Some explicit expressions of extended Stroh formalism for two-dimensional piezoelectric anisotropic elasticity

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Abstract

Since the extended Stroh formalism for two-dimensional piezoelectric anisotropic elasticity preserves essential features of Stroh formalism for pure elastic materials, it becomes important to get the corresponding explicit expressions of some important matrices frequently appeared in Stroh formalism. In this paper, explicit expressions are obtained for the fundamental matrix $N$, material eigenvector matrices $A$ and $B$, and Barnett–Lothe tensors $I$, $S$ and $H$. Although the explicit expressions are presented under the generalized plane strain and short circuit condition, by suitable replacement of the material constants they are still valid for the other two-dimensional states. To provide a clear picture of these expressions, two typical examples are presented, which are piezoelectric ceramics with two different poling axes.

1. Introduction

It is well known that Stroh formalism is an elegant and powerful tool for the study of two-dimensional deformation of anisotropic elastic materials (Ting, 1996). Since this is a complex variable formulation, most of the field solutions are in complex form and the complex material eigenvector matrices $A$ and $B$ usually play important roles on the final analytical solutions. In the establishment of Stroh formalism, there is an eigen-relation relating the complex material eigenvectors to real material properties. Several real matrices connecting through this eigen-relation, such as the fundamental elasticity matrix $N$ and Barnett–Lothe tensors $I$, $S$ and $H$, become crucial when the analytical solutions lead to real form expressions. Thus, to understand the effects of material properties on some problems of anisotropic elasticity, it is important to know the details of these matrices and hence many studies have been done in the literature to get the explicit expressions of $A$, $B$, $N$, $I$, $S$ and $H$ and most of them can be found in (Ting, 1996).

Due to the intrinsic coupling phenomenon between mechanical and electric fields, piezoelectric materials have been widely used as sensors and actuators in intelligent advanced structure design. To study their electromechanical behaviors, suitable mathematical modeling becomes important. Since the extended Stroh formalism for piezoelectric materials preserves most essential features of Stroh formalism, it becomes a popular tool for the study of piezoelectric anisotropic elasticity. Most of the analytical solutions presented in the literature such as (Barnett and Lothe, 1975; Pak, 1990; Sosa, 1991; Kuo and Barnett, 1991; Suo et al., 1992; Park and Sun, 1995; Liang et al., 1995; Liang and Hwu, 1996) show that the solutions for the problems of piezoelectric anisotropic materials can be purposely organized to have the same mathematical forms as those of the corresponding anisotropic elastic materials. This observation tells us the importance of getting the corresponding explicit expressions of $A$, $B$, $N$, $I$, $S$ and $H$ for piezoelectric materials. However, due to its possible complexity involving
piezoelectric effects, very few research efforts have been focused on this study. Suo et al. (1992) mentioned the structures of the inverse of impedance matrix $\mathbf{AB}^{-1}$ and provided explicit forms for some simple cases. Ting (1996) discussed the structures of $\mathbf{N}_1$ and $\mathbf{N}_2$ for general piezoelectric materials without detailed expressions. Soh et al. (2001) and Liou and Sung (2007) provided explicit expressions of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{H}$ for transversely isotropic piezoelectric materials whose poling axis coincides with $x_3$-axis and monoclinic piezoelectric materials, respectively. In this paper, solutions are provided for general piezoelectric materials covering all the possible two-dimensional states such as generalized plane strain/short or open circuit, and generalized plane stress/short or open circuit. The explicit expressions for $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{N}$ given in this paper are valid for all possible piezoelectric anisotropic materials. While for $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{H}$, standard procedure can be applied for all possible piezoelectric anisotropic materials and the explicit expressions are given only for two special examples: piezoelectric ceramics poling in two different axes.

2. Constitutive laws for piezoelectric materials in three-dimensional states

For an anisotropic and linearly electro-elastic solid, the constitutive relation between elastic field tensors (stresses $\sigma$ and strains $\varepsilon$) and electric field vectors (electric displacements or called induction $D$ and electric field $E$) can be represented by four equally important systems of piezoeffect equations. In tensor notation, they can be written as (Rogacheva, 1994).

$$
\begin{align*}
\sigma_{ij} &= C_{ijkl}^{E} \varepsilon_{kl} - e_{ij} E_k, \\
\varepsilon_{ij} &= S_{ijkl} \sigma_{jk} + d_{ij} D_k, \\
D_{i} &= C_{ijkl}^{D} \sigma_{jk} - h_{ij} D_k, \\
E_{j} &= -h_{ij} \sigma_{jk} + \beta_{ij} D_k,
\end{align*}
$$

(2.1)

where $S_{ijkl}$ and $C_{ijkl}^{E}$ are elastic compliances at constant electric field and induction; $C_{ijkl}^{D}$ and $d_{ij}$ are elastic stiffnesses at constant electric field and induction; $\varepsilon_{ij}$ and $\sigma_{ij}$ are dielectric permittivities and non-permittivities at constant strains and stresses; $d_{ij}$, $e_{ij}$, $g_{ij}$ and $h_{ij}$ are piezoelectric strain/charge, stress/charge, strain/voltage, stress/voltage tensors, respectively. Consideration of the symmetry of stresses and strains, and the path-independency of elastic strain energy, these constants have the following symmetry properties

$$
\begin{align*}
C_{ijkl}^{E} &= C_{iklj}^{E}, & e_{ij} &= e_{ji}, & \omega_{ij} &= \omega_{ji}, \\
S_{ijkl} &= S_{ijlk}, & d_{ij} &= d_{ji}, & \omega_{ij} &= \omega_{ji}, \\
C_{ijkl}^{D} &= C_{iklj}^{D}, & h_{ij} &= h_{ji}, & \beta_{ij} &= \beta_{ji}, \\
S_{ijkl} &= S_{ijlk}.
\end{align*}
$$

(2.2)

To express the constitutive laws in matrix form, the contracted notation assigning 11 to 1, 22 to 2, 33 to 3, 23 or 32 to 4, 13 or 31 to 5, 12 or 21 to 6, 14 or 41 to 7, 24 or 42 to 8, 34 or 43 to 9 are usually used in engineering expressions. With this assignment and the symmetry properties Eq. (2.2), certain transformations need add a factor of 2 or 4. They are

$$
\begin{align*}
2S_{ijk} &= S_{ipq}, & \text{if either } p &\text{ or } q > 3, \\
4S_{ijk} &= S_{ipq}, & \text{if both } p &\text{ and } q > 3, \\
2d_{ij} &= e_{ij}, & 2g_{ij} &= g_{ip}, & \text{if } p > 3.
\end{align*}
$$

(2.3)

No factors are needed for all other transformations.

By using the contracted notation, the constitutive laws Eq. (2.1) can be written in matrix form as

$$
\begin{align*}
\sigma &= [C_e \ e^{-\alpha_t}] [\varepsilon], \\
\varepsilon &= [S_e \ d^{-\alpha_t}] [-\sigma], \\
\sigma &= [C_d \ h^{-\beta_t}] [\varepsilon], \\
\varepsilon &= [S_d \ g^{-\beta_t}] [-\sigma],
\end{align*}
$$

(2.4)

where

$$
\begin{align*}
\sigma &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \ 0 & \sigma_{22} & \sigma_{23} \ 0 & \sigma_{32} & \sigma_{33} \end{bmatrix}, \\
\varepsilon &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \ 0 & \varepsilon_{22} & \varepsilon_{23} \ 0 & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \\
C_e &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \ 0 & C_{22} & C_{23} \ 0 & 0 & C_{33} \end{bmatrix}, \\
C_d &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \ 0 & C_{22} & C_{23} \ 0 & 0 & C_{33} \end{bmatrix},
\end{align*}
$$

(2.5)

$$
\begin{align*}
E &= \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \\
D &= \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}, \\
\varepsilon &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15} & \varepsilon_{16} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} & \varepsilon_{25} & \varepsilon_{26} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & \varepsilon_{34} & \varepsilon_{35} & \varepsilon_{36} \end{bmatrix}, \\
\sigma &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} \end{bmatrix}, \\
\omega &= \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix},
\end{align*}
$$

and similar expressions for $S_e$, $S_d$, $C_d$, $g$, $\alpha_t$, $\beta_t$. In the above, the superscript $T$ of the matrix denotes the transpose. Since the four equation sets shown in (2.4) describe the same materials from different bases, the matrices in different set
of (2.4) should have some relations. In other words, from any one of the four equation sets, one can obtain the other three sets by simple mathematical operation. For example, starting from the first set of (2.4) we can obtain the following relations

\[
\begin{align*}
S_E &= C_E^{-1}, & d &= eC_E^{-1}e^T + \omega, \\
C_D &= C_E + e^T\omega e, & h &= \omega e^T, & \beta_e &= \omega_e, \\
S_D &= C_E - C_E^{-1}e^T\omega eC_E^{-1}, & g &= \omega e\eta C_E^{-1}, & \beta_g &= \omega_g^{-1}.
\end{align*}
\]

(2.6)

Generally, their relations can be expressed by the following equations

\[
\begin{align*}
C_E &= S_E^{-1}, & C_D &= S_D^{-1}, & \beta_e &= \omega_e^{-1}, & \beta_g &= \omega_g^{-1}, \\
d &= eS_E e^T d + \omega e, & h &= dC_E e^T d = \eta e, & g &= hS_D e^T d = \eta g, \\
\omega_d - \omega_e &= dC_e d^T = eS_e e^T d = \eta d, & \beta_e - \beta_g &= hS_h e^T = gC_g g^T = \eta g, \\
C_D - C_e &= e^T\beta e = h^T\omega_h e = \eta g, & S_E - S_D &= g^T\omega_g g - d^T\beta_g d = d^T g.
\end{align*}
\]

(2.7)

3. Constitutive laws for piezoelectric materials in two-dimensional states

If we consider the most general anisotropic materials, the in-plane and anti-plane deformations will not be decoupled. Under this condition, the two-dimensional states are usually described by generalized plane strain ($\sigma_3 = 0$) or generalized plane stress ($\sigma_3 = 0$) without requiring the transverse shear strain or transverse shear stress to be zero. While for electric fields, open circuit condition ($D_3 = 0$) is considered when the faces of piezoelectric materials are in contact with non-conducting media and the top and bottom surfaces are free of charge; or short circuit condition ($E_3 = 0$) is considered if the top and bottom surfaces of the piezoelectric materials are held at the same electric potential. With the above consideration, the two-dimensional states will be divided into four different situations, i.e.,

I. Generalized plane strain and short circuit: $\sigma_3 = 0$ and $E_3 = 0$.

II. Generalized plane strain and open circuit: $\sigma_3 = 0$ and $D_3 = 0$.

III. Generalized plane stress and short circuit: $\sigma_3 = 0$ and $E_3 = 0$.

IV. Generalized plane stress and open circuit: $\sigma_3 = 0$ and $D_3 = 0$.

(3.1)

Under the above four different states, the constitutive laws (2.4) can be further reduced by eliminating the terms associated with zero values of $\sigma_3$ (or $E_3$) and $E_3$ (or $D_3$), and replacing $\sigma_3$ (or $E_3$) and $D_3$ (or $E_3$) by the other two-dimensional terms. By this way, the constitutive laws for piezoelectric materials in two-dimensional states can be written in matrix form as State I: $\sigma_3 = 0$ and $E_3 = 0$.

\[
\begin{align*}
\begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix} &= \begin{bmatrix}
C_E^0 & e^T \\
e^0 & \omega e^0
\end{bmatrix}
\begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix}, & \begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix} &= \begin{bmatrix}
S_E^0 & -d^T \\
-\omega d & -\omega e^T
\end{bmatrix}
\begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix}, & \begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix} &= \begin{bmatrix}
C_D^0 & h^T \\
-\omega_h & -\omega_g
\end{bmatrix}
\begin{bmatrix}
\sigma^0 \\
D^0
\end{bmatrix}.
\end{align*}
\]

(3.2a)

where

\[
\sigma^0 = \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}, \quad e^0 = \begin{bmatrix}
e_1 \\
e_2 \\
e_4 \\
e_5 \\
e_6
\end{bmatrix}, \quad C_E^0 = \begin{bmatrix}
C_{11}^0 & C_{12}^0 & C_{14}^0 & C_{15}^0 & C_{16}^0 \\
C_{12}^0 & C_{22}^0 & C_{24}^0 & C_{25}^0 & C_{26}^0 \\
C_{14}^0 & C_{24}^0 & C_{44}^0 & C_{45}^0 & C_{46}^0 \\
C_{15}^0 & C_{25}^0 & C_{45}^0 & C_{55}^0 & C_{56}^0 \\
C_{16}^0 & C_{26}^0 & C_{46}^0 & C_{56}^0 & C_{66}^0
\end{bmatrix}, \quad E_0 = \begin{bmatrix}
E_1 \\
E_2
\end{bmatrix}, \quad D_0 = \begin{bmatrix}
D_1 \\
D_2
\end{bmatrix},
\]

(3.2b)

\[
e^0 = \begin{bmatrix}
e_{11} \\
e_{12} \\
e_{14} \\
e_{15} \\
e_{16} \\
e_{21} \\
e_{22} \\
e_{24} \\
e_{25} \\
e_{26}
\end{bmatrix}, \quad \sigma_0^0 = \begin{bmatrix}
\sigma_0^0 \\
\sigma_0^0
\end{bmatrix} = \begin{bmatrix}
\omega_{11} & \omega_{12} \\
\omega_{12} & \omega_{22}
\end{bmatrix},
\]

(3.2c)

\[
\begin{bmatrix}
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0 \\
\tilde{S}_0
\end{bmatrix}, \quad \tilde{g} = \begin{bmatrix}
\tilde{g}_{11} \\
\tilde{g}_{12} \\
\tilde{g}_{14} \\
\tilde{g}_{15} \\
\tilde{g}_{16} \\
\tilde{g}_{21} \\
\tilde{g}_{22} \\
\tilde{g}_{24} \\
\tilde{g}_{25} \\
\tilde{g}_{26}
\end{bmatrix}, \quad \tilde{\beta}_g = \begin{bmatrix}
\tilde{\beta}_{11} \\
\tilde{\beta}_{12} \\
\tilde{\beta}_{22}
\end{bmatrix}.
\]

(3.2d)
and

\[
\bar{\mathbf{S}}_y^0 = \bar{\mathbf{S}}_y^0 + \bar{\mathbf{G}}_y^0 \quad \bar{\mathbf{g}}^0_y = \bar{\mathbf{g}}^0_y + \frac{\bar{\mathbf{b}} y^0}{\bar{\mathbf{b}}_y^0}, \quad \bar{\mathbf{p}}_y^0 = \bar{\mathbf{p}}_y^0 + \frac{\bar{\mathbf{b}} y^0}{\bar{\mathbf{b}}_y^0} = \bar{\mathbf{p}}_y^0.
\]  

(3.2d)

in which

\[
\bar{\mathbf{S}}_y^0 = \bar{\mathbf{S}}_y^0 + \frac{\delta y^0}{\bar{\mathbf{S}}_y^0} = \bar{\mathbf{S}}_y^0 \quad \bar{\mathbf{g}}^0_y = \bar{\mathbf{g}}^0_y - \frac{\bar{\mathbf{b}} y^0}{\bar{\mathbf{b}}_y^0}, \quad \bar{\mathbf{p}}_y^0 = \bar{\mathbf{p}}_y^0 - \frac{\bar{\mathbf{b}} y^0}{\bar{\mathbf{b}}_y^0} = \bar{\mathbf{p}}_y^0.
\]  

(3.2e)

To save the space of this paper, the expressions for \( \mathbf{S}_E, \mathbf{d}, \mathbf{\dot{\omega}}_E, \mathbf{C}_E, \mathbf{h}' \) and \( \mathbf{p}' \), which can be obtained by the same way, are not shown here. Similarly, only two types of constitutive laws are shown below for the other three states.

State II: \( \sigma_3 = 0 \) and \( D_3 = 0 \)

\[
\begin{align*}
\begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_E \\ \mathbf{e} - \alpha_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix}, \\
\begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^T \mathbf{- \beta}_e \end{bmatrix} \begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix}.
\end{align*}
\]

(3.3)

State III: \( \sigma_3 = 0 \) and \( E_3 = 0 \)

\[
\begin{align*}
\begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_E \\ \mathbf{e} - \alpha_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix}, \\
\begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^T \mathbf{- \beta}_e \end{bmatrix} \begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix}.
\end{align*}
\]

(3.4)

State IV: \( \sigma_3 = 0 \) and \( D_3 = 0 \)

\[
\begin{align*}
\begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_E \\ \mathbf{e} - \alpha_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix}, \\
\begin{bmatrix} \mathbf{e}^0 \\ \mathbf{g}^0 \\ -\mathbf{E}^0 \end{bmatrix} &= \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^T \mathbf{- \beta}_e \end{bmatrix} \begin{bmatrix} \mathbf{D}^0 \\ \mathbf{b}^0 \end{bmatrix}.
\end{align*}
\]

(3.5)

In the above,

\[
\begin{align*}
\mathbf{C}_y^E &= \mathbf{C}_y^E + \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \quad \mathbf{e}^0 = \mathbf{e} - \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \quad \mathbf{e}^0 = \mathbf{e} - \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \\
\mathbf{C}_y^E &= \mathbf{C}_y^E + \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \quad \mathbf{e}^0 = \mathbf{e} - \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \quad \mathbf{e}^0 = \mathbf{e} - \frac{\mathbf{e} y^0 \mathbf{e}^T}{\alpha_3}, \\
\mathbf{S}_y^0 &= \mathbf{S}_y^0 + \frac{\mathbf{g} y^0 \mathbf{g}^T}{\alpha_3}, \quad \mathbf{g}^0 = \mathbf{g} - \frac{\mathbf{g} y^0 \mathbf{g}^T}{\alpha_3}, \quad \mathbf{g}^0 = \mathbf{g} - \frac{\mathbf{g} y^0 \mathbf{g}^T}{\alpha_3}.
\end{align*}
\]

(3.6)

and \( \mathbf{S}_y^0, \mathbf{g}_y^0, \mathbf{p}_y^0, \mathbf{S}_y^0, \mathbf{g}_y^0, \mathbf{p}_y^0 \) are given in (3.2d) and (3.2e).

Similar to the three-dimensional states, some relations between the material constants can be obtained through simple inversion such as

\[
\begin{bmatrix} \mathbf{C}_y^E \\ \mathbf{e}^0 \\ \mathbf{e}^0 - \alpha_0^0 \end{bmatrix} \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^0 \\ \mathbf{g}^0 - \mathbf{E}^0 \end{bmatrix} = \mathbf{I}, \quad \begin{bmatrix} \mathbf{C}_y^E \\ \mathbf{e}^0 \\ \mathbf{e}^0 - \alpha_0^0 \end{bmatrix} \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^0 \\ \mathbf{g}^0 - \mathbf{E}^0 \end{bmatrix} = \mathbf{I}.
\]

(3.7a)

and

\[
\begin{bmatrix} \mathbf{C}_y^E \\ \mathbf{e}^0 \\ \mathbf{e}^0 - \alpha_0^0 \end{bmatrix} \begin{bmatrix} \mathbf{S}_d^0 \\ \mathbf{g}^0 \\ \mathbf{g}^0 - \mathbf{E}^0 \end{bmatrix} = \mathbf{I}.
\]

(3.7b)

4. Extended Stroh formalism for piezoelectric anisotropic elasticity

For two-dimensional linear anisotropic elasticity, there are two major complex variable formalisms in the literature. One is Lekhnitskii formalism (Lekhnitskii, 1963) which starts with the equilibrated stress functions followed by constitutive laws, strain-displacement relations and compatibility equations; the other is Stroh formalism (Stroh, 1958) which starts with the compatible displacements followed by strain-displacement relations, constitutive laws and equilibrium equations. With this understanding, to develop the extended Stroh formalism for piezoelectric anisotropic elasticity the most appropriate constitutive relation is the first equation set of Eq. (2.1). While for the extended Lekhnitskii formalism the most appropriate constitutive relation is the last equation set of Eq. (2.1). Thus, to describe the extended Stroh formalism for piezoelectric anisotropic elasticity, it is better to write the basic equations as

\[
\begin{align*}
\sigma_{ij} &= \mathbf{C}_{ijkl} \mathbf{e}^{kl} - \mathbf{e}_{ij} E_k, \\
\mathbf{D}_j &= \mathbf{e}_{ijkl} \mathbf{e}^{kl} + \mathbf{e}_{ij} E_k, \\
\mathbf{g}_{ij} &= \frac{1}{2} (\mathbf{u}_{ij} + \mathbf{u}_{ji}), \quad \mathbf{D}_{ij} = 0, \quad i,j,k,l = 1,2,3, \quad \mathbf{D}_{ij} = 0
\end{align*}
\]

(4.1)

where repeated indices imply summation, a comma stands for differentiation and \( u_i \) is the displacement in \( x_i \)-axis. By letting
the basic Eq. (4.1) can be rewritten in an expanded tensor notation as
\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \quad \sigma_{ij} = 0, \quad I, J, K, L = 1, 2, 3, 4.
\] (4.3)
where expanded elastic stiffness tensor \( C_{ijkl} \) has the following symmetry property
\[
C_{ijkl} = C_{jikl} = C_{iklj} = C_{jkl}.
\] (4.4)
Since the mathematical form of expanded expression (4.3) for piezoelectric anisotropic elasticity is exactly the same as that of pure anisotropic elasticity, the general solutions satisfying all basic Eq. (4.1) under two-dimensional deformation can therefore be written in the form of Stroh formalism and is usually called extended Stroh formalism. Following is the general solutions satisfying (4.1) presented in (Kuo and Barnett, 1991; Suo et al., 1992; Liang and Hwu, 1996; Ting, 1996)
\[
\mathbf{u} = 2 \text{Re}\{A \mathbf{f}(z)\}, \quad \phi = 2 \text{Re}\{B \mathbf{f}(z)\},
\] (4.5a)
where
\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}, \quad \mathbf{f}(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{bmatrix},
\]
\[
\mathbf{A} = [a_1, a_2, a_3, a_4], \quad \mathbf{B} = [b_1, b_2, b_1, b_4],
\]
\[
z_k = x_1 + \mu_k x_2, \quad k = 1, 2, 3, 4,
\]
and \( \text{Re} \) stands for the real part. The stress function \( \phi_i \) is related to the stresses by
\[
\sigma_{11} = -\phi_{12}, \quad \sigma_{22} = \phi_{11}, \quad i = 1, 2, 3, 4.
\] (4.6)
\( f_k(z_k), k = 1, 2, 3, 4 \) are four holomorphic functions of complex variables \( z_k \), which will be determined by the boundary conditions set for each particular problem. \( \mu_k \) and \( (a_k, b_k) \) are the material eigenvalues and eigenvectors which can be determined by the following eigenrelations:
\[
\mathbf{N} \xi = \mu^\xi \xi
\] (4.7a)
where \( \mathbf{N} \) is a 8 \times 8 fundamental matrix and \( \xi \) is a 8 \times 1 column vector defined by
\[
\mathbf{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \quad \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}
\] (4.7b)
and
\[
\mathbf{N}_1 = -\mathbf{T}^{-1} \mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^T, \quad \mathbf{N}_3 = \mathbf{R}^{-1} \mathbf{R}^T - \mathbf{Q} = \mathbf{N}_3^T.
\] (4.7c)
\( \mathbf{Q}, \mathbf{R}, \mathbf{T} \) are three 4 \times 4 real matrices defined by the elastic constants as
\[
Q_{ik} = C_{ijkl}, \quad R_{ik} = C_{nikl}, \quad T_{ik} = C_{ikjl}, \quad i, k = 1, 2, 3, 4.
\] (4.8)
Note that the general solutions (4.5) are obtained by considering the two-dimensional deformation in which \( u_i, i = 1, 2, 3, 4 \), depend on \( x_1 \) and \( x_2 \) only. Through the strain-displacement relation (4.1) and the relation for the electric field (4.2), we know that the two-dimensional state considered in the extended Stroh formalism is state I: generalized plane strain and short circuit \( (\varepsilon_3 = 0 \text{ and } E_3 = 0) \). From (3.2) we see that the material constants used in this state are \( C_{ij}^e, e_i^o, e_j^o, S_{ij}^e, \rho_{ij}^o, \rho_j^o, \text{ or } S_{ij}^e, d_i, d_j^o, \text{ or } C_{ij}^o, h_i^o, h_j^o. \) For the other two-dimensional states, to employ the general solution (4.5) the material constants should be replaced according to the relations shown in (3.3)–(3.5). For example, \( C_{ij}^e, e_i^o, e_j^o \) should be replaced by \( C_{ij}^o, e_i^o, e_j^o \) for state II, and replaced by \( C_{ij}^e, e_i^o, e_j^o \) for state III, and replaced by \( C_{ij}^c, e_i^c, e_j^c \) for state IV.
For the convenience of readers’ reference, we now show the matrix expressions \( \mathbf{Q}, \mathbf{R}, \mathbf{T} \) of (4.8) for state I.
\[
\mathbf{Q} = \begin{bmatrix} C_{11} & C_{12} & C_{15} & e_{11} \\ C_{16} & C_{16} & C_{15} & e_{16} \\ C_{15} & C_{15} & C_{15} & e_{15} \\ e_{11} & e_{16} & e_{15} & -e_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} C_{16} & C_{12} & C_{14} & e_{21} \\ C_{26} & C_{26} & C_{24} & e_{26} \\ C_{25} & C_{25} & C_{25} & e_{25} \\ e_{16} & e_{12} & e_{14} & -e_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{66} & C_{26} & C_{26} & e_{26} \\ C_{26} & C_{22} & C_{24} & e_{22} \\ C_{24} & C_{24} & C_{24} & e_{24} \\ e_{26} & e_{22} & e_{24} & -e_{22} \end{bmatrix}
\] (4.9)
Note that because the material eigenvalues \( \mu_k \) obtained from the eigen-relation (4.7) cannot be real if the strain energy is positive (Suo et al., 1992; Ting, 1996), \( \mu_k \) occurs as four pairs of complex conjugates. In the general solution (4.5), the material eigenvalues \( \mu_k \) and material eigenvectors \( \mathbf{a}, \mathbf{b} \) have been arranged to be \( \mu_{k,4} = \bar{\mu}_k, \text{ Im}(\mu_k) > 0, \) and \( \mathbf{a}_{k,4} = \bar{\mathbf{a}}, \mathbf{b}_{k,4} = \bar{\mathbf{b}}, k = 1, 2, 3, 4 \) where an overbar denotes the complex conjugate and \( \text{Im} \) stands for the imaginary part. Moreover, in the general solution (4.5), the material eigenvalues are assumed to be distinct and their associated eigenvectors are independent each other. For the cases that the material eigenvalues are repeated so that their associated eigenvectors are not independent each other, the general solution (4.5) should be modified (Ting, 1996) or one may introduce a small perturbation in the values of material properties to avoid the problem of degeneracy (Hwu and Yen, 1991).

From the above discussions we know that the fundamental matrix \( \mathbf{N} \) and its associated eigenvector \( \zeta = (\mathbf{a}, \mathbf{b}) \) play important roles in Stroh formalism. Due to their importance, several works have been done to get their explicit expressions for pure anisotropic elastic materials (Ting, 1996). Since \( \mathbf{a}_k, \mathbf{b}_k \) are the right eigenvectors of the fundamental matrix \( \mathbf{N} \), to have unique values of \( \mathbf{a}_k, \mathbf{b}_k \), normalization is necessary. Following is the orthogonality relation for the material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \) of anisotropic materials (Ting, 1996), which can also be extended to the piezoelectric anisotropic materials.

\[
\begin{bmatrix}
\mathbf{B}^T & \mathbf{A}^T
\end{bmatrix}
\begin{bmatrix}
\bar{\mathbf{A}} & \bar{\mathbf{B}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{I} & 0 \\
0 & 1
\end{bmatrix}.
\]  

(4.10a)

The two \( 8 \times 8 \) matrices on the left side of (4.10a) are the inverse of each other, and hence their products commute, i.e.,

\[
\begin{bmatrix}
\mathbf{A} & \bar{\mathbf{A}}
\end{bmatrix}
\begin{bmatrix}
\mathbf{B} & \bar{\mathbf{B}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{I} & 0 \\
0 & 1
\end{bmatrix}.
\]  

(4.10b)

From the relations (4.10), it has been observed that the following three matrices \( \mathbf{S}, \mathbf{H} \) and \( \mathbf{L} \) are real, which appear often in the final solutions to two-dimensional anisotropic elasticity problems,

\[
\mathbf{S} = 2i(\mathbf{A} \mathbf{B}) - \mathbf{I}, \quad \mathbf{H} = 2(\mathbf{A} \mathbf{A})^T, \quad \mathbf{L} = -2i(\mathbf{B} \mathbf{B})^T.
\]  

(4.11)

These three real matrices have also been proved to be the average values of \( \mathbf{N}_1(\theta), \mathbf{N}_2(\theta) \) and \( -\mathbf{N}_3(\theta) \) over the interval \( \theta = (0, \pi) \) (Barnett and Lothe, 1973), and hence are usually called Barnett–Lothe tensors.

5. Explicit expressions for fundamental matrix \( \mathbf{N} \)

Although the fundamental matrix \( \mathbf{N} \) is defined clearly in (4.7b, 4.7c), their calculation involves the matrix inversion. Therefore, if we do not pay special attention to get their explicit expressions, their results from pure numerical calculation can only provide their numerical values which are not appropriate for the understanding of the physical meaning of the analytical solutions found by using the extended Stroh formalism. Even it is possible to find the explicit expressions by using the symbolic computational software such as Mathematica, ignorant of the relations among the components may lead to complicated expressions. Although it has been indicated by Ting (1996) that certain elements of \( \mathbf{N}_1 \) and \( \mathbf{N}_2 \) are zero for piezoelectric materials, no explicit expressions for \( \mathbf{N} \) have been published in the literature. In this section, we will follow the steps described in (Ting, 1996) for pure anisotropic materials to get the explicit expressions of \( \mathbf{N} \) for piezoelectric materials.

In order to find the explicit expressions of \( \mathbf{N} \), we first re-organize (4.7c) into the following compact matrix form

\[
\begin{bmatrix}
\mathbf{I} & \mathbf{N}_l^T \\
0 & \mathbf{N}_r
\end{bmatrix}
\begin{bmatrix}
\mathbf{Q} & \mathbf{R} \\
\mathbf{R}^T & \mathbf{T}
\end{bmatrix} = \begin{bmatrix}
-\mathbf{N}_3 & 0 \\
0 & -\mathbf{N}_3
\end{bmatrix}.
\]  

(5.1)

Re-arrangement of (3.7a), and knowing the matrix expressions of \( \mathbf{Q}, \mathbf{R}, \mathbf{T} \) given in (4.9), we can get

\[
\begin{bmatrix}
\mathbf{Q} & \mathbf{R} \\
\mathbf{R}^T & \mathbf{T}
\end{bmatrix} = \begin{bmatrix}
\mathbf{I}_{2} & \mathbf{I}_{21} \\
0 & \mathbf{I}
\end{bmatrix}
\]  

(5.2a)

where

\[
\begin{bmatrix}
\mathbf{Q}^* \\
\mathbf{R}^* \\
\mathbf{T}
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
\mathbf{S}_{11} & 0 & \mathbf{S}_{15} & \mathbf{g}_{11} \\
0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{S}_{15} & \mathbf{g}_{15} \\
\mathbf{g}_{11} & \mathbf{g}_{15} & 0 & 0
\end{bmatrix}, & \begin{bmatrix}
\mathbf{S}_{16} & \mathbf{S}_{12} & \mathbf{S}_{14} & \mathbf{g}_{21} \\
0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{S}_{25} & \mathbf{g}_{25} \\
\mathbf{g}_{16} & \mathbf{g}_{12} & \mathbf{g}_{14} & \mathbf{g}_{25} \\
\end{bmatrix}, & \begin{bmatrix}
\mathbf{S}_{26} & \mathbf{S}_{22} & \mathbf{S}_{24} & \mathbf{g}_{22} \\
0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{S}_{45} & \mathbf{g}_{45} \\
\mathbf{g}_{26} & \mathbf{g}_{22} & \mathbf{g}_{24} & \mathbf{g}_{25} \\
\end{bmatrix}
\end{bmatrix}
\]  

(5.2b)

and

\[
\begin{bmatrix}
\mathbf{I}_{2} \\
\mathbf{I}_{21} \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]  

(5.2c)

Employing the relation (5.1), Eq. (5.2a) becomes
\[ \begin{bmatrix} -\mathbf{N}_i \mathbf{Q}^* \end{bmatrix} \begin{bmatrix} \mathbf{R}^T \end{bmatrix} = \begin{bmatrix} \mathbf{I}_2 \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{N}_i \end{bmatrix} \]

or

\[ -\mathbf{N}_i \mathbf{Q}^* = \mathbf{I}_2, \quad -\mathbf{N}_i \mathbf{R}^* = \mathbf{I}_2, \quad -\mathbf{N}_i \mathbf{Q}^* + \mathbf{R}^T = 0, \quad -\mathbf{N}_i \mathbf{R}^* + \mathbf{T}^- = \mathbf{N}_2. \]

From the above results, we see that the explicit expressions of \( \mathbf{N}_3 \) can be obtained directly from the first equation of Eq. (5.3b) since \( \mathbf{Q}^* \) shown in (5.2b) is a matrix of rank 3 not a full matrix of rank 4. Substituting the result of \( \mathbf{N}_3 \) into (5.3b) we can get \( \mathbf{N}_4 \), and then obtain \( \mathbf{N}_5 \) through (5.3b). Through this procedure, the explicit expressions of \( \mathbf{N}_1, \mathbf{N}_2, \) and \( \mathbf{N}_4 \) have been obtained as

\[
\mathbf{N}_1 = \begin{bmatrix} X_0 - 1 & Y_6 & Z_6 \\ X_2 & 0 & Y_2 & Z_2 \\ X_4 & 0 & Y_4 & Z_4 \\ X_6 & 0 & Y_8 & Z_6 \end{bmatrix}, \quad \mathbf{N}_2 = -\mathbf{N}_1 \mathbf{R}^* + \mathbf{T}^*, \quad \mathbf{N}_3 = -\frac{1}{\Delta} \begin{bmatrix} S_{11} & 0 & S_{15} & g_{11}^* \\ 0 & 0 & 0 & 0 \\ S_{15} & 0 & S_{55} & g_{15}^* \\ g_{15}^* & 0 & g_{15}^* & \beta_{15}^* \end{bmatrix}
\]

where

\[
\Delta = -S_{11} S_{55} \mu_{11} + 2S_{15} S_{15} \mu_{15} - S_{11} S_{15} \mu_{11} + S_{11} S_{15} \mu_{11} + \mu_{11} (S_{15} \mu_{15}),
\]

\[
S_{11} = -S_{11} S_{55} \mu_{11} - g_{15}^*, \quad S_{15} = S_{11} S_{55} \mu_{15} + g_{15}^*, \quad S_{55} = -S_{11} S_{55} \mu_{11} - g_{15}^*,
\]

\[
g_{11}^* = S_{15} g_{11}^* - S_{15} g_{11}^*, \quad g_{15}^* = -S_{11} g_{15}^* + S_{15} g_{15}^*, \quad \mu_{15}^* = S_{11} g_{15}^* - (S_{15} g_{15}^*),
\]

By using the relation obtained in (6.2), each displacement component can be expressed in terms of the stress functions through integration. With this relation, the following compatibility equations for two-dimensional problems,

\[
\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0, \quad -\varepsilon_{23,1} + \varepsilon_{13,2} = 0, \quad E_{1,2} - E_{2,1} = 0,
\]

will give us

\[
\mathbf{D} \mathbf{S}_p \mathbf{D}_p \mathbf{\phi} = 0.
\]
where
\[
\mathbf{D}_i = \begin{bmatrix}
\frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_2^2} & 0 & 0 & -\frac{\partial^2}{\partial x_1 \partial x_2} & 0 \\
0 & 0 & -\frac{\partial^2}{\partial x_1} & \frac{\partial^2}{\partial x_2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial^2}{\partial x_1} & -\frac{\partial^2}{\partial x_2} \\
\end{bmatrix}.
\]

(6.4b)

Since (6.4) is a system of homogeneous partial differential equations in two independent variables \(x_1\) and \(x_2\). A general solution for \(\phi_i\) depends on one complex variable that is a linear combination of \(x_1\) and \(x_2\), which is also applicable for \(u_i\) through (6.2a). Without loss of generality the coefficient of \(x_1\) is usually selected to be unity, i.e., \(z = x_1 + \mu x_2\). By comparison with the general solutions shown in (4.5a), we may now let
\[
u_i = ai f(z), \quad \phi_i = bi f(z),
\]
in which \(a_i\) and \(b_i\) are the material eigenvectors to be determined in this section. Substituting (6.5) into (6.4a) we obtain
\[
\mathbf{\Gamma}_s \mathbf{S}_m \mathbf{\Gamma}_s \mathbf{b} = \mathbf{0},
\]
(6.6a)

where
\[
\mathbf{\Gamma}_c = \begin{bmatrix}
\mu^2 & 1 & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & -1 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & -1 \\
\end{bmatrix}, \quad \mathbf{\Gamma}_s = \begin{bmatrix}
-\mu & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\mu \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
(6.6b)

Note that in the above \(\mathbf{\Gamma}_c\) and \(\mathbf{\Gamma}_s\) can be obtained directly from the matrices of differential operator, \(\mathbf{D}_c\) and \(\mathbf{D}_s\), with \(\partial / \partial x_1\) replaced by 1 and \(\partial / \partial x_2\) replaced by \(\mu\).

Eq. (6.6a) is a linear algebraic system of equations with four unknowns and three equations. To solve the unknown vector \(\mathbf{b}\), we need one more relation. From (4.6) we see that \(\phi_i, i = 1, 2, 3, 4\), are not independent each other because of the symmetry of stress \(\sigma_{12} = \sigma_{21}\), which will lead to
\[
-\phi_{22} = \phi_{11}.
\]
(6.7)

Substituting (6.5) into (6.7) we obtain
\[
-\mu b_2 = b_1,
\]
(6.8)

With the relation (6.8), the system of Eq. (6.6a) can be rewritten as
\[
\mathbf{\Gamma}_s \mathbf{S}_m \mathbf{\Gamma}_s \mathbf{b} = \mathbf{0},
\]
(6.9a)

where
\[
\mathbf{\Gamma}_s = \begin{bmatrix}
\mu^2 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\mu & 0 \\
-\mu & 0 & 0 \\
0 & 0 & -\mu \\
0 & 0 & 1 \\
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix},
\]
(6.9b)

After operating the matrix multiplication for (6.9), we get
\[
\begin{bmatrix}
-\ell_4 & \ell_3 & m_3 \\
-\ell_3 & \ell_2 & m_2 \\
-m_3 & m_2 & \rho_2 \\
\end{bmatrix} \begin{bmatrix}
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix} = \mathbf{0},
\]
(6.10a)

where
\[
\ell_2 = \tilde{S}_{15}^d \mu^2 - 2 \tilde{S}_{16}^d \mu + \tilde{S}_{17}^d,
\]
\[
\ell_3 = \tilde{S}_{16}^d \mu^3 - (\tilde{S}_{14}^d + \tilde{S}_{15}^d) \mu^2 + (\tilde{S}_{17}^d + \tilde{S}_{18}^d) \mu - \tilde{S}_{19}^d,
\]
\[
\ell_4 = \tilde{S}_{17}^d \mu^4 - 2 \tilde{S}_{16}^d \mu^3 + (2 \tilde{S}_{14}^d + \tilde{S}_{15}^d) \mu^2 - 2 \tilde{S}_{17}^d \mu + \tilde{S}_{18}^d,
\]
\[
m_2 = \tilde{S}_{20}^e \mu^2 - (\tilde{S}_{14}^e + \tilde{S}_{15}^e) \mu + \tilde{S}_{16}^e,
\]
\[
m_3 = \tilde{S}_{21}^e \mu^3 - (\tilde{S}_{14}^e + \tilde{S}_{16}^e) \mu^2 + (\tilde{S}_{17}^d + \tilde{S}_{18}^d) \mu - \tilde{S}_{19}^d,
\]
\[
\rho_2 = -\tilde{S}_{17}^d \mu^2 + 2 S_{16}^d \mu - \tilde{S}_{18}^d.
\]
(6.10b)

Non-trivial solutions of \(b_2, b_3, b_4\) exist only when the determinant of the coefficient matrix equal to zero, which will lead to the following characteristic equation for the determination of the material eigenvalues \(\mu\).
\[ \ell_2 \ell_4 \rho_2 + 2 \ell_2 m_2 m_3 - \ell_2 m_2^2 - \ell_3 m_2^2 - \rho_2 \ell_3^2 = 0. \] (6.11)

Eq. (6.11) is an 8th order polynomial which should lead to the same eigenvalues as those obtained from the eigen-relation (4.7). Furthermore, after obtaining the eigenvalues from (6.11), its associated eigenvectors can be obtained from (6.10a) and (6.8). The results are

\[ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = c \begin{bmatrix} \mu \ell_3 \\ -\ell_4 \\ \ell_3 \\ m_1^2 \end{bmatrix}, \quad \text{or} \quad c \begin{bmatrix} \mu \ell_3 \\ -\ell_3 \\ \ell_2 \\ m_2^2 \end{bmatrix}, \quad \text{or} \quad c \begin{bmatrix} \mu m_3^2 \\ -m_3^2 \\ \rho_2^2 \\ m_2^2 \end{bmatrix}. \] (6.12a)

where

\[ \ell_3 = \ell_4 \rho_2 - m_3^2, \quad \ell_2 = m_2 m_3 - \ell_3 \rho_2, \quad \ell_3 = \ell_2 \rho_2 - m_2^2, \]
\[ m_2 = \ell_3 m_3 - \ell_4 m_2, \quad m_3 = \ell_5 m_2 - \ell_3 m_3, \quad \rho_2 = \ell_2 \ell_4 - \ell_3^2. \] (6.12b)

In the above \( c \) is the scaling factor. Generally, the three different expressions shown in (6.12a) should be the same if they are non-trivial. If one or two of them is a trivial solution, i.e., zero, just take the non-trivial one as the eigenvector \( \mathbf{b} \). If all of them are trivial, one may take any three independent vectors of \( \mathbf{b} \) as the eigenvectors and employ the relation (6.8) to complete the eigenvector \( \mathbf{b} \). The solutions shown in (6.12) cover all the possible eigenvectors in which one of them agrees with that presented by Soh et al. (2001) whose solution will fail if its denominator equals to zero for some piezoelectric materials.

When \( \mathbf{b} \) is determined from (6.12), the other eigenvector \( \mathbf{a} \) can be obtained by the following relation, which is derived by substituting (6.5) into (6.2a),

\[ \Gamma \mathbf{a} = S \gamma \Gamma \mathbf{b}, \] (6.13)

where \( \Gamma \) is given in (6.6b), and \( \Gamma \) can be obtained from \( \mathbf{D} \) of (6.2b) with \( \partial / \partial x_1 \) replaced by 1 and \( \partial / \partial x_2 \) replaced by \( \mu \). By choosing an appropriate matrix \( \Gamma \), making \( \Gamma \gamma \Gamma = \mathbf{I} \), the eigenvector \( \mathbf{a} \) can then be determined by

\[ \mathbf{a} = \Gamma \gamma S \gamma \mathbf{b} = \Gamma \gamma S \gamma \Gamma \gamma \mathbf{b}_1 \] (6.14a)

where the second equality of Eq. (6.14a) comes from Eq. (6.8) and

\[ \Gamma \gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \] (6.14b)

After operating the matrix multiplication for (6.14a), we get

\[ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} p_1 b_2 + q_1 b_3 + r_1 b_4 \\ p_2 b_2 + q_2 b_3 + r_2 b_4 / \mu \\ p_4 b_2 + q_4 b_3 + r_4 b_4 / \mu \\ p_5 b_2 + q_5 b_3 + r_5 b_4 \end{bmatrix}, \] (6.15a)

where

\[ p_j = \mu^2 \tilde{S}^{0}_j + \tilde{S}^{0}_j - \mu \tilde{S}^{0}_j, \]
\[ q_j = \tilde{S}^{0}_j - \mu \tilde{S}^{0}_j, \]
\[ r_j = \tilde{S}^{0}_j - \mu \tilde{S}^{0}_j, \quad j = 1, 2, 4, 7 \] (6.15b)

Note that although the choice of \( \Gamma \gamma \) in (6.14b) may not be unique, different choice of \( \Gamma \gamma \) will lead to the same \( \mathbf{a} \). For example, if the 3\(^{rd} \) row of \( \Gamma \gamma \) is selected to be \((0 0 0 1 0 0)\), the 3\(^{rd} \) component of \( \mathbf{a} \) will be \( p_1 b_2 + q_1 b_3 + r_1 b_4 \) which can be proved to be identical to the one shown in (6.15a). After getting the explicit expressions of \( \mathbf{a} \) and \( \mathbf{b} \), through (6.12) and (6.15) for each material eigenvalue \( \mu_k \), \( k = 1, 2, 3, 4 \), the material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \) can be constructed as that shown in (4.5b) \( \mathbf{A}_5 \). To have a unique value for the eigenvectors, the scaling factors \( c_k \) should be normalized. The normalization has been defined through the orthogonality relation (4.10a), which shows thats

\[ C_k^2 = \frac{1}{2(a_{1k} b_{1k} + a_{2k} b_{2k} + a_{3k} b_{3k} + a_{4k} b_{4k})}, \quad k = 1, 2, 3, 4. \] (6.16)

where \( a_{jk} \) and \( b_{jk} \) are the components of material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \) before scaling.

In order to let readers have a clear picture about the explicit expressions of \( \mathbf{A} \) and \( \mathbf{B} \), two typical examples for the piezoelectric materials are shown below.
Example 1: Piezoelectric ceramics poling in $x_3$-axis

The constitutive relations for piezoelectric ceramics with poling direction parallel to $x_3$-axis can be written as

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} = 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix}
- \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}.
$$

(6.17)

If we consider the two-dimensional state of generalized plane strain and short circuit ($\epsilon_3 = 0$ and $E_3 = 0$), the explicit expressions of the eigenvectors $a$ and $b$ shown in (6.12) and (6.15) are written in terms of $\tilde{S}^{0}_{ij}, \tilde{g}^{0}_{ij}, \tilde{P}^{0}_{ij}$. Therefore, to get the explicit expressions for the material eigenvector matrices $A$ and $B$, the first thing we need to do is finding the inverse relation of (6.17), which gives us

$$
\begin{align}
\tilde{S}^{0}_{11} &= \frac{c_{11}^2}{c_{11}^2 - c_{12}^2}, & \tilde{S}^{0}_{12} &= \frac{-c_{12}^2}{c_{11}^2 - c_{12}^2}, \\
\tilde{S}^{0}_{22} &= \frac{c_{11}^2 + c_{12}^2}{c_{11}^2 - c_{12}^2}, & \tilde{S}^{0}_{66} &= \frac{2}{c_{11}^2 - c_{12}^2}, \\
\tilde{S}^{0}_{48} &= \tilde{g}^{0}_{24} = \frac{c_{13}^2}{c_{15}^2 + c_{16}^2}, & \tilde{S}^{0}_{57} &= \tilde{g}^{0}_{15} = \frac{c_{13}^2}{c_{15}^2 + c_{16}^2}, & \tilde{S}^{0}_{77} &= \tilde{g}^{0}_{15} = \frac{c_{13}^2}{c_{15}^2 + c_{16}^2}, \\
\tilde{S}^{0}_{16} &= \tilde{g}^{0}_{16} = \frac{c_{13}^2}{c_{15}^2 + c_{16}^2}, & \tilde{S}^{0}_{18} &= \tilde{g}^{0}_{18} = \frac{c_{13}^2}{c_{15}^2 + c_{16}^2}.
\end{align}
$$

(6.18a)

and all the other constants are zero, i.e.,

$$
\begin{align}
\tilde{S}^{0}_{45} &= \tilde{S}^{0}_{45} = \tilde{S}^{0}_{46} = \tilde{S}^{0}_{47} = \tilde{S}^{0}_{48} = \tilde{S}^{0}_{56} = \tilde{S}^{0}_{57} = \tilde{S}^{0}_{58} = \tilde{S}^{0}_{67} = \tilde{S}^{0}_{68} = \tilde{S}^{0}_{78} = 0.
\end{align}
$$

(6.18b)

Substituting (6.18a, 6.18b) into (6.10b), we get

$$
\begin{align}
\tilde{E}^{2}_{1} &= \tilde{S}^{0}_{44} (\mu_2 + 1), & \tilde{E}^{4}_{1} &= \tilde{S}^{0}_{11} (\mu_2 + 1)^2, & m_2 &= \tilde{g}(\mu_2 + 1), & \rho_2 &= \beta (\mu_2 + 1), \\
\tilde{E}^{3}_{1} &= \tilde{E}^{3}_{1} = 0.
\end{align}
$$

(6.19)

With this result, the characteristic Eq. (6.11) gives us the material eigenvalues with positive imaginary part, $\mu_1, k = 2, 3, 4$, as

$$
\begin{align}
\mu_1 &= \mu_2 = \mu_3 = \mu_4 = i.
\end{align}
$$

(6.20)

With the repeated eigenvalues obtained in (6.20), we see that the coefficient matrix of $b,1$ of (6.10a) is identical to zero and all the explicit expressions shown in (6.12a) provide trivial solution, i.e., $b=0$. As we explain in the paragraph following (6.12b), under this condition we may take any three independent vectors of $b,1$ as the eigenvectors and the relation $-\beta b_2 = b_1$ should be employed to complete the eigenvector $b$. With this understanding, the eigenvector matrix $B$ can be written as

$$
B = \begin{bmatrix}
-c_{1} \mu_1 & -c_{2} \mu_2 & 0 & 0 \\
c_{1} & c_{2} & 0 & 0 \\
0 & 0 & c_{3} & 0 \\
0 & 0 & c_{3} & 0
\end{bmatrix}.
$$

(6.21)

Note that in the above all four eigenvectors are independent each other depending on the assumption that $\mu_1 \neq \mu_2$. However, from (6.20) we know that it is not true since all the eigenvalues are the same. Therefore, the material eigenvector matrix $B$ for this special case does not exist owing to the fact that no enough independent eigenvectors exist for the repeated eigenvalues. This is the so called degenerate materials. For this special kind of materials, the general solution shown in (4.5) is not valid and should be modified. A modified formalism for degenerate materials has been proposed in (Ting and Hwu, 1988; Ting, 1996) for two-dimensional anisotropic elasticity, which may also be applied to the piezoelectric problems. Even (6.21) is not valid for the present case when $\mu_1 = \mu_2 = i$. In many applications, it is very useful by treating $\mu_1 = i$ and $\mu_2 = i + \epsilon$ where $\epsilon$ is a small perturbed value. Successful application of (6.21) can be seen in the next section when we derive the explicit expressions of Barnett–Lothe tensors $I$, $S$, and $H$. 
After getting the eigenvectors \( b_k, k = 1, 2, 3, 4 \), their associated eigenvector \( a_k \) can be obtained from (6.15). By using the constant values given in (6.18) and the results obtained in (6.20) and (6.21) as well as the assumption that \( \mu_1 \neq \mu_2 \), we can now write down the explicit expression for the material eigenvector matrix \( A \) as

\[
A = \begin{bmatrix}
    c_1(\mu_1^2 \hat{S}_{11} + \hat{S}_{11}^0) & c_2(\mu_1^2 \hat{S}_{12} + \hat{S}_{12}^0) & 0 & 0 \\
    c_1(\mu_2^2 \hat{S}_{12} + \hat{S}_{12}^0) & c_2(\mu_2^2 \hat{S}_{11} + \hat{S}_{11}^0) & 0 & 0 \\
    0 & 0 & -ic_3 \hat{S}_{24} & -ic_4 \hat{S}_{24} \\
    0 & 0 & ic_3 \hat{S}_{24} & ic_4 \hat{S}_{24}
\end{bmatrix}.
\]

The scaling factors \( c_k, k = 1, 2, 3, 4 \) shown in (6.21) and (6.22) can then be determined by the orthogonality relation (4.10a), or obtained directly from (6.16) with \( a_{b_k} \) given in (6.21) and (6.22), e.g., \( b_{11} = -\mu_1, b_{21} = 1, b_{31} = b_{41} = 0, \ldots, a_{44} = i\beta \).

Example 2: Piezoelectric ceramics poling in \( x_2 \)-axis

The constitutive relations for piezoelectric ceramics with poling direction parallel to \( x_2 \)-axis can be written as

\[
\begin{align*}
\sigma_1 &= \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{12} & c_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e_{16} \\ e_{21} & e_{22} & e_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{16} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e_{16} \\ 0 & 0 & 0 & 0 & 0 & e_{16} \\ 0 & 0 & 0 & 0 & 0 & e_{16} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}.
\end{align*}
\]

Similar to the previous case, we firstly find the inverse relation of (6.23) under the condition that \( e_3 = 0 \) and \( E_3 = 0 \), which gives us

\[
\begin{align*}
\hat{S}_{11}^0 &= \frac{c_{11}^2}{c_{11}^2 - c_{12}^2}, & (e_{21}^2C_{11}^e - e_{22}^2C_{12}^e)C^e, \\
\hat{S}_{12}^0 &= \frac{c_{12}^2}{c_{12}^2 - c_{12}^2}, & (e_{21}^2C_{11}^e - e_{22}^2C_{12}^e)C^e, \\
\hat{S}_{22}^0 &= \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, & (e_{21}^2C_{11}^e - e_{22}^2C_{12}^e)C^e, \\
\hat{S}_{44}^0 &= \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, & \hat{S}_{16}^0 = \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, \hat{S}_{66}^0 = \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, \\
\hat{S}_{18}^0 &= \hat{S}_{21}^0 = (e_{21}C_{11}^e - e_{22}C_{12}^e), & \hat{S}_{12}^0 = \hat{S}_{22}^0 = (e_{21}C_{11}^e - e_{22}C_{12}^e), \\
\hat{S}_{67}^0 &= \hat{S}_{16}^0 = \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, & \hat{S}_{77}^0 = \hat{S}_{11}^0 = \frac{c_{44}^2}{c_{44}^2 - c_{12}^2}, \\
-\hat{S}_{88}^0 &= \hat{S}_{22}^0 = (e_{21}C_{11}^e - e_{22}C_{12}^e), & C^e,
\end{align*}
\]

where

\[
C^e = (e_{21}^2C_{11}^e - e_{22}^2C_{12}^e - e_{22}^2C_{12}^e + e_{21}^2C_{11}^e, \quad \hat{S}_{11}^0 = \hat{S}_{12}^0 = \hat{S}_{16}^0 = \hat{S}_{17}^0 = \hat{S}_{24}^0 = \hat{S}_{25}^0 = \hat{S}_{26}^0 = \hat{S}_{27}^0 = \hat{S}_{28}^0 = \hat{S}_{88}^0 = 0.
\]

and all the other constants are zero, i.e.,

\[
\begin{align*}
\hat{S}_{14}^0 &= \hat{S}_{15}^0 = \hat{S}_{16}^0 = \hat{S}_{17}^0 = \hat{S}_{24}^0 = \hat{S}_{25}^0 = \hat{S}_{26}^0 = \hat{S}_{27}^0 = \hat{S}_{28}^0 = \hat{S}_{88}^0 = \hat{S}_{22}^0 = 0.
\end{align*}
\]

Substituting (6.24a–c) into (6.10b), we get

\[
\begin{align*}
\ell_2 &= \hat{S}_{12}^0 \mu_2 + \hat{S}_{24}^0, & \ell_4 &= \hat{S}_{11}^0 \mu_4 + (\hat{S}_{11}^0 + \hat{S}_{12}^0)\mu_2 + \hat{S}_{22}^0, \\
m_3 &= -\hat{g}_{21}^0 + \hat{g}_{18}^0, & \rho_2 &= -\hat{p}_{11}^0 \mu_2 + \hat{p}_{22}^0, \\
\ell_5 &= m_2 = 0.
\end{align*}
\]

With the results of (6.25), the characteristic Eq. (6.11) can be reduced to
\[ \ell_2(\ell_4 p_2 - m_2^2) = 0. \] (6.26)

Let \( \mu_1 \) be the root of \( \ell_2 = 0 \) and \( \mu_2, \mu_3, \mu_4 \) be the roots of \( \ell_4 p_2 - m_2^2 = 0 \), whose imaginary parts are positive. Since \( \ell_4 p_2 - m_2^2 = 0 \) is a 3rd order polynomial in \( \mu^2 \) whose coefficients are all real and \( \mu \) cannot be real, the most general expressions for the roots of \( \mu^2 \) are one pair of complex conjugates and one real. And hence, we may let

\[ \mu_1 = i\sqrt{S_{44}/S_{55}}, \quad \mu_2 = x_2 + iy_2, \quad \mu_3 = x_2 - iy_2, \quad \mu_4 = iy_4. \] (6.27)

The material eigenvector matrix \( \mathbf{B} \) can then be constructed through (6.12) with the values given in (6.25). Its final simplified expression is

\[ \mathbf{B} = \begin{bmatrix} 0 & -c_2 \mu_2 & -c_3 \mu_3 & -c_4 \mu_4 \\ 0 & c_2 & c_3 & c_4 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 \eta_2 & c_3 \eta_3 & c_4 \eta_4 \end{bmatrix}. \] (6.28a)

where

\[ \eta_k = \frac{\ell_4(\mu_k)}{m_3(\mu_k)} = \frac{m_3(\mu_k)}{\rho_2(\mu_k)}, \quad k = 2, 3, 4. \] (6.28b)

Note that through (6.25), (6.27) and (6.28b) we see that some relations exist for \( \mu_k \) and \( \eta_k \):

\[ \mu_3 = -\bar{\mu}_2, \quad \eta_3 = \eta_2, \quad \mu_4 = -\bar{\mu}_4, \quad \eta_4 = \eta_4. \] (6.29)

By the way similar to Example 1, the material eigenvector matrix \( \mathbf{A} \) can then be written as

\[ \mathbf{A} = \begin{bmatrix} 0 & c_2 a_{12} & c_3 a_{13} & c_4 a_{14} \\ 0 & c_2 a_{22} & c_3 a_{23} & c_4 a_{24} \\ c_1 \tilde{S}_{44}/\mu_1 & 0 & 0 & 0 \\ 0 & c_2 a_{42} & c_3 a_{43} & c_4 a_{44} \end{bmatrix}. \] (6.30a)

where

\[ a_{1k} = \tilde{S}_{11}^0 \mu_k^2 + \tilde{S}_{12}^0 + \tilde{g}_{31}^0 \eta_k, \quad a_{2k} = (\tilde{S}_{12}^0 \mu_k^2 + \tilde{S}_{22}^0 + \tilde{g}_{32}^0 \eta_k)/\mu_k, \]
\[ a_{3k} = -\tilde{g}_{16}^0 - \tilde{g}_{17}^0 \eta_k \mu_k, \quad k = 2, 3, 4. \] (6.30b)

The scaling factors \( c_k, k = 1, 2, 3, 4 \) shown in (6.28) and (6.30) can then be determined by the orthogonality relation (4.10a), or obtained directly from (6.16).

7. Explicit expressions for Barnett–Lothe tensors \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \)

In two-dimensional problems, three real matrices \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \) defined in Eq. (4.11) appear frequently in the final real form solutions. Although this definition is not valid for degenerate materials whose material eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \) may not exist, such as Example 1 shown in the last section, Barnett and Lothe (1973) devised an integral formalism to compute these matrices directly from the elastic stiffnesses. Their integral formalism shows that

\[ \mathbf{S} = \frac{1}{\pi} \int_0^\pi \mathbf{N}_1(\theta)d\theta, \quad \mathbf{H} = \frac{1}{\pi} \int_0^\pi \mathbf{N}_2(\theta)d\theta, \quad \mathbf{L} = -\frac{1}{\pi} \int_0^\pi \mathbf{N}_3(\theta)d\theta, \] (7.1)

which means that \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \) are, respectively, the average values of \( \mathbf{N}_1(\theta), \mathbf{N}_2(\theta), \) and \( -\mathbf{N}_3(\theta) \). By this integral formalism, the problems associated with degenerate materials disappear. Hence, \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \) sometimes are called Barnett–Lothe tensors. Due to the importance of \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \), it is always desirable to have their explicit expressions. Although the integral formalism (7.1) has its advantage to avoid the degenerate problems, it is not convenient for the calculation. If we have the explicit expressions of \( \mathbf{A} \) and \( \mathbf{B} \), it seems that a direct substitution by the definition (4.11) is a good approach. However, the presence of the normalization factors \( c_k \) for the eigenvector matrices \( \mathbf{A} \) and \( \mathbf{B} \) would lead the direct substitution to a unwieldy algebraic calculation. An alternative approach by employing \( \mathbf{AB}^{-1} \) is suggested by Ting (Ting, 1996). In the following, we will follow his steps to find the explicit expressions of \( \mathbf{S}, \mathbf{H}, \) and \( \mathbf{L} \) for the two examples discussed in the last section.

Knowing that \( \mathbf{AB}^{-1} = (\mathbf{AB})^t([\mathbf{B}]^{-1})^t \) and using the definitions given in (4.11), we have

\[ \mathbf{AB}^{-1} = -([\mathbf{SL}]^{-1} + i[\mathbf{L}]^{-1}). \] (7.2)

When \( \mathbf{A} \) is multiplied by \( \mathbf{B}^{-1} \), it is seen that the normalization factors cancel each other, which may prevent the unwieldy results by direct substitution into the definitions given in (4.11). Eq. (7.2) shows that \( \mathbf{L} \) and \( \mathbf{S} \) can be obtained, respectively, from the imaginary part and real part of \( \mathbf{AB}^{-1} \) by
\[ L = (L^{-1})^{-1}, \quad S = (SL^{-1})L. \]  

Once \( L \) and \( S \) are determined, \( H \) can be obtained by using the following identity (Ting, 1996)

\[ H = L^{-1} + S(SL^{-1}). \]

**Example 1: Piezoelectric ceramics poling in \( x_3 \)-axis**

Using the procedure outlined above, we first calculate \( AB^{-1} \) by (6.21) and (6.22). The result is

\[
AB^{-1} = \begin{bmatrix}
2i\tilde{S}_{11}^0 & -(\tilde{S}_{11}^0 + \tilde{S}_{12}^0) & 0 & 0 \\
(\tilde{S}_{11}^0 + \tilde{S}_{12}^0) & 2i\tilde{S}_{11}^0 & 0 & 0 \\
0 & 0 & i\tilde{S}_{44}^0 & ig \\
0 & 0 & ig & -i\beta
\end{bmatrix},
\]

which can be proved to be identical to that presented in (Suo et al., 1992). Note that during the derivation of (7.5), \( \mu_1 \neq \mu_2 \) was assumed for the calculation of \( B^{-1} \). After getting \( AB^{-1} \) in terms of \( \mu_1 \) and \( \mu_2 \), we insert their actual values, i.e., \( \mu_1 = \mu_2 = i \) to get (7.5). By this way the problem of degeneracy disappears, which means that although \( B^{-1} \) does not exist for the degenerate materials, \( AB^{-1} \) and hence \( L, S \) and \( H \) exist. With the result obtained in (7.5) and following the procedure described between (7.2) and (7.4), we now get

\[
L^{-1} = \begin{bmatrix}
2\tilde{S}_{11}^0 & 0 & 0 & 0 \\
0 & 2\tilde{S}_{11}^0 & 0 & 0 \\
0 & 0 & \tilde{S}_{44}^0 & g \\
0 & 0 & g & -\beta
\end{bmatrix}, \quad L = \begin{bmatrix}
(2\tilde{S}_{11}^0)^{-1} & 0 & 0 & 0 \\
0 & (2\tilde{S}_{11}^0)^{-1} & 0 & 0 \\
0 & 0 & k_1 & k_2g \\
0 & 0 & k_2g & -k_1\tilde{S}_{44}^0
\end{bmatrix}, \quad S = k_2 \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where

\[
k_1 = (g^2 + i\tilde{S}_{44}^0)^{-1} = e_{15}^e + \omega e_{15}^e C_{44}^e,
\]

\[
k_2 = (1 + k_0)/2, \quad k_3 = (3 + k_0)(1 - k_0)/2, \text{ and } k_0 = \tilde{S}_{11}^0/\tilde{S}_{11}^0.
\]

**Example 2: Piezoelectric ceramics poling in \( x_2 \)-axis**

By a similar approach as Example 1, we first calculate \( AB^{-1} \) by using the results given in (6.27)-(6.30). The result is

\[
AB^{-1} = \begin{bmatrix}
I_{11} & -Z_{21} & 0 & -Z_{41} \\
Z_{21} & I_{22} & 0 & I_{24} \\
0 & 0 & I_{33}^* & 0 \\
I_{41}^* & I_{24}^* & 0 & I_{44}^*
\end{bmatrix},
\]

in which \( L_0^* \) and \( \chi_0 \) are all real values and are related to the material constants by

\[
L_{11} = 2\tilde{S}_{11}^0 \text{Im}(\mu_2 \eta_2 + (\mu_2^* \eta_2 - \mu_2^0 \eta_4))/\lambda,
\]

\[
L_{22} = 2\text{Im}[(\gamma_2 \mu_4^* \eta_4 + (\gamma_2 \mu_4 - \gamma_4 \mu_2^0 \eta_4)/\mu_2 \mu_4))/\lambda,
\]

\[
L_{33}^* = \sqrt{\tilde{S}_{44}^0 \tilde{S}_{11}^0},
\]

\[
L_{44}^* = -2\text{Im}[(\gamma_4 \mu_4^* \eta_4 + (\gamma_4 \mu_4^* - \gamma_4 \mu_4^0 \eta_4))/\mu_2 \mu_4))/\lambda,
\]

\[
\chi_{21} = \tilde{S}_{12}^0 + 2\tilde{S}_{11}^0 \text{Re}(\mu_2 \mu_4 \eta_4 + \mu_2 \mu_4 \eta_2 (\mu_2 - \mu_4))/\lambda,
\]

\[
\chi_{41} = \tilde{S}_{21}^0 + 2\tilde{S}_{11}^0 \text{Re}(\mu_2 \mu_4 \eta_4 + \mu_2 \mu_4 \eta_2 (\mu_2 - \mu_4))/\lambda,
\]

and

\[
\lambda = 2\text{Re}(\mu_2 \eta_2 + (\mu_4 \eta_2 - \mu_2 \eta_4)), \quad \gamma_k = \tilde{S}_{22}^0 + \tilde{g}_{22} \eta_k, \quad k = 2, 4.
\]

Note that in Eq. (7.7b) two equalities are given for \( L_{24}, Z_{21} \) and \( Z_{41} \), which come from the fact that \( L^{-1} \) is symmetric and \( SL^{-1} \) is skew-symmetric. With the result of Eq. (7.7) and the procedure described in Eqs. (7.2)–(7.4), we get
\[
L^{-1} = \begin{bmatrix}
L_{11} & 0 & 0 & 0 \\
0 & L_{22} & 0 & L_{24}' \\
0 & 0 & L_{33}' & 0 \\
0 & L_{24}' & 0 & L_{44}'
\end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix}
(L_{11}')^{-1} & 0 & 0 & 0 \\
0 & k_{i}L_{44} & 0 & -k_{i}L_{24}' \\
0 & 0 & (L_{33}')^{-1} & 0 \\
0 & -k_{i}L_{24}' & 0 & k_{i}L_{44}'
\end{bmatrix}.
\]

where
\[
S_{12} = -k_{i}(\alpha_{21}L_{24}' - \alpha_{41}L_{24}), \quad S_{14} = k_{i}(\alpha_{21}L_{24}' - \alpha_{41}L_{22}), \\
S_{21} = \alpha_{21}L_{24}, \quad S_{41} = \alpha_{41}L_{11}, \\
H_{11} = L_{11}' - k_{i}(\alpha_{21}L_{44}' - 2\alpha_{21}\alpha_{41}L_{24} + \alpha_{41}L_{22}), \quad H_{22} = L_{22}' - (\alpha_{21}/L_{11}), \\
H_{24} = L_{24}' - (\alpha_{21}\alpha_{41}/L_{11}), \quad H_{33} = L_{33}', \quad H_{44} = L_{44}' - (\alpha_{41}/L_{11}),
\]

and
\[
k_{i} = [L_{22}'L_{44}' - (L_{24}')^{2}]^{-1}.
\]

It can be proved that the explicit solutions shown in Eqs. (7.7), (7.8) agree with those presented by Soh et al. (2001), although they are different in outward appearance. Note that one typing error occurs in eqn. (40b) of (Soh et al., 2001), in which Im of the first term should be corrected as Re.

8. Conclusions

Four types of constitutive laws for piezoelectric materials in three-dimensional state are briefly reviewed in this paper. Four different two-dimensional states are discussed to get the reduced constitutive laws for two-dimensional problems. Even the extended Stroh formalism for piezoelectric anisotropic elasticity was derived under one of the four two-dimensional states: the generalized plane strain and short circuit condition, by suitable replacement of the material constants it can still be applied to the other two-dimensional states. With this understanding, in this paper only the explicit expressions for the state of generalized plane strain and short circuit are presented. The explicit expressions for the fundamental matrix \(\mathbf{N}_{1}, \mathbf{N}_{2}\), and \(\mathbf{N}_{3}\), and material eigenvectors \(\mathbf{a}\) and \(\mathbf{b}\) are presented in (5.4) and (6.12) and (6.15) for all possible piezoelectric anisotropic materials. Standard procedure for getting the explicit expressions of Barnett–Lothe tensors \(\mathbf{L}\), \(\mathbf{S}\) and \(\mathbf{H}\) is shown in (7.2)–(7.4). Two typical examples for piezoelectric ceramics poling in two different axes, \(x_{3}\)-axis or \(x_{2}\)-axis, are shown in Sections 6 and 7 for getting their explicit expressions of \(\mathbf{A}, \mathbf{B}, \mathbf{L}, \mathbf{S}\) and \(\mathbf{H}\).

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References


