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On the structure of categories of coalgebras

Peter Johnstone[a,*], John Power[b], Toru Tsujishita[c], Hiroshi Watanabe[d], James Worrell[e]

[a] Department of Pure Mathematics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, UK
[b] Department of Computer Science, University of Edinburgh, Scotland, UK
[c] Department of Mathematics, Hokkaido University, Japan
[d] Semantics Group, Electrotechnical Laboratory, Tsukuba, Japan
[e] Computing Laboratory, Oxford University, UK

Abstract

Consideration of categories of transition systems and related constructions leads to the study of categories of F-coalgebras, where F is an endofunctor of the category of sets, or of some more general 'set-like' category. It is fairly well known that if $\mathscr{E}$ is a topos and $F:\mathscr{E} \to \mathscr{E}$ preserves pullbacks and generates a cofree comonad, then the category of F-coalgebras is a topos. Unfortunately, in most of the examples of interest in computer science, the endofunctor F does not preserve pullbacks, though it comes close to doing so. In this paper we investigate what can be said about the category of coalgebras under various weakenings of the hypothesis that F preserves pullbacks. It turns out that almost all the elementary properties of a topos, except for effectiveness of equivalence relations, are still inherited by the category of coalgebras; and the latter can be recovered by embedding the category in its effective completion. However, we also show that, in the particular cases of greatest interest, the category of coalgebras is not itself a topos.

Keywords: Coalgebra; Topos; Subobject; Classifier; Weak pullback; Cofree comonad

0. Introduction

A labelled transition system is traditionally taken to consist of a set S of states, a set L of labels, and a ternary relation $T \subseteq S \times L \times S$, the interpretation being that $(s, l, s') \in T$ if and only if the system, when in state s, can make a transition labelled l and arrive at state s'. However, in most contexts the appropriate notion of 'morphism of transition systems', even if we assume a fixed set of labels, is not simply a function

* Corresponding author. Tel.: +44-1223-338680.
E-mail address: ptj@pmms.cam.ac.uk (P. Johnstone).

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between the state-sets which preserves the transition relations. From this point of view, it makes sense to replace the relation $T \subseteq S \times L \times S$ by its transpose $t : S \to P(L \times S)$ (where P denotes the covariant power-set functor); that is, we regard a transition system as a coalgebra for the endofunctor $P(L \times -)$ of **Set**. Then a morphism $(S,t) \to (S',t')$ becomes a function $f : S \to S'$ such that, for any $s \in S$, we have

$$(l,s') \in t'(f(s)) \Leftrightarrow (\exists \bar{s} \in S)(f(\bar{s}) = s' \text{ and } (l,\bar{s}) \in t(s)).$$

In other words, a morphism of L-labelled transition systems is what is usually called a functional bisimulation.

Of course, functional bisimulations are not enough: we need to consider general bisimulations, which are not functions but relations. A bisimulation from (S,t) to (S',t') is a relation $R \subseteq S \times S'$ such that, if s_1 is R-related to s_1' and s_1' can make an l-labelled transition to s_2', then s_1 can make an l-labelled transition to some state s_2 which is R-related to s_2', and conversely. Under these conditions, the instances of the relation R can themselves be viewed as the states of a transition system $(R,\bar{t})$ equipped with a jointly monic pair of functional bisimulations to (S,t) and (S',t'), that is, we may regard R as a relation in the category of functional bisimulations. This strategy (which was exploited by Joyal et al. [18]; cf. also [2]) is the one which we shall adopt: that is, we aim first to study the category of functional bisimulations, and once we have a good enough understanding of it (in particular, once we know that it is regular, and so admits a good calculus of relations [12]), we may then pass to the associated category of relations in order to study general bisimulations. (This approach may be contrasted with that of Rutten [33], who first passes from the category of sets – or some other base category such as metric spaces – to its category of relations, and then considers transition systems as coalgebras for a suitable endofunctor of the latter. The two approaches are fairly obviously equivalent in some sense, but we shall not give a detailed comparison here.)

It has been recognized in computer science for some time that it is fruitful to view various other kinds of dynamical systems, besides labelled and unlabelled transition systems, as coalgebras for appropriate endofunctors. For instance, deterministic automata, probabilistic transition systems, transducers, resumptions and objects (in the sense of object oriented programming) can all be handled in this way. See [10, 29, 32, 34] for further information and examples.

We are thus led to study categories of coalgebras for endofunctors of **Set** similar to $P(L \times -)$. Actually, there is a technical reason why this particular endofunctor (even in the case when L is a singleton) is a bad one to study: namely that the category of $P(L \times -)$-coalgebras has no terminal object, as a consequence of well-known results of Cantor and Lambek (see Example 2.2), and it is therefore unlikely to have many other good categorical properties. However, this difficulty is relatively easily circumvented. In any particular context, it is usually easy to place a bound on the size of the set of transitions which can be made from a given state, and so we may consider our transition system as a coalgebra for the functor $P_\kappa(L \times -)$, where κ is a suitable (regular) cardinal and $P_\kappa(A)$ denotes the set of subsets of A of cardinality less than κ. (This is not essentially different from the approach of Aczel and Mendler [2], who considered P as an endofunctor of the category of classes: their 'sets' are our 'sets of cardinality less than κ'.) For any functor of this kind, it turns out that the category of coalgebras not only has a terminal object, but also inherits many other good categorical properties from **Set**; and these in turn lead to good categorical properties of its category of relations.

In practice, in the present paper we shall mostly take κ to be ω, so that we are concerned with finitely branching transition systems. However, this is simply a matter of convenience: we emphasize that our main results remain true, with the same proofs, if we allow branching up to any fixed cardinality. We shall also simplify matters by assuming, for most of the time, that the set L of labels is a singleton, so that we are (in effect) dealing with unlabelled transition systems.

Our approach is axiomatic: that is, we work with endofunctors $F:\mathscr{E}\to\mathscr{E}$ of a general category $\mathscr{E}$, and see which properties of $\mathscr{E}$ and of F are required to deduce good categorical properties of the category $\mathscr{E}_F$ of F-coalgebras. Our reason for doing this is the belief that, for a full understanding of categories of transition systems, it will be necessary not only to consider endofunctors more general than $P_\kappa(L\times -)$, but also to consider systems whose states form not just abstract sets but objects of some more general category (for example, a category of domains); cf. [34, 10]. Thus, we do not wish to assume from the outset that our base category $\mathscr{E}$ is the classical category of sets. On the other hand, we shall assume whenever we find it necessary that $\mathscr{E}$ has any of the categorical properties of a topos: this assumption may be reconciled with our domain-theoretic intentions by recalling the general philosophy of synthetic domain theory (cf. [13]) that domains ought to be viewed as 'variable sets', that is as objects of some topos.

We recall that, in any topos $\mathscr{E}$, the construction of power-objects may be made into a covariant functor $P:\mathscr{E}\to\mathscr{E}$, and that this functor has a subfunctor, traditionally denoted K, such that $K(A)$ may be thought of as 'the object of finite subobjects of A' [14, 9.13] – in particular, when $\mathscr{E}$ is the classical topos of sets, then $K(A)$ is exactly $P_\omega(A)$ as defined earlier. (The notion of finiteness encapsulated by this functor is commonly called *Kuratowski-finiteness*, or K-finiteness for short.) Thus our leading example will be the endofunctor K of an arbitrary topos.

However, there are two other endofunctors, closely related to K, which we shall also be interested in studying; both of these, unlike K, require the assumption that $\mathscr{E}$ has a natural number object (i.e. 'satisfies the axiom of infinity'). One is the *list functor* L: for a set A, $L(A)$ is the set of all finite lists $(a_1,a_2,\ldots,a_n)$ of elements of A (including the empty list) – equivalently, it is the underlying set of the free monoid generated by A. For the construction of the analogous endofunctor of an arbitrary topos with natural number object, see [14, 6.41]. We may also form the free commutative monoid $M(A)$ as a quotient of $L(A)$, in which we identify two lists if they contain the same elements of A with the same multiplicities (but not necessarily in the same order); equivalently, we may think of the elements of $M(A)$ as finite multisets of elements of A. (Of course $K(A)$ is in turn a quotient of $M(A)$, in which we 'forget' the multiplicities with which elements appear in a list, as well as their order: as observed in [14, 9.16], it is the free semilattice generated by A.)

When $\mathscr{E}$ is the topos of sets, we may think of M-coalgebras as 'transition systems with multiplicities', in which we may have several different ways of making the transition from one given state to another, and the corresponding notion of bisimulation has to keep track of the number of possible transitions from each state (i.e., if $f : A \to A'$ is a functional bisimulation and there are n possible transitions, counted with multiplicities, from a state a, then there must be exactly n possible transitions from $f(a)$.) L-coalgebras are similar except that we also have a total ordering (which we may think of as 'preference') on the set of possible transitions from each state, and once again the notion of bisimulation must respect these orderings. (Note that we do not assert that either of these notions of 'transition system' is computationally significant; merely that they give rise to mathematically interesting categories of coalgebras, and that understanding how things work in these cases – which are in some respects simpler – will help us to understand the case of K-coalgebras.)

We may now briefly describe the contents of this paper. The key to understanding the structure of $\mathscr{E}_F$ turns out to be the fact that, although the functors F under consideration do not in general preserve finite limits, they 'come close' to preserving pullbacks – and these 'near-preservation' properties are essential for lifting properties of $\mathscr{E}$ to properties of $\mathscr{E}_F$. We therefore begin with a section in which we discuss these weak preservation properties in general terms, and establish the relations between them. The second essential ingredient of our results, and the one which distinguishes the finite-powerset functor K from the full-powerset functor P, is the fact that the former generates a cofree comonad, enabling us to transfer our attention from categories of 'mere' coalgebras for an endofunctor to categories of Eilenberg–Moore coalgebras for a comonad; we discuss this phenomenon, and give sufficient conditions for it to happen, in Section 2. Section 3 contains our main positive results on the transfer of properties from $\mathscr{E}$ to $\mathscr{E}_F$: we show that, provided F preserves weak pullbacks and generates a cofree comonad, then $\mathscr{E}_F$ inherits 'almost all' of the structure of a topos from $\mathscr{E}$. It does not inherit all the structure, unless F actually preserves pullbacks; in particular, it fails to inherit cartesian closedness or effectiveness of equivalence relations. However, there is a well-known 'effectivization' technique for repairing the latter defect, which can be applied to $\mathscr{E}_F$; we consider this in Section 4, and show that for suitable functors F it yields a topos (in which $\mathscr{E}_F$ is fully embedded). Finally, Section 5 considers the relationships between categories of coalgebras and categories obtained by Artin glueing; using results of Carboni and Johnstone [6], this enables us to show that $\mathscr{E}_F$ is not a topos in general (in particular, it is not a topos for our leading example $F = K$), so that the effectivization process of Section 4 cannot be omitted.

A few words should be said about the process which led to the five-author collaboration that produced this paper. An initial collaboration between JP, TT and HW, during JP's visit to Japan in late 1997, and building on the earlier work of TT and

HW [35–37], resulted in a submission [28] to the CMCS '98 workshop, which turned out to have a considerable overlap with an independent submission [38] by JW. When this was discovered, there began an extended electronic correspondence, which soon came to involve PJ, in an attempt to find the best possible synthesis of the results. By the time a clear picture began to emerge, our various contributions had become so thoroughly intertwined that there was no alternative to writing them up as a five-author paper. In addition to CMCS'98 (Lisbon, March 1998), versions of the material in the paper were presented by one or other of the authors at the following meetings: PSSL 66 (Birmingham, March 1998), Seminar on Theory and Applications of Domains (Dagstuhl, May 1998), MFPS 14 (London, May 1998) and LICS 13 (Indianapolis, June 1998). The five-author extended abstract [16] was written for the last of these: it may be regarded as a preliminary version, without detailed proofs, of the present paper, but it also contains some material on monoidal closed structures which has been omitted from the present version.

1. Weak preservation of pullbacks

We recall that a cone over a diagram in a category is said to be a *weak limit* if it satisfies the 'existence' but not necessarily the 'uniqueness' clause in the definition of a limit.

Lemma 1.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be categories with limits of some particular shape J. Then the following conditions on a functor $F:\mathscr{E}\to\mathscr{F}$ are equivalent*:

(i) *F preserves weak limits of diagrams of shape J.*
(ii) *F sends limits of diagrams of shape J to weak limits.*
(iii) *For any diagram D of shape J in $\mathscr{E}$, the canonical comparison map $F(\varprojlim D)\to \varprojlim F(D)$ is a split epimorphism.*

Proof. The equivalence of (ii) and (iii) is immediate when we recall that, in a category where limits exist, a cone over a diagram is a weak limit iff its factorization through the limiting cone is split epic. The equivalence of (i) and (ii) follows from the fact that any functor preserves split epimorphisms. □

We are also interested in a still weaker condition than preservation of weak limits. Following [7], we shall say that a functor $F:\mathscr{E}\to\mathscr{F}$ *nearly preserves* limits of shape J if, for every diagram D of shape J in $\mathscr{E}$, the canonical morphism $F(\varprojlim D)\to \varprojlim F(D)$ is a cover (also called an extremal epimorphism); that is, it does not factor through any proper subobject of $\varprojlim F(D)$. Since split epimorphisms are covers, it is clear that 'F preserves weak limits' implies 'F nearly preserves limits'. (Of course, if $\mathscr{F}$ is a category – such as **Set**, under the assumption of the axiom of choice – in which every extremal epimorphism is split, then the converse holds.)

Lemma 1.2. *If F nearly preserves pullbacks, then it preserves pullbacks of monomorphisms: that is, if*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

is a pullback square with h (and therefore g) monic, then the comparison map $FA \to FB \times_{FD} FC$ is an isomorphism.

Proof. First we show that F preserves monomorphisms. Let $f : A \rightarrowtail B$ be a monomorphism; then

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback, so its near-preservation implies that the diagonal map from FA to the kernel-pair $R \rightrightarrows FA$ of Ff is epic. Hence the two projections $R \rightrightarrows FA$ are equal; so Ff is monic.

Now consider the general pullback diagram in the statement. Since g is monic, so is Fg, and so the canonical map $FA \to FB \times_{FD} FC$ must be monic since Fg factors through it. By assumption, it is a cover; so it must be an isomorphism. □

We thus have the following hierarchy of conditions on a functor $F : \mathscr{E} \to \mathscr{F}$ between categories with pullbacks:

(i) F preserves pullbacks;
(ii) F preserves weak pullbacks;
(iii) F nearly preserves pullbacks;
(iv) F preserves pullbacks of monomorphisms;
(v) F preserves monomorphisms.

Each of the conditions in this list implies the ones below it, but none of the implications are reversible. In most of our work on coalgebras, we shall assume that F satisfies condition (iv), but at some points we shall need to assume (iii) or (ii).

Examples 1.3. (a) Let $F : \mathbf{Set} \to \mathbf{Set}$ be the subfunctor of the covariant power-set functor which sends a set A to the set of all subsets of A of cardinality at most three. It is easy to see that F, like the power-set functor itself, preserves pullbacks of monomorphisms. But it does not nearly preserve the pullback

$$\begin{array}{ccc} A=\{a,b,c,d\} & \xrightarrow{f} & B=\{a,c,d\} \\ \Big\downarrow g & & \Big\downarrow h \\ C=\{a,b,d\} & \xrightarrow{k} & D=\{a,d\} \end{array},$$

where f and k send b to a, g and h send c to d and all other elements are mapped to themselves: the pair (B,C) is an element of $FB\times_{FD}FC$ which is not in the image of the comparison map from FA.

(b) Let $G:\mathbf{Set}\to\mathbf{Set}$ be the quotient of the functor $\mathbf{Set}(2,-)$ (where 2 denotes a two-element set) by the equivalence relation which identifies all non-injective maps $2\to A$, for each A. It is clear that G preserves monomorphisms, but it does not preserve pullbacks of monomorphisms: for example, in the pullback

$$\begin{array}{ccc} A'=\{a\} & \xrightarrow{f'} & B'=\{a\} \\ \Big\downarrow & & \Big\downarrow \\ A=\{a,b,c\} & \xrightarrow{f} & B=\{a,b\} \end{array}$$

where $f(c)=b$ and all other elements are mapped to themselves, the injection $2\to A$ sending 0 and 1 to b and c, respectively, and the unique map $2\to B'$ define an element of $GA\times_{GB}GB'$ which is not in the image of the comparison map from GA'.

Of rather more direct relevance to us is the following.

Example 1.4. In any Boolean topos $\mathscr{E}$, the finite-powerset functor K preserves weak pullbacks. To see this, let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow g & & \Big\downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

be a pullback square in $\mathscr{E}$: we have to construct a splitting for the morphism $KA\to KB\times_{KD}KC$ which, in 'elementwise' terms, sends a K-finite subobject $A'\subseteq A$ to the pair consisting of its images under f and g. But, given K-finite subobjects $B'\subseteq B$ and $C'\subseteq C$ with the same image D' in D, the object $B'\times_{D'}C'$ is K-finite since it is a subobject of $B'\times C'$, and it is a subobject of A mapping onto B' and C'. Applying this construction to the generic element of $KB\times_{KD}KC$ yields the required splitting.

The thing which makes the above argument work is the fact that, in a Boolean topos, every subobject of a K-finite object is K-finite (cf. [15]). In a general topos, this is not true; however, it is still the case that K nearly preserves pullbacks. To show this, one proves in the internal logic of $\mathscr{E}$ that "the set of subobjects $C' \subseteq C$ such that, for all K-finite $B' \subseteq B$ with $Kh(B') = Kk(C')$, there exists a K-finite $A' \subseteq A$ with $Kf(A') = B'$ and $Kg(A') = C'$" contains singletons and is closed under finite unions, so must be the whole of KC. We omit the details.

Nevertheless, in a general topos the functor K does *not* preserve weak pullbacks. To give a counterexample, we shall work in the *Sierpiński topos* whose objects are morphisms $(A_0 \to A_1)$ in **Set** and whose morphisms are commutative squares. For such an object, we may identify $K(A_0 \to A_1)$ with $(KA_0 \to KA_1)$. Now consider the pullback square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array},$$

where D_1 is a singleton, B_1 and C_1 are two-element sets $\{b_1, b_2\}$ and $\{c_1, c_2\}$, respectively, and A_1, A_0, B_0, C_0 and D_0 are all copies of $B_1 \times C_1$ with identity maps between them. Consider the finite subsets $\{(b_1, c_1), (b_2, c_2)\}$ and $\{(b_1, c_2), (b_2, c_1)\}$ of A_0; these define elements of KA_0 with distinct images in KA_1, KB_0 and KC_0, but the same image in both KB_1 and KC_1. So the comparison map $KA \to KB \times_{KD} KC$ contains a copy of the non-split epimorphism $(2 \to 2) \to (2 \to 1)$; hence it is itself a non-split epimorphism.

Remark 1.5. By considering the finite sets in the counterexample just given as finite multisets (in which each element has multiplicity 1), we may obtain the same conclusion about the functor M: in general, it does not preserve weak pullbacks. However, it does nearly preserve pullbacks (in any topos $\mathscr{E}$): the proof is an easy exercise in manipulating finite cardinals. Hence, in a topos (such as **Set**) satisfying the axiom of choice, it preserves weak pullbacks.

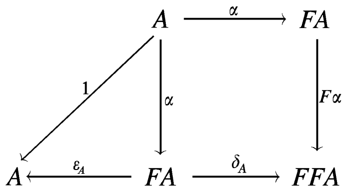
2. Coalgebras and coalgebras

As indicated in the Introduction, we are interested in studying the category of F-coalgebras for certain functors $F : \mathscr{E} \to \mathscr{E}$; that is, the category whose objects are objects A of $\mathscr{E}$ equipped with a structure map $\alpha : A \to FA$, and whose morphisms $(A, \alpha) \to (B, \beta)$ are morphisms $f : A \to B$ of $\mathscr{E}$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ FA & \xrightarrow{Ff} & FB \end{array}$$

commutes. But there is another and older meaning of the word 'coalgebra', that of an Eilenberg–Moore coalgebra for a comonad: if the functor F carries a comonad structure

(F, ε, δ), then a morphism $\alpha : A \to FA$ is called an Eilenberg–Moore coalgebra structure if the diagrams



commute. Since we shall need to consider both types of coalgebra in what follows, we shall adopt the convention of spelling 'Coalgebra' with a capital C when we mean an Eilenberg–Moore coalgebra for a comonad, and with a small c when we refer to a 'mere' coalgebra for an endofunctor. We write $\mathscr{E}_F$ for the category of F-coalgebras in $\mathscr{E}$, and $\mathscr{E}_{\mathbb{F}}$ for the category of Coalgebras for a comonad $\mathbb{F} = (F, \varepsilon, \delta)$; the use of different typefaces should suffice to distinguish between them.

Given a functor $F : \mathscr{E} \to \mathscr{E}$, we say that a comonad $\mathbb{G} = (G, \varepsilon, \delta)$ is *cofree* on F if there is a natural transformation $\theta : G \to F$ such that, for any object A, composition with θ_A induces a bijection from Coalgebra structure maps $\alpha : A \to GA$ to 'mere' coalgebra structures $\beta = \theta_A \alpha : A \to FA$, and such that, given two $\mathbb{G}$-Coalgebras (A, α) and (B, β), a morphism $f : A \to B$ is a Coalgebra homomorphism (that is, satisfies $(Gf)\alpha = \beta f$) iff it is a coalgebra homomorphism (that is, satisfies $(Ff)\theta_A \alpha = \theta_B \beta f$). (This is a stronger condition than merely demanding that $(\mathbb{G}, \theta)$ should be a universal arrow from the forgetful functor (comonads on $\mathscr{E}$) $\to$ (endofunctors of $\mathscr{E}$) to F, which is what one would ordinarily mean by saying that $\mathbb{G}$ is cofree on F; however, the latter condition seems too weak to be of much practical use. In [19, Section 22], Kelly discusses the difference between the two notions (in the dual case of monads) in detail; his name for what we have called a cofree comonad would be a 'coalgebraically cofree comonad'.)

Proposition 2.1. *The following conditions on a functor $F : \mathscr{E} \to \mathscr{E}$ are equivalent*:

(i) *There exists a cofree comonad on F.*
(ii) *The forgetful functor $\mathscr{E}_F \to \mathscr{E}$ is comonadic.*
(iii) *The forgetful functor $\mathscr{E}_F \to \mathscr{E}$ has a right adjoint.*
(iv) *For every object A of $\mathscr{E}$, the category whose objects are F-coalgebras equipped with a map to A has a terminal object.*
(v) (*if $\mathscr{E}$ has finite products*) *For every object A of $\mathscr{E}$, the functor $B \mapsto FB \times A$ has a terminal coalgebra.*

Proof. (i) $\Rightarrow$ (ii) holds since (i) contains the statement that there is a comonad $\mathbb{G}$ such that $\mathscr{E}_{\mathbb{G}} \cong \mathscr{E}_F$, by an isomorphism identifying the two forgetful functors to $\mathscr{E}$. Conversely, if (ii) holds, then we obtain a natural transformation $\theta : G \to F$ as the composite

$$GA \xrightarrow{\overline{\delta_A}} FGA \xrightarrow{F\varepsilon_A} FA,$$

where $\overline{\delta_A}$ is the coalgebra structure corresponding to the Coalgebra structure $\delta_A : GA \to GGA$, and it is easy to verify that it has the universal property stated in (i). (ii) $\Rightarrow$ (iii) is trivial; (iii) $\Leftrightarrow$ (iv) since (iv) is a well-known criterion for the existence of a right adjoint to a given functor; and (iv) $\Leftrightarrow$ (v) is a simple restatement of the condition. So the only part requiring any work is (iii) $\Rightarrow$ (ii). But even this is easy, since the forgetful functor $\mathscr{E}_F \to \mathscr{E}$ creates any limits which are preserved by F: in particular, it creates equalizers for parallel pairs of morphisms whose images in $\mathscr{E}$ have an absolute equalizer. So the result follows from the Precise Comonadicity Theorem [24, p. 147]. □

Example 2.2. Not every endofunctor generates a cofree comonad. If $P : \mathscr{E} \to \mathscr{E}$ is the covariant power-object functor on a topos $\mathscr{E}$, then by a well-known result of Lambek [22] a terminal P-coalgebra would have to be an isomorphism $A \to PA$; but no such isomorphism can exist in a non-degenerate topos (cf. [14, Exercise 5.7]).

So we need to restrict the class of functors we consider in such a way as to exclude P. One way of doing this, in the context of a Grothendieck topos $\mathscr{E}$, was exploited in [4, 28]. Note that the category $\mathscr{E}_F$ may be viewed as the inserter of the diagram

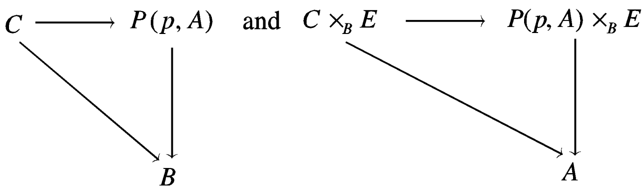
$$\mathscr{E} \underset{F}{\overset{1}{\rightrightarrows}} \mathscr{E}$$

in the meta-2-category of categories (that is, the universal solution to the problem of finding a functor $U : \mathscr{D} \to \mathscr{E}$ together with a natural transformation $1_{\mathscr{E}} \circ U \to F \circ U$). In particular, it is a weighted limit (cf. [20]). So by Theorem 5.1.6 of Makkai and Pare [27], if $\mathscr{E}$ is an accessible category and F is an accessible functor (that is, preserves κ-filtered colimits for some cardinal κ), then $\mathscr{E}_F$ is also accessible. But we also know that the forgetful functor $\mathscr{E}_F \to \mathscr{E}$ creates all colimits which exist in $\mathscr{E}$: in particular, if $\mathscr{E}$ is cocomplete, then so is $\mathscr{E}_F$. Now a category is locally presentable iff it is accessible and cocomplete [27, 6.1.4]; and any such category has a generating set and is well-copowered [27, 6.1.3], so that the Special Adjoint Functor Theorem [24, p. 125] may be used to construct right adjoints for colimit-preserving functors between locally presentable categories. In particular, the forgetful functor $\mathscr{E}_F \to \mathscr{E}$ has a right adjoint. Thus we have verified the following result, first noted by Barr [4]:

Proposition 2.3. *Let $\mathscr{E}$ be a locally presentable category. Then any accessible functor $F : \mathscr{E} \to \mathscr{E}$ generates a cofree comonad.*

In particular, Proposition 2.3 implies that the functor $P_\kappa(X \times -) : \mathbf{Set} \to \mathbf{Set}$ generates a cofree comonad for any cardinal κ and any fixed set X, as we claimed in the Introduction.

The following more elementary result is also of interest, though it covers a much more restricted class of functors. We recall the notion of *partial product* studied in [11]: given an object A of $\mathscr{E}$ and a morphism $p : E \to B$, a partial product of p and A is an object $P(p,A)$ equipped with projections $P(p,A) \to B$ and $P(p,A) \times_B E \to A$, which is universal among such – that is, given any object C equipped with morphisms $C \to B$ and $C \times_B E \to A$, there is a unique morphism $C \to P(p,A)$ yielding commutative diagrams



In a category with finite limits, the partial product $P(p,A)$ exists for all A iff p is exponentiable as an object of $\mathscr{E}/B$; in this case it is readily seen to be functorial in A. By a *partial product functor*, we shall mean a functor $\mathscr{E} \to \mathscr{E}$ of the form $P(p,-)$ for some p. Partial product functors were also studied in [6]: as observed there, the list functor L is a partial product functor for any topos $\mathscr{E}$ with a natural number object, of the form $P(s+,-)$ where $s+ : N \times N \to N$ is the *generic finite cardinal* in the sense of Johnstone and Wraith [17].

Lemma 2.4. *Let $\mathscr{E}$ be a topos with a natural number object, and $F : \mathscr{E} \to \mathscr{E}$ a partial product functor. Then the cofree comonad generated by F exists, and is also a partial product functor.*

Proof. For simplicity, we shall give the proof in the case when $\mathscr{E}$ is the classical topos **Set** of constant sets: the recursion-theoretic techniques needed to formulate it in a general topos with natural number object may be found, for example, in [17]. Suppose F is the partial product functor induced by $p : E \to I$; that is, an element of FA is a pair (i,h) where $i \in I$ and $h : p^{-1}(i) \to A$. It is convenient to think of the elements of I as 'shapes' or 'templates', and the elements of $p^{-1}(i)$ as 'holes' in the template i which have to be 'filled' or 'labelled' by elements of A to produce an element of FA. (Thus E may be identified with the set of pairs (i,x), where i is a template and x is a distinguished hole in it.)

For each natural number n, the nth iterate F^n is also a partial product functor: its templates are 'trees of I-templates of height n', that is (rooted) trees of height n where every node is labelled by an element of I, together with, for each node at height less than n, a bijection from its children to the holes in the template which is its label. The holes in such a tree are the holes in the templates labelling nodes at height n. Let T_n denote the set of all trees of templates of height n (equivalently, the set $F^n(1)$, where 1 is a singleton set); then we have a truncation map $T_n \to T_{n-1}$, which makes $T_{(-)}$ into a diagram of shape $\mathbb{N}^{\mathrm{op}}$. Let T_∞ denote the inverse limit of this diagram; we think of an element t of T_∞ as an 'infinite tree of templates' possessing truncations $t|_n$ of all possible heights n. We may regard such a tree t as a single template whose holes are all its nodes (including the root): this defines a map $q: E_\infty \to T_\infty$, where E_∞ is the set of pairs (t,x) with t an infinite tree and x a distinguished node of t. Let G denote the partial product functor associated with this map.

Given an A-labelled infinite tree (t,h) (that is, an element of GA), we have a distinguished element of A, namely the element labelling the root of t; this defines a natural transformation $\varepsilon: G \to 1_{\mathbf{Set}}$. We also have an element of FA, namely the template attached to the root of t together with the elements of A labelling the height-1 nodes: this defines a natural transformation $\theta: G \to F$. And we have a natural transformation $\delta: G \to GG$; δ_A sends an A-labelled tree (t,h) to the same tree t, with each node x now labelled by the entire A-labelled tree t_x of nodes above x in t. It is not hard to see that $\mathbb{G} = (G, \varepsilon, \delta)$ satisfies the equations for a comonad.

Moreover, given any F-coalgebra (A,$\beta: A \to FA$), there is a unique $\mathbb{G}$-Coalgebra structure map $\alpha: A \to GA$ such that $\theta_A \alpha = \beta$. The equations for a Coalgebra imply that, for any $a \in A$, the label at the root of $\alpha(a)$ must be a itself, and that for any node x of $\alpha(a)$ the labelled tree $\alpha(a)_x$ must coincide with $\alpha(b)$ where b is the label at x; thus we see that the entire tree and its labelling can be reconstructed from the knowledge, for all $b \in A$, of of the template at the root and the labels at height 1 in $\alpha(b)$ (that is, of the labelled template $\theta_A \alpha(b)$). Thus we have shown that $\mathbb{G}$ is the cofree comonad generated by F. □

Remark 2.5. Although, as we stated earlier, Lemma 2.4 is much more special than Proposition 2.3, they are actually not far apart in the case when $\mathscr{E}$ is the classical category of sets. For any accessible functor $F: \mathbf{Set} \to \mathbf{Set}$, being the left Kan extension of its restriction to a small full subcategory of **Set**, may be expressed as a quotient of a coproduct of representable functors: that is, we have an epimorphic natural transformation $\alpha: \tilde{F} \to F$ where $\tilde{F}$ is of the form $\sum_{i \in I} (-)^{E_i}$, and is thus a partial product functor. Assuming the axiom of choice, this epimorphism is pointwise split (that is, α_A has a splitting β_A for each A; we do not assert that the β_A form a natural transformation of functors). It now follows that, for any set A, a terminal $\tilde{F}(-) \times A$-coalgebra is a weakly terminal $F(-) \times A$-coalgebra, and we may construct a terminal $F(-) \times A$-coalgebra by factoring it by a suitable congruence. Thus F inherits the property of generating a cofree comonad from $\tilde{F}$. (We are indebted to Gordon Plotkin for this observation.)

Example 2.6. As mentioned in the Introduction, we are mainly interested in the endofunctors K, L and M of an arbitrary topos with natural number object, whose coalgebras correspond to various notions of finitary transition systems, and with the functors $K(X \times -)$, $L(X \times -)$ and $M(X \times -)$ (where X is a fixed 'object of labels') which correspond to labelled transition systems. Since the list functor L is a partial product functor, we may read off the description of its cofree comonad from the proof of Lemma 2.4: in the set-theoretic notation employed there, an element of $G_L(1)$ is a finitely branching tree (not necessarily well-founded, but with each node at finite height), together with a total ordering of the children of each node. An element of $G_L(A)$ is a tree as above, together with a labelling of its nodes by elements of A (that is, a mapping h from the set of nodes to A). For the functor $L(X \times -)$, the cofree comonad may be similarly described, except that each edge of the tree must be labelled by an element of X.

The finite-multiset functor M is naturally a quotient of L, and its cofree comonad G_M may similarly be described as a quotient of G_L: $G_M(A)$ is the object we obtain by 'forgetting the ordering' on the nodes of the trees in $G_L(A)$, or more formally by identifying two trees if one can be obtained from the other by a (height- and label-preserving) permutation of its nodes.

For the finite-powerset functor K, the cofree comonad G_K may again be obtained as a quotient of G_M. In order to define it, let us first define an equivalence relation R on (the nodes of) an A-labelled tree (t,h) to be a *congruence* if it satisfies the following three properties:

(i) if $(x,y)\in R$, then x and y are at the same height and have the same label;
(ii) if $(x,y)\in R$, then $(x',y')\in R$, where x' and y' are the parents of x and y;
(iii) if $(x,y)\in R$, then for every child x'' of x there exists a child y'' of y such that $(x'',y'')\in R$.

Clearly, if these three properties are satisfied, then the quotient t/R can be given the structure of an A-labelled tree. We say two A-labelled trees (t,h) and (t',h') are *bisimilar* if there exist congruences R and R' such that $(t/R, h/R) \cong (t'/R', h'/R')$; this coincides with the relation defined (for unlabelled trees) by Barr [4], though his description of it was different from ours. We then take $G_K(A)$ to be the set of bisimilarity classes of A-labelled trees.

However, it is also useful to think of $G_K(A)$ as a subset rather than a quotient of $G_M(A)$ (although G_K is *not* a subfunctor of G_M). We may do this because, for any A-labelled tree (t,h), the set of congruences on (t,h) has a greatest member: it is clear that if $(R_i \mid i\in I)$ is a family of congruences, then the transitive closure of $\bigcup_{i\in I} R_i$ is again a congruence. If we form the quotient of t by this largest congruence, we obtain a minimal representative for its bisimilarity class: that is, a tree $(\bar{t},\bar{h})$ which is bisimilar to (t,h) and whose only congruence is the identity. We call an A-labelled tree *strongly extensional* if it has the latter property: we may now identify $G_K(A)$ with the set of strongly extensional A-labelled trees. (In this identification, the action of G_K on morphisms is as follows: given $f : A \to B$ and a strongly extensional A-labelled tree (t,h), we define $G_K f(t,h)$ to be the quotient of the B-labelled tree (t, fh) by its largest congruence.)

Remark 2.7. An A-labelled tree (t,h) is called *extensional* if, for any two children x and y of any node of t, the information that the subtrees rooted at x and y are isomorphic (as labelled trees) forces $x = y$. (An equivalent condition is that the only label-preserving automorphism of (t,h) is the identity.) It is clear that a strongly extensional tree is extensional, since if we are given two nodes x and y as above we can find a congruence which identifies them; but the converse is false for non-well-founded trees – a counterexample (with trivial labelling) is given in [4]. Any labelled tree has a largest extensional quotient, obtained by identifying a pair of nodes iff there is a label-preserving automorphism sending the first to the second; thus the assignment

$$A \mapsto \{\text{extensional } A\text{-labelled trees}\}$$

can be made into a functor H, which is a quotient of G_M and has G_K as a quotient: given $f : A \to B$, $Hf(t,h)$ is the largest extensional quotient of the B-labelled tree (t, fh). This functor again carries a comonad structure similar to those on G_K, G_L and G_M, although it is not cofree on any endofunctor.

Given a functor $F : \mathscr{E} \to \mathscr{E}$, we have already noted that the forgetful functor $\mathscr{E}_F \to \mathscr{E}$ creates any types of limits which are preserved by F. If F generates a cofree comonad, then the functor part G of the latter is the composite of this forgetful functor with its right adjoint (which of course preserves limits); so it inherits all the limit-preservation properties enjoyed by F. However, the weak limit preservation properties studied in Section 1 are rather more delicate. In the first place, knowing that F preserves pullbacks of monomorphisms is not enough to deduce that the forgetful functor $U : \mathscr{E}_F \to \mathscr{E}$ does so, for the simple reason that it need not preserve monomorphisms. If U does preserve monomorphisms, then it inherits the preservation of their pullbacks from F, and hence so does the cofree comonad generated by F if it exists.

Rather less obviously, we have

Lemma 2.8. *If $\mathscr{E}$ has pullbacks and F preserves weak pullbacks, then $U : \mathscr{E}_F \to \mathscr{E}$ preserves weak pullbacks.*

Proof. Given two morphisms $f : (A,\alpha) \to (C,\gamma)$ and $g : (B,\beta) \to (C,\gamma)$ which possess a weak pullback in $\mathscr{E}_F$, we may form the pullback

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

in $\mathscr{E}$. Since F sends this to a weak pullback, it is clear that P inherits an F-coalgebra structure map $\pi : P \to FP$ making p and q into coalgebra homomorphisms. So we get a comparison map $h : (P,\pi) \to (Q,\psi)$, where (Q,ψ) is the weak pullback of f and g in $\mathscr{E}_F$. But we also have a comparison map $k : Q \to P$ since P is a pullback in $\mathscr{E}$; and the composite $kh: P \to P$ is the identity for the same reason. Thus P is a retract of Q, and the latter is a weak pullback in $\mathscr{E}$. $\square$

Remark 2.9. Under the hypotheses of Lemma 2.8, if we also know that F generates a cofree comonad $\mathbb{G}$, then we may conclude that G preserves weak pullbacks, since the right adjoint of the forgetful functor certainly does so. It follows that G preserves pullbacks of monomorphisms, and hence (by Proposition 3.4 below) that $\mathscr{E}_F \cong \mathscr{E}_{\mathbb{G}}$ actually has pullbacks in this case.

It is also worth remarking that if, in the situation of Lemma 2.8, F itself carries a (not necessarily cofree) comonad structure and the coalgebras (A,α), (B,β) and (C,γ) appearing in the proof are actually Coalgebras, then so is (P,π) where π is constructed as above. So we may conclude that the forgetful functor $\mathscr{E}_{\mathbb{F}} \to \mathscr{E}$ also preserves weak pullbacks in this situation.

It seems unlikely that the property of nearly preserving pullbacks is inherited by G from F, but we do not have an explicit counterexample.

3. Transfer of properties to Coalgebras

From now on, we shall assume that the functors $F: \mathscr{E} \to \mathscr{E}$ in which we are interested generate cofree comonads, in the sense of Proposition 2.1, and so we are able to work with categories of Coalgebras rather than of coalgebras. One of the oldest theorems of elementary topos theory is the result that, if $\mathscr{E}$ is a topos and $\mathbb{G}$ is a comonad on $\mathscr{E}$ whose functor part G preserves finite limits, then $\mathscr{E}_{\mathbb{G}}$ is a topos: the proof of this theorem occupies a whole chapter of the earliest published account [21] of elementary topos theory, and it can also be found as Theorem 2.32 in [14], as Theorem V 8.4 in [25], and so on. (In [40, Theorem 50.5], the corresponding result is proved for quasitoposes.)

However, it seems to be less well known that the hypothesis 'G preserves finite limits' of this theorem can be weakened to 'G preserves pullbacks'. For the particular case of Artin glueing (see Section 5 below), this weakening was noticed quite early (cf. [6, p. 451]), but the fact that the same weakening can be made in general was apparently first observed by Rosebrugh and Wood [30]. We give a proof here which is essentially the same as theirs, because we shall need it in order to obtain a slightly more general result.

Proposition 3.1. *Let $\mathbb{G} = (G, \varepsilon, \delta)$ be a comonad on a category $\mathscr{E}$ with finite limits, such that the functor G preserves pullbacks of monomorphisms, and let (A,α) be a particular $\mathbb{G}$-Coalgebra. Then there is a comonad $\mathbb{G}' = (G', \varepsilon', \delta')$ on $\mathscr{E}/A$, such that G' preserves the terminal object, and such that the category of Coalgebras $(\mathscr{E}/A)_{\mathbb{G}'}$ is isomorphic to $(\mathscr{E}_{\mathbb{G}})/(A,\alpha)$. Moreover, if G preserves all pullbacks, then G' preserves all finite limits.*

Proof. To define G' at an object $f: B \to A$ of $\mathscr{E}/A$, we form the pullback

$$\begin{array}{ccc} G'B & \xrightarrow{h} & GB \\ \Big\downarrow{\scriptstyle G'f} & & \Big\downarrow{\scriptstyle Gf} \\ A & \xrightarrow{\alpha} & GA \end{array}$$

this is clearly functorial. Moreover, since $\varepsilon_A \alpha = 1_A$, the composite $\varepsilon_B h : G'B \to GB \to B$ is a morphism over A, and thus defines a natural transformation ε' from G' to the identity. And since G preserves the pullback square above (because α is a split monomorphism), $G'G'B$ is the pullback of GGB along the composite $(G\alpha)\alpha : A \to GGA$; but this composite equals $\delta_A \alpha$, and so the morphism δ_B induces a morphism $\delta'_f : G'B \to G'G'B$ over A, which is also natural in f. It is straightforward to verify that ε' and δ' form a comonad structure on G'. Moreover, G' preserves the terminal object by construction; and it preserves pullbacks if G does, since the pullback functor $\alpha^* : \mathscr{E}/GA \to \mathscr{E}/A$ preserves them, so in the latter case it preserves all finite limits. Finally, for any $f: B \to A$, there is a bijective correspondence between morphisms $f \to G'f$ in $\mathscr{E}/A$ and morphisms $\beta : B \to GB$ satisfying $(Gf)\beta = \alpha f$; and it is again easy to see that this restricts to a correspondence between $\mathbb{G}'$-Coalgebra structures on f and $\mathbb{G}$-Coalgebra structures on B which make f a Coalgebra homomorphism. $\square$

Taking (A, α) to be the terminal $\mathbb{G}$-Coalgebra $(G1, \delta_1)$, we immediately obtain

Corollary 3.2. *If $\mathscr{E}$ is a topos and $\mathbb{G}$ is a comonad on $\mathscr{E}$ whose functor part preserves pullbacks, then $\mathscr{E}_{\mathbb{G}}$ is a topos.*

Corollary 3.3. *Let $\mathscr{E}$ be a topos and $F : \mathscr{E} \to \mathscr{E}$ a functor which preserves pullbacks and generates a cofree comonad* (*for example, a partial product functor*). *Then $\mathscr{E}_F$ is a topos.*

Corollary 3.3 tells us, in particular, that for any topos $\mathscr{E}$ with a natural number object the category of coalgebras for the list functor $L : \mathscr{E} \to \mathscr{E}$ is a topos. However, we are also interested in K-coalgebras and M-coalgebras, and these functors do not preserve pullbacks. So our main task in this section is to investigate how much of Corollary 3.3 remains true under the weaker hypotheses on F considered in Section 1.

Proposition 3.4. *Let $\mathscr{E}$ be a category with finite limits, and $\mathbb{G} = (G, \varepsilon, \delta)$ a comonad on $\mathscr{E}$ whose functor part G preserves pullbacks of monomorphisms. Then $\mathscr{E}_{\mathbb{G}}$ has finite limits.*

Proof. The functor G preserves equalizers of coreflexive pairs, since these can be seen as a special case of pullbacks of monomorphisms; so the forgetful functor $\mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ creates them, and in particular $\mathscr{E}_{\mathbb{G}}$ has them. But, by Linton's theorem [23], the existence of such equalizers suffices to 'lift' finite products from $\mathscr{E}$ to $\mathscr{E}_{\mathbb{G}}$; and then we may construct all finite limits from products and coreflexive equalizers.

In fact, we may give an explicit construction of products in $\mathscr{E}_{\mathbb{G}}$, as follows: given Coalgebras (A, α) and (B, β), form the diagram

$$\begin{array}{ccccc}
P & \longrightarrow & Q & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} \\
R & \longrightarrow & G(A\times B) & \xrightarrow{G\pi_2} & GB \\
\downarrow & & \downarrow{\scriptstyle G\pi_1} & & \\
A & \xrightarrow{\alpha} & GA & &
\end{array}$$

in which all the squares are pullbacks. Since α and β are (split) monic, these pullbacks are all preserved by G; moreover, since the named morphisms are all Coalgebra homomorphisms (where GA, GB and $G(A \times B)$ are regarded as cofree Coalgebras), we may equip P, Q and R uniquely with Coalgebra structures so that the squares become pullbacks in $\mathscr{E}_{\mathbb{G}}$. We claim that P, with this structure, is a product of (A, α) and (B, β) in $\mathscr{E}_{\mathbb{G}}$.

To see this, let (C, γ) be a third Coalgebra, and $f : (C, \gamma) \to (A, \alpha)$ and $g : (C, \gamma) \to (B, \beta)$ two Coalgebra homomorphisms. Then the diagram

$$\begin{array}{ccccc}
A & \xleftarrow{f} & C & \xrightarrow{g} & B \\
 & & \downarrow{\scriptstyle \gamma} & & \\
\downarrow{\scriptstyle \alpha} & & GC & & \downarrow{\scriptstyle \beta} \\
 & & \downarrow{\scriptstyle G(f,g)} & & \\
GA & \xleftarrow{G\pi_1} & G(A\times B) & \xrightarrow{G\pi_2} & GB
\end{array}$$

commutes, and so the triple $(f, G(f, g)\gamma, g)$ induces a morphism $C \to P$, which is moreover a Coalgebra homomorphism since each of f, g and $G(f, g)\gamma$ is. On the other hand, if $h: C \to G(A \times B)$ is any Coalgebra homomorphism making the above diagram commute, then

$$\pi_1 \varepsilon_{A\times B} h = \varepsilon_A (G\pi_1) h = \varepsilon_A \alpha f = f$$

and similarly $\pi_2 \varepsilon_{A\times B} h = g$, so

$$\varepsilon_{A\times B} h = (f, g) = (f, g)\varepsilon_C \gamma = \varepsilon_{A\times B} G(f, g)\gamma;$$

but a Coalgebra homomorphism into $G(A \times B)$ is uniquely determined by its composite with $\varepsilon_{A\times B}$. So the pair (f, g) factors uniquely through the pair of projections $(P \to A, P \to B)$.

We remark in passing that the pullback of a diagram

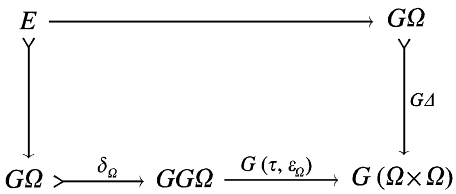
$$(A, \alpha) \longrightarrow (C, \gamma) \longleftarrow (B, \beta)$$

in $\mathscr{E}_{\mathbb{G}}$ may be constructed in exactly the same way, replacing the object $G(A \times B)$ by $G(A \times_C B)$. □

Example 3.5. We give an explicit description of binary products in $\mathbf{Set}_{\mathbb{G}}$, where $\mathbb{G}$ is the cofree comonad generated by the finite-multiset functor M (cf. 2.6). Following the prescription above, let (A, α) and (B, β) be two $\mathbb{G}$-Coalgebras; then an element of (the underlying set of) the product $(A, \alpha) \times (B, \beta)$ is a triple (a, b, t), where $a \in A$ and $b \in B$ are elements such that the labelled trees αa and βb have the same underlying unlabelled tree, and $t \in G(A \times B)$ is an $(A \times B)$-labelled tree which 'specializes' to αa and to βb when the two projections are applied to its labels. Equivalently, we may think of it as a triple $(a, b, [h])$, where a and b are as before and $[h]$ is an equivalence class of bijections h from the underlying unlabelled tree of αa to that of βb (the notion of equivalence being induced by composition with label-preserving automorphisms, i.e. $h \equiv ghf$ whenever f and g are label-preserving automorphisms of αa and βb, respectively). Equivalently again, we may think of $[h]$ as a double coset of the pair of subgroups $(\mathrm{Aut}(\beta b), \mathrm{Aut}(\alpha a))$ in the automorphism group of the unlabelled tree. The M-coalgebra structure map on this set sends $(a, b, [h])$ to the multiset of all triples $(a', b', [h'])$, where a' and b' are labels attached to children of the roots of αa and βb which are identified by h, and h' is the restriction of h to the subtrees rooted at these nodes – it is easy to see that this description is independent of the choice of h within the double coset $[h]$.

Proposition 3.6. *In addition to the hypotheses of Proposition* 3.4, *suppose that* $\mathscr{E}$ *has a subobject classifier, and that the forgetful functor* $\mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ *preserves monomorphisms. Then* $\mathscr{E}_{\mathbb{G}}$ *has a subobject classifier.*

Proof. Let $\top : 1 \to \Omega$ be a subobject classifier in $\mathscr{E}$. The construction of a subobject classifier in $\mathscr{E}_{\mathbb{G}}$ is a direct transcription of that in the case when G preserves all finite limits (see, for example, [14, 2.32]); we simply have to observe that the construction given there, and the proof that it works, uses only pullbacks of monomorphisms and their preservation by G. Explicitly, let $\tau : G\Omega \to \Omega$ classify $G\top : G1 \rightarrowtail G\Omega$, and form the pullback



We note that $G\Delta$ is monic since Δ is split monic, so E inherits a coalgebra structure making it a subcoalgebra of $(G\Omega, \delta_\Omega)$.

Now let (A, α) be a Coalgebra. Since the forgetful functor $\mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ preserves monomorphisms (and also reflects them, because it is faithful), the subobjects of (A, α) in $\mathscr{E}_{\mathbb{G}}$ are exactly the subobjects $A' \rightarrowtail A$ in $\mathscr{E}$ for which the composite $A' \rightarrowtail A \to GA$ factors through $GA' \rightarrowtail GA$. But any subobject $A' \rightarrowtail A$ corresponds to a morphism $f : A \to \Omega$ in $\mathscr{E}$ and hence to a Coalgebra morphism $\bar{f} = (Gf)\alpha : (A, \alpha) \to (G\Omega, \delta_\Omega)$. We claim that $\bar{f}$ factors through the subcoalgebra E iff A' has a (necessarily unique) Coalgebra structure making it into a subCoalgebra of A, i.e. iff $A' \leqslant \alpha^*(GA' \rightarrowtail GA)$ in $\mathrm{Sub}(A)$. Note that, by naturality of ε, we always have $GA' \leqslant \varepsilon_A^*(A')$ in $\mathrm{Sub}(GA)$ and hence $\alpha^*(GA') \leqslant \alpha^*\varepsilon_A^*(A') = A'$; so the inequality above is equivalent to the equality $A' = \alpha^*(GA')$. Next, observing that the composite $G(\tau, \varepsilon_\Omega)\delta_\Omega \bar{f}$ is determined by its composite with $\varepsilon_{\Omega \times \Omega}$, we see that it factors through $G\Delta$ iff $\tau\varepsilon_{G\Omega}\delta_\Omega \bar{f} = \varepsilon_\Omega \varepsilon_{G\Omega}\delta_\Omega \bar{f}$. But the right-hand side of this equation reduces to $\varepsilon_\Omega \bar{f} = f$, which classifies $A' \rightarrowtail A$, and the left-hand side to $\tau \bar{f} = \tau(Gf)\alpha$, which classifies $\alpha^*(GA')$. So the result is established. □

The existence of a subobject classifier for the category of F-coalgebras was proved, under some additional non-elementary hypotheses (local presentability, etc.) besides those of Proposition 3.6, in [28] (and see also [38]). That proof is now superseded by the one above.

In passing, we note that, even in the absence of cartesian closedness, the existence of a subobject classifier may be used to derive certain topos-like properties of $\mathscr{E}_{\mathbb{G}}$: for example, the fact that all its monomorphisms are regular (cf. [14, Lemma 1.21]), and hence that every epimorphism is a cover.

Example 3.7. Once again, we give an explicit description of the subobject classifier for $\mathbf{Set}_M$, where M is the finite-multiset functor. As in Example 3.5, we write $\mathbb{G}$ for the cofree comonad generated by M; then the proof of Proposition 3.6 tells us that the subobject classifier should be a subCoalgebra of the cofree Coalgebra $(G\Omega, \delta_\Omega)$, where $\Omega = \{\bot, \top\}$ is the subobject classifier for $\mathbf{Set}$. It is not hard to verify that, in this case, the bottom edge of the pullback square defining E in the proof of Proposition 3.6 sends an Ω-labelled tree to the same tree with each node labelled by the pair of truth-values (p, p'), where p is its original label and p' is the infimum of the labels of all the nodes above the one under consideration. Thus an Ω-labelled tree belongs to E iff its labelling is increasing, in the sense that the label of any node is less than or equal to the labels of all its children. Of course, given a subCoalgebra (A', α') of a Coalgebra (A, α), the corresponding classifying map $A \to E$ sends an element a to the Ω-labelled tree obtained from the underlying unlabelled tree of αa by labelling each node with the truth-value of the assertion that the element of A labelling it belongs to A'.

Lemma 3.8. *In addition to the hypotheses of Proposition* 3.4, *suppose $\mathscr{E}$ has finite* (*resp. arbitrary set-indexed*) *disjoint coproducts which are stable under pullback. Then so has $\mathscr{E}_{\mathbb{G}}$.*

Proof. Existence of coproducts in $\mathscr{E}_{\mathbb{G}}$ follows from the fact that the forgetful functor creates them. Their disjointness is immediate from the fact that it also creates intersections of subobjects. So we need only verify pullback-stability. Suppose $A = A_1 + A_2$, where A_1 and A_2 have $\mathbb{G}$-coalgebra structures α_1 and α_2; then the Coalgebra structure α on A factors through $GA_1 + GA_2 \rightarrowtail G(A_1 + A_2)$ (which is monic, because GA_1 and GA_2 are subobjects of $G(A_1 + A_2)$, which must be disjoint since the initial object 0 of $\mathscr{E}$ is strict initial and so $G(0) \cong 0$). Now suppose we are given a Coalgebra map $f : (A, \alpha) \to (C, \gamma)$ and we wish to form the pullback of f along $g : (B, \beta) \to (C, \gamma)$. The construction of pullbacks in the proof of Proposition 3.4 tells us first to form the pullback in $\mathscr{E}$ of α along $G\pi_1 : G(A \times_C B) \to GA$; but we may factor this pullback square as

$$\begin{array}{ccccc}
Q_1 + Q_2 & \rightarrowtail & G(A_1 \times_C B) + G(A_2 \times_C B) & \rightarrowtail & G(A \times_C B) \\
\downarrow & & \downarrow & & \downarrow \\
A_1 + A_2 & \rightarrowtail & GA_1 + GA_2 & \rightarrowtail & GA
\end{array}$$

and then if we compose the top edge with $G\pi_2 : G(A \times_C B) \to GB$ and pull back along β, we get a similar coproduct decomposition of the pullback $(A, \alpha) \times_{(C,\gamma)} (B, \beta)$. The argument for infinite coproducts is similar. □

As we indicated in the Introduction, we are ultimately interested in studying the category $\mathbf{Rel}(\mathscr{E}_{\mathbb{G}})$ of relations in $\mathscr{E}_{\mathbb{G}}$; so it is important for us to know that $\mathscr{E}_{\mathbb{G}}$ is a regular category, since the latter is a necessary condition for composition of relations to be associative [12, 1.569]. Fortunately, this is not difficult to establish under appropriate hypotheses.

Lemma 3.9. *In addition to the hypotheses of Proposition* 3.4, *suppose that* $\mathscr{E}$ *is regular*, *that the forgetful functor* $\mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ *preserves monomorphisms*, *and that* G *nearly preserves pullbacks. Then* $\mathscr{E}_{\mathbb{G}}$ *is regular*; *moreover the forgetful functor* $\mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ *preserves image factorizations.*

Proof. Let $f : (A, \alpha) \to (B, \beta)$ be a morphism of $\mathscr{E}_{\mathbb{G}}$, and write $A \to C \rightarrowtail B$ for the image factorization of f in $\mathscr{E}$. Since G preserves monomorphisms, there is a unique morphism $\gamma : C \to GC$ making

$$\begin{array}{ccccc} A & \twoheadrightarrow & C & \rightarrowtail & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ GA & \longrightarrow & GC & \rightarrowtail & GB \end{array}$$

commute; it is straightforward to verify that γ is a $\mathbb{G}$-Coalgebra structure, and that (C, γ) is the image of f in $\mathscr{E}_{\mathbb{G}}$. Thus we need only show that covers are stable under pullback in $\mathscr{E}_{\mathbb{G}}$.

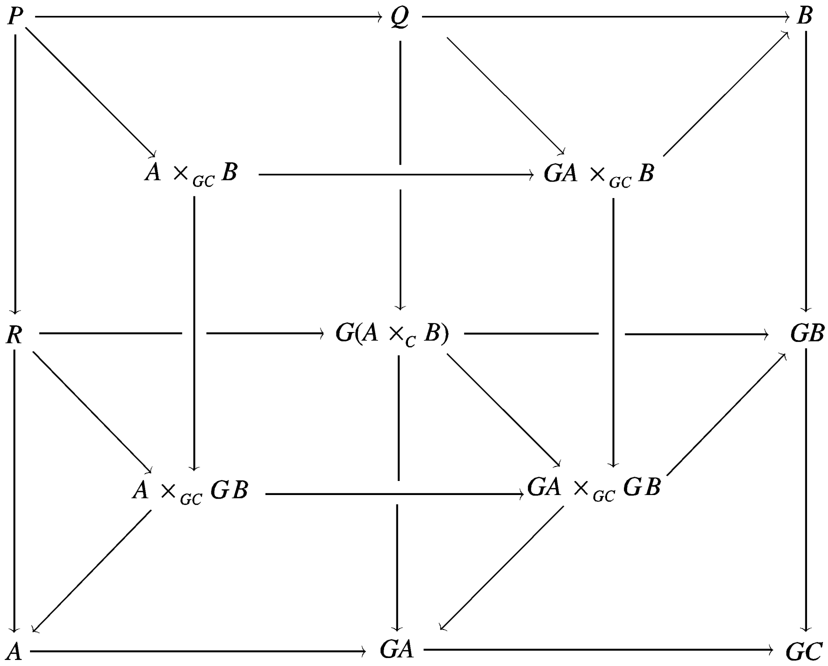
Suppose given a diagram

$$(A, \alpha) \xrightarrow{f} (C, \gamma) \xleftarrow{g} (B, \beta)$$

in $\mathscr{E}_{\mathbb{G}}$, where f is a cover. To form the pullback of f along g, we need to form the diagram

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} \\ R & \longrightarrow & G(A \times_C B) & \longrightarrow & GB\,, \\ \downarrow & & \downarrow & & \\ A & \xrightarrow{\alpha} & GA & & \end{array}$$

where the three squares are pullbacks in $\mathscr{E}$. But we may enlarge this to



where all faces of the cube, and the square faces of the two triangular prisms, are pullbacks; hence the morphism $P \to A \times_{GC} B$ is a cover because it is a pullback of the comparison map $G(A \times_B C) \to GA \times_{GC} GB$. Moreover, we may identify the codomain of this morphism with $A \times_C B$, since the composites $A \to GC$ and $B \to GC$ both factor through the monomorphism $\gamma : C \rightarrowtail GC$; and hence the morphism $A \times_{GC} B \to B$ is also a cover, because it is a pullback of f. So the composite $P \to B$ is a cover (in $\mathscr{E}$, and hence in $\mathscr{E}_{\mathbb{G}}$), as required. □

Remark 3.10. If we make the stronger assumption that G preserves weak pullbacks, then we may simplify the proof of Lemma 3.9 considerably. For in this case we know by Remark 2.9 that the forgetful functor $U : \mathscr{E}_{\mathbb{G}} \to \mathscr{E}$ also preserves weak pullbacks; and it preserves and reflects covers by the first part of the proof of Lemma 3.9. So, given a cover $f : (A, \alpha) \to (C, \gamma)$ in $\mathscr{E}_{\mathbb{G}}$, the image under U of its pullback along $g : (B, \beta) \to (C, \gamma)$ is the composite of the comparison map $U((A, \alpha) \times_{(C,\gamma)} (B, \beta)) \to A \times_C B$ (which is split epic, and hence a cover) with the pullback of Uf along Ug (which is a cover, since $\mathscr{E}$ is regular).

Putting together the last few results, we have:

Proposition 3.11. *Let $\mathscr{E}$ be a Grothendieck topos, and let $F : \mathscr{E} \to \mathscr{E}$ be an accessible functor which preserves weak pullbacks. Then the category $\mathscr{E}_F$ of F-coalgebras* *satisfies all the hypotheses of Giraud's theorem* [14, 0.45; 25, p. 575] *except possibly for effectiveness of equivalence relations.*

Proof. By Proposition 2.3, F generates a cofree comonad $\mathbb{G}$, which preserves weak pullbacks by Lemma 2.9. So by Proposition 3.4 the category $\mathscr{E}_F \cong \mathscr{E}_{\mathbb{G}}$ has finite limits; by Lemma 3.8 it has arbitrary disjoint coproducts which are stable under pullback, and by Lemma 3.9 it is regular and all its epimorphisms are covers. It is clearly locally small since $\mathscr{E}$ is; and it is accessible by Proposition 2.3, so it has a set of generators. Thus we have verified all but one of the Giraud hypotheses as stated in [14]. □

Nevertheless, under the hypotheses of Proposition 3.11, the category $\mathscr{E}_F$ need not be a topos. We shall give counterexamples in Example 5.6 below.

4. Effectivization

We have seen that, under suitable hypotheses on F, the category $\mathscr{E}_F$ inherits from $\mathscr{E}$ all the exactness properties of a topos except for effectiveness of equivalence relations. For any regular category $\mathscr{C}$, there is a standard way of remedying this defect, which is described in detail in [12] (and see also [26, Section 25]): we follow Freyd and Scedrov in using the name *effective regular category* for a regular category in which all equivalence relations are effective (other authors commonly call this an *exact* or *Barr-exact* category, cf. [3]).

Theorem 4.1. *For any regular category* $\mathscr{C}$, *there is an effective regular category* $\mathbf{Eff}(\mathscr{C})$ *and a full embedding* $\mathscr{C} \to \mathbf{Eff}(\mathscr{C})$ *which is universal among regular functors from* $\mathscr{C}$ *to effective regular categories.*

Proof. The standard construction of $\mathbf{Eff}(\mathscr{C})$ begins by embedding $\mathscr{C}$ in its allegory of relations $\mathbf{Rel}(\mathscr{C})$, then splitting the equivalence relations (which appear as idempotents in this category), and finally cutting back to the category of maps in the resulting allegory. More explicitly, an object of $\mathbf{Eff}(\mathscr{C})$ is a pair (A,R) where A is an object of $\mathscr{C}$ and $R \rightarrowtail A \times A$ is an equivalence relation on A, and a morphism $(A,R) \to (B,S)$ is a relation $F : A \looparrowright B$ which is 'functional relative to R and S', in the sense that

$$SFR = F, \quad R \leqslant F^{\circ}F \quad \text{and} \quad FF^{\circ} \leqslant S.$$

Composition in $\mathbf{Eff}(\mathscr{C})$ is the usual composition of relations; the identity morphism on (A,R) is R itself. The embedding $I : \mathscr{C} \to \mathbf{Eff}(\mathscr{C})$ sends an object A to (A,Δ) where Δ denotes the diagonal relation on A, and a morphism $f : A \to B$ to the graph of f. For the remaining details, see [12] or [26]. □

The above construction should not be confused with the exact completion of a category with finite limits, as studied in works such as [8]: the question of when the latter is cartesian closed has been studied by Rosický [31] and Birkedal et al. [5]. The exact completion freely adjoins images as well as quotients, in such a way that existing image factorizations are not necessarily preserved. See also [9] for a comparison of the two constructions.

As is shown in [12, 2.213], the passage from $\mathscr{C}$ to $\mathbf{Eff}(\mathscr{C})$ preserves the property of having disjoint pullback-stable coproducts; and it also clearly preserves local smallness and the possession of a set of generators (the image under I of any set of generators for $\mathscr{C}$ serves as a set of generators for $\mathbf{Eff}(\mathscr{C})$). Thus we may immediately conclude from Corollary 3.11.

Corollary 4.2. *Let $\mathscr{E}$ be a Grothendieck topos, and let $F : \mathscr{E} \to \mathscr{E}$ be an accessible functor which preserves weak pullbacks. Then the category $\mathbf{Eff}(\mathscr{E}_F)$ is a Grothendieck topos.*

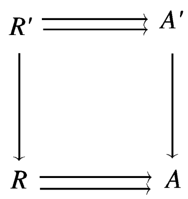
In particular, this corollary applies when F is the finite-powerset functor on any Boolean Grothendieck topos, by Example 1.4. However, it would clearly be desirable to have an 'elementary' version of the result, not making use of the set-theoretic hypotheses in Giraud's theorem. Although we have not yet succeeded in proving such an elementary result, we devote the rest of this section to discussing how it might be proved. The next few results may also be found in [26, 25.21, 25.23, 25.25].

Lemma 4.3. *Every subobject of an object of the form IA in $\mathbf{Eff}(\mathscr{C})$ is (isomorphic to one) in the image of I.*

Proof. Suppose $F : (B,R) \to IA$ is a monomorphism in $\mathbf{Eff}(\mathscr{C})$. (The assertion that F is monic is equivalent to saying that the inequality $R \leqslant F^{\circ}F$ is an equality.) Let $A' \rightarrowtail A$ be the image of the composite $F \rightarrowtail B \times A \to A$; then F still defines a relation $B \looparrowright A'$, which is an isomorphism $(B,R) \cong IA'$ in $\mathbf{Eff}(\mathscr{C})$. □

Corollary 4.4. *If Ω is a subobject classifier in a regular category $\mathscr{C}$, then $I\Omega$ is a subobject classifier in $\mathbf{Eff}(\mathscr{C})$.*

Proof. By Lemma 4.3 and the fact that I is full and faithful, we know that morphisms $IA \to I\Omega$ in $\mathbf{Eff}(\mathscr{C})$ correspond to subobjects of IA. For a general object (A,R), a morphism $(A,R) \to I\Omega$ in $\mathbf{Eff}(\mathscr{C})$ corresponds to a morphism $A \to \Omega$ in $\mathscr{C}$ having equal composites with the projections $R \rightrightarrows A$; that is, to a subobject $A' \rightarrowtail A$ for which we have a diagram



in which both squares are pullbacks. But R' is then an equivalence relation on A', and (A', R') defines a subobject of (A, R) in $\mathbf{Eff}(\mathscr{C})$ (the monomorphism $(A', R') \rightarrowtail (A, R)$ being R' regarded as a relation $A' \looparrowright A$); and every subobject of (A, R) arises in this way, since we have a pullback-stable coequalizer diagram

$$IR \rightrightarrows IA \twoheadrightarrow (A, R)$$

in $\mathbf{Eff}(\mathscr{C})$. □

In much the same way, we may prove

Lemma 4.5. *Let $\mathscr{C}$ be a regular category, and A an object of $\mathscr{C}$. Suppose there exists an object PA of $\mathbf{Eff}(\mathscr{C})$ such that, for any $B \in \text{ob}\, \mathscr{C}$, subobjects of $B \times A$ in $\mathscr{C}$ correspond, naturally in B, to morphisms $IB \to PA$ in $\mathbf{Eff}(\mathscr{C})$. Then PA is a power-object for IA in $\mathbf{Eff}(\mathscr{C})$; that is, for any $C \in \text{ob}\, \mathbf{Eff}(\mathscr{C})$, morphisms $C \to PA$ correspond to subobjects of $C \times IA$.*

Proof. Suppose $C = (B, S)$; then subobjects of $C \times IA$ in $\mathbf{Eff}(\mathscr{C})$ correspond to subobjects of $IB \times IA \cong I(B \times A)$ in $\mathbf{Eff}(\mathscr{C})$ (equivalently, by Lemma 4.3, to subobjects of $B \times A$ in $\mathscr{C}$) whose pullbacks along the two projections $I(S \times A) \rightrightarrows I(B \times A)$ are isomorphic. So this again follows from the fact that $IS \rightrightarrows IB \to (B, S)$ is a coequalizer diagram in $\mathbf{Eff}(\mathscr{C})$. □

Proposition 4.6. *Let $\mathscr{C}$ be a regular category, and suppose that for each object A of $\mathscr{C}$ there exists an object PA of $\mathbf{Eff}(\mathscr{C})$ with the property indicated in Lemma 4.5. Then $\mathbf{Eff}(\mathscr{C})$ is a topos.*

Proof. After Lemma 4.5, we have to show that the possession of power-objects is inherited by arbitrary objects of $\mathbf{Eff}(\mathscr{C})$ from objects of the form IA. But, once again, it follows from the fact that $IR \rightrightarrows IA \to (A, R)$ is a pullback-stable coequalizer diagram that we may define $P(A, R)$ to be the equalizer of

$$PA \underset{Pb}{\overset{Pa}{\rightrightarrows}} PR,$$

where $Pa : PA \to PR$ is, as usual, the morphism corresponding to the subobject obtained by pulling back the universal subobject of $PA \times IA$ along $1_{PA} \times Ia : PA \times IR \to PA \times IA$. □

Thus, to give an elementary proof that $\mathbf{Eff}(\mathscr{E}_F)$ is a topos, it would suffice to construct for each F-coalgebra A a 'pre-power object' PA equipped with an equivalence relation $R : PA \looparrowright PA$ and a relation $PA \looparrowright A$ having a suitable universal property. Sadly, we have not yet been able to find such a construction for a general F. However, we conclude this section by giving an explicit description of how PA and R may be constructed for the finite-multiset functor $M : \mathbf{Set} \to \mathbf{Set}$.

Example 4.7. As in Example 3.5, we write $\mathbb{G}$ for the cofree comonad generated by M; given a $\mathbb{G}$-Coalgebra (A,α), we shall write $\alpha_0 : A \to MA$ for the corresponding M-coalgebra (so $\alpha_0 a$ is the multiset of labels of height-1 nodes in the labelled tree αa), and $[\alpha a]$ for the underlying unlabelled tree of αa.

In order to define the 'pre-power-object' of a Coalgebra (A,α), we need a method of 'removing the multiplicities' from A by labelling its nodes with 'memories of how they were reached'. We do this by means of the notion of a 'good $\mathbb{N}^*$-labelled tree' (where $\mathbb{N}^*$ denotes the set of finite sequences of natural numbers): we call an $\mathbb{N}^*$-labelled tree *good* if, whenever a node x is labelled by a sequence s, there exists a natural number n such that the children of x are labelled by the sequences $s.0, s.1, \ldots, s.(n-1)$ (without repetitions). Note that we do *not* require the root of the tree to be labelled by the empty sequence. We identify a good $\mathbb{N}^*$-labelled tree T with the set of labels of its nodes (note that these suffice to reconstruct the tree up to isomorphism), and regard it as an M-coalgebra (T,τ) by setting $\tau(s)$ to be the set of all sequences $s.n$ which occur in T. (Thus (T,τ) is in fact a K-coalgebra, although this is not relevant for our present purposes.)

Given (A,α), we define its 'pre-power-object' to be the Coalgebra (P,π), where elements of P are pairs (T,f) such that T is a good $\mathbb{N}^*$-labelled tree and f is an increasing function from the underlying set of the product M-coalgebra $(T,\tau)\times(A,\alpha_0)$ to $2=\{\bot,\top\}$. (By 'increasing', we mean that if x and y are in the domain of f and x makes a transition to y then $f(x)\leqslant f(y)$.) The transition map is defined as follows: if s is the sequence labelling the root of T, then

$$\pi_0(T,f) = \{(T/s.n, f|_{T/s.n\times A}) \mid s.n \in T\},$$

where $T/s.n$ denotes the subtree of T rooted at $s.n$. (Note once again that this is a set rather than a multiset, i.e., none of its elements has multiplicity greater than 1.)

Elements of the product $(P,\pi)\times(P,\pi)$ may be identified with quintuples (T,f,T',f',h) where $h:[T]\to[T']$ is an isomorphism of unlabelled trees. (We do not have to worry about double cosets here, because (T,τ) and (T',τ') have no non-identity automorphisms as labelled trees.) We define R to be the set of such quintuples for which $f=f'(h\times 1_A)$; it is straightforward to verify that this is a subCoalgebra of $(P,\pi)\times(P,\pi)$, and that it is an equivalence relation on (P,π) in the category of $\mathbb{G}$-coalgebras, so we may regard $((P,\pi),R)$ as an object of $\mathbf{Eff}(\mathbf{Set}_M)$.

Now suppose given another Coalgebra (B,β) and a subCoalgebra (C,γ) of $(B,\beta)\times(A,\alpha)$. We define a new subCoalgebra (F,ϕ) of $(B,\beta)\times(P,\pi)$, as follows: F consists of those quadruples $(b,T,f,[g])$ (where $b\in B$, $(T,f)\in P$ and $g:[\beta b]\to[T]$ is an isomorphism of unlabelled trees) for which, whenever $a\in A$ and $u:[\alpha a]\to[\beta b]$ is an embedding, we have

$$(b',a,[u^{-1}])\in C \Leftrightarrow f(a,s,gu)=\top,$$

where b' is the label attached to the node of $[\beta b]$ which is the image of the root of $[\alpha a]$ under u, and s is the sequence labelling the corresponding node of T. It is easy to verify that F is a subCoalgebra of $(B,\beta)\times(P,\pi)$, that is, a morphism $(B,\beta)\looparrowright(P,\pi)$ in $\mathbf{Rel}(\mathbf{Set}_M)$.

Moreover, if $(b,T,f,[g])\in(B,\beta)\times(P,\pi)$ and $(T,f,T',f',h)\in R$, we have

$$\begin{aligned}(b,T,f,[g])\in F &\Leftrightarrow f(a,s,gv^{-1})=\top \quad \text{for all } (b',a,[v])\in C\\ &\Leftrightarrow f'(a,hgv^{-1})=\top \quad \text{for all } (b',a,[v])\in C\\ &\Leftrightarrow (b,T',f',[hg])\in F,\end{aligned}$$

from which it follows that $FF^\circ = R$. Also, given $b\in B$, we can choose a good $\mathbb{N}^*$-labelled tree T and a particular isomorphism $g:[\beta b]\to[T]$, and then define f by

$$f(a,s,u)=[\![(b',a,[u^{-1}g])\in C]\!]$$

(where b' is the label attached to the node of $[\beta b]$ mapped by g to s); then it is clear that $(T,f)\in P$, and that $(b,T,f,[g])\in F$, so $\Delta_B\leqslant F^\circ F$. Thus we see that F defines a morphism $I(B,\beta)\to((P,\pi),R)$ in $\mathbf{Eff}(\mathbf{Set}_M)$.

Conversely, suppose given a relation $F:(B,\beta)\looparrowright(P,\pi)$ with $F^\circ=R$ and $\Delta_B\leqslant F^\circ F$. We define $C\subseteq(B,\beta)\times(A,\alpha)$ by

$$\begin{aligned}(b,a,[h])\in C \;\Leftrightarrow\; &\text{whenever } (b,T,f,[g])\in F, \text{ we have } f(a,s_0,gh^{-1})=\top\\ &(\Leftrightarrow \text{there exists } (b,T,f,[g])\in F \text{ with } f(a,s_0,gh^{-1})=\top),\end{aligned}$$

where s_0 denotes the label attached to the root of T. Once again, it is easy to verify that C is a subCoalgebra of $(B,\beta)\times(A,\alpha)$, and that the two constructions given above are inverse to each other. Thus we have a bijection between subCoalgebras of $(B,\beta)\times(A,\alpha)$ and morphisms $I(B,\beta)\to((P,\pi),R)$ in $\mathbf{Eff}(\mathbf{Set}_M)$, and this bijection is also easily seen to be natural in (B,β). So $((P,\pi),R)$ satisfies the hypothesis of Lemma 4.5.

5. Coalgebras and Artin glueing

Let $F:\mathscr{E}\to\mathscr{F}$ be a functor. We recall that the category $\mathbf{Gl}(F)$ obtained by *Artin glueing* along F is simply the comma category $(\mathscr{F}\downarrow F)$; in other words, it is the category whose objects are triples (A,B,f) with $f:B\to FA$, and whose morphisms are pairs $(A\to A', B\to B')$ giving rise to commutative squares

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow f & & \downarrow f' \\ FA & \longrightarrow & FA' \end{array}$$

in $\mathscr{F}$. We further recall from [39] that it may be identified with the category of Coalgebras $(\mathscr{E}\times\mathscr{F})_{\mathbb{G}}$, where the functor G is defined by $G(A,B)=(A,B\times FA)$, and its counit and comultiplication are induced by the projection $B\times FA\to B$ and the diagonal map $FA \to FA \times FA$, respectively. Note that the functor G inherits any limit-preservation property enjoyed by F.

In [6], the question of when $\mathbf{Gl}(F)$ is a topos was investigated: the answer turns out to be simple.

Proposition 5.1. $\mathbf{Gl}(F)$ *is a topos if and only if* $\mathscr{E}$ *and* $\mathscr{F}/F1$ *are toposes, and* F *preserves pullbacks.*

Proof. See [6, Theorem 4.3]. □

If we consider the case when $\mathscr{E} = \mathscr{F}$, there is an obvious similarity between $\mathbf{Gl}(F)$ and the category of F-coalgebras $\mathscr{E}_F$. However, the latter is not a full subcategory of the former, and its topos structure (if it possesses one!) is substantially different. We shall find it more profitable to compare $\mathbf{Gl}(F)$ not with $\mathscr{E}_F$ but with $\mathscr{E}_{(1+F)}$, where $1 + F$ denotes the coproduct of F and the constant functor with value 1. In what follows we shall assume that $\mathscr{E}$ is a topos: this is more than enough to ensure that F and $1+F$ share the same connected-limit-preservation properties. We shall also assume that F preserves pullbacks of monomorphisms.

We may define a functor $U : \mathbf{Gl}(F) \to \mathscr{E}_{(1+F)}$ by $U(A,B,f) = (A + B, g)$, where g is the composite

$$A + B \to 1 + FA \to 1 + F(A + B).$$

Lemma 5.2. *The functor* U *just defined has a right adjoint.*

Proof. Given a $(1+F)$-coalgebra (C,γ), we define $R(C,\gamma)$ to be $(C_0, C_1, \gamma|_{C_1})$, where

$$\begin{array}{ccccccc}
C_0 & \rightarrowtail & C & & \longleftarrow\!\!\!< & & C_1 \\
\downarrow & & \downarrow{\scriptstyle \gamma} & & & & \downarrow \\
1 & \rightarrowtail & 1+FC & \longleftarrow\!\!\!< & FC & \longleftarrow\!\!\!< & FC_0
\end{array}$$

are pullbacks. It is easy to verify that a coalgebra homomorphism $U(A,B,f) \to (C,\gamma)$ must map A into C_0 and B into C_1, and hence that R is right adjoint to U. □

Lemma 5.3. *The image of* U *is a sieve in* $\mathscr{E}_{(1+F)}$: *that is, if a morphism of* $\mathscr{E}_{(1+F)}$ *has its codomain in the image, then the morphism itself* (*and its domain*) *lies in the image.*

Proof. Suppose given a coalgebra morphism $h : (C,\gamma) \to U(A,B,f)$. Then C decomposes as a coproduct $C_0 + C_1$, where $C_0 = h^*(A)$ and $C_1 = h^*(B)$; and the commutativity of

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & 1+FC \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle 1+Fh} \\ A+B & \longrightarrow & 1+F(A+B) \end{array}$$

ensures that γ maps C_0 into the first summand of $1+FC$, and C_1 into FC_0 (which we may identify with $(Fh)^*(FA \rightarrowtail F(A+B))$, since F preserves pullbacks of monomorphisms). So we have $C \cong U(C_0, C_1, \gamma|_{C_1})$, by an isomorphism identifying h with $U(h|_{C_0}, h|_{C_1})$. □

Since U is clearly full and faithful, it follows immediately that it preserves limits of arbitrary non-empty diagrams. (In fact it is not hard to see that it identifies $\mathbf{Gl}(F)$ with the slice category $(\mathscr{E}_{(1+F)})/U(1)$.) We may now conclude

Corollary 5.4. *If $\mathscr{E}_{(1+F)}$ is cartesian closed, then so is $\mathbf{Gl}(F)$.*

Proof. From the last two lemmas, we know that U is full and faithful, preserves binary products and has a right adjoint R. It follows immediately that the object $R(UA^{UB})$ has the universal property of an exponential A^B, for any two objects A and B of $\mathbf{Gl}(F)$. □

Putting everything together, we have

Corollary 5.5. *Let $\mathscr{E}$ be a topos, and $F : \mathscr{E} \to \mathscr{E}$ a functor which preserves pullbacks of monomorphisms, and such that $1+F$ generates a cofree comonad. Then the following conditions are equivalent.*
(i) *F preserves pullbacks.*
(ii) *$\mathbf{Gl}(F)$ is a topos.*
(iii) *$\mathbf{Gl}(F)$ is cartesian closed.*
(iv) *$\mathscr{E}_{(1+F)}$ is a topos.*
(v) *$\mathscr{E}_{(1+F)}$ is cartesian closed.*

Proof. Clearly, the functors $(1+F)$ and $(A,B) \mapsto (A, B \times FA)$ both preserve pullbacks of monomorphisms; and they preserve arbitrary pullbacks iff F does. So by Propositions 3.4 and 3.6, we know that both $\mathbf{Gl}(F)$ and $\mathscr{E}_{(1+F)}$ have finite limits and subobject classifiers, and the equivalences (ii) ⇔ (iii) and (iv) ⇔ (v) are immediate. (i) ⇔ (ii) is Proposition 5.5 above; and (i) ⇒ (iv) follows from Corollary 3.3 since $(1+F)$ generates a cofree comonad. Finally, (v) ⇒ (iii) is Corollary 5.4. □

Example 5.6. On any topos $\mathscr{E}$, the Kuratowski functor $K : \mathscr{E} \to \mathscr{E}$ admits a coproduct decomposition $K \cong 1 + K^+$, where K^+ is the 'non-empty finite subobjects' functor. Similarly, M admits a decomposition as $1 + M^+$. Since neither K^+ nor M^+ preserves pullbacks (unless $\mathscr{E}$ is degenerate), it follows that the categories $\mathscr{E}_K$ and $\mathscr{E}_M$ cannot be toposes, in contrast to $\mathscr{E}_L$.

For functors F which do not admit a coproduct decomposition as $1+F^+$, the question whether $\mathscr{E}_F$ can be cartesian closed without F preserving pullbacks remains open. Another case in which we may compare $\mathscr{E}_F$ with a category obtained by glueing occurs when F is a pointed endofunctor in the sense of Kelly [19]; that is, when there is a natural transformation η from the identity functor to F (for example, the singleton map for the functor K^+). Then we may define $U: \mathbf{Gl}(F) \to \mathscr{E}_F$ by

$$U(A, B, f: B \to FA) = (A + B, A + B \to FA \to F(A + B)),$$

where the first component of the structure map is induced by η_A and f. As in Lemma 5.2 above, we may prove that this functor has a right adjoint R if F preserves pullbacks of monomorphisms: $R(C, \gamma) = (C_0, C_1, \gamma|_{C_1})$ where $C_0 \rightarrowtail C$ is the equalizer of γ and η_C, and $C_1 \rightarrowtail C$ is the pullback of $FC_0 \rightarrowtail FC$ along γ. However, U is not full in this case, and it need not preserve binary products, so it does not seem possible to deduce that $\mathbf{Gl}(F)$ inherits cartesian closedness from $\mathscr{E}_F$.

References

[1] P. Aczel, Non-well-founded sets, CSLI Lecture Notes, vol. 14, Stanford University, 1988.
[2] P. Aczel, P.F. Mendler, A final coalgebra theorem, in: Category Theory and Computer Science, Lecture Notes in Computer Science, vol. 389, Springer, Berlin, 1989, pp. 357–365.
[3] M. Barr, Exact categories, in: Exact Categories and Categories of Sheaves, Lecture Notes in Math., vol. 236, Springer, Berlin, 1971, pp. 1–120.
[4] M. Barr, Terminal coalgebras in well-founded set theory, Theoret. Comput. Sci. 114 (1993) 299–315.
[5] L. Birkedal, A. Carboni, G. Rosolini, D.S. Scott, Type theory via exact categories, Proc. 13th IEEE Symp. on Logic in Computer Science, IEEE Computer Soc. Press, Silver Springs, MD, 1998.
[6] A. Carboni, P.T. Johnstone, Connected limits, familial representability and Artin glueing, Math. Struct. Comput. Sci. 5 (1995) 441–459.
[7] A. Carboni, G.M. Kelly, R.J. Wood, A 2-categorical approach to change of base and geometric morphisms I, Cahiers Topologie Géom. Differentiable Catégoriques 32 (1991) 47–95.
[8] A. Carboni, R.C. Magno, The free exact category on a left exact one, J. Austral. Math. Soc., Series A 33 (1982) 295–301.
[9] A. Carboni, E.M. Vitale, Regular and exact completions, J. Pure Appl. Algebra 125 (1998) 79–116.
[10] E. De Vink, J.J.M.M. Rutten, Bisimulation for probabilistic transition systems: a coalgebraic approach, Theoret. Comput. Sci., to appear; extended abstract in Proc. ICALP '97, Lecture Notes in Computer Science, vol. 1256, Springer, Berlin, 1997, pp. 460–470.
[11] R. Dyckhoff, W. Tholen, Exponentiable morphisms, partial products and pullback complements, J. Pure Appl. Algebra 49 (1987) 103–116.
[12] P. Freyd, A. Scedrov, Categories, Allegories, North-Holland Mathematical Library, vol. 39, Elsevier, 1990.
[13] J.M.E. Hyland, First steps in synthetic domain theory, in: Category Theory, Lecture Notes in Math., vol. 1488, Springer, Berlin, 1991, pp. 131–156.
[14] P.T. Johnstone, Topos Theory, LMS Mathematical Monographs, vol. 10, Academic Press, New York, 1977.
[15] P.T. Johnstone, F.E.J. Linton, Finiteness and decidability: II, Math. Proc. Cambridge Philos. Soc. 84 (1978) 207–218.

[16] P.T. Johnstone, A.J. Power, T. Tsujishita, H. Watanabe, J. Worrell, An axiomatics for categories of transition systems as coalgebras, in Proc. 13th IEEE Symp. on Logic in Computer Science, IEEE Computer Soc. Press, Silver Spring, MD, 1998.
[17] P.T. Johnstone, G.C. Wraith, Algebraic theories in toposes, in: Indexed Categories and their Applications, Lecture Notes in Math., vol. 661, Springer, Berlin, 1978, pp. 141–242.
[18] A. Joyal, M. Nielsen, G. Winskel, Bisimulation and open maps, Inform and Comput. 127 (1996) 164–185.
[19] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Austral. Math. Soc. 22 (1980) 1–83.
[20] G.M. Kelly, Elementary observations on 2-categorical limits, Bull. Austral. Math. Soc. 39 (1989) 301–317.
[21] A. Kock, G.C. Wraith, Elementary Toposes, Aarhus Universitet Lecture Notes Series, vol. 30, 1971.
[22] J. Lambek, Subequalizers, Canad. Math. Bull. 13 (1970) 337–349.
[23] F.E.J. Linton, Coequalizers in categories of algebras, in: Seminar on Triples and Categorical Homology Theory, Lecture Notes in Math., vol. 80, Springer, Berlin, 1969, pp. 75–90.
[24] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics, vol. 5, Springer, Berlin, 1971.
[25] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic, Springer, Berlin, 1992.
[26] C. McLarty, Elementary Categories, Elementary Toposes, Oxford Logic Guides, vol. 21, Oxford University Press, Oxford, 1992.
[27] M. Makkai, R. Paré, Accessible Categories: the Foundations of Categorical Model Theory, Contemporary Mathematics, vol. 104, Amer. Math. Soc., Providence, RI, 1989.
[28] A.J. Power, H. Watanabe, An axiomatics for categories of coalgebras, Electron. Notes Theoret. Comput. Sci. 11 (1998).
[29] H. Reichel, An approach to object semantics based on terminal co-algebras, Math. Struct. Comput. Sci. 5 (1995) 129–152.
[30] R.D. Rosebrugh, R.J. Wood, Pullback preserving functors, J. Pure Appl. Algebra 73 (1991) 73–90.
[31] J. Rosický, Cartesian closed exact completions, J. Pure Appl. Algebra 142 (1999) 261–270.
[32] J.J.M.M. Rutten, Universal coalgebra: a theory of systems, Theoret. Comput. Sci. 249 (2000) 3–80.
[33] J.J.M.M. Rutten, Relators and metric bisimulations, Electron. Notes Theoret. Comput. Sci. 11 (1998) 1–7.
[34] J.J.M.M. Rutten, D. Turi, On the foundations of final semantics: non-standard sets, metric spaces, partial orders, in: Semantics: Foundations and Applications, Lecture Notes in Computer Science, vol. 666, Springer, Berlin, 1993, pp. 477–530.
[35] T. Tsujishita, H. Watanabe, Monoidal closedness of the category of simulations, Hokkaido Univ. Preprint Series in Mathematics, vol. 392, 1997.
[36] H. Watanabe, A criterion for the existence of subobject classifiers, Hokkaido Math. J. 28 (1999) 117–132.
[37] H. Watanabe, The subobject classifier of the category of functional bisimulations, Hokkaido Math. J. 28 (1999) 301–314.
[38] J. Worrell, Toposes of coalgebras and hidden algebras, Electron. Notes Theoret. Comput. Sci. 11 (1998).
[39] G.C. Wraith, Artin glueing, J. Pure Appl. Algebra 4 (1974) 345–348.
[40] O. Wyler, Lecture Notes on Topoi and Quasitopoi, World Scientific, Singapore, 1991.