Physics Letters B 746 (2015) 276–280

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Thermodynamics of static dyonic AdS black holes in the ω-deformed Kaluza–Klein gauged supergravity theory

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ARTICLE INFO

Article history:
Received 19 March 2015
Received in revised form 26 April 2015
Accepted 7 May 2015
Available online 11 May 2015
Editor: M. Cvetič

Keywords:
Black hole thermodynamics
Dyonic AdS black hole
Conserved charge
Kaluza–Klein supergravity

ABSTRACT

We study thermodynamical properties of static dyonic AdS black holes in four-dimensional ω-deformed Kaluza–Klein gauged supergravity theory, and find that the differential first law requires a modification via introducing a new pair of thermodynamical conjugate variables (X, Y). To ensure such a modification, we then apply the quasi-local ADT formalism developed in Kim et al. (2013) [20] to calculate the quasi-local conserved charge and identify that the new pair is precisely the one previously introduced to modify the differential form of the first law.

1. Introduction

Black holes are the most important compact objects predicted soon after Albert Einstein built his General Theory of Relativity [1] just one century ago. With lots of exact solutions being found, there are many researches on various different kinds of black holes and a large amount of great success has been achieved in the area of black hole physics. One of the remarkable achievements is the four laws of black hole thermodynamics established by Bardeen, Carter and Hawking [2] via the analogy with those of ordinary thermodynamical system. In particular, the differential first law is expressed as $dM = T\,dS + \Omega\,dJ + \Phi\,dQ + \Psi\,dP$ for a stationary asymptotically flat black hole. In this formula, (M, J, Q, P) are the mass, angular momentum, electric and magnetic charge measured at infinity, while $(T, S, \Omega, \Phi, \Psi)$ represent the temperature, entropy, angular velocity of the horizon, electro-static and magnetic potentials at the horizon, respectively.

However, it has been demonstrated in certain cases [3–9] that thermodynamics might receive necessary modifications due to the presence of nontrivial matter fields. If the asymptotic behavior of a scalar field at infinity is $\phi = \phi_\infty + \phi_1/r + \phi_2/r^2 + \cdots$ in the asymptotically flat spacetimes, it was shown in Ref. [3] that under variation of moduli field ϕ_∞, the first law of black hole thermodynamics becomes $dM = T\,dS + \Omega\,dJ + \Phi\,dQ + \Psi\,dP - \Sigma\,d\phi_\infty$, where $\Sigma = \phi_1$ is the scalar charge. In the cases of asymptotically AdS spacetimes with $\phi_\infty = 0$, the contribution of a scalar field to black hole thermodynamics was previously studied in Refs. [4,5]. In recent researches [6–8] on thermodynamics of spherically symmetric static AdS black holes, such as the solution given in conformal gravity [6] and the one presented in Einstein–Proca theory [7] as well as dyonic black holes found in Kaluza–Klein (KK) gauge supergravity theory [8], the first law should be modified by extra hairs. It was found that in Ref. [6], the modification of the first law is due to the spin-2 hair, while in Ref. [7] the contribution comes from a massive spin-1 modes, rather than the massive spin-2 modes. In four-dimensional asymptotically AdS spacetimes, it was shown that a massless scalar with the large-r boundary behavior $\phi = \phi_1/r + \phi_2/r^2 + \cdots$ can break some of the boundary AdS symmetries unless one of the following three conditions is satisfied: (1) $\phi_1 = 0$, or (2) $\phi_2 = 0$, or (3) ϕ_2/ϕ_1^2 is a fixed constant [4]. (See also Refs. [5] and [10].) However, it should be pointed out that if the mass squared of the four-dimensional scalar field is -2, the aforementioned cases are the ones preserving the full SO(2, 3) symmetry.

In this Letter, we will focus on the case where the modification of the first law is due to the scalar hair of static dyonic AdS black holes [8,9] in (ω-deformed) KK gauged supergravity theory. In Ref. [8], a static dyonic AdS black hole solution in four-dimensional KK gauged supergravity was constructed, for which the scalar boundary behavior violates the above-mentioned three

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http://dx.doi.org/10.1016/j.physletb.2015.05.013

criteria [11]. It is proposed in Ref. [8] that the first law should be rescued via introducing a new pair of thermodynamical conjugate variables (X, Y), so that the differential first law can be rewritten as $dM = T\,dS + \Omega\,dJ + \Phi\,dQ + \Psi\,dP - X\,dY$. On the contrary, Chow and Compere [9] presented general static AdS black hole solution in four-dimensional $\mathcal{N} = 2$, STU gauged supergravity and insisted that the first law is unchanged while the mass is non-integrable due to the non-existence of a conserved symplectic structure in covariant phase space. Obviously, it deserves a deeper investigation of this issue, see Ref. [12] for a recent discussion on this subject.

On the other hand, the maximal $\mathcal{N} = 8$, SO(8) gauged supergravity theory constructed in Ref. [13] has been regarded uniquely for a long time. But a recent evidence [14] demonstrates that there is a one-parameter family of inequivalent SO(8) gauged supergravity theories characterized by an angular parameter ω. In recent years, there has been a lot of great interest to study various different consistent truncations [15–17] of the ω-deformed maximal $\mathcal{N} = 8$, SO(8) gauged supergravity, especially the truncations of scalar fields. Some black hole solutions in ω-deformed gauged $\mathcal{N} = 8$ supergravity was constructed recently in Ref. [18]. What most interested us here is the truncation to the ω-deformed KK gauged supergravity theory and exact solutions to it [17].

The aim of this Letter is to investigate thermodynamical properties of spherically symmetric static dyonic AdS black holes in four-dimensional ω-deformed KK gauged supergravity theory [17]. Similar to the analysis done in the un-deformed case [8], we find that the differential first law needs a modification by introducing a new pair of thermodynamical conjugate variables (X, Y). To ensure such a modification, it is necessary to calculate the conserved charge via another different approach. Compared with the Wald's formalism [19] adopted in Ref. [8], in this work we will apply the quasi-local ADT formalism [20] to derive the conserved charge and deduce that the new conjugate pair (X, Y) is precisely that previously introduced [8] to modify the differential form of the first law.

The remaining part of this Letter is organized as follows. In Section 2, we study the thermodynamics of static dyonic AdS black holes in the ω-deformed KK gauged supergravity theory. We check the differential first law and the Bekenstein–Smarr formula. Inferred from the modification given in [8], we obtain precisely the same modification of the differential first law. In Section 3, the conserved charge is calculated by using the quasi-local ADT formalism to identify the modification with the quantity introduced in Section 2. Finally, we present our conclusions with some comments.

2. Thermodynamics of static dyonic AdS black holes in ω-deformed KK gauged supergravity

In Ref. [17], the ω-deformed KK gauged supergravity theory is obtained via a series of consistent truncations of the ω-deformed maximal $\mathcal{N} = 8$, SO(8) gauged supergravity. The Lagrangian for this theory is given by

$$e^{-1}\mathcal{L} = R + 6g^2\cosh\phi - \frac{3}{2}(\partial\phi)^2 - \frac{2F_{\mu\nu}F^{\mu\nu} + \sin 2\omega \sinh 3\phi\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}{8(e^{3\phi}\cos^2\omega + e^{-3\phi}\sin^2\omega)}\,, \tag{1}$$

in which g is the cosmological constant and ω is a deformed parameter. Note that in the un-deformed case where $\omega = 0$, the above theory returns to the KK gauged supergravity theory. If we set $g = 0$ further, then it reduces to the standard KK supergravity theory obtained from the S^1 reduction of the five-dimensional pure Einstein's gravity theory.

A four-dimensional static dyonic AdS black hole solution is also presented there [17], for which the metric, the dilaton scalar, the U(1) gauge potential and its dual are given below:

$$ds^2 = \sqrt{H_1(r)H_2(r)}\Big[-\frac{f(r)\,dt^2}{H_1(r)H_2(r)} + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\Big]\,, \tag{2}$$

$$\phi = \frac{1}{2}\ln\Big[\frac{H_2(r)}{H_1(r)}\Big]\,, \qquad f(r) = 1 - \frac{2m}{r} + g^2r^2H_1(r)H_2(r)\,, \tag{3}$$

$$A = \Big[\frac{h_1(r)}{H_1(r)}\cos\omega - \frac{h_2(r)}{H_2(r)}\sin\omega\Big]dt + 4(P\cos\omega + Q\sin\omega)\cos\theta\,d\varphi\,, \tag{4}$$

$$\widetilde{A} = \Big[\frac{h_1(r)}{H_1(r)}\sin\omega + \frac{h_2(r)}{H_2(r)}\cos\omega\Big]dt + 4(P\sin\omega - Q\cos\omega)\cos\theta\,d\varphi\,, \tag{5}$$

where

$$h_1(r) = \frac{4Q}{q}\Big(\frac{p}{r} + \frac{p+q}{q-2m}\Big)\,, \quad h_2(r) = \frac{4P}{p}\Big(\frac{q}{r} + \frac{p+q}{p-2m}\Big)\,,$$
$$H_1(r) = 1 + \frac{q-2m}{r} + \frac{q(p-2m)(q-2m)}{2(p+q)r^2}\,, \tag{6}$$
$$H_2(r) = 1 + \frac{p-2m}{r} + \frac{p(p-2m)(q-2m)}{2(p+q)r^2}\,,$$

in which

$$P = \frac{\sqrt{p(p^2-4m^2)}}{4\sqrt{p+q}}\,, \quad Q = \frac{\sqrt{q(q^2-4m^2)}}{4\sqrt{p+q}}\,, \tag{7}$$

are electric and magnetic charges of the static dyonic AdS black hole in the un-deformed KK gauged supergravity theory [8], expressed in terms of the parameters $p \ge 2m$ and $q \ge 2m$. The relation between (p, q) and (β_1, β_2) used in Ref. [8] is

$$\beta_1 = \frac{p(q-2m)}{q(p+2m)}\,, \quad \beta_2 = \frac{q(p-2m)}{p(q+2m)}\,. \tag{8}$$

Compared with the solution given in Ref. [8], the only modification is the U(1) gauge potentials obtained via a duality rotation of those in [8], while the metric and the dilaton scalar remain unchanged. If we set $\omega = 0$ and $p = q$, then the dilaton vanishes and the solution recovers to the dyonic Reissner–Nordström AdS black hole in four-dimensional Einstein–Maxwell theory.

The event horizon is defined through $f(r_+) = 0$. The temperature and entropy of the horizon are easily calculated as

$$T = \frac{f'(r_+)}{4\pi\sqrt{H_1(r_+)H_2(r_+)}}\,, \quad S = \pi r_+^2\sqrt{H_1(r_+)H_2(r_+)}\,. \tag{9}$$

The electro-static and magnetic potentials are given by

$$\Phi_\omega = A_t\big|_{r_+} - A_t\big|_\infty = \Phi\cos\omega - \Psi\sin\omega\,,$$
$$\Psi_\omega = \widetilde{A}_t\big|_{r_+} - \widetilde{A}_t\big|_\infty = \Phi\sin\omega + \Psi\cos\omega\,, \tag{10}$$

where

$$\Phi = \frac{h_1(r_+)}{H_1(r_+)} - \frac{\sqrt{(q+2m)(p+q)}}{\sqrt{q(q-2m)}}\,,$$
$$\Psi = \frac{h_2(r_+)}{H_2(r_+)} - \frac{\sqrt{(p+2m)(p+q)}}{\sqrt{p(p-2m)}}\,. \tag{11}$$

The electric and magnetic charges can be computed as

$$Q_\omega = Q\cos\omega - P\sin\omega\,, \quad P_\omega = Q\sin\omega + P\cos\omega\,. \tag{12}$$

Note that the above expressions are expressed in terms of their counterparts in the un-deformed case [8] after considering the relation (8).

Using the conformal Weyl tensor method [21], it is not difficult to calculate the mass

$$M = \frac{p+q}{4}\,. \tag{13}$$

Now let's check the differential first law of thermodynamics. The first law does hold for $g = 0$, and is written as $dM = T\,dS + \Phi_\omega\,dQ_\omega + \Psi_\omega\,dP_\omega$. For $g \neq 0$, it is apparent that the first law still holds true in the cases: (1) $p = 2m$, or (2) $q = 2m$, or (3) $p = q$ if g is not viewed as a thermodynamical viable. In other case, the differential first law no longer holds true and must be compensated via introducing a new conjugate pair (X, Y)

$$dM = T\,dS + \Phi_\omega\,dQ_\omega + \Psi_\omega\,dP_\omega - X\,dY\,, \tag{14}$$

where

$$X = g^2\frac{(p-q)(p^2-4m^2)^{3/2}(q^2-4m^2)^{1/2}}{16(p+q)^2}\,, \quad Y = \sqrt{\frac{q^2-4m^2}{p^2-4m^2}}\,, \tag{15}$$

are the same expressions as those given in Ref. [8]. This is easily verified by using (8), and it can be proved that

$$\begin{aligned} -X\,dY &= dM - T\,dS - \Phi_\omega\,dQ_\omega - \Phi_\omega\,dP_\omega \\ &= dM - T\,dS - \Phi\,dQ - \Psi\,dP\,, \end{aligned} \tag{16}$$

which means that ω doesn't change the first law of thermodynamics.

One can further treat the cosmological constant as a generalized "pressure" $\mathcal{P} = 3g^2/(8\pi)$, its conjugate quantity $\mathcal{V}$ as a thermodynamical volume [22], then the differential first law reads

$$\begin{aligned} dM &= T\,dS + \Phi_\omega\,dQ_\omega + \Psi_\omega\,dP_\omega - X\,dY + \mathcal{V}\,d\mathcal{P} \\ &= T\,dS + \Phi\,dQ + \Psi\,dP - X\,dY + \mathcal{V}\,d\mathcal{P}\,, \end{aligned} \tag{17}$$

where

$$\mathcal{V} = \frac{4\pi}{3}\Big[r_+^3 + \frac{3}{4}(p+q-4m)r_+^2 + \frac{1}{4}(p-2m)(q-2m)\Big(3r_+ - m + \frac{pq}{p+q}\Big)\Big]. \tag{18}$$

One can also verify that the Bekenstein–Smarr formula is given by

$$\begin{aligned} M &= 2T\,S + \Phi_\omega\,Q_\omega + \Psi_\omega\,P_\omega - 2\mathcal{V}\mathcal{P} \\ &= 2T\,S + \Phi\,Q + \Psi\,P - 2\mathcal{V}\mathcal{P}\,, \end{aligned} \tag{19}$$

and is independent of the deformation parameter ω. Moreover, the (X, Y) pair doesn't appear in the integral first law.

3. Quasi-local conserved charge

In the last section, we have approved the viewpoint proposed in Ref. [8] and followed the same recipe to modify the differential first law by introducing a new pair of thermodynamical variables (X, Y). However, the precise physics origin of (X, Y) still remains a mystery. In order to make an in-depth analysis of the (X, Y) pair, we shall adopt the quasi-local ADT formalism [20] rather than the covariant phase space approach [19] used in [8] to calculate the conserved charge and study the fall-off behavior of the scalar field at infinity.

As far as the conserved charge of AdS black hole is concerned, up to date there are many different methods available to calculate it, such as the covariant phase space approach [19], cohomology method [23], Ashtekar–Magnon–Das formalism [24], Abbott–Deser–Tekin (ADT) formalism [25], and quasi-local ADT formalism [20]. For an earlier review on the quasi-local conserved charge, see Ref. [26]. The quasi-local ADT formalism [20] is a novel way to calculate the conserved charge at finite spacetime domains and receives a lot of recent attention [27,28]. Speaking roughly, one can establish a one-to-one correspondence between the ADT potential and the off-shell linear Noether potential by considering an appropriate variation of metric. From the off-shell ADT potential, one can easily construct the quasi-local charge. Note that the same result can be arrived at by varying Bianchi identity [28].

Taking the variation of the action, we get

$$\delta S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\big(E_\Psi\delta\Psi + \nabla_\mu\Theta^\mu\big)\,, \tag{20}$$

where $E_\Psi\delta\Psi = E_{(g)\mu\nu}\delta g^{\mu\nu} + E_\phi\delta\phi + E_A^\nu\delta A_\nu$ and $\Theta^\mu = \Theta_{(g)}^\mu + \Theta_\phi^\mu + \Theta_A^\mu$ denote the equation of motion and the surface term, respectively. Considering the infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$, one can deduce the off-shell Noether current $\mathcal{J}^\mu$ through equating the diffeomorphism to the general variation as

$$\mathcal{J}^\mu = 2\mathcal{E}^{\mu\nu}\xi_\nu + \xi^\mu\mathcal{L} - \Theta^\mu\,, \tag{21}$$

where we have used the off-shell identity $2\xi_\nu\nabla_\mu E_{(g)}^{\mu\nu} + E_\phi\delta_\xi\phi + E_A\delta_\xi A = \nabla_\mu(Z^{\mu\nu}\xi_\nu)$, and denoted $\mathcal{E}^{\mu\nu} = E_{(g)}^{\mu\nu} - Z^{\mu\nu}/2$. Then one can introduce the off-shell Noether potential $K^{\mu\nu}$ by using $\mathcal{J}^\mu = \nabla_\nu K^{\mu\nu}$, in which

$$K^{\mu\nu} = 2\nabla^{[\mu}\xi^{\nu]} + \big[k(\phi)F^{\mu\nu} - 4\hat{k}(\phi)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\big]A_\lambda\xi^\lambda\,, \tag{22}$$

where

$$k(\phi) = \frac{1}{e^{3\phi}\cos^2\omega + e^{-3\phi}\sin^2\omega}\,,$$
$$\hat{k}(\phi) = \frac{-\sin 2\omega\sinh 3\phi}{2(e^{3\phi}\cos^2\omega + e^{-3\phi}\sin^2\omega)}\,.$$

Now let's define the ADT current $\mathcal{J}_{ADT}^\mu$ which reads

$$\mathcal{J}_{ADT}^\mu = \xi_\nu\delta\mathcal{E}^{\mu\nu} + \frac{1}{2}g^{\alpha\beta}\mathcal{E}^{\mu\nu}\xi_\nu\delta g_{\alpha\beta} + \mathcal{E}^{\mu\nu}\xi^\rho\delta g_{\nu\rho} + \frac{1}{2}\xi^\mu E_\Psi\delta\Psi\,, \tag{23}$$

and write it in a compact form: $\sqrt{-g}\mathcal{J}_{ADT}^\mu = \delta(\sqrt{-g}\mathcal{E}^{\mu\nu}\xi_\nu) + \sqrt{-g}\xi^\mu E_\Psi\delta\Psi/2$. The corresponding off-shell ADT potential $\mathcal{Q}_{ADT}^{\mu\nu}$ is introduced by $\mathcal{J}_{ADT}^\mu = \nabla_\nu\mathcal{Q}_{ADT}^{\mu\nu}$. To find out the relationship between the off-shell Noether current for the infinitesimal diffeomorphism and the linearized conserved current for a Killing vector, we now take $\xi^\mu = (\partial_t)^\mu$ and consider the change in the Noether potential $K^{\mu\nu}$, then we can get

$$\mathcal{Q}_{ADT}^{\mu\nu} = \frac{1}{2}\delta K^{\mu\nu} + \frac{1}{4}K^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta} - \xi^{[\mu}\Theta^{\nu]}\,, \tag{24}$$

from which the off-shell ADT potential $\mathcal{Q}_{ADT}^{\mu\nu}$ is rewritten as [23]

$$\mathcal{Q}_{ADT}^{\mu\nu} = \mathcal{Q}_{(g)}^{\mu\nu} + \mathcal{Q}_\phi^{\mu\nu} + \mathcal{Q}_F^{\mu\nu} + \mathcal{Q}_{CS}^{\mu\nu}\,, \tag{25}$$

where

$$\mathcal{Q}^{\mu\nu}_{(g)} = \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} + \xi^{[\mu}\nabla^{\nu]}h + \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} - h^{\alpha[\mu}\nabla_\alpha\xi^{\nu]} - \xi^{[\mu}\nabla_\alpha h^{\nu]\alpha}\,, \tag{26a}$$

$$\mathcal{Q}^{\mu\nu}_{F} = \frac{1}{2}\Big[F^{\mu\nu}\frac{\partial k(\phi)}{\partial\phi^a}\delta\phi^a + k(\phi)\delta F^{\mu\nu} + \frac{h}{2}k(\phi)F^{\mu\nu} - 2k(\phi)h^{\mu\lambda}F^{\nu}_{\lambda}\Big](A_\lambda\xi^\lambda + Cst) + k(\phi)\big(\xi^\lambda F^{\mu\nu} + \xi^{[\mu}\nabla_\alpha F^{\nu]\alpha}\big)\delta A_\lambda\,, \tag{26b}$$

$$\mathcal{Q}^{\mu\nu}_{CS} = -2\epsilon^{\mu\nu\rho\sigma}\Big[F_{\rho\sigma}\frac{\partial\hat{k}(\phi)}{\partial\phi^a}\delta\phi^a + \hat{k}(\phi)\delta F_{\rho\sigma}\Big](A_\lambda\xi^\lambda + Cst) - 2\big[\epsilon^{\mu\nu\rho\sigma}\xi^\lambda + 2\xi^{[\mu}\epsilon^{\nu]\lambda\rho\sigma}\big]\hat{k}(\phi)F_{\rho\sigma}\delta A_\lambda\,, \tag{26c}$$

$$\mathcal{Q}^{\mu\nu}_{\phi} = 3\xi^{[\mu}\nabla^{\nu]}\phi\,\delta\phi\,, \tag{26d}$$

in which $h_{\mu\nu} = \delta g_{\mu\nu}$, $h^{\mu\nu} = -\delta g^{\mu\nu}$, $h = g^{\mu\nu}\delta g_{\mu\nu}$, $\delta F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\delta F_{\alpha\beta}$.

One can choose the gauge $\delta A_\mu = \mathcal{L}_\xi A_\mu = 0$ so that $d(A_\mu\xi^\mu)$ is gauge invariant. For our aim to calculate the conserved mass, a convenient choice for the gauge constant Cst that makes $(A_\mu\xi^\mu + Cst)|_\infty = 0$ yields

$$Cst = \frac{\sqrt{(q+2m)(p+q)}}{\sqrt{q(q-2m)}}\cos\omega - \frac{\sqrt{(p+2m)(p+q)}}{\sqrt{p(p-2m)}}\sin\omega\,, \tag{27}$$

which is equivalent to making a gauge transformation on A_μ so that the electro-static potential or the t-component of the shifted potential vanishes at infinity.

According to Ref. [23], for a class of one-parameter path in the solution space, one can define a path-independent quasi-local conserved charge as

$$\mathcal{Q} = \frac{1}{8\pi}\int\sqrt{-g}\,\mathcal{Q}^{\mu\nu}_{ADT}d\Sigma_{\mu\nu}\,. \tag{28}$$

For our final aim to obtain the conserved charge, we can introduce a new radial coordinate ρ which is convenient for us to study the asymptotic fall-off behaviors of the metric and the matter fields. Then the line element becomes

$$ds^2 = -f(\rho)dt^2 + \frac{d\rho^2}{h(\rho)f(\rho)} + \rho^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\,. \tag{29}$$

For large ρ, we get

$$\begin{aligned} f(\rho) &= g^2\rho^2 + 1 - \frac{p+q}{2\rho} + \frac{p^2+q^2-pq-4m^2}{4\rho^2} + \cdots\,,\\ h(\rho) &= 1 + \frac{3(p-q)^2}{16\rho^2} + \frac{(p-q)^2(8m^2-p^2-q^2)}{8(p+q)\rho^3} + \cdots\,,\\ \phi &= \frac{p-q}{2\rho} + \frac{(p-q)(8m^2-p^2-q^2)}{8(p+q)\rho^2} + \cdots\,, \end{aligned} \tag{30}$$

and choose an infinitesimal parametrization of a one-parameter path in the solution space by letting

$$m \to m + dm\,,\quad p \to p + dp\,,\quad q \to q + dq\,. \tag{31}$$

After some tedious algebra manipulations and using the gauge choice (27), we can obtain the infinitesimal charge

$$d\mathcal{Q} = \frac{1}{4}(dp+dq) - \frac{g^2(p-q)^2}{4(p+q)}m\,dm + \frac{g^2(p-q)}{16(p+q)^2}\big[(p^2-4m^2)q\,dq - (q^2-4m^2)p\,dp\big]\,. \tag{32}$$

Using Eq. (13), we finally get

$$X\,dY = d\mathcal{Q} - dM = \frac{g^2(p-q)}{32(p+q)^2}\big[(p^2-4m^2)\,d(q^2-4m^2) - (q^2-4m^2)\,d(p^2-4m^2)\big]\,. \tag{33}$$

In the general cases, $\mathcal{Q}$ is not integrable because $d^2\mathcal{Q} = dX\wedge dY \neq 0$ unless one of the following conditions is satisfied: (1) $g=0$, or (2) $p=2m$, or (3) $q=2m$, or (4) $p=q$, or (5) $p^2-4m^2 = c(q^2-4m^2)$ for an arbitrary constant c. Note that in the above analysis, we have treated the cosmological constant g as a true constant ($dg=0$). If g is viewed as a variable, then $\mathcal{Q}$ is still not integrable and one must add a counter-term $\mathcal{V}\,d\mathcal{P}$ to cancel the corresponding divergent term in the expression of $d\mathcal{Q}$. Therefore it is reasonable to infer that the differential first law of thermodynamics needs a modification via the (X, Y) pair.

At last, we would like to check that $X\,dY$ is the exact modification given before. According to the analysis made in Ref. [8], we have

$$\begin{aligned} h(\rho)f(\rho) - f(\rho) &= \frac{1}{4}g^2\phi_1^2 + \frac{2}{3\rho}g^2\phi_1\phi_2 + \cdots\,,\\ \phi &= \frac{\phi_1}{\rho} + \frac{\phi_2}{\rho^2} + \cdots\,, \end{aligned} \tag{34}$$

where

$$\phi_1 = \frac{p-q}{2}\,,\quad \phi_2 = \frac{(p-q)(8m^2-p^2-q^2)}{8(p+q)}\,, \tag{35}$$

so we can re-express [8]

$$X\,dY = \frac{1}{12}g^2(2\phi_2\delta\phi_1 - \phi_1\delta\phi_2)\,, \tag{36}$$

which vanishes when (1) $\phi_1 = 0$, or (2) $\phi_2 = 0$, or (3) $\phi_2 = c\phi_1^2$.

4. Conclusion

In this Letter, we have studied thermodynamical properties of static dyonic AdS black hole in four-dimensional ω-deformed KK gauged theory, and obtained the expression of the modified term for the differential first law. Although the similar problem in the un-deformed theory has been tackled previously in Ref. [8] by using the Wald's procedure, here we have adopted a different method based on the quasi-local ADT formalism to compute the conserved charge. We approve the proposal in Ref. [8] to modify the differential first law by adding a new term $X\,dY$. Furthermore, we find that the deformation parameter ω doesn't change the first law.

Although the differential first law can be satisfactorily restored via the introduction of the (X, Y) pair, this is somehow by heart. Do these two quantities have a general universal definition, and what is the genuine physics hidden behind them?

Acknowledgements

S.Q. Wu is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11275157. S.L. Li is grateful to Dr. J.J. Peng for helpful discussions and our referee for good advice to improve the presentation of this work.

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