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$\aleph_0$-CATEGORICAL, $\aleph_0$-STABLE STRUCTURES*

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Introduction

It was long conjectured that a complete $\aleph_1$-categorical theory is not finitely axiomatizable. More precisely, there were two quite distinct conjectures, the first already mentioned in [21] as having occurred to a number of people:

(1) A theory which is $\aleph_1$-categorical but not $\aleph_0$-categorical cannot be finitely axiomatizable.

(2) A totally categorical theory must have the finite submodel property.

This last condition signifies that any first-order property of a model of the given theory is shared by one of its finite substructures. In particular, if the theory is itself finitely axiomatizable it will then have finite models. The first of these conjectures was demolished by Peretyatkin in [22] in a direct construction reminiscent of the domino models of [20], simplified subsequently by Morley and by Parigot. The second was proved for almost strongly minimal theories in [20] and in general by Zil'ber in [29, 33] overcoming formidable technical obstacles; the second reference fills in a gap caused by an error in the first. Zil'ber's approach is to develop a general structural analysis of models of $\aleph_1$-categorical theories, already quite highly developed in [27], and then to bring in number theoretic relationships in the $\aleph_0$-categorical case. He suggested a different way to complete the analysis in [28], based on a then conjectural classification of $\aleph_0$-categorical strongly minimal sets. (We learned later that Zil'ber proved this conjecture in summer 1980, that is, shortly before our own work began.)

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Our own work originated primarily in an attempt to repair the error in [29], before learning of Zil'ber's amplification. We came to realize that the conjecture in [28] was in fact a rather direct consequence of known results on finite permutation groups (an observation made also by C. Mills, independently), and that this could serve as the basis for a systematic structure theory for $\aleph_0$-categorical, $\aleph_0$-stable models. It is that general structure theory and its applications which will be presented here.

A second motivation for this work came independently from the work on homogeneous stable structures for a finite relational language reported in [15] and completed recently by [7]. The work reported here may be viewed either as a refinement of Zil'ber's structure theory for $\aleph_1$-categorical structures in the $\aleph_0$-categorical case, or as a generalization of Lachlan's theory of homogeneous stable structures for a finite relational language. To fully appreciate the 'state of the art' of each of these three theories, some acquaintance with the other two is desirable.

Let us now consider the main features of the structure theory of $\aleph_0$-categorical, $\aleph_0$-stable models as it will be presented here. We begin with the classification of the 'strictly minimal' sets, that is, of strongly minimal, $\aleph_0$-categorical structures having some minor additional properties. These can be completely classified, and they may be thought of as classical geometries (affine, projective, or degenerate) of infinite dimension over a finite field.

From this Classification Theorem various useful results can be derived. Call a structure *primitive* if it carries no non-trivial 0-definable equivalence relation. Since any $\aleph_0$-categorical structure carries only finitely many 0-definable equivalence relations, many problems can be reduced to the primitive case. The Coordinatization Theorem states that any $\aleph_0$-categorical, $\aleph_0$-stable primitive structure is constructible in a very explicit way (as a 'grassmannian' structure) from a rank one set. There is another result quite closely related to the Coordinatization Theorem. Consider combinatorial geometries $\mathcal{P}=(P, F; I)$ where P, F are arbitrary disjoint sets of 'points' and 'flats' and $I \subseteq P\times F$. Assume that $\mathcal{P}$ is definable in an $\aleph_0$-stable structure $\mathcal{M}$, and let 'rank' denote Morley rank as computed in $\mathcal{M}$. Assume also that the flats $f \in F$ are of constant rank r (construed as subsets of P), and are *almost disjoint*:

$$\operatorname{rank}(f \cap f') < r \quad \text{for } f, f' \in F \text{ distinct.}$$

Then if $\mathcal{M}$ is $\aleph_0$-categorical, we have

$$(*) \qquad \operatorname{rank} P \geqslant \operatorname{rank} F + r.$$

This is a strong constraint. If, for example, $\mathcal{P}=(F^2, L; \in)$ where F is an algebraically closed field and L consists of all lines in the plane F^2, we have

$$r=1, \qquad \operatorname{rank} F^2 = \operatorname{rank} L = 2.$$

We refer to $(*)$ as the Fundamental Rank Inequality. It seems to express the idea that definable sets are very rigid. (They can be moved by parallel translation but not twisted.)

The essential step in our analysis will be the derivation of (*) when $r = \text{rank}\, P - 1$, using the Classification Theorem; it will then be easy to derive both the Coordinatization Theorem and the general case of (*) by formal manipulations. A more subtle point which deserves mention at this juncture is that we will work initially under the assumption that all structures under consideration are of *finite* Morley rank. Once the Coordinatization Theorem has been derived under this hypothesis, it will be applied to prove that *all* $\aleph_0$-categorical $\aleph_0$-stable structures are of finite Morley rank.

As applications of this theory we will prove in Section 6 that there are no $\aleph_0$-categorical $\aleph_0$-stable pseudoplanes and that the fundamental order of an $\aleph_0$-categorical $\aleph_0$-stable theory is finite (confirming a conjecture of Poizat). We conclude with a proof of the finite submodel property which yields some extra information; it has been refined by Loveys [16].

For readers acquainted with the subject who have not made a detailed study of Zil'ber's work, the following comments may be of interest. Zil'ber shows in [28] that the nonexistence of totally categorical rank 2 pseudoplanes implies the Classification Theorem; we show the converse, as mentioned. Zil'ber shows by arguments reminiscent of finite geometry that totally categorical rank 2 pseudoplanes do not exist in [30]; that proof is a technical *tour de force.* If we combine the present results with [12] we arrive at the *equivalence* of the following conjectures:

(A) There is no $\aleph_0$-categorical, stable pseudoplane.

(B) An $\aleph_0$-categorical, stable theory is always $\aleph_0$-stable.

This is particularly intriguing, since it has been conjectured that there are no $\aleph_0$-categorical pseudoplanes at all. At the same time Zil'ber has conjectures about general strongly minimal sets in [33], with no hypothesis of $\aleph_0$-categoricity.

The theory developed here supports the view that $\aleph_0$-categorical, $\aleph_0$-stable structures cannot be very complex. At the same time, if M_0 is any finite module, then the direct sum $M_0^{(\alpha)}$ of α copies of M_0 is an example of such a structure. One might conjecture that an $\aleph_0$-categorical, $\aleph_0$-stable structure should always be constructible from finite structures in a reasonably transparent way; for primitive structures the Coordinization Theorem says something of this kind. If this seems too vague, the following will do as a test question:

Is every totally categorical theory finitely axiomatizable *relative to* the axiom of infinity (the axiom scheme stating that a model is infinite)?

(See the *Notes added in proof* at the end of the paper.)

1. Preliminaries

In this section we will review some notation, terminology, and technical results needed in the sequel. We will not distinguish notationally between structures and their universes.

If M is a structure, a set $A \subseteq M^n$ is *definable* if it is parametrically definable in M (as a relation) and B-*definable* if it is definable using parameters from B. We write 'a-definable' and '0-definable' when $B = \{a\}$ or $\emptyset$ respectively.

Morley rank and degree (of a type, a formula, or a definable set) are denoted rnk and deg respectively, and we write (rnk, deg)(p) for the pair $\langle \mathrm{rnk}(p), \deg(p)\rangle$; these pairs are ordered lexicographically. We write $\mathrm{rnk}(\bar{a}/B)$ for $\mathrm{rnk}(\mathrm{tp}(\bar{a}/B))$. We write $\bar{a} \vDash p$ to mean $p \subseteq \mathrm{tp}(\bar{a})$.

We will need some special properties of Morely rank in $\aleph_0$-categorical structures. Let $\oplus$ denote the natural sum of ordinals.

Lemma 1.1. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable. Let $\phi(\bar{x}; \bar{y})$ be a formula defined in M.*

(i) $\mathrm{rnk}(\exists \bar{x}\, \phi(\bar{x}; M)) \leqslant \mathrm{rnk}(\exists \bar{y}\, \phi(M; \bar{y})) \oplus \sup\{\mathrm{rnk}(\phi(\bar{a}; M)) \colon \bar{a} \in M\}$.

(ii) *If the sets $\phi(\bar{a}; M)$ are pairwise disjoint and of constant rank, then we have equality in* (i).

(iii) *If $\exists \bar{y}\, \phi(M; \bar{y})$ is strongly minimal and the sets $\phi(\bar{a}; M)$ are of constant rank r and degree* 1, *with $\mathrm{rank}(\phi(\bar{a}; M) \cap \phi(\bar{b}; M)) < r$ for $\bar{a} \neq \bar{b}$, then* $\deg(\exists \bar{x}\, \phi(x; M)) = 1$.

Part (i) is found in [13, §3]. For (ii) the reverse inequality is easily proved by induction on $\mathrm{rnk}(\exists \bar{y}\, \phi(M; \bar{y}))$; and (iii) is easily deduced from (i).

Corollary 1.2. *For M $\aleph_0$-categorical and $\aleph_0$-stable, and $A, B \subseteq M$ definable subsets of finite ranks m, n respectively:*

(i) $\mathrm{rnk}(A \times B) = m + n$.

(ii) $\mathrm{rnk}(A^k) = km$.

We assume a basic knowledge of forking [25, III] or equivalent results on rank. For $\bar{a} \in M$, $A \subseteq B \subseteq M$ we say $\mathrm{tp}(\bar{a}/B)$ *forks* over A if $\mathrm{rnk}(\bar{a}/B) < \mathrm{rnk}(\bar{a}/A)$. For A, B, $C \subseteq M$ we say that A is *independent* from B over C if $\mathrm{rnk}(\bar{a}/B \cup C) = \mathrm{rnk}(\bar{a}/C)$ for all $a \in A$. A key result (forking symmetry) says that if A is independent from B over C, then B is independent from A over C. In a slightly different vein, a set $A \subseteq M$ is said to be independent over C (or just independent, for $C = \emptyset$) if for each $a \in A$, a is independent from $A - \{a\}$ over C. The first part of the following lemma is useful in calculations of rank.

Lemma 1.3. *Let $B = \{b_1, b_2, \ldots\}$ be independent over C and $\bar{a}$ arbitrary.*

(i) *If* $\mathrm{tp}(b_i/\bar{a}, C)$ *forks over C for $i = 1, \ldots, n$, then* $\mathrm{rnk}(\bar{a}/C) \geqslant n$.

(ii) *We cannot have* $\mathrm{tp}(b_i/\bar{a}C)$ *forking for all i.*

Here we work in an $\aleph_0$-stable structure M; but (ii) holds already in the superstable case [25, V.1.7]. Part (i) is easy.

If M is a structure and $A \subseteq M$ is nonempty, then by $M \mid A$ we denote the structure whose universe is A and whose given relations are all those of the form $R \cap A^n$ where $1 \leq n < \omega$ and R is a 0-definable relation on M. The following lemma has its origin in the work of Shelah [24, 3.1(A)] and independently Baldwin [1, §4].

Lemma 1.4. *Let M be a stable structure and $A \subseteq M$ be nonempty and definable. If $n < \omega$ and $B \subseteq A^n$, then B is definable in M iff B is definable in $M \mid A$. Also* (rnk, deg)(B) *is the same whether it be computed in M or in $M \mid A$.*

Some knowledge of the theory of strongly minimal sets is required such as may be found in [3, §1], together with the following. A family $\{H_0, \ldots, H_{n-1}\}$ of 0-definable strongly minimal sets in a structure M is *orthogonal over* $A \subseteq M$ if $\bar{a}_0 {}^\frown \bar{a}_1 {}^\frown \cdots {}^\frown \bar{a}_{n-1}$ is independent over A whenever $a_i \in H_i$ is independent over A for each $i < n$. The family is called *orthogonal* if it is orthogonal over $\emptyset$. We say that H_0 is orthogonal to H_1 if $\{H_0, H_1\}$ is orthogonal.

Lemma 1.5. *Let M be a stable structure, $A \subseteq M$, and let $\mathcal{H} = \{H_0, \ldots, H_{n-1}\}$ be a family of 0-definable strongly minimal sets in M.*

(i) *$\mathcal{H}$ is orthogonal if and only if it is orthogonal over A.*

(ii) *$\mathcal{H}$ is orthogonal if and only if $\{H_i, H_j\}$ is orthogonal for all $i < j < n$.*

(iii) *The relation of nonorthogonality is an equivalence relation on $\mathcal{H}$.*

Part (i) is a special case of a general principle which can be found in [18, 2.2(i)]. Part (ii) is an application of [25, V, 1.4(1)].

An element of a structure is called *algebraic* if it is in acl($\emptyset$), the algebraic closure of $\emptyset$. Let $\{H_0, \ldots, H_{n-1}\}$ be a family of 0-definable strongly minimal sets. From the definition of orthogonality, if $a_i \in H_i$ is non-algebraic for each $i < n$, then $\mathrm{tp}(a_0, \ldots, a_{n-1})$ does not depend on the choice of the a_i.

We shall need the finite equivalence relation theorem [25, III, 2.8] as it applies to $\aleph_0$-stable structures:

Lemma 1.6. *If M is an $\aleph_0$-stable structure and $A \subseteq M$ is 0-definable, then there is a 0-definable equivalence relation E on A with A/E finite such that for each equivalence class C either* $\deg(C) = 1$ *or* $\mathrm{rnk}(C) < \mathrm{rnk}(A)$.

A closely related result inspired by 1.6 is the normalization lemma [12, §1] which we formulate only for $\aleph_0$-categorical $\aleph_0$-stable structures. Let M be an $\aleph_0$-stable structure and $\phi(\bar{x}; \bar{y})$ be a formula. We call $\phi(\bar{x}; \bar{y})$ *normal* if for all $\bar{b}_0, \bar{b}_1 \in M$ such that $l(\bar{b}_0) = l(\bar{b}_1) = l(\bar{y})$ we have $\phi(M; \bar{b}_0) = \phi(M; \bar{b}_1)$ whenever $\mathrm{rnk}(\phi(M; \bar{b}_0) \dot{-} \phi(M; \bar{b}_1)) < \mathrm{rnk}\, \phi(M; \bar{b}_0)$. Here $A_0 \dot{-} A_1$ denotes the symmetric difference of A_0 and A_1.

Lemma 1.7. *If M is an $\aleph_0$-categorical $\aleph_0$-stable structure and $\phi(\bar{x}; \bar{y})$ is a formula,* *there exists a normal formula* $\phi^*(\bar{x}; \bar{y})$ *such that for all* $\bar{b} \in M$ *with* $l(\bar{b}) = l(\bar{y})$ *either* $\phi(M; \bar{b}) = \phi^*(M; \bar{b}) = 0$ *or* $\operatorname{rnk}(\phi(M; \bar{b}) \dot{-} \phi^*(M; \bar{b})) < \operatorname{rnk}(\phi(M; \bar{b}))$.

If M is a structure and $\bar{a}_0, \bar{a}_1 \in M$ have the same length we say that $\bar{a}_0$ and $\bar{a}_1$ are *conjugate* if $\operatorname{tp}(\bar{a}_0) = \operatorname{tp}(\bar{a}_1)$. Similarly if $n < \omega$ and $A_0, A_1 \subseteq M^n$ are definable we say that A_0 and A_1 are *conjugate* if there exist a formula $\phi(\bar{x}; \bar{y})$ and $\bar{b}_0, \bar{b}_1 \in M$ such that $\bar{b}_0$, $\bar{b}_1$ are conjugate and $A_i = \phi(M; \bar{b}_i)$ for $i < 2$. Here we have in mind strongly $\aleph_0$-homogeneous structures in which case two sequences or definable subsets are conjugate iff there is an automorphism of M taking one onto the other. If all $a_0, a_1 \in M$ are conjugate, i.e. if there is only one 1-type over 0, then we call M *transitive*.

We shall need the following observation which is equivalent to the remark that a theory with the strict order property [25, II, 4.3] is unstable.

Lemma 1.8. *If* A_0, A_1 *are conjugate definable subsets of the structure* M *and* $A_0 \subsetneq A_1$, *then* M *is unstable.*

If M is a structure and $A \subseteq M$, by (M, A) we denote the structure obtained from M by naming each element of A.

We need to use imaginary elements of the structures we consider; see [25, III, §6]. Let M be a structure, $n < \omega$ and E be a 0-definable equivalence relation on M_n. For simplicity suppose $L(M)$, the language of M, is relational, and that $M \cap (M^n/E) = 0$. Form a new structure M^* as follows. $L(M^*)$ is obtained by ajoining to $L(M)$ new relation symbols U and V, the first being unary and the second $(n+1)$-ary. Let M^* have universe $M \cup (M^n/E)$, $U^{M^*} = M$,

$$V^{M^*} = \{(\bar{a}, \bar{a}/E) : \bar{a} \in M^n\},$$

and $R^{M^*} = R^M$ for each relation symbol R of $L(M)$. Any structure obtained from M by repeating this process a finite number of times is called an *extension by definitions* of M. If M^* is an extension by definitions of M and $A \subseteq M^*$ is definable in M^*, then we say that A is *attached* to M. It is easy to see that M^* is $\aleph_0$-stable iff M is, and $\aleph_0$-categorical iff M is. Further, if M^* is an extension by definitions of M, then $M^* \mid M$ is essentially the same as M. We say 'essentially' because formally M and $M^* \mid M$ can have different languages. Also, M is 0-definable in M^*.

If M is a stable structure and $A \subseteq M$ let $M(A)$ denote

$$\{a \in M : \operatorname{tp}(a/A) \text{ does not fork over } 0\}.$$

If $\bar{a} \in M$ let $M(\bar{a})$ denote $M(\operatorname{rng}(\bar{a}))$. If M is $\aleph_0$-categorical, then $M(\bar{a})$ is $\bar{a}$-definable.

Unless stated otherwise, type means complete type, and types are over $\emptyset$ unless specified to be over some other set. If M is a structure and $E \subseteq M$, then $\operatorname{Tp}(E)$ denotes the set of all finite types over $\emptyset$ which are realized in E. Here the types are types of Th(M) and if necessary we write $\mathrm{Tp}_M(\bar{a})$ to make it clear which structure we have in mind.

We occasionally need strong types. If $\bar{a} \in M^n$, let stp($\bar{a}$) be the set of equivalence classes $\bar{a}/E$ for E an equivalence relation on M^n with M^n/E finite. Trivially $\mathrm{stp}(\bar{a}) = \mathrm{stp}(\bar{b})$ implies $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$.

2. Strictly minimal sets

If H is a strongly minimal set in an $\aleph_0$-categorical $\aleph_0$-stable structure M, then the restriction of M to H (as in Section 1) is a strongly minimal $\aleph_0$-categorical structure in its own right; the latter will also be denoted H. Such a structure will be said to be *strictly* minimal if, in addition, it is *primitive*, i.e. carries no nontrivial 0-definable equivalence relation.

Strictly minimal sets may be classified in terms of certain geometries associated with them. For our purposes a geometry is a set of points S together with a *closure operator* $\langle\ \rangle$ operating on arbitrary finite subsets of S, construed as a relational system in the following *canonical* way. For each n we introduce a relation $R_n(x_1, \ldots, x_n, y)$ defined by:

$$y \in \langle\{x_1, \ldots, x_n\}\rangle.$$

Then $(S; \{R_n\})$ is the relational system associated with S. We do not need to axiomatize the notion further, since we deal only with examples. We will refer mainly to the *affine* and *projective* geometries over fields, and the *degenerate* geometry ($\langle A \rangle = A$ for all $A \subseteq S$).

The canonical closure operator on a strongly minimal structure H is the operator $\langle A \rangle = \mathrm{acl}(A)$, and if H is $\aleph_0$-categorical, then the associated geometry is 0-definable over H. In this context we see that H is *strictly* minimal iff points are closed. Our point of departure will be the following Classification Theorem, conjectured by Zil'ber in [28] and proved independently by Cherlin, Mills, and Zil'ber.

Theorem 2.1. *If H is strictly minimal, then its associated geometry is isomorphic to one of the following:*

(i) *The degenerate geometry on H.*
(ii) *An affine geometry of infinite dimension over a finite field.*
(iii) *A projective geometry of infinite dimension over a finite field.*

The 'easy' proof of Cherlin and Mills amounts to the observation that by looking at a sufficiently large algebraically closed subset of H, one gets a problem in finite permutation groups whose solution is implicit in the literature. This proof, however, relies ultimately on the classification of the finite simple groups, in a qualitative formulation; the nature and the precise number of the sporadics is irrelevant. The 'hard' proof by Zil'ber is actually relatively direct. The group-theoretic context of the 'easy' proof is discussed in an appendix.

The Classification Theorem can be refined slightly. In order to avoid purely linguistic entanglements, let the *canonical structure* associated with H be the *set* H equipped with *all* 0-definable relations. By an abuse of language, call two structures M_1, M_2 (for possibly distinct languages L_1, L_2) *isomorphic* if there is an identification of the languages (induced by a bijection between the relation symbols) with respect to which they become isomorphic in the usual sense. When $L_1 = L_2$ this is weaker than the customary notion.

Lemma 2.2. *Let G be a geometry of the type referred to in Theorem* 2.1. *Then the canonical structures associated to strictly minimal sets whose geometries are of type G fall into finitely many isomorphism types.*

We will not give the proof here, since it drops out automatically from the group-theoretic proof of Theorem 2.1. (There are more direct arguments available, but in any case everything comes down to determining $\mathrm{Aut}\, H$ as a subgroup of $\mathrm{Aut}\, G$.) The result is quite explicit: a degenerate geometry corresponds to a unique canonical structure, whereas affine or projective geometries over F_q are classified by subfields F_0 of F_q. More explicitly, elements of F_q are encoded by equivalence classes of colinear triples or quadruples (respectively) in the geometry, and we can adjoin a relation picking out the class corresponding to a generator of F_0. All of this is reasonably clear if one is familiar with the structure of $\mathrm{Aut}\, G$.

In the case of a projective or degenerate geometry, the lattice of closed sets is modular; accordingly we will call a strictly minimal set modular if its geometry is of one of these two types. As is well known, this condition is equivalent to the familiar dimension formula:

$$\dim X + \dim Y = \dim X \cap Y + \dim \langle X, Y \rangle,$$

and this is the important condition in practice. Similarly we call a strictly minimal set affine if its geometry is; the affine strictly minimal sets are the sources of the most significant technical difficulties.

One way to evade difficulties associated with affine strictly minimal sets H runs as follows. Let H' denote the set of equivalence classes of lines of H with respect to the parallelism relation. This may be represented as a quotient of the 0-definable set $H^{(2)} = \{(a, b) : a, b \in H \text{ distinct}\}$ by a 0-definable equivalence relation. If H is affine over F_q, then H' is a strictly minimal set, projective over F_q (with the same field of constants).

The more generally useful approach involves the introduction of constants. Quite generally, if H is a strongly minimal 0-definable subject of the stable, $\aleph_0$-categorical structure M and $A \subseteq M$ is finite, we construct a quotient structure H/A as follows. Let $H_1 = H - \mathrm{acl}(A)$ and define $E(a, b)$ by '$\mathrm{acl}(a, A) = \mathrm{acl}(b, A)$'

for $a, b \in H$. Let $H/A = H_1/E$ equipped with the geomery induced by the closure operator $\langle X \rangle = \mathrm{acl}(X \cup A) \cap H_1/E$. This will be a strictly minimal set. In particular $H/\emptyset$ is a strictly minimal set canonically associated with H (and with a canonically isomorphic lattice of closed sets). If H is modular and $A \subseteq H$ is finite, then as geometries $H/A \simeq H$ (not canonically), while if H is affine and $A \subseteq H$ is finite and nonempty, then $H/A \simeq H'$ (as is clear if A has one element, and hence also in general by the modular case). All of this is clear by inspection and also follows rapidly from first principles, that is, from the dimension formula and the uniqueness of parallel lines in affine geometry.

Any strictly minimal set is either already modular, or becomes so after naming *one* element; and this is all that really matters.

We will need to examine quotients of the form H/A more closely when A is not necessarily contained in H; in fact, this is a crucial issue, and modular H will be remarkably well behaved. The first remark is quite evident.

Lemma 2.3. *For H affine and $A \subseteq H'$ finite, H/A is affine and $(H/A)' \simeq H'/A$ canonically (even 0-definably).*

Proof. We leave this to the reader, remarking only that as a set $H/A = H/E$, where $E(a, b)$ means "$a = b$ or $\langle a, b \rangle \in \langle A \rangle$."

We now come to the most important results of this section.

Lemma 2.4. *Let M be stable, $H \subseteq M$ modular and strictly minimal, $A \subseteq M$ finite, $A^* = \mathrm{acl}(A) \cap H$. Then $H/A = H/A^*$ as geometries.*

Proof. Replacing H by H/A^*, we may assume that $A^* = \emptyset$. We claim then that $H = H/A$. Suppose on the contrary that:

$$(*) \qquad y \in \mathrm{acl}(x_1, \ldots, x_n, A) - \langle x_1, \ldots, x_n \rangle$$

for some $\bar{x}, y \in H$ with n minimal. The relation $(*)$ is definable on H in M, hence also H-definable (see 1.4). Let $\phi(\bar{x}, y, \bar{c})$ define $(*)$, with $\bar{c}$ in H.

If $x'_1, \ldots, x'_n$ in H are independent over $A\bar{c}$, then for some $y \in H$ we have $y \in \mathrm{acl}(\bar{x}', A) - \langle \bar{x}' \rangle$ and hence $y \in \langle \bar{x}', \bar{c} \rangle - \langle \bar{x}' \rangle$. By the dimension formula applied to $\bar{x}'$, y and to $\bar{c}$,

$$(n+1) + \dim \bar{c} = (n + \dim \bar{c}) + \dim (\langle \bar{x}', y \rangle \cap \langle \bar{c} \rangle).$$

Hence $\langle \bar{x}', A \rangle \cap \langle \bar{c} \rangle \neq \emptyset$. Since $\bar{x}'$ is independent over $A\bar{c}$, easily $\langle A \rangle \cap \langle \bar{c} \rangle \neq \emptyset$, a contradiction. □

Corollary 2.5. *If H_0, H_1 are nonorthogonal, modular, strictly minimal 0-definable sets, then there is a (unique) 0-definable bijection between them.*

Proof. Take $A \subseteq H_0$, $B \subseteq H_1$ finite and minimal such that A and B are separately each independent, but $A \cup B$ is not independent. Then $H_0 \neq H_0/B$, $H_1 \neq H_1/A$, so $\mathrm{acl}(A) \cap H_1$ and $\mathrm{acl}(B) \cap H_0$ are nonempty. By minimality, A and B are singletons $\{a\}$ and $\{b\}$, algebraic in each other. Since $\langle a \rangle = \{a\}$ and $\langle b \rangle = \{b\}$, everything follows. □

Corollary 2.6. *If H is a modular strictly minimal set in the stable structure M and $A \subseteq M$ is finite, then $H/A \simeq H$.*

This is evident by Lemma 4 (but it is the main ingredient in the key proof in Section 3).

Lemma 2.7. *Let H be affine and strictly minimal in the stable structure M. Let $A \subseteq M$ be finite, $A^* = \mathrm{acl}(A) \cap H$, $A' = \mathrm{acl}(A) \cap H'$.*

(i) *If $A^* \neq \emptyset$, then $H/A = H/A^*$,*

(ii) *If $A^* = \emptyset$, then $H/A = H/A'$.*

Proof. If $A^* \neq \emptyset$ replace H by H/A^* and apply Lemma 2.4 to conclude. If $A^* = \emptyset$ replace H by H/A'. So now assume $A' = \emptyset$. We claim $H = H/A$. Otherwise, as in the proof of Lemma 2.4 we obtain $\bar{x}$, y, $\bar{c}$ in H with $\bar{x}$ independent over $A\bar{c}$ so that:

$$y \in \langle \bar{x}, \bar{c} \rangle - \langle \bar{x} \rangle.$$

We can apply the dimension formula to $\langle \bar{x}, y \rangle$ and $\langle \bar{c}, y \rangle$ (since they meet), getting:

$$(\dim \bar{x} + 1) + \dim\langle \bar{c}, y \rangle = (\dim \bar{x} + \dim \bar{c}) + \dim(\langle \bar{x}, y \rangle \cap \langle \bar{c}, y \rangle).$$

If $y \in \langle \bar{c} \rangle$ conclude as before that $y \in \mathrm{acl}(A) \cap \langle \bar{c} \rangle$. If $y \notin \langle \bar{c} \rangle$ conclude that $\langle \bar{x}, y \rangle \cap \langle \bar{c}, y \rangle$ contains a line l. Again, if l meets $\langle \bar{c} \rangle$ conclude that $\mathrm{acl}(A)$ meets $\langle \bar{c} \rangle$. Hence we may suppose $l \cap \langle \bar{c} \rangle = \emptyset$, so $\dim \bar{c} \geqslant 2$ and l is parallel to $\bar{c}$. It is easy to see then that $l \in A'$ (by varying $\bar{x}$). □

Corollary 2.8. *Let H_0, H_1 be nonorthogonal strictly minimal 0-definable sets and let A be such that for $i = 0$ or 1, H_i/A is modular. Then:*

(i) *If H_0, H_1 are both affine or both modular, then $\mathrm{acl}(A) \cap H_0$, $\mathrm{acl}(A) \cap H_1$ have the same dimension.*

(ii) *If H_0 is affine and H_1 is modular, then*

$$\dim(\mathrm{acl}(A) \cap H_0) = \dim(\mathrm{acl}(A) \cap H_1) + 1.$$

Proof. If both are modular, this follows from Corollary 2.5. If H_0 is affine and $H_1 = H_0'$ it is evident, as in this case $\mathrm{acl}(A)$ meets H_0. The general case follows by combining these two. □

Lemma 2.9. *Let H_0, H_1 be strictly minimal and 0-definable in the stable structure M, with H_0 modular. Let $A \subseteq H_0$ with $\mathrm{acl}(A) \cap H_1 \neq \emptyset$. Then H_1 is modular.*

Proof. Suppose that H_1 is affine. We may take A finite, $A = B \cup \{a\}$ with $\mathrm{acl}(B) \cap H_1 = \emptyset$. Then H_1/B is affine, by 2.3 and 2.7, and H_0/B is modular. Replacing H_0 by H_0/B, we may suppose that $A = \{a\}$. Then as in the proof of Corollary 2.5 there is a 0-definable bijection between H_0 and H_1, and this gives a contradiction, since only one is modular. □

Corollary 2.10. *Let H_0, H_1 be strictly minimal, with H_0 modular. Let $a \in H_1$, $\mathrm{acl}(a) \cap H_0 \neq \emptyset$. Then H_1 is modular.*

Proof. $a \in \mathrm{acl}(\mathrm{acl}(a) \cap H_0)$. Apply Lemma 2.9.

3. The main technical lemma

This section is devoted to the proof of one powerful result, Proposition 3.3 below, in a rather technical form. In the next section we will see a more concrete formulation of this result, the Coordinatization Theorem, as well as an extensive generalization, the Fundamental Rank Inequality. These more useful versions of the result will be seen to follow easily from the special case treated below.

We begin with a minor preparatory lemma.

Lemma 3.1. *Let P be a transitive $\aleph_0$-categorical, $\aleph_0$-stable structure of finite rank n equipped with two families H_1, H_2 of definable subsets of M of rank $n-1$ and degree 1. Assume that H_1, H_2 are 0-definable and strongly minimal. Then:*

(1) For a in P, there is a finite and nonzero number of sets $A \in H_1$ (or H_2) with a in A.

(2) If $A, B \in H_1 \cup H_2$ are distinct and B is algebraic in A, then $A \cap B = \emptyset$.

A word of explanation is in order here. To say that H is a definable family of subsets of a structure P is to say that there are formulas $\phi(\bar{x}, \bar{y})$ and $\delta(\bar{y})$ (where δ may involve parameters from P) so that:

$$H = \{\phi(P, \bar{a}) : M = \delta(\bar{a})\}.$$

In this situation, identify H with $\delta(M)/\sim$, where $\bar{a} \sim \bar{a}'$ means $\phi(P, \bar{a}) = \phi(P, \bar{a}')$. Thus in particular H is said to be strongly minimal if $\delta(M)/\sim$ is.

Proof Lemma 3.1. (1) The nonempty set $\bigcup H_1$ is 0-definable and therefore equals P. Thus each $a \in P$ belongs to some element of H_1. On the other hand, if a belongs to an infinite number of sets in H_1, then a belongs to all but a finite number k of the sets in H_1, and this number k is independent of a. However, for any $k+1$ sets A_i in H_1 we can find some b in $P - \bigcup A_i$, a contradiction. Thus our first claim is proved.

(2) Let X be the union of all those sets of the form $A \quad B$ or $A-B$ which are of rank less than $n-1$, where A, B belong to $H_1 \; H_2$ and $B \in \operatorname{acl}(A)$. Then X is the union of a rank one (or less) collection of sets each of which is of rank less than $n-1$, so rank $X \leqslant n-1$. Hence $X \neq P$, and since X is 0-definable we find $X = \emptyset$. Our second claim follows.

Our next lemma contains the main technical point.

Lemma 3.2. *Let $(P; H_1, \ldots, H_m)$ be a transitive $\aleph_0$-categorical, $\aleph_0$-stable structure equipped with m definable strongly minimal families of definable subsets of M of rank $n-1$, degree* 1. *($n = \operatorname{rank} P$.) Suppose that $H_1, \ldots, H_m$ are indiscernible in the sense that $(P; H_1, \ldots, H_m) \equiv (P; H_{\sigma 1}, \ldots, H_{\sigma m})$ for any permutation σ, and that the strictly minimal sets $\bar{H}_i$ obtained from H_i by factoring out algebraic closures of points are modular. Let n^* be the cardinality of the algebraically closed subsets of the $\bar{H}_i$ of dimension n. Then $m \leqslant n^*$.*

Proof. There are two cases to be considered:

Case 1: *The H_i are pairwise orthogonal.*

Fix $a \in P$. By Lemma 3.1 we can find sets $A_i \in H_i$ with $a \in A_i$. By the orthogonality, $\operatorname{rank}(\operatorname{tp}(A_1, \ldots, A_m)) = m$, and since $A_1, \ldots, A_m \in \operatorname{acl}(a)$ we conclude that $\operatorname{rank}(\operatorname{tp}(a)) \geqslant m$. Thus $m \leqslant n \leqslant n^*$.

Case 2: *The H_i are pairwise nonorthogonal.*

By 2.8 there is a unique 0-definable bijection between each pair $\bar{H}_i$, $\bar{H}_j$. Fix $a \in P$. Select $A_i \in H_i$ so that $a \in A_i$ and let $B_i \in \bar{H}_1$ correspond to A_i naturally (via projection followed by bijection). By Lemma 3.1(2) no A_i depends algebraically on any other as they all intersect, and it follows that the B_i are all distinct. Thus we have:

$$|\operatorname{acl}(a) \cap \bar{H}_1| \geqslant m.$$

If $m > n^*$, then $\dim(\operatorname{acl}(a) \cap \bar{H}_1) > n$, forcing rank $P > n$, a contradiction. Thus $m \leqslant n^*$.

Proposition 3.3. *Let M be $\aleph_0$-categorical, $\aleph_0$-stable of finite rank n. Let F be a definable family of definable subsets of M, each of rank $n-1$. Suppose that the elements of F are almost disjoint in the following sense:*

$$A, B \in F \textit{ distinct } \Rightarrow \operatorname{rank}(A \cap B) < n-1.$$

Then rank $F \leqslant 1$.

Proof. To fix notation, suppose that $F=\{\phi(M,\bar a): M\models\delta(\bar a)\}$ and let $\bar a\sim\bar a'$ mean $\phi(M,\bar a)=\phi(M,\bar a')$. We may suppose that F is 0-definable (adjoining necessary constants to the language) and that M is transitive and of degree 1 (for the last point, use the Finite Equivalence Relation Theorem 1.6). We may also suppose that F is an atom over $\emptyset$.

Assuming that rank $F\geq 1$, choose a strongly minimal subset I of F. More explicitly, choose an atom $\psi(\bar y;\bar z)$ so that $\psi(\bar y;\bar z)\Rightarrow\delta(\bar y)$ and so that for some $\bar b$ in M, $\psi(M,\bar b)/\sim$ is strongly minimal, and let $I=I(\bar b)$ be the corresponding subset of F. By a suitable (padded) choice of the parameters we may suppose that the strictly minimal set $H(\bar b)$ obtained from $I(\bar b)$ by factoring out the algebraic closures of points is modular (cf. 2.3), and that the type p of $\bar b$ is not algebraic.

Let q be the strong type of $\bar b$. The bulk of our argument is devoted to proving:

(1) If $\bar b_1$, $\bar b_2$ is a Morley sequence in $q(M)$, then $I(\bar b_1)-I(\bar b_2)$ is finite.

To see this, suppose the contrary, and then form a Morley sequence $\bar b_1,\ldots,\bar b_m$ of length $m=n^*+1$ where n^* is the cardinality of an algebraically closed subset of $H(\bar b)$ of dimension n. Let $\bar c=\bar b_1,\ldots,\bar b_m$ and let $P=M(\bar c)$ be $\{a\in M:\text{rank}(\text{tp}(a/\bar c))=n\}$. Then rank $P=n$ and P is an atom over $\bar c$.

Adjoin $\bar c$ to the language (thereby altering the notion of algebraic closure). Let $H_i=I(\bar b_i)-\text{acl}(\emptyset)$ and let $\bar H_i$ be the strictly minimal set obtained from H_i by factoring out the algebraic closures of points. We now need to check that:

(2) for $A\in H_i$, rank $A\cap P=n-1$.

Let $J_i=\{A\in H_i:\text{rank } A\cap P<n-1\}$. Since rank $M-P<n$ it follows easily that J_i is finite, hence contained in acl($\emptyset$), hence empty, and (2) follows. Therefore H_i can be identified with the set:

$$H_i^P=\{A\cap P: A\in H_i\}$$

which we will again denote H_i.

Now the structure $(P; H_1,\ldots,H_m)$ can be seen to satisfy the conditions of the previous lemma. The indiscernibility of the H_i (in the sense of that lemma) reflects the indiscernibility of the $\bar b_i$. Corollary 2.6 implies that $\bar H_i$ is again modular and that our current definition of n^* agrees with the one used in the statement of Lemma 3.2. Thus Lemma 3.2 applies, and yields a contradiction, proving the claim (1).

The rest is easy. Let $I^*(\bar b)$ denote a normalization of (the definition of) $I(\bar b)$, so that:

(3) If $\bar b_1$, $\bar b_2$ is a Morley sequence in $q(M)$, then $I^*(\bar b_1)=I^*(\bar b_2)$.

It then follows easily that in fact $I^*(\bar b_1)=I^*(\bar b_2)$ for any $\bar b_1$, $\bar b_2$ in $q(M)$, and hence that $\{I^*(\bar c):\bar c\models p\}$ is finite. Let $I=\bigcup\{I^*(\bar c):\bar c\models p\}$. then I is 0-definable, so $I=F$ (we assumed that F is an atom). Thus, rank $F=$ rank $I=1$, as claimed.

4. Coordinatization; rank inequality

We will first exploit Proposition 3.3 to prove the Coordinatization Theorem, which is simply a very convenient (and fundamental) corollary. We will then turn to a generalization of Proposition 3.3, the Fundamental Rank Inequality (the special case treated in Proposition 3.3 is the hard one). Throughout this section we observe the convention:

M is an $\aleph_0$-categorical and $\aleph_0$-stable structure of finite rank n.

However, it will be seen in Section 5 that the rank of an $\aleph_0$-categorical, $\aleph_0$-stable structure is always finite, so our hypotheses are redundant in retrospect.

For the Coordinatization Theorem it will be convenient to have the following terminology. If P, A are two infinite 0-definable subsets of M, then we will say that A *cordinatizes* P if:

(1) A is an atom.

(2) For all x in P, $\mathrm{acl}(x) \cap A \neq 0$.

In this situation we write $\mathrm{crd}(x)$ for $\mathrm{acl}(x) \cap A$.

In fact, in such situations we have generally a mixture of a coordinatization with a decomposition. To see this, let us define an equivalence relation on P by $x \sim y$ iff $\mathrm{crd}(x) = \mathrm{crd}(y)$. When this relation is equality we have a true coordinatization. Otherwise it is the quotient $P/{\sim}$ which is coordinatized. (If P is an atom and A is strongly minimal, then we can identify $P/{\sim}$ with the set of k-dimensional algebraically closed subsets of A, where $k = \mathrm{rank}(P/{\sim})$.)

Theorem 4.1 (Coordinatization Theorem). *If M is transitive, then there is an extension by definitions M^* of M, 0-interpretable over M (i.e., constructed without using parameters from M), in which there is a rank one set A which coordinatizes M.*

Note. In general A cannot be taken to be of degree 1 (this would require the introduction of parameters).

Proof of 4.1. We may suppose that $\deg M = 1$, using the Finite Equivalence Relation Theorem 1.6 to return to the general case.

Let $\phi(\bar{x}, \bar{y})$ be a normal formula which fixes the type p of $\bar{y}$ and such that $(\mathrm{rnk}, \deg)\phi(x', \bar{b}) = (n-1, 1)$ when $\bar{b} \models p$. Let $F = \{\phi(M, \bar{b}) : \bar{b} \models p\}$. By Proposition 3.3, $\mathrm{rank}\, F \leq 1$ and it follows readily from Lemma 3.1 that $\mathrm{rank}\, F = 1$ and that F coordinatizes M. $\square$

Definition. Let A be an $\aleph_0$-categorical structure, $X \subseteq A$ a finite algebraically closed set, and $\mathrm{Gr}(X, A)$ the set of subsets of A conjugate to X under $\mathrm{Aut}\, A$. We make $\mathrm{Gr}(X, A)$ into a structure, called a grassmannian structure, by imposing on it all relations corresponding to 0-definable relations on A. Then $\mathrm{Aut}\, \mathrm{Gr}(X, A) = \mathrm{Aut}\, A$ canonically, so we may consider $\mathrm{Gr}(X, A)$ to be A in disguise.

If A coordinatizes P, we have a natural map $P \to \mathrm{Gr}(X, A)$ for some X, and if P is *primitive*, that is, carries no nontrivial 0-definable equivalence relation, then this is an isomorphism (up to a change in language). Thus the Coordinatization Theorem is expressed equally well by:

(*) A primitive, $\aleph_0$-categorical, $\aleph_0$-stable structure of finite rank is isomorphic to a grassmannian over a rank one set.

Corollary 4.2. *Either M is strictly minimal, or M realizes more than one nontrivial 2-type.*

Proof. If only one nontrivial 2-type is realized in M, then M is certainly primitive and hence isomorphic to some grassmannian $\mathrm{Gr}(X, A)$ with A of rank 1. If X meets one of its conjugates, there would be at least two 2-types. Hence $\mathrm{Gr}(X, A)$ *partitions* A, and is therefore also of rank 1. As M is primitive it is also of degree 1, by 1.6. It follows directly that M is strictly minimal. □

Corollary 4.3. *If* $k = \mathrm{card}\,(S_2\emptyset)$, *then* $\mathrm{rank}\, M \leq k$.

The proof is left to the reader. (The result is not needed.)

The following lemma will be needed for the proof of the Fundamental Rank Inequality.

Lemma 4.4. *There is a definable collection C of strongly minimal subsets of M such that C has rank $n-1$ and such that the intersection of any two elements of C is finite.*

Proof. We may suppose that M is transitive. Let $\Phi = \{\phi(M, \bar{a}) : \bar{a} \models p\}$ be a 0-definable collection of strongly minimal sets, with $\phi(x, \bar{y})$ normal. As M is transitive, $\bigcup \Phi = M$ and hence $\mathrm{rnk}\, \Phi \geq n-1$. Let C be a definable subset of Φ of rank $n-1$.

Theorem 4.5 (Fundamental Rank Inequality). *Let F be a definable collection of definable subsets of M. Suppose that the elements of F are of constant rank r, and are almost disjoint in the sense that the intersection of any two has rank less than r. Then* $r + \mathrm{rank}\, F \leq n$.

Proof. Proceed by downward induction on r, beginning with the special case $r = n-1$, which is Proposition 3.3.

For the inductive step, we may suppose the elements of F to be of degree 1. Let $f = \mathrm{rank}\, F$. By the previous lemma there is a definable family C of almost disjoint strongly minimal subsets of F with $\mathrm{rank}\, C = f-1$.

For c in C let $F(c) = \bigcup c$. Each set $F(c)$ is of rank $r+1$ and of degree 1. Let $F^*(c)$ be a normalization of $F(c)$ (or more precisely, a normalization of the definition of $F(c)$). If the family $\{F^*(c): c \in C\}$ is of rank $f-1$, then by induction $r+f=(r+1)+(f-1) \leq n$ as desired.

Suppose accordingly that $\{F^*(c): c \in C\}$ is of rank less than $f-1$. Then there is a set X such that $\{c: F^*(c)=X\}$ is infinite. Of course rank $X=r+1$. Let F' be $\{A \in F: \operatorname{rank}(A \cap X)=r\}$. Then by Proposition 3.3, rank $F' \leq 1$. On the other hand, if $F^*(c)=X$, then $\operatorname{rank}(F(c)-X)<r$, and hence $c \cap F'$ is infinite. Since there are infinitely many such c, and any two of them are almost disjoint, we have a contradiction. $\square$

5. The finiteness of rank

In this section it is shown that an $\aleph_0$-categorical, $\aleph_0$-stable structure has finite rank. Thus the Coordinatization Theorem applies to all $\aleph_0$-categorical, $\aleph_0$-stable structures.

Theorem 5.1. *Let M be an $\aleph_0$-categorical $\aleph_0$-stable structure. Then* $\operatorname{rnk}(M)$ *is finite.*

Proof. Let M be a counterexample to the theorem. There exist $\bar{a} \in M$ and $A \subseteq M$ such that A is the locus of a 1-type over $\bar{a}$ and $(\operatorname{rnk}, \deg)(A)=(\omega, 1)$. By 1.4 the structure $M \mid A$ has rank ω and degree 1 and has only one 1-type. Thus we can suppose $(\operatorname{rnk}, \deg)(M)=(\omega, 1)$ and that M has only one 1-type. Let $M'(\bar{a})$ denote $M-M(\bar{a})$ for any $\bar{a} \in M$.

Claim. For all $\bar{a} \in M$, $M'(\bar{a})$ is closed in the sense that if $\bar{b} \in M'(\bar{a})$, then $M'(\bar{b}) \subseteq M'(\bar{a})$.

Proof of Claim. Fix $\bar{b} \in M'(\bar{a})$ and let

$$A=\bigcup\{M'(\bar{c}): c \in M'(\bar{a}) \text{ and } \operatorname{tp}(\bar{c}/\bar{a})=\operatorname{tp}(\bar{b}/\bar{a})\}.$$

From 1.1, 1.2 it follows that $\operatorname{rnk}(A)$ is finite. By inspection A is $\bar{a}$-definable. Therefore $M'(\bar{b}) \subseteq A \subseteq M'(\bar{a})$, so we are done.

If $a \in M'(a_0) \cap M'(a_1)$, then $M'(a) \subseteq M'(a_0) \cap M'(a_1)$, whence $M'(a)=M'(a_0)=M'(a_1)$ by 1.8. Therefore the sets $M'(a)$, $a \in M$, are the classes of an infinite 0-definable equivalence relation E. By 1.1, $\operatorname{rnk}(M/E)=\omega$. Thus replacing M by M/E we can suppose $M'(a)=\{a\}$, i.e. there is a unique nontrivial 2-type. By the claim, for every $\bar{b} \in M$ there exists an independent sequence $\bar{a} \in M$ such that $M'(\bar{b}) \subseteq M'(\bar{a})$. Therefore $\operatorname{rank}(M'(\bar{a}))$ is unbounded as $\bar{a}$ runs through the independent sequences of all finite lengths. Fix an independent sequence $\bar{a} \in M$ of least possible length such that $\operatorname{rnk}(M'(\bar{a}))>1$. For any $b_0, b_1 \in M'(\bar{a})$ there exists an independent sequence $\bar{b} \in M'(a)$ such that $\bar{b}$ extends $\langle b_0, b_1\rangle$ and $l(\bar{b})=l(\bar{a})$. (If not, we have $M'(\bar{a}) \subseteq M'(\bar{b})$ for some independent sequence $\bar{b} \in M'(\bar{a})$ with $l(\bar{b})<l(\bar{a})$, contradicting the choice of $\bar{a}$.) Moreover, $M'(\bar{b})=M'(\bar{a})$ by 1.8 again.

It follows that for any $b_0, b_1 \in M'(\bar{a})$ there is an automorphism of M taking $\langle b_0, b_1\rangle$ to $\langle a_0, a_1\rangle$ and $M'(\bar{a})$ onto itself. Hence the structure $M \mid M'(\bar{a})$ has a unique nontrivial 2-type and finite rank>1. This contradicts 4.2 so the theorem is proved.

6. Pseudoplanes; the fundamental order

This section is devoted to two easy applications of the Fundamental Rank Inequality.

As a concrete application of the Rank Inequality we give an easy proof of a result of Zil'ber. Recall that a pseudoplane is an incidence geometry consisting of points and lines subject only to the following axioms.

(A) Every point is incident with infinitely many lines.

(B) Every line is incident with infinitely many points.

(C) Every two points are incident with only finitely many common lines.

(D) Every two lines are incident with only finitely many common points.

It was conjectured in [12] that there are no $\aleph_0$-categorical pseudoplanes. This is a combinatorial assertion of great power, which would settle rapidly many questions connected with stable $\aleph_0$-categorical theories. See [12, 27] for examples.

Theorem 6.1. *There is no* $\aleph_0$-*categorical and* $\aleph_0$-*stable pseudoplane.*

Proof. Let P be the set of points and let L be the set of lines in a supposed counterexample. Decompose P into atoms $P_1, \ldots, P_k$ over 0. Associate to P_i the set L_i of all lines which are incident with an infinite subset of P_i. In this way we obtain incidence geometries (P_i, L_i) which satisfy all of the pseudoplane axioms except possibly A.

We will show first that one of these geometries is a pseudoplane. Let $i \to j$ mean that for some (and hence for every) a in P_i there are infinitely many lines in L_j incident with a. For every i there is a j with $i \to j$. Let n_i be the cardinality of the algebraic closure in P of any element of P_i. Observe:

(1) If $i \to j$ and (P_i, L_i), (P_j, L_j) are not pseudoplanes, then $n_i < n_j$.

To see this, make the following choices:

$a \in P_i$,

a line $l \in L_j$ incident with a,

a point $b \in P_j$ incident with l.

We may take $l \notin L_i$, and then we have $a \in \mathrm{acl}(l)$. Since $l \in L_j$ therefore $l \in \mathrm{acl}(b)$. Thus $a \in \mathrm{acl}(b)$ and hence $n_i \leq n_j$. Since we may also take $b \notin \mathrm{acl}(a)$, claim (1) follows.

It follows that one of the geometries (P_i, L_i) is a pseudoplane and realizes a unique type of point. Noticing that the axioms for pseudoplanes are selfdual, we may suppose similarly (with a change of notation):

(2) The pseudoplane (P, L) contains a unique type of point and a unique type of line.

Now let us compute the rank of the set:

$$I = \{(a, l) : a \in P, l \in L, a \text{ incident with } l\}.$$

This can be done in two ways. If m is the rank of the set of lines incident with a given point, and m' is the rank of the set of points incident with a given line, then we find:

(3) $\quad \operatorname{rank} I = \operatorname{rank} P + m = \operatorname{rank} L + m'.$

The fundamental Rank Inequality applies with $M = P$ and with $F = L$ viewed as a definable collection of subsets of P. Therefore we have the inequality: $\operatorname{rank} L + m' \leqslant \operatorname{rank} P$, which combines with the above to give $m = 0$, a contradiction.

Remark. Zil'ber found a direct proof of this result in summer 1980. He showed previously in [27] that this result implies a statement which is essentially the classification theorem for strictly minimal structures. It seems quite possible that one could extract from his work a theorem about all sufficiently 'large' Jordan groups which would not depend on the classification theorem for finite simple groups.

We turn now to our second topic, the so-called 'fundamental order' studied in [17]. In its terminology the theorem we are about to prove would be formulated as follows: for $\aleph_0$-categorical, $\aleph_0$-stable theories the fundamental order is finite. For a more immediately accessible formulation, let M by any structure, and notice that its automorphism group $\operatorname{Aut} M$ acts naturally on the Stone space SM.

Theorem 6.2. *If M is countable, $\aleph_0$-categorical, and $\aleph_0$-stable, then* $\operatorname{Aut} M$ *has a finite number of orbits on S_1M.*

We are grateful to B. Poizat for drawing our attention to this problem in various formulations. In fact we will prove something stronger.

Theorem 6.3. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable. Then any type p over M is definable over a pair of elements of M.*

Of course, Theorem 6.2 follows from Theorem 6.3 just using the $\aleph_0$-categoricity of M.

Proof of 6.3. Proceed by induction on $n = \operatorname{rank} M$.

We may suppose that M is transitive. Let $r = \operatorname{rank} p$. If $r = n$, then the type p is definable over a singleton as a consequence of the Finite Equivalence Relation Theorem 1.6. Suppose therefore that $r < n$.

Fix a normalized formula $\phi(x, \bar{a})$ in p of rank r and degree 1 and let q be the type of $\bar{a}$ over $\emptyset$. Let $F = \{\phi(M, \bar{b}) : \bar{b} \vDash q\}$. We claim:

(1) For each x in M, x belongs to finitely many members of F.

To see this, compute the rank of:

$$I = \{(x, X) : x \in M, X \in F, x \in X\}.$$

We find:

$$n + \operatorname{rank}\{X \in F : x \in X\} = \operatorname{rank} F + \operatorname{rank} X \leq n$$

by the Fundamental Rank Inequality, and (1) follows.

Now fix $\alpha \in \phi(M, \bar{a})$ and $\beta \in \phi(M, \bar{a}) - \bigcup \{X \in F : \alpha \in X, X \neq \phi(M, \bar{a})\}$. Then $\phi(M, \bar{a})$ is $\{\alpha, \beta\}$-definable as the unique element of F containing both α and β, and p is $\{\alpha, \beta\}$ definable as the unique type of rank r containing $\phi(x, \bar{a})$.

7. Homogeneous substructures

In this section we shall examine the envelopes of Zil'ber [29] from a different point of view. We shall show that countable envelopes are unique up to isomorphism and that they are homogeneous substructures in a sense to be defined below. As a corollary we shall see that if M is any $\aleph_0$-stable structure and ϕ is a sentence true in M, then there is a finite homogeneous substructure of M in which ϕ is true. These results have been extended by J. Loveys [16].

Throughout this section let M be an $\aleph_0$-categorical, $\aleph_0$-stable structure. Let L denote the canonical language of M which for each $n \geq 1$ and each $p \in S_n(0)$ has an n-ary relation symbol R whose interpretation is $p(M)$. Until further notice assume (as we may) that M is an L-structure. Notice that M admits elimination of quantifiers with respect to L.

We introduce some new terminology. Call $J \subseteq M$ *strictly rank 1* if $\operatorname{rnk} J = 1$, J is a finite union of infinite atoms, and there is no nontrivial 0-definable equivalence relation on J which has a finite class. By 1.6 if J is strictly rank 1 there is a unique 0-definable equivalence relation on J which partitions J into strictly minimal pieces, i.e., each equivalence class H is strictly minimal in $(M, \{H\})$. By $(A)_J$ we denote $J \cap \operatorname{acl}(A)$ and we say that $A \subseteq J$ is *closed* if $A = (A)_J$. We call $B \subseteq M$ *J-compatible with* A if $(A \cup B)_J = (A)_J$.

We call $E \subseteq M$ a *J-envelope of* A if E is maximal among subsets of M J-compatible with A. When J is strictly minimal and either J is modular or $(A)_J \neq 0$ our definition is equivalent to Zil'ber's. This follows from 2.4 and 2.7.

A subset $A \subseteq M$ is called *homogeneous* if for all finite tuples $\bar{b}_0 \cup \langle c_0 \rangle$, $\bar{b}_i \in A$

such that $\mathrm{tp}(\bar{b}_0) = \mathrm{tp}(\bar{b}_1)$ there exists $c_1 \in A$ such that $\mathrm{tp}(\bar{b}_1 \cup \langle c_1 \rangle) = \mathrm{tp}(\bar{b}_0 \cup \langle c_0 \rangle)$. Notice that, if M is countable and $A \subseteq M$ is homogeneous, then the isomorphism type of $M \mid A$ is determined by $\mathrm{Tp}(A)$.

Let H be a component of a strictly rank 1 set $J \subseteq M$. For any $A \subseteq M$ let $\dim_H(A)$ denote $\dim((A)_H)$. Call $A \subseteq M$ *J-balanced* if either $\dim_{H_0}(A) = \dim_{H_1}(A)$ or both dimensions are infinite for any conjugate strictly minimal components H_0, H_1 of J. Call $A \subseteq M$ *J-adequate* if for each strictly minimal component H of J, either H is modular or $(A)_H \neq 0$.

The main result of this section, Theorem 7.3, says that if A is J-balanced and J-adequate then any J-envelope E of A is homogeneous and $\mathrm{Tp}(E) = \{\mathrm{tp}(\bar{a}) : \bar{a}$ is J-compatible with $A\}$. This fixes $\mathrm{Th}(M \mid E)$ and if M is countable it fixes the isomorphism type of $M \mid E$ as an L-structure.

In order to prove Theorem 7.3 we need two technical lemmas which we now present.

Lemma 7.1. *Let $J \subseteq M$ be strictly rank 1 and $A \subseteq J$ be closed.*

(i) *If $\bar{b}_0{}^\frown\langle e_0 \rangle$, $\bar{b}_1 \in M$ are finite sequences J-compatible with A and $\mathrm{stp}(\bar{b}_0) = \mathrm{stp}(\bar{b}_1)$, then there exists $e_1 \in M$ such that $\mathrm{tp}(\bar{b}_1, e_1) = \mathrm{tp}(\bar{b}_0, e_0)$ and $\bar{b}_1{}^\frown\langle e_1 \rangle$ is J-compatible with A.*

(ii) *If A is J-balanced, $\bar{b}_0{}^\frown\langle e_0 \rangle$, $\bar{b}_1 \in M$ are finite sequences J-compatible with A, and $\mathrm{tp}(\bar{b}_0) = \mathrm{tp}(\bar{b}_1)$, then there exists $e_1 \in M$ such that $\mathrm{tp}(\bar{b}_1, e_1) = \mathrm{tp}(\bar{b}_0, e_0)$ and $\bar{b}_1{}^\frown\langle e_1 \rangle$ is J-compatible with A.*

Proof. (i) Since $\mathrm{stp}(\bar{b}_0) = \mathrm{stp}(\bar{b}_1)$ we may suppose that every component of J is 0-definable in the given structure M. This reduces (i) to (ii). (J-balance becomes vacuous.)

(ii) Let $\mathscr{H}$ denote

$$\{H : H \text{ is a component of } J \text{ and } A \cap H \neq \emptyset\}.$$

Recall that the components of J are imaginary elements of M. Let $\bar{c}_i \in A$ $(i<2)$ be sequences meeting each $H \in \mathscr{H}$ just once such that $\mathrm{tp}(\bar{b}_0 \cup \bar{c}_0) = \mathrm{tp}(\bar{b}_1 \cup \bar{c}_1)$. A counting argument shows that such $\bar{c}_0$, $\bar{c}_1$ exist. Choose maximal sequences $\langle H_{i,j} : j<n \rangle$ $(i<2)$ in $\mathscr{H}$ such that $H_{i,j}$, $H_{i,k}$ are orthogonal $(j<k<n)$ and such that

$$\mathrm{tp}(\bar{b}_0 \cup \bar{c}_0 \cup \langle H_{0,j} : j<n \rangle) = \mathrm{tp}(\bar{b}_1 \cup \bar{c}_1 \cup \langle H_{1,j} : j<n \rangle).$$

For each $j<n$ choose $\bar{d}_0(j) \in A \cap H_{0,j}$ independent over $\bar{b}_0 \cup \bar{c}_0$ such that

$$(\bar{b}_0, \bar{c}_0, \bar{d}_0(j))_{H_{0,j}} = (\bar{b}_0, \bar{c}_0, e_0)_{H_{0,j}}.$$

For each $j<n$ choose $\bar{d}_i(j) \in A \cap H_{1,j}$ independent over $\bar{b}_1 \cup \bar{c}_1$ such that $l(\bar{d}_1(j)) = l(\bar{d}_0(j))$. Such $\bar{d}_i(j)$ exist; otherwise we should have $|A \cap H_{0,j}| \neq |A \cap H_{1,j}|$ which is impossible since $H_{0,j}$ and $H_{1,j}$ are conjugate components of J. Let $\bar{d}_i$ denote the sequence

$$\bar{d}_i(0) \cdots \bar{d}_i(n-1) \quad (i<2).$$

Since for fixed $i<2$ the $H_{i,j}$ ($j<n$) are pairwise orthogonal, $\bar{d}_i$ is independent over $\bar{b}_i\bar{c}_i$. From (1) it follows that $\mathrm{tp}(\bar{b}_0, \bar{c}_0, \bar{d}_0)=\mathrm{tp}(\bar{b}_1, \bar{c}_1, \bar{d}_1)$. Choose e_1 such that

$$\mathrm{tp}(e_1/\bar{b}_1\bar{c}_1\bar{d}_1)=\mathrm{tp}(e_0/\bar{b}_0\bar{c}_0\bar{d}_0).$$

Towards a contradiction suppose $\bar{b}_1{}^\frown\langle e_1\rangle$ is not J-compatible with A.

Choose minimal $C\subseteq A$ such that

$$(C\cup \mathrm{rng}((\bar{b}_1{}^\frown\bar{c}_1{}^\frown\bar{d}_1)\cup\{e_1\}))_J\nsubseteq A.$$

Clearly C is finite and nonempty. Let $c\in C$ and

$$E=(C-\{c\})\cup \mathrm{rng}(\bar{b}_1{}^\frown\bar{c}_1{}^\frown\bar{d}_1)\cup\{e_1\}.$$

Let

$$a\in(C\cup \mathrm{rng}(\bar{b}_1{}^\frown\bar{c}_1{}^\frown\bar{d}_1)\cup\{e_1\})_J-A$$

and H be the component of J containing a. Let $c\in H^*\in\mathscr{H}$. There exists $j<n$ such that H^* and $H_{1,j}$ are not orthogonal. By 1.5(i), 2.4 and 2.5 there is a $\bar{c}_1$-definable bijection between $H^*/\bar{c}_1$ and $H_{1,j}/\bar{c}_1$. Therefore we may suppose $c\in H_{1,j}$. There are two cases.

Case 1: $H\notin\mathscr{H}$. By 2.4, $H_{1,j}/E$ is modular. Let c/E denote the element of $H_{1,j}/E$ represented by c. Since $(c/H)_{H/E}\neq\emptyset$, H/E is modular by 2.9, and H/E and $H_{1,j}/E$ are not orthogonal. Since $(E)_H=\emptyset$, H is modular by 2.3 and 2.8. By 1.5(i) and 2.5 there is a 0-definable bijection between H and $H_{1,j}$. Hence $H\cap A\neq\emptyset$, contradiction.

Case 2: $H\in\mathscr{H}$. Since $(E\cup\{c\})_H\subseteq(E)_H$, H and $H_{1,j}$ are not orthogonal. Therefore we may suppose that $H=H_{1,j}$. Since C was chosen minimal, $(E)_H\subseteq A$ and so $H/A=(H/E)/A$ by 2.4 and 2.8. But this contradicts $(E\cup\{c\})_H\subseteq A$. This completes the proof of the lemma.

Lemma 7.2. *Let $J\subseteq M$ be strictly rank 1 and M^* be an extension by definitions of M. Let $A\subseteq M$, $b\in M^*$, $B\subseteq M^*$ and $\bar{b}\in B$ be finite. Let $(\mathrm{rng}\,\bar{b})\cup\{b\}$ be J-compatible with A, and B be J-compatible with A. For each component H of J let H be modular or $(A)_H\neq 0$. Then there exists $c\in M^*$ such that $B\cup\{c\}$ is J-compatible with A and $\mathrm{tp}(c/\bar{b})=\mathrm{tp}(b/\bar{b})$.*

Proof. By naming the components of J we may suppose that they are 0-definable. By naming $a\in(A)_H$ if necessary we may suppose that each component H of J is modular. Replacing A by $(A)_J$ and B by $B\cup A$ we may suppose that $A=(a)_J\subseteq J$.

We proceed by induction on $\mathrm{rnk}(\mathrm{tp}(b\mid\bar{b}))$. If $\mathrm{rnk}(\mathrm{tp}(b\mid\bar{b}))=0$ we may take $c=b$. Accordingly we suppose $\mathrm{rnk}(\mathrm{tp}(b\mid\bar{b}))>0$. By naming the elements of $\bar{b}$ we may suppose that $\bar{b}=0$. Let $C=\{c\in M^*:\mathrm{tp}(c)=\mathrm{tp}(b)\}$. There are two cases.

Case 1: $\mathrm{rnk}\,C>1$. Applying 4.1 to $M^*\mid C$ there is an extension by definitions of M^* (which by a change of notation will also be denoted M^*) in which there is a rank 1 set D coordinatizing C. Let $d\in\mathrm{crd}(b)\cup D$; then $\mathrm{rnk}(\mathrm{tp}(b\mid d))<\mathrm{rnk}(\mathrm{tp}(b))$.

Since $\{b\}$ is J-compatible with A, so is $\{d\}$. By the induction hypothesis there exists $e \in D$ such that $\mathrm{stp}(e) = \mathrm{stp}(d)$ and $B \cup \{e\}$ is J-compatible with A. By 7.1(i) there exists $b_e \in C$ such that $\mathrm{tp}(b_e, e) = \mathrm{tp}(b, d)$ and $\{b_e, e\}$ is J-compatible with A. Notice that $\mathrm{rnk}(\mathrm{tp}(b_e \mid e)) = \mathrm{rnk}(\mathrm{tp}(b \mid d)) < \mathrm{rnk}(\mathrm{tp}(b))$. By the induction hypothesis we have the conclusion of the theorem with b_e for b, $\langle e \rangle$ for $\bar{b}$, and $B \cup \{e\}$ for B. Hence there exists $c \in M^*$ such that $B \cup \{e, c\}$ is J-compatible with A and $\mathrm{tp}(c/e) = \mathrm{tp}(b_e/e)$. Since $\mathrm{tp}(c) = \mathrm{tp}(b)$ we are done.

Case 2: $\mathrm{rnk}\, C = 1$. Let E_0 be the coarsest 0-definable equivalence relation on C all of whose classes are finite. Let $D = C/E_0$. We may suppose $D \subseteq M^*$. By 1.6 there is a 0-definable equivalence relation E_1 on D which partitions it into strictly minimal pieces. Let F be one of the pieces such that $\mathrm{acl}(\{b\}) \cap F \neq 0$ and let $f \in \mathrm{acl}(\{b\}) \cap F$. Let $M_1 = (M, \{F\})$ and $M_1^* = (M^*, \{F\})$. Note that J is strictly rank 1 in M_1, F is strictly minimal in M_1^*, and each of $\{f\}$ and B is J-compatible with A in M_1.

We will now choose $\langle b_1, f_1 \rangle$ such that $\mathrm{tp}(b_1, f_1) = \mathrm{tp}(b, f)$ in M_1 and $\{b_1, f_1\}$ is J-compatible with A.

Subcase 1: $(A \cup B)_F \neq \emptyset$. Choose $f_1 \in (A \cup B)_F$ and then b_1 such that $\mathrm{tp}(b_1, f_1) = \mathrm{tp}(b, f)$. Since b_1 and f_1 are mutually algebraic, $\{b_1, f_1\} \cup B$ is J-compatible with A.

Subcase 2: Otherwise. Let $f_1 = f$ and $b_1 = b$.

Claim. $B \cup \{f_1\}$ is J-compatible with A.

Proof of the Claim. In Subcase 1 this is immediate. Accordingly suppose that $(A \cup B)_F = \emptyset$. Then $f_1 = f$. Towards a contradiction suppose $B \cup \{f\}$ is not J-compatible with A. Then $(A \cup B \cup \{f\})_J \not\subseteq A$. Recall that $(A \cup B)_J \subseteq A$. Let $P \subseteq A \cup B$ be a finite set such that $(P \cup \{f\})_H \not\subseteq A$ and $(P)_H \subseteq P$ for one of the components H of J. From 2.4, H/P is modular. Applying 2.10 to H/P and F/P we see that F/P is modular. Since $(P)_F = \emptyset$, F is modular by 2.3 and 2.7. Since H/C and F/C are not orthogonal, neither are H and F. By 2.5 there is a 0-definable bijection between H and F. Since $\{b\}$ is J-compatible with A, $f \in (A)_F$ and so $(A \cup B)_F \neq 0$, contradiction. This completes the proof of the claim.

Since $b_1 \in \mathrm{acl}(\{f_1\})$, $B \cup \{b_1\}$ is J-compatible with A. Taking $c = b_1$ completes Case 2 and the proof.

Theorem 7.3. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable. Let $J \subseteq M$ be strictly rank* 1 *and $A \subseteq M$ be J-balanced and J-adequate. Then any nonempty J-envelope E of A is a homogeneous substructure of M and*

$$\mathrm{Tp}(E) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in M,\ \bar{a} \text{ is } J\text{-compatible with } A\}.$$

Remark. If J is strictly minimal the expression for $\mathrm{Tp}(E)$ may be simplified. For by the results of Section 2, if J is a modular strictly minimal set, then

$$\mathrm{Tp}(E) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in M,\ ((\bar{a})_J \subseteq (A)_J\},$$

while if J is affine, then

$$\mathrm{Tp}(E) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in M, (\bar{a})_{J'} \text{ and } (\bar{a})_J \subseteq (A)_{J'}\}.$$

Proof. A J-envelope E of A is a J-envelope of $(A)_J$ which contains A. Thus we can suppose that $A \subseteq J$ and A is closed. Since A is J-adequate, 7.1 is applicable. Since A is J-balanced, 7.2 is applicable. Let $\bar{b}_0 \cup \langle e_0 \rangle \in M$ be J-compatible with A, $\bar{b}_1 \in E$, and $\mathrm{tp}(\bar{b}_0) = \mathrm{tp}(\bar{b}_1)$. By 7.1(ii) there exists $e_1 \in M$ such that $\mathrm{tp}(\bar{b}_0 \cup \langle e_1 \rangle) = \mathrm{tp}(\bar{b}_0 \cup \langle e_0 \rangle)$ and $\bar{b}_1 \cup \langle e_1 \rangle$ is J-compatible with A. By 7.2, e_1 may be chosen such that $E \cup \{e_1\}$ is J-compatible with A, i.e., e_1 may be chosen in E. This is sufficient to prove the theorem.

Corollary 7.4. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable, and ϕ be a sentence true in M. There is a finite homogeneous substructure $N \subseteq M$ in which ϕ is true.*

Remark. In this corollary we are still assuming that M has the canonical language. However, whatever the language of M, provided there are only a finite number of function symbols the result remains true. This is because in the canonical language there is a sentence which asserts that the structure is closed under the functions represented by the function symbols of the given language of M.

Proof of 7.4. Instead of finding finite $N \subseteq M$ it is enough to ensure that $\mathrm{rnk}(N) < \mathrm{rnk}(M)$, because then repeating the process sufficiently often we obtain a finite structure. Suppose $\mathrm{rnk}(M) = n > 0$ and by 4.1 let M^* be an extension by definitions of M in which there is a rank 1 set J coordinatizing M. By appropriate choice of M^* we may suppose that J is strictly rank 1.

It is easy to find a finite set $\mathcal{T}$ of finite types over $\emptyset$ such that, if N is a homogeneous substructure of M with $\mathrm{Tp}(N) \supseteq \mathcal{T}$, then $N \models \phi$.

Let $A \subseteq M^*$ be finite such that $\mathrm{Tp}(A) \supseteq \mathcal{T}$ and $A \cap H \neq \emptyset$ for every nonmodular strictly minimal piece H of J. Then A is J-adequate. Let H_0 and H_1 be orthogonal strictly minimal pieces of J. If $a \in H_0 - (A)_J$, then by 2.4, or by 2.8, if H_0 is affine, $\dim_{H_0}(A \cup \{a\}) = \dim_{H_0}(A) + 1$. Since H_0 and H_1 are orthogonal over A, $\dim_{H_1}(A \cup \{a\}) = \dim_{H_1}(A)$. If H_0 and H_1 are not orthogonal and both of the same kind, then by 2.6, $\dim_{H_0}(A) = \dim_{H_1}(A)$. Therefore by adjoining a finite number of elements of J to A if necessary we may suppose that A is J-balanced.

Let N^* be a J-envelope of A in M^* and $N = M \cap N^*$. By 7.3, N is a homogeneous substructure of M and $\mathrm{Tp}(N) \supseteq \mathcal{T}$. Hence $N \models \phi$. Let $\sim$ be the equivalence relation on M defined by the coordinatization. Let C be an infinite atom of M. Then $C/\sim$ is infinite. Since A is finite only a finite number of the $\sim$-classes meet N. Let $D \in C/\sim$; then $\mathrm{rnk}(D) < \mathrm{rnk}(C)$ by 1.1. Also, since N is a homogeneous substructure of M, every $E \subseteq N$ definable in N has the form $F \cap N$, where F is a definable subset of M. Therefore $\mathrm{rnk}_N(D \cap N) \leq \mathrm{rnk}_M(D) <$

$\mathrm{rnk}_M(C)$. Since $C \cap N$ is covered by a finite number of D's we have $\mathrm{rnk}_N(C \cap N) < \mathrm{rnk}_M(C)$. Thus $\mathrm{rnk}(N) < \mathrm{rnk}(M)$ as required.

8. Totally categorical structures

Lemma 8.1. *Let M be a totally categorical structure. Attached to M there is a 0-definable strictly minimal set H.*

Proof. Applying 4.1 to any non-algebraic 1-type of M we obtain a rank 1 set A which is attached to M and which is the solution set of a type over $\emptyset$. By 1.6, A is partitioned into a finite number of strongly minimal sets by a 0-definable equivalence relation E. Factoring out the algebraic closures of points if necessary we can suppose that these strongly minimal sets are strictly minimal. Since M is totally categorical so is any extension by definitions. Therefore any two strictly minimal components of A are non-orthogonal. We can suppose that each component of A is modular, because in the other case we can replace each component by the associated modular strictly minimal set. Recall from 2.5 that between any two modular non-orthogonal strictly minimal sets there is a unique 0-definable bijection. From the uniqueness these bijections must commute. It is now clear that by factoring out the algebraic closures of points in A we obtain a strictly minimal set H as required.

Lemma 8.2. *Let M be totally categorical and H be a strictly minimal set attached to M. There exists $k < \omega$ such that if $N \subseteq M$ is k-saturated and algebraically closed in M, then $(N)_H \neq \emptyset$ and N is an H-envelope of $(N)_H$ in M.*

Proof. We proceed by induction on $(\mathrm{rnk}, \deg)(M)$. Let $\mathrm{rnk}(M) = n$. The case $n = 1$ is easy, so we leave it to the reader. Thus suppose $n > 1$ and let C be an atom of rank n. Applying 4.1 we obtain A attached to M such that A is an atom of rank 1, and $\mathrm{acl}(c) \cap A \neq \emptyset$ for all $c \in C$. We can suppose that on A there is no nontrivial 0-definable equivalence relation with finite classes. Let M^* be an extension by definitions of M in which both A and C exist. For $a \in A$ let (M^*, a) be given its canonical language and let M_a denote the substructure of (M^*, a) whose universe is

$$(M - C) \cup \{c \in C : a \in \mathrm{acl}_M(c)\}.$$

Let $N_a = M_a \mid (M_a \cap N)$ whenever $M_a \cap N \neq 0$. Notice that $(\mathrm{rnk}, \deg)(M_a) < (\mathrm{rnk}, \deg)(M)$ because the sets $\{c \in C : a \in \mathrm{acl}_M(c)\}$ as a runs through A form a family which is i-disjoint for some $i < \omega$.

Let H_a be a strictly minimal set attached to M_a in some uniform way and 0-definable in M_a. Notice that if $N_a \cap C \neq 0$, then N_a is algebraically closed in M_a, because N is algebraically closed in M and $a \in \mathrm{acl}_M(c)$ for any $c \in M_a \cap C$. Let B

denote $(N)_H$ and B_a denote $(N_a)_{H_a}$. Let $(N)_A$ denote $A \cap \mathrm{acl}_M(N)$. By the induction hypothesis there exists $m < \omega$ such that, if N_a is an m-saturated substructure of M_a, then N_a is an H_a-envelope in M_a of B_a and $B_a \neq 0$.

Choose $k < \omega$ such that, if $N \subseteq M$ is k-saturated and algebraically closed in M, then $(N)_A$ meets each strictly minimal component of A, $B \neq 0$, and for each $a \in (N)_A$ $B_a \neq 0$ and N_a is an m-saturated substructure of M_a which meets C. It is easy to see that such k exists. Consider any $N \subseteq M$ which is k-saturated and algebraically closed. Towards a contradiction let N not be an H-envelope of B. Since $B \neq \emptyset$ and $B = (N)_H$ we see from 2.4 or 2.7, as the case may be, that $H/B = (H/N)/B$ which means that N is H-compatible with B. Therefore there exists $d \in M - N$ such that $N \cup \{d\}$ is H-compatible with B. There are two cases.

Case 1: There exists $a \in (N)_A$ such that $d \in M_a$. By the induction hypothesis N_a is an H_a-envelope of B_a in M_a. Therefore $(N \cup \{d\})_H \not\subseteq B_a$. Since M is totally categorical, H and H_a are non-orthogonal. It follows that $(N \cup \{d\})_H \not\subseteq B$, contradiction.

Case 2: Otherwise. Then there exists $a \in (\mathrm{acl}_M(d) \cup A) - (N)_A$. Now H is not orthogonal to any of the strictly minimal components of A because M is totally categorical. It follows that $(N \cup \{a\})_H \not\subseteq B$, contradiction.

This completes the proof of the lemma.

It is easy to verify by the same induction that, if $(N)_H$ is finite in the statement of 8.2, so is N. Combining this remark, 7.3, and the last two lemmas we have

Theorem 8.3. *Let M be a countable totally categorical structure. There exists $k < \omega$ such that if $N \subseteq M$ is k-saturated and algebraically closed, then either N is isomorphic to M or N is a finite homogeneous substructure of M.*

Can the hypothesis that N be algebraically closed in M be dropped? Using the Classification Theorem we can see that if $\mathrm{rnk}(M) = 1$, then k-saturation implies algebraical closure for sufficiently large $k < \omega$. For $\mathrm{rnk}(M) > 1$ the question is open.

The property of totally categorical structures expressed by 8.3 actually characterizes them among the $\aleph_0$-categorical, $\aleph_0$-stable structures. This is immediate from

Lemma 8.4. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable but not totally categorical. For every $j < \omega$ there exists an infinite j-saturated substructure N of M such that N is algebraically closed in M and* $(\mathrm{rnk}, \mathrm{deg})(N) < (\mathrm{rnk}, \mathrm{deg})(M)$.

Proof. Applying 4.1 and naming some elements of M if necessary we obtain an atom C of M and a modular strictly minimal set H attached to M which coordinatizes C such that $\mathrm{rnk}(C) = \mathrm{rnk}(M)$. We may suppose that $H \subseteq M$. Since M is $\aleph_0$- but not $\aleph_1$-categorical, by [21, 5.3] there exists $I \subseteq M$ such that $|I| = \aleph_0$ and I

is indiscernible over H. Now $(I)_H$ is finite because M is superstable (see [14, Proof of Lemma 3]). Let N be an H-envelope of A; then $(\mathrm{rnk}, \deg)(N) < (\mathrm{rnk}, \deg)(M)$ as in the proof of 7.4. From 7.4 if A is large enough, then $\mathrm{Tp}(N)$ is large enough to guarantee that N is j-saturated. Since $(I)_H \subseteq A$, I is H-compatible with A by 2.7 and so N may be chosen to contain I. Therefore N is infinite which completes the proof.

Appendix I. The classification theorem

We will prove Theorem 2.1: Let H be a strictly minimal set in an $\aleph_0$-categorical structure M. For $2 \leq i < \omega$ define a relation R_i by:

$$R_i(a_1, \ldots, a_i, b) \quad \text{iff} \quad b \in \mathrm{acl}(\bar{a}).$$

Then this imposes a geometry on H satisfying one of the following:

(0) H is degenerate.

(1) H is an infinite-dimensional affine geometry over a finite field, and $R_i(\bar{a}, b)$ iff b is in the affine span of $a_1, \ldots, a_i$.

(2) H is an infinite-dimensional projective geometry over a finite field, and $R_i(\bar{a}, b)$ iff b is in the linear span of $a_1, \ldots, a_i$.

In fact, this result is implicit in the literature on finite permutation groups, modulo easy reductions. (As it happens, the relevant results depend on the classification of finite simple groups. Zil'ber's proof avoids this.) The basic idea is to determine the automorphism group of H, from which the structure of H itself is easily recovered.

More precisely, we consider *finite sections* of H, by which we mean the substructures $\mathscr{X} = \langle X; \{R_i \mid X\}\rangle$ where X is a finite algebraically closed subset of H, and their automorphism groups $\Gamma = \mathrm{Aut}\,\mathscr{X}$. We consider Γ simply as a group of permutations of the finite set X. Then Γ has the following fundamental properties:

Lemma 1. *If $\mathscr{X}$ is a finite section of H and $\Gamma = \mathrm{Aut}\,\mathscr{X}$, then:*

(1) *Γ operates doubly transitively on X.*

(2) *If $Y \leq X$ is algebraically closed, then $\Gamma(Y)$ operates transitively on $X - Y$.* (Here $\Gamma(Y)$ denotes the subgroup of Γ fixing Y pointwise.)

Of course, the first assertion is a transparent reformulation of our explicit assumption that H contains a unique nontrivial 2-type, while the second assertion is a version of the characteristic property of strongly minimal sets: if $a, b \in H - Y$ with Y algebraically closed, then $\mathrm{tp}(a/Y) = \mathrm{tp}(b/Y)$.

Definition. Let Γ be a group of permutations of a finite set X. Γ is a *Jordan* group iff:

(1) Γ operates doubly transitively on X.

(2) There is a set $Y \leq X$ with $1 < |Y| < |X| - 1$ so that $\Gamma(Y)$ operates transitively on $X - Y$.
Actually, we should require $|Y| > 2$ to eliminate PSL$(2, q)$.

Corollary. *If $\mathscr{X}$ is a finite section of H of dimension at least* 4, *then* $\Gamma = \operatorname{Aut} \mathscr{X}$ *is a Jordan group.*

Jordan groups were considered originally by Jordan himself, and there was a renewal of activity in the past two decades (cf. [11]). Apparently no definitive classification was available prior to the classification of the finite simple groups, while on the other hand once the classification of all finite simple groups is in hand far better results are available. In other words, the theory of Jordan groups has been suddenly — and accidentally — trivialized. As a consequence of this, the following precise information can be obtained by inspecting the literature on finite groups:

Lemma 2. *If $\mathscr{X}$ is a finite section of H of sufficiently large dimension, then* $\Gamma = \operatorname{Aut} \mathscr{X}$ *is of one of the following three types:*

(0) *'Trivial'*: $A_n \leq \Gamma \leq S_n$ $(n = |X|)$.
(1) *Affine:* $\mathrm{ASL}(n, q) \leq \Gamma \leq \mathrm{A\Gamma L}(n, q)$ (*some* n, q).
(2) *Projective:* $\mathrm{PSL}(n+1, q) \leq \Gamma \leq \mathrm{P\Gamma L}(n+1, q)$ (*some* n, q).

Here S_n is the full symmetric group on a set, $\mathrm{A\Gamma L}(n, q)$ is the full automorphism group of the affine geometry of dimension n over a finite field F_q, and $\mathrm{P\Gamma L}(n+1, q)$ is the full automorphism group of the corresponding projective geometry. Our lemma states that Γ is isomorphic with a large subgroup of one of these three standard groups — not merely abstractly, but as a *permutation group.*

Some justification of Lemma 2 will be given in the second appendix. For the present we continue with the proof of Theorem 2.1. The proof of the following is trivial:

Lemma 3. *Let $\mathscr{X}$ be a finite section of H of dimension d, $Y \subseteq X$,* $\dim Y < d$, *and* $\Gamma = \operatorname{Aut} \mathscr{X}$. *Suppose that (Γ, X) satisfies the conclusion of Lemma 2 with $n > 1$. Then in the three respective cases we have* $\mathrm{acl}(Y) =$

(0) Y,
(1) *the affine subspace generated by* Y,
(2) *the projective subspace generated by* Y.

Proof. If $|X - Y| = 2$, $\Gamma = A_n$ special (trivial) arguments are needed. We will omit this case.

Let $\bar{Y}$ be the algebraic closure of Y, and let Y' be either Y (in case (0)) or the subspace spanned by Y. $\Gamma(Y)$ is transitive off Y' in each case, so that if $\bar{Y}$ meets

$X-Y'$ then:

(i) $\bar{Y}\supseteq X-Y'$.

On the other hand if $\bar{Y}\subseteq Y'$, since $\Gamma(Y)$ is also transitive off $\bar{Y}$, conclude $\bar{Y}=Y'$. Thus it suffices to eliminate (i). We assume (i), and in all cases we prove $\bar{Y}=X$, contradicting dim $Y<d$.

In case (0) our claim is immediate, so we turn to cases (1, 2). If dim $Y=1$, then $\bar{Y}=Y'=Y$, and if dim $Y=2$, we may give a special argument, as follows. As $n>1$, $X\neq Y'$. If $Y'=Y\subseteq\bar{Y}$, then $\bar{Y}=X$, as desired. If $Y'\neq Y$, then $|Y'|\geqslant 3$. Thus if we pick $p\in Y$ and a line l through p, $l\neq Y'$, then there are at least two points $p', p''\in l-\{p\}$. By (i) and the double transitivity of Γ:

(ii) $\mathrm{acl}(p', p'')\supseteq X-l$.

But then $\bar{Y}\supseteq(X-Y')\cup(X-l)\cup Y=X$, a contradiction.

Thus if dim $Y=2$, then $\bar{Y}=Y'$. But then for Y arbitrary; $\bar{Y}$ contains all lines through any two of its points, hence $\bar{Y}\supseteq Y'$. Together with (i), this yields $\bar{Y}=X$, a contradiction. (For affine geometry over F_2 the details are slightly different.)

Proof of Theorem 2.1. Assume H is not degenerate, and take a finite section $\mathscr{X}$ of H of dimension $d+1$, where:

(i) $d\geqslant 3$.

(ii) Some set of d elements is not algebraically closed.

(iii) d is large enough so that Lemma 2 (and hence Lemma 3) applies.

The relation $R_2(a_1, a_2, b)$ gives a notion of colinearity, and by Lemma 3, $\mathscr{X}$ (and hence H) satisfies exactly one of the following axiom systems:

(1) affine geometry, with lines of cardinality q,

(2) projective geometry, with lines of cardinality $q+1$,

for some fixed prime power q. Now let $d\to\infty$ and apply Lemma 3 to all finite sections of H. Our claim follows.

Further comments on the group-theoretic results needed are in the following appendix. Notice also that Lemma 2.2 can be derived from Lemma 2 in a similar fashion.

Appendix II. On permutation groups

As a result of the classification of finite simple groups, it is possible to give a list of *all* doubly transitive finite permutation groups, and *a fortiori* of all finite Jordan groups. The experts have known this for a few years, and Professor Mann of Hebrew University kindly supplied us with the relevant references to this fact, namely [8, 9].

There is an important division into cases at the outset.

Definition. A permutation group acts *regularly* on a set if it acts transitively, and only the identity fixed a point.

Fact 1. *Let Γ be a doubly transitive permutation group on a finite set X and let N be a minimal normal subgroup. Then either:*

(I) *N is nonabelian, in which case N is simple and the natural map $\Gamma \to \mathrm{Aut}\, N$ is injective.*

(II) *N is abelian and regular, and if Γ_0 is the stabilizer in Γ of one fixed point, then Γ is a semi-direct product $N \times \Gamma_0$. Also Γ_0 is a transitive subgroup of $\mathrm{Aut}\, N$ and the study of (Γ, X) can be reduced to the study of (Γ_0, N) where Γ_0 acts on N by conjugation.*

This is an entirely straightforward result (cf. [26]).

In case 1, $N \leqslant \Gamma \leqslant \mathrm{Aut}\, N$ with N a nonabelian simple group. The classification of these groups is now known, namely:

Type 1. The alternating groups A_n.

Type 2. The Chevalley groups — various matrix groups over finite fields, including 'twisted' Chevalley groups, cf. [4].

Type 3. The 26 sporadic groups.

For our application in the preceding appendix, we need a list of Jordan groups up to *finitely many* exceptions. Hence we can get by using only a fraction of the available information. (We can also allow for errors in the classification theorem, as long as only finitely many sporadic groups have been 'lost'.)

In type 1, $\Gamma = A_n$ or S_n, and this has only its usual doubly transitive permutation representation for $n > 7$ [19, 1]. Case 2 is more complicated, and the possibilities were worked out in [8], based mainly on character theory. If we simplify the result as much as possible by omitting finitely many special cases, the remaining infinite families are:

Class I (Chevalley type)

(1) $\mathrm{PSL}(n+1, q) \leqslant \Gamma \leqslant \mathrm{P\Gamma L}(n+1, q)$ acting on points or hyperplanes of the projective geometry $P(n, q)$.

(2) A group Γ_2 lying between $G = \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{Sz}(q)$, or $G_2(q)$ and its automorphism group, acting so that the stabilizer of one point in G is a Borel subgroup.

(3) The symplectic group $\mathrm{Sp}(2n, 2)$ acting on the cosets of one of the orthogonal groups $\mathrm{GO}^+(2n, 2)$ or $\mathrm{GO}^-(2n, 2)$. The number of elements being permuted is $2^{n-1}(2^n + 1)$ or $2^{n-1}(2^{n-1})$ respectively.

Now it is necessary to check that none of these groups is acting as a Jordan group (with $|Y| > 2$ and $q > 4$ if $G = \mathrm{PSL}(2, q)$ and for $n > 2$ or $n = 2$ and $\mathrm{GO}^+(2n, 2)$ if $G = \mathrm{Sp}(2n, 2)$). I found it convenient to inspect $\mathrm{PSL}(2, q)$ and $\mathrm{Sp}(2n, 2)$ rather concretely (see the end for a better way), while eliminating the other three groups in case (2) by coarse numerical considerations.

Intuitively case (3) is the most attractive, as it contains a dimension parameter n that might go to infinity. On the other hand, the only structure naturally associated with this example is a vector space over F_2 equipped with a nondegenerate quadratic form, and the infinite-dimensional analog is not even stable, much less strongly minimal, so it is not surprising that this example turns out to be irrelevant.

Examples of elementary arguments serving to eliminate such groups will be given at the end. Let us continue now with the second half of the story. In our second class of permutation groups, we essentially have to consider an elementary abelian p-group V and a group Γ_0 of automorphisms of V, with Γ_0 acting transitively on V. Such pairs (V, Γ_0) were studied in [9].

Some preliminaries from §6 of [9, II] will clarify the situation. Let $\bar{\Gamma}$ be the quotient of Γ_0 by its largest solvable normal subgroup. If $\bar{\Gamma} = 1$, that is if Γ_0 is solvable, then the classification goes back to 1957 [10], and there are only finitely many examples apart from the rather trivial case of subgroups of $\Gamma\mathrm{L}(1, q) = F_q^\times \times \mathrm{Gal}(F_q/F_p)$ where q is a power of p.

So we may assume that $\bar{\Gamma} \neq 1$. In this case [9, I §6] it may be seen that $\bar{\Gamma}$ contains a unique minimal normal subgroup, which we will call $E(\Gamma_0)$, and that $E(\Gamma_0)$ is simple. Furthermore if $n = |V|$, then $n < |\mathrm{Aut}(E(\Gamma))|^2$ [9, I p. 456].

In short, even in Class II the group Γ_0 is associated with a single nonabelian simple group $E(\Gamma_0)$, and to each particular simple group E correspond only finitely many possible permutation groups Γ_0. Then in view of the classification of finite simple groups, it suffices for our purposes to know all the groups Γ_0 for which $E(\Gamma_0)$ is either an alternating group of a (possibly twisted) Chevalley group. This is exactly the problem treated in [9], and after we remove finitely many examples the final list is:

Class II (Γ_0 acting on V)

V is given a vector space structure over a certain finite field so that Γ_0 acts semilinearly and we have one of the following:

(1) $\mathrm{SL}(V/F) \leq \Gamma_0$.

(2) $\mathrm{Sp}(V/F) \lhd \Gamma_0$ (and V carries a symplectic structure).

(3) $\dim V = B$, $G_2(2^m) < G$, $|F| = 2^m$.

The first case of course corresponds to affine geometry, and the other two can be eliminated. Case (2) is easy to deal with concretely and case (3) leads to a numerical contradiction.

In this way it is possible to check Lemma 2 of Appendix I.

Two examples (*of Type* I)

Example 1. Let $G \leq \Gamma \leq \mathrm{Aut}\, G$ with $G = \mathrm{PSU}(3, q)$, $\mathrm{Sz}(q)$, or ${}^2G_2(q)$, permuting the cosets of a Borel subgroup. The orders of G, Aut G, and a Borel subgroup can be found in [6] and we conclude that the degree of the permutation representation will be:

$$V = q^3+1,\ q^2+1, \text{ or } q^3+1 \text{ respectively}$$

while $|\text{Aut } G| =$

$$(q^3+1)q^3(q^2-1)n, \qquad (q^2+1)q^2(q-1)n, \quad \text{or} \quad (q^3+1)q^3(q-1)n,$$

so that the order of the stabilizer of two points is at most:

$$s = n(q^2-1),\ n(q-1),\ n(q-1) \text{ respectively.}$$

If x is the order of the algebraic closure of a set of two points and v is the degree (as above), the Jordan property would yield:

$$v - x \leqslant s.$$

On the other hand by [11] we have always $x < v/6$ which by arithmetic yields $p^n < (6/5)n$, a contradiction.

Example 2. $\Gamma = \text{Sp}(2n, 2)$ acting on cosets of $\text{GO}^{\pm}(2n, 2)$. Let us make this more concrete. On the F_2-space V we have a nondegenerate quadratic form $Q = Q^+$ or Q^- as well as a symplectic bilinear form $(\ ,\ )$. Let $X = \{x \in V : Q(x) = 0\}$. Then:

$$|X| = 2^{n-1}(2^n \pm 1) = [\text{Sp}(2n, 2) : \text{GO}^{\pm}(2n, 2)].$$

Since $\text{GO}(2n, 2)$ stabilizes X, there are at most $|X|$ transforms of X under $\text{Sp}(2n, 2)$.

Now for $x \in V$ define the symplectic *transvection* $T_x \in \text{Sp}(2n, 2)$ by:

$$T_x(v) = V + (v, x)\, x.$$

Check that $T_x \in \text{GO}(2n, 2)$ iff $x \in X$, or explicitly:

$$Q(T_x v) = Q(v) + (x, v)[Q(x) + 1]$$

T_x is an involution.

Remark. For $x, y \in X$ distinct, $T_x X \neq T_y X$.

Let $z = x + T_x y$. Choose $v \in X$ with $(v, z) \neq 0$ (easy). By an easy computation $T_x(v) \in T_y X$.

Thus we have found $|X|$ distinct transforms of X so $\text{GO}(2, 2)$ is the exact stabilizer of X in $\text{Sp}(2n, 2)$ and our permutation representation acts on $\{\Gamma_x X : x \in X\}$. For $\gamma \in \text{Sp}(2n, 2)$ we have $\gamma \Gamma_x \gamma^{-1} = \Gamma \gamma x$ and in particular for $\gamma \in \text{GO}(2n, 2)$:

$$\gamma \Gamma_x X = \Gamma_{\gamma x} X.$$

Thus $\text{GO}(2n, 2)$ acts on $\{\Gamma_x X : x \in X\}$ as it acts on X. So now we may identify our algebraically closed set X on which Γ supposedly acts with this set X of vectors.

Suppose $n > 2$ or $n = 2$, $O = O^+$. Let 0, $x \in X$, and let $O = \mathrm{GO}(2n, 2)$. The stabilizer O_x has two orbits on $X = \{0, x\}$, namely:

$$A_\varepsilon = \{v \in X : (x, v) = \varepsilon\} \quad \text{for } \varepsilon = 0, 1.$$

Let $B = \mathrm{acl}(0, x)$. One of these orbits A_ε is the complement of B. This holds (with the same ε) for any $y \in B - (0)$. If $\varepsilon = 0$, then B is a strictly isotropic subspace, a contradiction. If $\varepsilon = 1$ look at specific vectors:

$$x = (1, 0, 0, \ldots), \quad y = (0, 1, 0, \ldots), \quad z = (1, 0, 1, 0, \ldots)$$

to find $\mathrm{acl}(0, x) = \mathrm{acl}(0, y) = \mathrm{acl}(0, z)$ but $(x, z) = 0$, a contradiction.

Remark. Kantor points out that he showed in [11], (3.7)] that the stabilizer of a point in a Jordan group again has a representation as a Jordan group (in nontrivial cases); and this is obvious in the case at hand. So this provides an easier way to wipe out 'small' candidates.

Notes added in proof

(1) Ahlbrandt and Ziegler made enormous progress on the "quasi-finite axiomatizability" problem alluded to in the Introduction. An elaboration of their method may well handle the general case.

(2) Chatzidakis [5] found more interesting $\aleph_0$-categorial, $\aleph_0$-stable structures in nature.

(3) David Evans' thesis (Oxford) will contain a quite direct proof of Theorem 2.1, involving coherent configurations and the idea of [28].

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