A}_I f(x) = \int_I f\big(x - \mathbf{P}(t)\big)|t-b|^{k/3}\,dt := \int_I f\big(x - \mathbf{P}(t)\big)\,d\mu_I(t),$$

for each fixed I. We set

$$\langle \mathcal{A}_I \chi_E, \chi_F\rangle = \alpha|F|, \qquad \langle \mathcal{A}_I \chi_E, \chi_F\rangle = \beta|E|,$$

with $|E| \neq 0$, $|F| \neq 0$, and observe it suffices to establish[5]

$$\langle \mathcal{A}_I \chi_E, \chi_F\rangle \lesssim |E|^{2/3}|F|^{2/3} \quad \Leftrightarrow \quad |E| \gtrsim \alpha^2\beta \quad \Leftrightarrow \quad |F| \gtrsim \beta^2\alpha, \tag{24}$$

uniformly in I. As discussed in Section 2 we will apply Christ's argument to prove

$$|E| \gtrsim \alpha^2\beta \quad \text{in the range } \alpha \leqslant \beta \tag{25}$$

and similarly $F| \gtrsim \beta^2\alpha$ in the range $\beta \leqslant \alpha$. But from the relation $\alpha|F| = \beta|E|$, we see that (25) implies (24). This only works since we are proving an estimate on the line of duality. We shall concentrate on the estimate in (25) (the proof of the second estimate is similar) and so we assume from now on that $\alpha \leqslant \beta$.

By the discussion in Section 2 we can find a point $x_0 \in E$ and

$$\begin{aligned}
&S \subset I \text{ so that } \mu(S) \gtrsim \beta;\\
&\text{for each } s \in S \text{ there is } T_s \subset I \text{ so that } \mu(T_s) \gtrsim \alpha;\\
&\text{if } \mathcal{P} = \big\{(s,t) \in I^2\colon\ s \in S, t \in T_s,\big\} \ \Rightarrow\ x_0 + \Phi_{\mathbf{P}}(\mathcal{P}) \subset E.
\end{aligned}$$

Therefore (see Section 2)

$$|E| \gtrsim \iint_{\mathcal{P}} \big|J_{\Phi_{\mathbf{P}}}(s,t)\big|\,ds\,dt \gtrsim \int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2}|s-t|\,dt\,ds. \tag{26}$$

As before we use a simple pigeonhole argument to reduce to various cases where the factors $|s-b|$, $|t-b|$ and $|s-t|$ in the integrand of the interated integral in (26) have a definite size relationship.

We shall use similar dynamic notation as in Section 4: $B_\alpha = \{t \in I\colon\ |t-b| \leqslant \delta\alpha^{3/(k+3)}\}$ and $B_{s,\alpha} = \{t \in I\colon\ |t-s| \geqslant \delta\alpha|s-b|^{-k/3}\}$ with analogous conclusions as before if $\delta > 0$ is chosen small enough in any particular situation.

[5] We shall again abuse notation and relabel the measures μ_I as μ.

Case 1. On T_s, $|t-b| \leqslant (1/2)|s-b|$ holds; in this case $|s-b| \sim |t-s|$. Thus

$$\int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2}|s-t|\,dt\,ds \gtrsim \int_{S\setminus B_\beta} |s-b|^{k/2+1} \int_{T_s\setminus B_\alpha} |t-b|^{k/2}\,dt\,ds$$

$$\gtrsim \int_{S\setminus B_\beta} |s-b|^{k/3} \int_{T_s\setminus B_\alpha} |t-b|^{2k/3+1}\,dt\,ds \gtrsim \beta\alpha^2.$$

Here we have not used the relation $\alpha \leqslant \beta$. In addition,

$$\int_{S\setminus B_\beta} |s-b|^{k/2+1} \int_{T_s\setminus B_\alpha} |t-b|^{k/2}\,dt\,ds \gtrsim \beta^{\frac{3}{2}\frac{k+4}{k+3}}\alpha^{\frac{3}{2}\frac{k+2}{k+3}}.$$

Notice that $\beta^{\frac{3}{2}\frac{k+4}{k+3}}\alpha^{\frac{3}{2}\frac{k+2}{k+3}} \gtrsim \alpha^2\beta$ for $\alpha \leqslant \beta$. The former of these two estimates suffices for the proof of Theorem 3. However, both estimates will be required in order to obtain Lorentz space bounds.

Case 2. On T_s, $(1/2)|s-b| \leqslant |t-b| \leqslant 2|s-b|$ holds.

$$\int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2}|s-t|\,dt\,ds \geqslant \int_{S\setminus B_\beta} |s-b|^{k/2} \int_{T_s\setminus B_{s,\alpha}} |t-b|^{k/2}|s-t|\,dt\,ds$$

$$\gtrsim \alpha \int_{S\setminus B_\beta} |s-b|^{k/6} \int_{T_s\setminus B_{s,\alpha}} |t-b|^{k/2}\,dt\,ds.$$

We make the important observation here that, in this case, $|t-b| \lesssim |s-b|$ on $B_{s,\alpha}$ and therefore $\mu(T_s \setminus B_{s,\alpha}) \gtrsim \alpha$ if $\delta > 0$ is chosen appropriately. Therefore the last iterated integral is bounded below by

$$\alpha \int_{S\setminus B_\beta} |s-b|^{k/6+k/6} \int_{T_s\setminus B_{s,\alpha}} |t-b|^{k/3}\,dt\,ds \gtrsim \alpha^2\beta.$$

Case 3. On T_s, $2|s-b| \leqslant |t-b|$ holds; in this case $|t-s| \sim |t-b|$. Thus

$$\int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2}|s-t|\,dt\,ds \gtrsim \int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2+1}\,dt\,ds$$

$$\gtrsim \int_{S\setminus B_\beta} |s-b|^{k/2} \int_{T_s\setminus B_\alpha} |t-b|^{k/2+1}\,dt\,ds$$

$$\gtrsim \beta^{\frac{3}{2}\frac{k+2}{k+3}}\alpha^{\frac{3}{2}\frac{k+4}{k+3}} \gtrsim \alpha^2\beta$$

since $\alpha \leqslant \beta$. This completes the proof of (25) and hence the proof of Theorem 3. □

To prove the Lorentz estimates for the operator $\mathcal{A}_I$ we put ourselves back in the setting of Section 5, with the (obvious) difference that we must consider the estimates just proven. Recall the appropriate setup:

- there are 4 sets $E(=E_i)$, $E'(=E_j)$, $G(=G_i \cap G_j)$, F with $|E| \sim \eta 2^{-3i/2}$, $|E'| \sim \eta 2^{-3j/2}$, and $G \subset F$,
- four parameters $\epsilon > 0$, $\beta = \epsilon |E|^{2/3}|F|^{-1/3}$, $\delta = \epsilon |E'|^{2/3}|F|^{-1/3}$, $\beta' = \beta|G|/|E|$,
- we may assume $|G| > \epsilon^3|F|$, $\mathcal{A}_I \chi_E \gtrsim \beta$ on G, $\mathcal{A}_I \chi_{E'} \gtrsim \delta$ on G,
- we further assume $\beta \leqslant \delta$, which is equivalent to $|E| \leqslant |E'|$,[6]

and we wish to show that (22) leads to a contradiction; this will manifest itself in two possible forms, the inequality

$$|E| \gtrsim \epsilon^c |E'| \quad \text{or the inequality} \quad |G| \leqslant K^{-1}\epsilon^3|F|,$$

for some $c \geqslant 0$ and for a sufficiently large K. Clearly $|G| \leqslant K^{-1}\epsilon^3|F|$ contradicts (22). The inequality $|E| \gtrsim \epsilon^c|E'|$ is equivalent to $2^{3(i-j)/2} \lesssim (1/\epsilon)^c$ which in turn is equivalent to $0 \leqslant i-j \lesssim c\log(1/\epsilon)$ which contradicts our basic assumptions on i and j. As indicated at the end of Section 2 the arguments in Section 5 break down in the two-dimensional setting and a slightly more elaborate argument is needed here. To carry out our arguments, we define two refinements

$$E^1 = \big\{y \in E\colon \mathcal{A}_I^* \chi_G(y) \geqslant \beta'/2\big\}, \qquad G^1 = \big\{x \in G\colon \mathcal{A}_I \chi_{E^1}(x) \geqslant \beta/4\big\}.$$

The standard argument shows that $G^1 \neq \emptyset$, thus we pick $x_0 \in G^1$ and set

$$\begin{aligned} S &= \big\{s \in I\colon x_0 - \mathbf{P}(s) \in E^1\big\} &\Rightarrow\quad& \mu(S) = \mathcal{A}_I \chi_{E^1}(x_0) \geqslant \beta/4, \\ T_s &= \big\{t \in I\colon x_0 - \mathbf{P}(s) + \mathbf{P}(t) \in G\big\} &\Rightarrow\quad& \mu(T_s) = \mathcal{A}_I^* \chi_G\big(x_0 - \mathbf{P}(s)\big) \geqslant \beta'/2, \\ U_{s,t} &= \big\{u \in I\colon x_0 - \mathbf{P}(s) + \mathbf{P}(t) - \mathbf{P}(u) \in E'\big\} &\Rightarrow\quad& \mu(U_{s,t}) = \mathcal{A}_I \chi_{E'}\big(x_0 - \mathbf{P}(s) + \mathbf{P}(t)\big) \geqslant \delta. \end{aligned}$$

Case A. $|G| \geqslant \epsilon^p |E|$, where $p > 0$ will be determined later.

For fixed $s \in S$ we have

$$\psi_s(T_s \times U_{s,t}) \subset E', \quad \text{where } \psi_s(t,u) = x_0 - \mathbf{P}(s) + \mathbf{P}(t) - \mathbf{P}(u),$$

therefore

$$|E'| \gtrsim \int_{T_s} |t-b|^{k/2} \int_{U_{s,t}} |u-b|^{k/2}|u-t|\,du\,dt \gtrsim \delta^C {\beta'}^D$$

thanks to Cases 1, 2 and 3 in this section; here $(C,D) = (2,1)$, (A,B) or (B,A), where $(A,B) := (\frac{3}{2}\frac{k+4}{k+3}, \frac{3}{2}\frac{k+2}{k+3})$, and in all instances $C + D = 3$. Hence

[6] Since our arguments are completely symmetrical, this assumption does not pose any restrictions, as the roles of E and E' can be interchanged.

$$|E'| \gtrsim \delta^C \beta'^D = \delta^C \beta^D |G|^D |E|^{-D} \geqslant \delta^C \beta^D \epsilon^{p(D-1)} |G||E|^{-1}$$
$$\gtrsim \epsilon^C |E'|^{2C/3} |F|^{-C/3} \epsilon^D |E|^{2D/3} |F|^{-D/3} \epsilon^{p(D-1)} |G||E|^{-1},$$

which is equivalent to

$$|E|^{1-2D/3} \gtrsim \epsilon^{3+p(D-1)} |E'|^{2C/3-1} |F|^{-1} |G| \gtrsim \epsilon^{6+p(D-1)} |E'|^{2C/3-1},$$

the contradiction we wished to find.

Case B. $|G| \leqslant \epsilon^p |E|$. This case is more involved and will be split into subcases. Let

$$\mathcal{Q} = \{(s,t) \in I^2\colon\ s \in S,\ t \in T_s\}, \qquad \Phi_{\mathbf{P}}(s,t) = x_0 - \mathbf{P}(s) + \mathbf{P}(t).$$

Clearly $\Phi_{\mathbf{P}}(\mathcal{Q}) \subset G$, hence

$$|G| \gtrsim \int_S |s-b|^{k/2} \int_{T_s} |t-b|^{k/2} |s-t|\, dt\, ds.$$

Let

$$T_s = T_s^1 \cup T_s^2 \cup T_s^3,$$

where

$$T_s^1 = T_s \cap \{t \in I\colon\ |t-b| \leqslant (1/2)|s-b|\},$$
$$T_s^2 = T_s \cap \{t \in I\colon\ (1/2)|s-b| < |t-b| \leqslant 2|s-b|\},$$
$$T_s^3 = T_s \cap \{t \in I\colon\ |t-b| \geqslant 2|s-b|\}.$$

Also let

$$S^1 = \{s \in S\colon\ \mu(T_s^2) \geqslant \beta'/6\}, \qquad S^2 = \{s \in S\colon\ \mu(T_s^1) \geqslant \beta'/6\},$$
$$S^3 = \{s \in S\colon\ \mu(T_s^3) \geqslant \beta'/6\}.$$

Case B1. $\mu(S^1) \leqslant \beta/12$. Then either $\mu(S^2) \geqslant \beta/12$ or $\mu(S^3) \geqslant \beta/12$.

Case (B1a). $\mu(S^2) \geqslant \beta/12$. In this case, by Case 1,

$$|G| \gtrsim \int_{S^2} |s-b|^{k/2} \int_{T_s^1} |t-b|^{k/2} |s-t|\, dt\, ds \gtrsim \beta^A \beta'^B = \epsilon^3 |E|^2 |F|^{-1} (|G|/|E|)^B.$$

This implies

$$|F| \gtrsim \epsilon^3 |E|^{2-B} |G|^{B-1} \geqslant \epsilon^3 \epsilon^{-p(2-B)} |G|^{2-B+B-1} \quad \Leftrightarrow \quad |G| \lesssim \epsilon^{p(2-B)-3} |F|,$$

contradicting $|G| \gtrsim \epsilon^3 |F|$ for p chosen sufficiently large (note $B < 2$).

Case (B1b). $\mu(S^3) \geqslant \beta/12$. Here, by Case 3,

$$|G| \gtrsim \int_{S^3} |s-b|^{k/2} \int_{T_s^3} |t-b|^{k/2}|s-t|\,dt\,ds \gtrsim \beta'^A \beta^B = \epsilon^3 |E|^2 |F|^{-1}\big(|G|/|E|\big)^A.$$

This leads to

$$|F| \gtrsim \epsilon^3 |E|^{2-A}|G|^{A-1} \geqslant \epsilon^3 \epsilon^{-p(2-A)}|G|^{2-A+A-1} \quad \Leftrightarrow \quad |G| \lesssim \epsilon^{p(2-A)-3}|F|,$$

contradicting $|G| \gtrsim \epsilon^3|F|$ for sufficiently large p (note $A<2$ if $k \neq 0$[7]).

Case B2. $\mu(S^1) > \beta/12$. To take care of this case we shall define subsets $T_s^{2,1}, T_s^{2,2}$ of T_s^2 as

$$T_s^{2,1} = \big\{t \in T_s^2\colon\ \mu\big(\{u \in U_{s,t}\colon\ |u-b| \leqslant 2|t-b|\}\big) \geqslant \delta/2\big\},$$
$$T_s^{2,2} = \big\{t \in T_s^2\colon\ \mu\big(\{u \in U_{s,t}\colon\ |u-b| > 2|t-b|\}\big) \geqslant \delta/2\big\}.$$

Case (B2a). There exists $s_0 \in S^1$ so that $\mu(T_{s_0}^{2,1}) \geqslant \beta'/12$. Hence, we bound the measure of E' by integrating over $T_{s_0}^{2,1}$. By Cases 1 and 2, we have

$$|E'| \gtrsim \int_{T_{s_0}^{2,1}} |t-b|^{k/2} \int_{U_{s_0,t}} |u-b|^{k/2}|u-t|\,du\,dt \gtrsim \beta'\delta^2 = \epsilon^3|E|^{-1/3}|E'|^{4/3}|G||F|^{-1}.$$

This implies $|E|^{1/3} \gtrsim \epsilon^3|E'|^{1/3}|G||F|^{-1} \gtrsim \epsilon^6|E'|^{1/3}$, giving us the desired contradiction.

Case (B2b). For every $s \in S^1$ we have $\mu(T_s^{2,1}) < \beta'/12$. Thus, we must have that $\mu(T_s^{2,2}) \geqslant \beta'/12$. Now the integration occurs over $T_s^{2,2}$; fixing an $s \in S^1$, we have

$$|E'| \gtrsim |s-b|^{k/6} \int_{T_s^{2,2}} |t-b|^{k/3} \int_{U_{s,t}} |u-b|^{k/2}|u-t|\,du\,dt \gtrsim |s-b|^{k/6}\delta^A\beta'.$$

Now, if we choose $\mathfrak{S} \subset S^1$, so that $\mu(\mathfrak{S}) = \beta/100$ we have

$$|E'| \int_{\mathfrak{S}} |s-b|^{k/3}\,ds \gtrsim \delta^A\beta' \int_{\mathfrak{S}\setminus B_\beta} |s-b|^{k/3+k/6}\,ds \gtrsim \delta^A\beta'\beta \times \beta^{\frac{k}{6}\frac{3}{k+3}} = \delta^A\beta'\beta^B,$$

and this implies

$$\begin{aligned}
&\beta|E'| \gtrsim \delta^A\beta'\beta^B \\
&\quad\Leftrightarrow\quad |E'| \gtrsim \delta^A\beta^B|G||E|^{-1} = \epsilon^3|E'|^{2A/3}|E|^{2B/3-1}|F|^{-1}|G| \geqslant \epsilon^6|E'|^{2A/3}|E|^{2B/3-1} \\
&\quad\Leftrightarrow\quad |E|^{1-2B/3} \gtrsim \epsilon^6|E'|^{2A/3-1},
\end{aligned}$$

which is the required contradiction. This completes the proof of Theorem 1.

[7] The case $k=0$ is simpler and is dealt with in [10].

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