Lines of vortices for solutions of the Ginzburg–Landau equations

Hassen Aydi a,b,*

a Laboratoire d’analyse et de mathématiques appliquées, Université Paris XII, 61 Avenue du Général de Gaulle, 94010 Créteil Cedex, France

b Université de Monastir, Institut Supérieur d’Informatique de Mahdia, Route de Réjiche, BP 35, Mahdia 5121, Tunisia

Received 22 May 2007
Available online 26 September 2007

Abstract

For disc domains and for periodic models, we construct solutions of the Ginzburg–Landau equations which verify in the limit of a large Ginzburg–Landau parameter specified qualitative properties: the limit density of the vortices concentrates on lines.

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Résumé

Pour des disques et pour des modèles périodiques, on construit des solutions des équations de Ginzburg–Landau qui vérifient à la limite d’un grand paramètre de Ginzburg–Landau des propriétés spécifiques qualitatives : la densité limite des vortex se concentre sur des lignes.

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Keywords: Ginzburg–Landau equations; Vorticity; Periodic model

Mots-clés : Equations de Ginzburg–Landau ; Vorticité ; Modèle périodique

1. Introduction

The Ginzburg–Landau energy of superconductivity in a regular bounded simply connected domain \( \Omega \subset \mathbb{R}^2 \) is

\[
J_\Omega(u, A) = \frac{1}{2} \int_\Omega |\nabla u - iAu|^2 + \frac{1}{2} \int_\Omega |h - h_{ex}|^2 + \frac{\kappa^2}{4} \int_\Omega (1 - |u|^2)^2.
\]

(1.1)

Where \( h_{ex} \) is the intensity of the applied magnetic field. \( A : \Omega \mapsto \mathbb{R}^2 \) is the vector potential and \( h = \text{curl} A \) is the induced magnetic field. \( u \) is a complex valued function called the order parameter. \( \kappa = \frac{1}{\epsilon} \) is the Ginzburg–Landau parameter of the material for which we assume \( \kappa \to +\infty \). We say that \( (u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) \) is a critical point of \( J_\Omega \) if it is solution of the Ginzburg–Landau equations, namely:

\[
\begin{align*}
\nabla^2 u &= \frac{1}{\epsilon^2} u(1 - |u|^2) \quad \text{in } \Omega, \\
-\nabla \perp h &= \langle iu, \nabla u - iAu \rangle \quad \text{in } \Omega,
\end{align*}
\]

(1.2)

* Correspondence to: Laboratoire d’analyse et de mathématiques appliquées, Université Paris XII, 61 Avenue du Général de Gaulle, 94010 Créteil Cedex, France.
E-mail address: aydi@univ-paris12.fr.

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doi:10.1016/j.matpur.2007.09.007
with the boundary conditions on $\partial \Omega$,
\[
\begin{cases}
h = h_{\text{ex}}, \\
(\nabla u - iAu) \cdot \nu = 0,
\end{cases}
\]
where $\nu$ is the unit outward normal to the boundary $\partial \Omega$.

Many papers have made clear the mathematical mechanisms of the apparition of vortices of $u$, i.e. isolated zeros of $|u|$ with nonzero winding number $d \in \mathbb{Z}$ of $\frac{u}{|u|}$ around such a zero, or in other words topological singularities of $u$, and the definitions of what can be called a “vortex-structure” of minimizers or even of nonminimizing critical points. The first and main work was the book of Béthuel, Brezis and Hélein [4], and then the paper of Bethuel and Rivière [5]. There has also been a lot of research on the full Ginzburg–Landau functional $J_{\Omega}$. Particularly, in the article [8], E. Sandier and S. Serfaty gave necessary conditions on the limit density of the vortices for arbitrary solutions of the Ginzburg–Landau equations. These conditions included the possibility of densities that are supported on uni-dimensional sets, i.e. lines. However the existence of such solutions remained an open question. In this work, we present construction of solutions of which the vortices concentrate in the limit $\varepsilon \to 0$ along lines. Such construction will follow from the minimization of the Ginzburg–Landau energy over appropriate spaces. It is the first time that such solutions are described because the limit densities of vorticity of all the known solutions obtained by local or global minimization were supported on bi-dimensional sets [7]. In this paper, we deal with applied fields $h_{\text{ex}}$ given by the following limit
\[
\lambda = \lim_{\varepsilon \to 0} \frac{\log \varepsilon}{h_{\text{ex}}}. \tag{1.4}
\]
We assume that this limit exists, is finite and does not vanish.

2. Main results

The first part of the work is devoted to the study of the periodic model, while we take in the second part the case of a disc domain.

2.1. The periodic case

Let $K$ be any open square in $\mathbb{R}^2$ of sidelength 1 and $p_\varepsilon \in \mathbb{N}$ be a function of $\varepsilon$ such that the following limit exists, is finite and does not vanish
\[
\alpha = 2\pi \lim_{\varepsilon \to 0} \frac{p_\varepsilon}{h_{\text{ex}}}. \tag{2.1}
\]
We define the space where we minimize the Ginzburg–Landau energy $J_K$. Let $(u, A) \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, then $(u, A)$ belongs to $Q_{\varepsilon}$ if there exists $(f, g) \in H^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}) \times H^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$ such that $\forall (x, y) \in \mathbb{R}^2$
\[
\begin{cases}
u(x + \frac{1}{p_\varepsilon}, y) = u(x, y)e^{if(x,y)}, \\
u(x, y + 1) = u(x, y)e^{ig(x,y)},
\end{cases} \tag{2.2}
\]
\[
\begin{cases}
u(x + \frac{1}{p_\varepsilon}, y) = A(x, y) + \nabla f(x, y), \\
u(x, y + 1) = A(x, y) + \nabla g(x, y).
\end{cases} \tag{2.3}
\]
Now we state some notations and definitions that will be used in the sequel. First, letting $(\tilde{i}, \tilde{j})$ be an orthonormal basis of $\mathbb{R}^2$ and giving $F \subset \mathbb{R}^2$, we define $F_{n, n'}$ to be the image of $F$ by translation of vector $n\tilde{i} + n'\tilde{j}$ where $n, n' \in \mathbb{Z}$.

Given a function $T$ on $\mathbb{R}^2$, then we say that

(i) $T$ is $K$-periodic if $\forall (x, y) \in \mathbb{R}^2$, $T(x + 1, y) = T(x, y + 1)$.
(ii) $T$ is $R$-periodic if $T(x + \frac{1}{p_\varepsilon}, y) = T(x, y)$.
(iii) $T$ is $KR$-periodic if both (i) and (ii) hold.

Proceeding similarly as in [2] (see also [1]), the infimum of $J_K$ over $Q_{\varepsilon}$ is achieved. We denote by $(u_\varepsilon, A_\varepsilon)$ a sequence of minimizers, then it is a periodic critical point, i.e. solution of the Ginzburg–Landau equations (1.2) in $\mathbb{R}^2$. 

\[
\]
We restrict our attention to the asymptotic behavior of \((u_\varepsilon, A_\varepsilon)\) in the limit \(\varepsilon \to 0\), in particular the vortex-structure. Before all, we need the following construction of vortex balls

**Proposition 2.1.** For \(h_{\text{ex}} \leq C|\log \varepsilon|\), there exists \(\varepsilon_0\) such that if \(\varepsilon < \varepsilon_0\) and \((u_\varepsilon, A_\varepsilon)\) a minimizer of \(J\) over \(Q_\varepsilon\), then there exist a rectangle \(R_1^1\) of the form \([x, x + \frac{1}{p\varepsilon}] \times [y, y + 1]\) \(x, y \in \mathbb{R}\) (without loss of generality the rectangle is \(R_1^1 = [0, \frac{1}{p\varepsilon}] \times [0, 1]\)) and a family of disjoint balls \((B_i = B_i(a_i, r_i))_{i \in N_\varepsilon}\) of center \(a_i\) and of radii \(r_i\) satisfying:

\[
\begin{aligned}
\{ x \in R^1, |u_\varepsilon(x)| < \frac{3}{4} \} & \subset \bigcup_{i \in N_\varepsilon} B_i, \\
\bigcup_{i \in N_\varepsilon} B_i(a_i, r_i) & \subset R^1, \quad \sum_{i \in N_\varepsilon} r_i \leq C|\log \varepsilon|e^{-\sqrt{|\log \varepsilon|}}, \\
\text{card}(N_\varepsilon) & \leq C|\log \varepsilon|h_{\text{ex}}, \\
J_{B_i}(u_\varepsilon, A_\varepsilon) & \geq \pi |d_i||\log \varepsilon|(1 - o(1)),
\end{aligned}
\]  

(2.4)  

where \(d_i\) is the degree of the map \(\frac{u_\varepsilon}{|u_\varepsilon|}\) restricted to \(\partial B_i\).

Thanks to the definition of \(R^1\), the open square of sidelength 1 is \(K = [0, 1] \times [0, 1]\). Now taking \(B_j^1(a_j, r_j) = B_1(a_j, r_j)\), we let for \(1 \leq j \leq q_\varepsilon\) the ball \(B_j^1(a_j, r_j)\) be the extended of \(B_j^1(a_j, r_j)\) by \(R\)-periodicity to the rectangle \(R^1 = [j - 1, j] \times [0, 1]\). The main result is

**Theorem 1.** Assume that \(\lambda > 0\) and let \((u_\varepsilon, A_\varepsilon)\) be a minimizer of the energy \(J_K\) over the space \(Q_\varepsilon\) and \(h_\varepsilon = \text{curl} A_\varepsilon\) be the associated magnetic field, then taking \(v_\varepsilon\) the extended measure by \(K\)-periodicity to \(\mathbb{R}^2\) of \(\frac{2\pi \sum_{i \in N_\varepsilon} d_i(\sum_{j = 1}^{\infty} \delta_{a_j})}{h_{\text{ex}}}\), there exist a \(K\)-periodic \(f_\infty \in H^1_{\text{loc}}(\mathbb{R}^2)\) and a Radon measure \(v_\infty\) on \(\mathbb{R}^2\) such that up an extraction of \(\varepsilon_n\) from \(\varepsilon\),

\[
\begin{aligned}
h_{\varepsilon_n} & \to h_\varepsilon \quad \text{weakly locally in } H^1, \\
v_{\varepsilon_n} & \to v_\infty = -\Delta f_\infty + f_\infty \quad \text{in the sense of measures.}
\end{aligned}
\]  

(2.7)  

Moreover \(x \to f_\infty(x, y)\) is constant and the restriction of the measure \(v_\infty\) on \(K\) is supported on a finite number of horizontal lines such that the mass of \(v_\infty\) on each one belongs to \(\alpha \mathbb{Z}\). Suppose in addition that \(\lambda < 2\) then if \(1 - \frac{1}{2} > \alpha \frac{\lambda}{4(\varepsilon_{\text{ex}} - 1)}\), we have \(v_\infty \neq 0\).

2.2. The case of a disc

In this paragraph, the domain is taken to be the disc \(B_R = B(0, R)\). We define \(q_\varepsilon \in \mathbb{N}\) to be a function of \(\varepsilon\) such that the following limit exists, is finite and does not vanish,

\[
\beta = \lim_{\varepsilon \to 0} \frac{q_\varepsilon}{h_\varepsilon}.
\]  

(2.9)

The natural space where we perform the minimization of the energy \(J_{B_R}\) is denoted by \(G_\varepsilon\) and it is defined as follows. Let \((u, A) \in H^1(B_R, \mathbb{C}) \times H^1(B_R, \mathbb{R}^2)\), then \((u, A)\) belongs to the space \(G_\varepsilon\) if there exists \(f \in H^2(B_R, \mathbb{C})\) such that for any \(x \in B_R\):

\[
\begin{aligned}
u(e^{i\frac{2\pi}{q_\varepsilon}} x) & = u(x)e^{i\beta f(x)}, \\
A(e^{i\frac{2\pi}{q_\varepsilon}} x) & = e^{i\frac{2\pi}{q_\varepsilon}} A(x) + e^{i\frac{2\pi}{q_\varepsilon}} \nabla f(x).
\end{aligned}
\]  

(2.10)  

(2.11)

Let us choose the Coulomb gauge,

\[
\begin{aligned}
\text{div } A & = 0 \quad \text{in } B_R, \\
A \cdot v & = 0 \quad \text{on } \partial B_R.
\end{aligned}
\]  

(2.12)

In the presence of this gauge, the infimum of the energy \(J_{B_R}\) over the space \(G_\varepsilon\) is achieved and we denote it by \((u_\varepsilon, A_\varepsilon)\). In particular, it is a critical point hence solution of the Ginzburg–Landau equations (1.2) and (1.3). Similar to Proposition 2.1, we can state
Proposition 2.2. For \( h_{\text{ex}} \leq C|\log \varepsilon| \), there exists \( \varepsilon_0 \) such that if \( \varepsilon < \varepsilon_0 \) and \((u_\varepsilon, A_\varepsilon)\) a minimizer of \( J \) over \( G_\varepsilon \), then there exist \( r_\varepsilon \in \left[ \frac{1}{4|\log \varepsilon|}, \frac{2}{2|\log \varepsilon|} \right] \), \( \theta_1 \in [0, 2\pi] \) and a family of disjoint balls \((B_i = B(a_i, r_i))_{i \in L_\varepsilon \cup T_\varepsilon}\) of center \( a_i \) and of radii \( r_i \) such that

\[
\bigcup_{i \in L_\varepsilon} B_i(a_i, r_i) \subset B(0, r_\varepsilon),
\]

\[
\bigcup_{i \in T_\varepsilon} B_i(a_i, r_i) \subset \left\{ r e^{i \theta}, \ r < R, \ \theta_1 < \theta < \theta_1 + \frac{2\pi}{q_\varepsilon} \right\},
\]

\[
\sum_{i \in L_\varepsilon \cup T_\varepsilon} r_i \leq C|\log \varepsilon|e^{-\sqrt{|\log \varepsilon|}}, \quad \text{card}(L_\varepsilon \cup T_\varepsilon) \leq C|\log \varepsilon|h_{\text{ex}},
\]

\[
J_{B_i}(u_\varepsilon, A_\varepsilon) \geq \pi |d_i| |\log \varepsilon| \left(1 - o(1)\right),
\]

where \( d_i \) is the degree of the map \( \frac{u_\varepsilon}{|u_\varepsilon|} \) restricted to \( \partial B_i \) if \( \overline{B_i} \subset B_R \) and \( d_i = 0 \) if \( \partial B_i \cap \partial B_R \neq \emptyset \).

Notation. Following the above proposition, we take the sector

\[
S_{r_\varepsilon, \theta_1} = \left\{ r e^{i \theta}, \ r < R, \ \theta_1 < \theta < \theta_1 + \frac{2\pi}{q_\varepsilon} \right\}.
\]

\( S_{r_\varepsilon, \theta_1} \) is of angle \( \frac{2\pi}{q_\varepsilon} \). Thanks to Proposition 2.2, \( \{(a_i, d_i)_{i \in L_\varepsilon}\} \) and \( \{(a_i, d_i)_{i \in T_\varepsilon}\} \) are respectively the associated families of vortices in the ball \( B_{r_\varepsilon} = B(0, r_\varepsilon) \) and in the sector \( S_{r_\varepsilon, \theta_1} \). For simplification setting \( S^i_{r_\varepsilon, \theta_1} = S_{r_\varepsilon, \theta_1} \) and \( B^i_1(a^i_1, r^i_1) = B_i(a_i, r_i), \ i \in T_\varepsilon \), we define \( B^i_1(a^i_1, r^i_1) \) to be the extended of \( B^i_1(a^i_1, r^i_1) \) by \( S \)-periodicity to the sector \( S^i_{r_\varepsilon, \theta_1} \) such that \( \theta_j = \theta_1 + \frac{2\pi(j-1)}{q_\varepsilon} \) where \( 1 \leq j \leq q_\varepsilon \). We define also the measure:

\[
\mu_\varepsilon = \frac{2\pi \left( \sum_{i \in L_\varepsilon} d_i \delta_{a_i} + \sum_{i \in T_\varepsilon} d_i \left( \sum_{j=1}^{q_\varepsilon} \delta_{a^j_i} \right) \right)}{h_{\text{ex}}}.
\]

We take \( H^1_1(B_R) \) to be the space of functions \( f \) in \( H^1(B_R) \) such that \( f = 1 \) on the boundary \( \partial B_R \). The main result is

Theorem 2. Assume that \( \lambda > 0 \) and let \((u_\varepsilon, A_\varepsilon)\) be a minimizer of the energy \( J_{B_R} \) over the space \( G_\varepsilon \). Then up to extraction of \( \varepsilon_n \) from \( \varepsilon \), there exist \( h_\infty \in H^1_1(B_R) \) and \( \mu_\infty \in \mathcal{M}(B_R) \) such that

\[
\frac{h_{\varepsilon_n}}{h_{\text{ex}}} \rightharpoonup h_\infty \quad \text{weakly in} \ H^1_1(B_R),
\]

\[
\mu_{\varepsilon_n} \rightharpoonup \mu_\infty = -\Delta h_\infty + h_\infty \quad \text{in the sense of measures}.
\]

Again, \( h_\infty \) is radial and \( \mu_\infty \) is supported on a finite number of concentric circles of positive radii such that the mass of \( \mu_\infty \) on each circle belongs to \( 2\pi \beta \mathbb{Z} \). In addition, for any \( R > 0 \) there exists \( \beta_0(R) > 0 \) such that \( \forall \beta < \beta_0(R) \) there exists \( \lambda_0(R) > 0 \) such that if \( \lambda < \lambda_0(R) \), we have \( \mu_\infty \neq 0 \).

Remark 2.1. Unfortunately, Theorems 1 and 2 don’t give us the appropriate expression of the limit measure of vorticity. We have just found that it is different to 0 and so there is at least one line of vortices.

Notation. When it is not necessary, we will write in both cases \( J \) instead of \( J_{B_R} \) or \( J_K \).

3. The periodic model

We assume that the applied field \( h_{\text{ex}} \) is such that \( \lambda > 0 \) and we let \( K \) be any open square in \( \mathbb{R}^2 \) of sidelength 1. Consider \((u_\varepsilon, A_\varepsilon)\) a family of minimizers of the energy \( J_K \) over the space \( Q_\varepsilon \) and \( h_\varepsilon = \text{curl} \ A_\varepsilon \) the associated magnetic field.
3.1. Proof of Proposition 2.1

First, let us set \( \Omega \subset \mathbb{R}^2 \) and we can adjust the ball-construction which was used in [3,6,8] to the following lemma

**Lemma 3.1.** For \( h_{ex} \leq C |\log \varepsilon| \), there exists \( \varepsilon_0 \) such that if \( \varepsilon < \varepsilon_0 \) and \( (u_{\varepsilon}, A_{\varepsilon}) \) satisfies \( |\nabla u_{\varepsilon} - iA_{\varepsilon} u_{\varepsilon}| < \frac{C}{\varepsilon} \) and \( J_K(u_{\varepsilon}, A_{\varepsilon}) \leq C |\log \varepsilon|^2 \), then there exists a family of disjoint balls \( (B_i = B_i(a_i, r_i))_{i \in I_\varepsilon} \) of center \( a_i \) and of radii \( r_i \) such that

\[
\begin{align*}
\left\{ x \in \Omega, \ |u_{\varepsilon}(x)| < \frac{3}{4} \right\} & \subset \bigcup_{i \in I_\varepsilon} B_i, \\
\sum_{i \in I_\varepsilon} r_i & \leq C |\log \varepsilon| e^{-\sqrt{|\log \varepsilon|}}, \\
\text{card}(I_\varepsilon) & \leq C |\log \varepsilon|h_{ex}, \\
J_{B_i}(u_{\varepsilon}, A_{\varepsilon}) & \geq \pi |d_i||\log \varepsilon|(1 - o(1)),
\end{align*}
\]

where \( d_i \) is the degree of the map \( \frac{u_{\varepsilon}}{|u_{\varepsilon}|} \) restricted to \( \partial B_i \) if \( |u_{\varepsilon}| \leq 0 \) and \( d_i = 0 \) otherwise.

Now we take \( \Omega = ]0, 1[ \times ]0, 2[ \) in Lemma 3.1. Let \( (u_{\varepsilon}, A_{\varepsilon}) \) is a minimizer of \( J_K \) over \( Q_\varepsilon \), it is then clear that the hypotheses of Lemma 3.1 are verified, so there exists a family of balls defined on \( \Omega \) depending on \( \varepsilon \) denoted by \( (B_i)_{i \in I_\varepsilon} = (B_i(a_i, r_i))_{i \in I_\varepsilon} \) such that the three assertions (3.2), (3.3) and (3.4) hold. Thanks to (3.2), we have

\[
\sum_{i \in I_\varepsilon} r_i \leq C |\log \varepsilon| e^{-\sqrt{|\log \varepsilon|}}
\]

and by definition of \( p_\varepsilon \), \( \sum_{i \in I_\varepsilon} r_i = o\left(\frac{1}{p_\varepsilon}\right) \). Consequently, if we project the balls \( (B_i(a_i, r_i))_{i \in I_\varepsilon} \) onto the horizontal line of equation \( y = \frac{x}{2} \) and contained in \( \Omega \), then for a sufficiently small \( \varepsilon \), there exist \( 0 < x_1 < 1 \) such that the two lines of equation \( x = x_1 \) and \( x = x_1 + \frac{1}{p_\varepsilon} \) don’t intersect any ball of the family \( (B_i(a_i, r_i))_{i \in I_\varepsilon} \). For a small enough \( \varepsilon \), there exists also \( 0 < y_1 < 1 \) such that there is no intersection between the two lines of equation \( y = y_1 \) and \( y = y_1 + 1 \), and the balls \( (B_i(a_i, r_i))_{i \in I_\varepsilon} \). Let us define

\[
N_\varepsilon = \left\{ i \in I_\varepsilon, \ B_i(a_i, r_i) \subset R^1 = ]x_1, x_1 + \frac{1}{p_\varepsilon}[ \times ]y_1, y_1 + 1[ \right\}.
\]

In particular, the balls \( (B_i(a_i, r_i))_{i \in N_\varepsilon} \) defined on the rectangle \( R^1 \) are disjoint. In addition, Lemma 3.1 implies that the other assertions of Proposition 2.1 hold. Without loss of generality, taking \( R^1 = ]0, \frac{1}{p_\varepsilon}[, 0, 1[ \) completes the proof of Proposition 2.1.

From now on we take \( K = ]0, 1[ \times ]0, 1[ \) and we define:

\[
D_\varepsilon := \sum_{i \in N_\varepsilon} |d_i|.
\]

Let \( (u_{\varepsilon}, A_{\varepsilon}) \) be a minimizer of \( J_K \) over \( Q_\varepsilon \), then in particular \( J_K(u_{\varepsilon}, A_{\varepsilon}) \leq J_K(1, 0) = \frac{1}{2} h_{ex}^2 \), so we obtain from the second Ginzburg–Landau equation:

\[
\frac{1}{2} \|h_{\varepsilon} - h_{ex}\|_{H^1(K)}^2 \leq J_K(u_{\varepsilon}, A_{\varepsilon}) \leq Ch_{ex}^2.
\]

Then, \( \frac{h_{\varepsilon}}{h_{ex}} \) is bounded in \( H^1(K) \), so thanks to the \( K \)-periodicity of \( h_{ex} \), we can find a subsequence \( \varepsilon_n \to 0 \) and a \( K \)-periodic \( f_{\infty} \in H^1_{loc}(\mathbb{R}^2) \) such that

\[
\frac{h_{\varepsilon_n}}{h_{ex}} \rightharpoonup f_{\infty} \text{ weakly locally in } H^1.
\]

Combining \( h_{ex} \leq C |\log \varepsilon| \) together with (2.6) we get:

\[
\pi p_\varepsilon \sum_{i \in N_\varepsilon} |d_i||\log \varepsilon|(1 - o(1)) \leq J_K(u_{\varepsilon}, A_{\varepsilon}) \leq C |\log \varepsilon|h_{ex}.
\]

By definition of \( p_\varepsilon \), there exists \( C > 0 \) independently of \( \varepsilon \) such that for a sufficiently small \( \varepsilon \)

\[
D_\varepsilon \leq C.
\]
Referring to the definition of \( v_\varepsilon \) given in Theorem 1, it is then clear from (3.8) that \( (v_\varepsilon)_n \) is a bounded sequence of measures, and extracting again if necessary, we can assume that there exists a Radon measure \( v_\infty \) on \( \mathbb{R}^2 \) such that as \( n \to +\infty \)

\[
v_\varepsilon_n \to v_\infty \quad \text{in the sense of measures.}
\]

Finally, proceeding similarly as in [1], Proposition 5.9, the relation between \( v_\infty \) and \( f_\infty \) is:

\[
v_\infty = -\Delta f_\infty + f_\infty \quad \text{in } \mathbb{R}^2.
\]  

(3.9)

Thanks to [8], Lemma 4.1, we have \(|\nabla f_\infty| \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)\) for \( 1 \leq p < +\infty \). In particular \( f_\infty \in W^{1,p}_\text{loc}(\mathbb{R}^2) \), and by Sobolev injection, \( f_\infty \in C^{0,\alpha}_\text{loc}(\mathbb{R}^2) \) with \( 0 \leq \alpha < 1 \).

3.2. Properties of \( (f_\infty, v_\infty) \)

(i) \( x \to f_\infty(x, y) \) is constant on \( \mathbb{R} \)

We take any smooth compactly supported function \( g \) and any real number \( \alpha \), then there exists a sequence of integers \( k_\varepsilon \) such that \( \frac{k_\varepsilon}{p_\varepsilon} \to \alpha \). We denote \( t_\varepsilon \) the translation \( (x, y) \to (x + \frac{k_\varepsilon}{p_\varepsilon}, y) \) and \( t_a \) the translation \( (x, y) \to (x + a, y) \). We know that \( f_\varepsilon(x + \frac{k_\varepsilon}{p_\varepsilon}, y) = f_\varepsilon(x) \), then using change of variables,

\[
\int f_\varepsilon g = \int (f_\varepsilon \circ t_\varepsilon) g = \int f_\varepsilon(g \circ t_\varepsilon^{-1}),
\]

where \( t_\varepsilon^{-1} \) is the translation \( (x, y) \to (x - \frac{k_\varepsilon}{p_\varepsilon}, y) \). Passing to the limit, we find:

\[
\int f_\infty g = \int f_\infty(g \circ t_a^{-1}) = \int (f_\infty \circ t_a) g,
\]

and therefore \( f_\infty = f_\infty \circ t_a \).

(ii) The restriction of \( v_\infty \) on \( K \) is concentrated on a finite number of horizontal lines such that the mass of \( v_\infty \) on each line belongs to \( \alpha \mathbb{Z} \)

The vortex balls \( (B_i(a^j_i, r_i))_{i \in N_\varepsilon, \ 1 \leq j \leq p_\varepsilon} \) defined on \( K \) depends on \( \varepsilon \), hence from now on we will write:

\[
d_i(\varepsilon) = d_i \quad \text{and} \quad a^j_i(\varepsilon) = a^j_i \quad \text{for } i \in N_\varepsilon \quad \text{and} \quad 1 \leq j \leq p_\varepsilon,
\]

where \( d_i = \deg(\frac{a_i}{a^j_i}, \partial B_i(a^j_i, r_i)) \). (3.8) gives us \( \sum_{i \in N_\varepsilon} |d_i(\varepsilon)| \leq C \), thus the cardinal of \( \{i \in N_\varepsilon, d_i(\varepsilon) \neq 0\} \) is bounded independently of \( \varepsilon \). First, if for any \( \varepsilon < \varepsilon_0, d_i(\varepsilon) = 0, \forall i \in L_\varepsilon \), then \( v_\infty = 0 \) so \( v_\infty = 0 \). Second, if there exist points with nonzero degrees, then without loss of generality, there exists \( m \in \mathbb{N}^* \) such that these points are denoted \( \{a^j_i(\varepsilon), 1 \leq i \leq m, 1 \leq j \leq p_\varepsilon\} \). Now, up to extraction from \( \varepsilon \to 0 \) there exist \( q_i \in \mathbb{Z} \) and \( b^j_i \in R^1 \) with \( 1 \leq i \leq m \) such that \( d_i(\varepsilon_n) \to q_i \) and \( a^j_i(\varepsilon_n) \to b^j_i \). For simplification, let

\[
\forall 1 \leq i \leq m, \quad b^j_i = (x_i, y_j) \quad \text{with } 0 < y_1 < \cdots < y_m < 1.
\]

Note that \( y_i \) is constant and does not depend on \( \varepsilon \). The extended points of \( (b^j_i)_{1 \leq i \leq m} \) by \( R \)-periodicity to \( K \) are \( \{b^j_i = (x_i + \frac{(j-1)}{p_\varepsilon}, y_j), 1 \leq i \leq m, 1 \leq j \leq p_\varepsilon\} \). It is easy that as \( n \to +\infty \),

\[
\frac{\sum_{j=1}^{p_\varepsilon} \delta_{a^j_i(\varepsilon_n)}}{p_\varepsilon} \to \delta_{[(0,1) \times \{y_j\}]}, \quad \text{in the sense of measures.}
\]  

(3.10)

Using \( d_i(\varepsilon_n) \to q_i \) together with \( \alpha h_{\text{ex}} \simeq 2\pi p_\varepsilon \), we have

\[
2\pi \sum_{i=1}^{m} d_i(\varepsilon_n) \frac{\sum_{j=1}^{p_\varepsilon} \delta_{a^j_i(\varepsilon_n)}}{h_{\text{ex}}} \to \alpha \sum_{i=1}^{m} q_i \delta_{[(0,1) \times \{y_j\}]} \quad \text{in the sense of measures.}
\]  

(3.11)

We define \( \Sigma^j_i \) to be the horizontal line contained in \( K \) and of equation \( y = y_j \). The left-hand side of (3.11) is the restriction of the measure \( v_\varepsilon \) on \( K \), hence the restriction of the limit measure on \( K \) is equal to \( \sum_{i=1}^{m} \alpha q_i \delta_{\Sigma^j_i} \).
It is clear from (i) and (ii) that $f_\infty \in V$ where $V$ is such that

$$V := \left\{ h \in H^{1}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}), \ h \text{ is } K\text{-periodic, } x \to h(x, y) \text{ is constant and the restriction of} \right.$$ \[ \left. \text{the measure } v = -\Delta h + h \text{ on } K \text{ is supported on a finite number of horizontal lines such that the mass of } v \text{ on each one belongs to } \alpha \mathbb{Z} \right\}. \tag{3.12} \]

Moreover, proceeding as [7], Lemma 3.2 we get

$$\liminf_{n \to \infty} \frac{J_K(u_{\epsilon_n}, A_{\epsilon_n})}{h_{\epsilon_n}^2} \geq E(f_\infty), \tag{3.13}$$

where $E$ is defined for any $\lambda > 0$ and over $V$ as

$$E(f) = \frac{\lambda}{2} \int_K |v| + \frac{1}{2} \int_K |\nabla f|^2 + \frac{1}{2} \int_K |f - 1|^2. \tag{3.14}$$

**Proposition 3.1.** Assume that $0 < \lambda < 2$, then if $1 - \frac{\lambda}{2} > \alpha \frac{e + 1}{4(e - 1)}$, we have

$$\limsup_{\epsilon \to 0} \frac{J(u_\epsilon, A_\epsilon)}{h_{\epsilon_0}^2} < \frac{1}{2}. \tag{3.15}$$

**Corollary 3.1.** Assume that $0 < \lambda < 2$, then if $1 - \frac{\lambda}{2} > \alpha \frac{e + 1}{4(e - 1)}, v_\infty \neq 0$.

**Proof.** Assume that $v_\infty = 0$, then $f_\infty = 0$, so in particular from (3.13),

$$\liminf_{n \to \infty} \frac{J_K(u_{\epsilon_n}, A_{\epsilon_n})}{h_{\epsilon_n}^2} \geq E(0) = \frac{1}{2}. \tag{3.16}$$

Proposition 3.1 contradicts (3.16). \[\square\]

The proof of Theorem 1 is then completed.

### 3.3. Proof of Proposition 3.1

For any $f \in V$, we take the measure $v = -\Delta f + f$ and for $y \in K$ we define $G$ to be the Green function, solution of

$$-\Delta_x G(x, y) + G(x, y) = \delta_y \quad \text{in } \mathbb{R}^2. \tag{3.17}$$

Remark that $G$ exists, is unique and symmetric, i.e. $G(x, y) = G(y, x)$. We can refer to [1], Lemma 5.7 for more properties on the function $G$. In particular, we have:

$$(f - 1)(y) = \int_{\mathbb{R}^2} G(y, x) \text{d}(v - 1)(x). \tag{3.18}$$

The measure $(v - 1)$ denotes the difference between of the measure $v$ and the Lebesgue measure on $\mathbb{R}^2$. Let $I$ be the functional:

$$I(v) = \frac{\lambda}{2} \int_K |v| + \frac{1}{2} \int_{K \times \mathbb{R}^2} G(x, y) \text{d}(v - 1)(y) \text{d}(v - 1)(x). \tag{3.19}$$

Using (3.18), then for any $f \in V$ such that $-\Delta f + f = v$ we can prove that

$$I(v) = E(f). \tag{3.20}$$
Proposition 3.2. Set $h_{\text{ex}}$ be such that $\lim_{\varepsilon \to 0} \frac{\log|\varepsilon|}{h_{\text{ex}}} = \lambda$ exists, is finite and does not vanish. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $Q_\varepsilon$, then for any Radon measure $\nu$ on $\mathbb{R}^2$ which is $K$-periodic and constant on horizontal lines such that its restriction on $K$ is supported on a finite number of horizontal lines and its mass on each line belongs to $\alpha N$, we have

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \leq I(\nu).$$

(3.21)

Thanks to (3.20), Proposition 3.2 can be stated differently

Corollary 3.2. Assume $\lambda > 0$ and let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over $Q_\varepsilon$, then for any $f \in V$ such that $(-\Delta f + f)$ is positive,

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \leq E(f).$$

Proof. Let $f \in V$, then by definition of the space $V$, $f$ is $K$-periodic, so in particular the measure $\nu = -\Delta f + f$, is $K$-periodic. Again, $\nu$ is constant on horizontal lines and its restriction on $K$ is concentrated on a finite number of horizontal lines. Moreover, $\nu$ is taken to be positive, so the mass of $\nu$ on each line belongs to $\alpha N$. Combining all the above, Proposition 3.2 implies that

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \leq I(\nu).$$

The identity (3.20) completes the proof. □

Proof of Proposition 3.2. Suppose that the assumptions of Proposition 3.2 hold, then without loss of generality we assume that the restriction of the measure $\nu$ on $K$ is supported on $m$ horizontal lines denoted by $\{\Sigma_i \mid 1 \leq i \leq m\}$. The mass of $\nu$ on each one belongs to $\alpha N$, hence there exist $(y_i)_{1 \leq i \leq m}$ with $0 < y_1 < y_2 < \cdots < y_m < 1$ and $(n_i)_{1 \leq i \leq m}$ with $n_i \in \mathbb{N}$ such that the restriction of $\nu$ on $K$ is equal to $\alpha \sum_{i=1}^{m} n_i \delta_{\Sigma_i}$ where $\delta_{\Sigma_i}$ is the measure of arclength along $\Sigma_i$ and the equation of $\Sigma_i$ is $y = y_i$.

The upper bound (3.21) is obtained by a construction of a test configuration $(v_\varepsilon, B_\varepsilon)$ in the space $Q_\varepsilon$. For this, we need to describe the vortices of $(v_\varepsilon, B_\varepsilon)$. We split the proof into two steps.

Step 1. We consider the sequence $p_\varepsilon$ defined by (2.1). For $1 \leq j \leq p_\varepsilon$, let $R_j = \frac{j-1}{p_\varepsilon}, \frac{j}{p_\varepsilon} \times ]0,1[$. We place in the rectangle $R_j$ the points

$$a_j^k = \begin{cases} j - 1/2, & y_k \text{ if } 1 \leq k \leq m, \\ \frac{j}{p_\varepsilon}, & \text{otherwise} \end{cases}$$

(3.22)

We define $v_\varepsilon$ to be the extended measure to $\mathbb{R}^2$ by $K$-periodicity of $\frac{2\pi}{h_{\text{ex}}} \sum_{k=1}^{m} (n_k \sum_{i=1}^{p_\varepsilon} \delta_{a_j^k})$. Let $1 \leq k \leq m$ be fixed, then as $\varepsilon \to 0$,

$$\frac{\sum_{i=1}^{p_\varepsilon} \delta_{a_j^k}}{p_\varepsilon} \to \delta_{\Sigma_k},$$

in the sense of measures, where $\Sigma_k$ is the horizontal line of equation $y = y_k$. Using the fact that $\alpha h_{\text{ex}} \simeq 2\pi p_\varepsilon$ as $\varepsilon \to 0$, the measure $v_\varepsilon$ converges to $\nu$. Now, we refer to [1], Chapter 5 to have:

$$\limsup_{\varepsilon \to 0} \frac{1}{2} \int_{K \times \mathbb{R}^2} G(x, y) d(v_\varepsilon - 1)(x) d(v_\varepsilon - 1)(y) \leq \frac{\lambda}{2} v(K) + \int_{K \times \mathbb{R}^2} G(x, y) d(v - 1)(x) d(v - 1)(y).$$

(3.23)

Step 2. Here we construct a test configuration $(v_\varepsilon, B_\varepsilon)$ to be in the space $Q_\varepsilon$. First, we construct a function $h_\varepsilon$ $KR$-periodic by letting:

$$h_\varepsilon(x) = h_{\text{ex}} \int_{\mathbb{R}^2} G(x, y) dv_\varepsilon(y).$$
so that
\[-\Delta h_\epsilon + h_\epsilon = h_{\text{ex}} v_\epsilon \quad \text{in } \mathbb{R}^2.\]
(3.24)

$h_\epsilon$ is taken as the magnetic field. Having defined $h_\epsilon$ on $\mathbb{R}^2$, we let $B_\epsilon$ be a solution of curl $B_\epsilon = h_\epsilon$. $B_\epsilon$ is taken to be the magnetic potential. To drop the subscripts, we take for any $1 \leq k \leq m$ and $1 \leq i \leq p_\epsilon$, $B_i^k = B_i^k(\alpha_i^k, \epsilon)$. Let us then choose $\rho_\epsilon$ such that $0 \leq \rho_\epsilon \leq 1$, $\rho_\epsilon = 0$ in $\bigcup_{1 \leq k \leq m} B_i^k$, $\rho_\epsilon = 1$ in $R_1 \setminus \bigcup_{1 \leq k \leq m} B_i^k(\alpha_i^k, 2\epsilon)$, and $\rho_\epsilon = \frac{|x-x_i^k|}{\epsilon} - 1$ otherwise. We may extend $\rho_\epsilon$ by $KR$-periodicity to $\mathbb{R}^2$, so we get:

$$
\rho_\epsilon \left( x + \frac{1}{\rho_\epsilon}, y \right) = \rho_\epsilon (x, y) = \rho_\epsilon (x, y + 1) \quad \forall (x, y) \in \mathbb{R}^2.
$$

Next, we define the function $\phi_\epsilon$ only modulo $2\pi$ where $\rho_\epsilon \neq 0$. Letting $x_0$ be on $\mathbb{R}^2 \setminus \bigcup_{1 \leq i \leq p_\epsilon, 1 \leq k \leq m} (B_i^k(\alpha_i^k, \epsilon))_{n,n'}$, we define the function $\phi_\epsilon$ on $\mathbb{R}^2 \setminus \bigcup_{1 \leq i \leq p_\epsilon, 1 \leq k \leq m} (B_i^k(\alpha_i^k, \epsilon))_{n,n'}$,

$$
\phi_\epsilon (x) = \int_{(x_0, x)} B_\epsilon \cdot \tau - \nabla h_\epsilon \cdot v,
$$
(3.25)

where $(x_0, x)$ is any curve joining $x_0$ to $x$ in $\mathbb{R}^2 \setminus \bigcup_{1 \leq i \leq p_\epsilon, 1 \leq k \leq m} (B_i^k(\alpha_i^k, \epsilon))_{n,n'}$. Let us take $v_\epsilon = \epsilon^0 \phi_\epsilon$. It is easy that $(v_\epsilon, B_\epsilon) \in Q_\epsilon$, and

$$
\frac{J(v_\epsilon, B_\epsilon)}{h_{\text{ex}}^2} \leq \frac{1}{2} \int_K |\nabla h_\epsilon|^2 + \frac{1}{2} \int_K |h_\epsilon - h_{\text{ex}}|^2 + o_\epsilon(1),
$$
(3.26)

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. Then, following again the proof of [1], Proposition 5.8 and using (3.23) yields:

$$
\limsup_{\epsilon \to 0} \frac{1}{2} \int_K |\nabla h_\epsilon|^2 + \frac{1}{2} \int_K |h_\epsilon - h_{\text{ex}}|^2 \leq I(v).
$$

Combining this with (3.26) allows to conclude that

$$
\limsup_{\epsilon \to 0} \frac{J(v_\epsilon, B_\epsilon)}{h_{\text{ex}}^2} \leq I(v).
$$
(3.27)

This inequality is true for the test configuration $(v_\epsilon, B_\epsilon)$, so it is true in particular for any minimizer of the energy $J$ over the space $Q_\epsilon$ and (3.21) is proved. □

**Proof of Proposition 3.1, completed.** Here we take $f \in V$ such that the measure $(-\Delta f + f)$ is concentrated on one horizontal line where the mass is equal to $\alpha$. Hence, there exists $y_0 \in ]0, 1]$ such that the restriction of the measure $(-\Delta f + f)$ on $K$ is:

$$
-\Delta f + f = \alpha \delta_\Sigma,
$$
(3.28)

where $\Sigma$ is the horizontal line contained in $K$ and of equation $y = y_0$. Again $x \to f(x, y)$ is constant in particular on $[0, 1]$, hence to drop the subscripts we set for $y \in [0, 1]$, $g(y) = f(x, y)$. We denote by $g'_l$ (resp. $g'_r$) the left (resp. right) derivative of $g$, then

$$
\alpha = g'_l(y_0) - g'_r(y_0).
$$
(3.29)

We deduce in particular that $\int_K \left| -\Delta f + f \right| = \int_K f = \alpha$, and

$$
\int_K \left( -\Delta f + f \right)(f - 1) = \alpha(g - 1)(y_0).
$$

Consequently

$$
E(f) = \frac{1}{2} + \alpha \left( \frac{g(y_0)}{2} - \left( \frac{1}{2} - \frac{\lambda}{2} \right) \right) - \frac{1}{2} \int_K f.
$$
From (3.28), (3.29), it is easy that \( g(y_0) = \frac{e+1}{2(e-1)} \alpha \), and then
\[
E(f) = \frac{1}{2} + \left(1 - \frac{\lambda}{2}\right) \alpha + \frac{e+1}{4(e-1)} \alpha^2.
\]
(3.30)

Going back to the upper bound given by Corollary 3.2 we can conclude
\[
\limsup_{\epsilon \to 0} \frac{J_K(u_\epsilon, A_\epsilon)}{h_{ex}^2} \leq \frac{1}{2} + \left(1 - \frac{\lambda}{2}\right) \alpha + \frac{e+1}{4(e-1)} \alpha^2.
\]
(3.31)

Now assume that \( 0 < \lambda < 2 \), so choosing \( 1 - \frac{\lambda}{2} > \alpha \frac{e+1}{4(e-1)} \), completes the proof of Proposition 3.1. \( \square \)

4. The case of disc domains

In this section the applied field \( h_{ex} \) is such that \( \lambda > 0 \).

4.1. Proof of Proposition 2.2

Let \((u_\epsilon, A_\epsilon)\) be a minimizer of \( J \) over \( G_\epsilon \), then it is simply that the hypotheses of Lemma 3.1 are verified, so there exists a family of balls in \( B_R \) depending on \( \epsilon \) denoted by \((B_i)_{i \in I_\epsilon} = (B(a_i, r_\epsilon))_{i \in I_\epsilon}\) such that the assertion (3.4) holds. We start by the proof of the assertions (2.13), (2.14). First
\[
\sum_{i \in I_\epsilon} r_i \leq C |\log \epsilon| e^{-\sqrt{|\log \epsilon|}}.
\]

Therefore, \( \sum_{i \in I_\epsilon} 2 r_i = o\left(\frac{1}{|\log \epsilon|}\right) \). Hence, if \( \epsilon \) is small enough, there exists \( 1 < c < 2 \) such that when we take \( r_\epsilon = \frac{c}{|\log \epsilon|} \), the boundary of the ball of center 0 and of radius \( r_\epsilon \) does not intersect any ball of the family \((B_i)_{i \in I_\epsilon}\). We define \( L_\epsilon = \{ i \in I_\epsilon, B_i \subseteq B(0, r_\epsilon) \} \), then (2.13) is satisfied. In addition in view of the fact that \( q_\epsilon \simeq \beta h_{ex} \) as \( \epsilon \to 0 \) and \( r_\epsilon = \frac{c}{|\log \epsilon|} \), we can write:
\[
\sum_{i \in I_\epsilon} \frac{2 r_i}{r_\epsilon} = o\left(\frac{2 \pi}{q_\epsilon}\right).
\]
(4.1)

Let us project \((B_i)_{i \in I_\epsilon}\) on \( \{r_\epsilon e^{i \theta}, \theta \text{ belongs to an interval of length } \frac{2 \pi}{q_\epsilon}\} \), then thanks to (4.1) and if \( \epsilon \) is small enough, there exists \( 0 \leq \theta_1 \leq 2 \pi \) such that the two lines \( \{r_\epsilon e^{i \theta_1}, r \in [r_\epsilon, R]\} \) and \( \{r_\epsilon e^{i (\theta_1 + \frac{2 \pi}{q_\epsilon})}, r \in [r_\epsilon, R]\} \) don’t intersect any ball of the family \((B_i)_{i \in I_\epsilon \setminus L_\epsilon}\). These two lines together with \( \{r_\epsilon e^{i \theta}, \theta_1 < \theta < \theta_1 + \frac{2 \pi}{q_\epsilon}\} \) form in the disc \( B_R \) the boundary of the sector \( S_{\theta_1, \theta_1}\) which is defined by (2.17). Now, let us define:
\[
T_\epsilon = \left\{ i \in I_\epsilon, B_i \subset \left\{r_\epsilon e^{i \theta}, r_\epsilon < r \leq R, \theta_1 < \theta < \theta_1 + \frac{2 \pi}{q_\epsilon}\right\} = S_{\theta_1, \theta_1}\right\}.
\]

By definition of \( L_\epsilon \) and \( T_\epsilon \), the balls \((B_i)_{i \in I_\epsilon \cup T_\epsilon}\) are disjoint. Moreover, it is clear that the three assertions (2.14), (2.15) and (2.16) hold. This completes the proof of Proposition 2.2.

Now using the fact that \( |u_\epsilon(x e^{i \frac{2 \pi}{q_\epsilon}})| = |u_\epsilon(x)| \) we obtain from (3.1):
\[
\left\{ x \in B_R, |u_\epsilon| < \frac{3}{4}\right\} \subset \left( \bigcup_{1 \leq j \leq q_\epsilon, i \in T_\epsilon} B_i^j \right) \cup \left( \bigcup_{i \in L_\epsilon} B_i \right).
\]
(4.2)

Set \( D_\epsilon := \sum_{i \in T_\epsilon} |d_i| \). Similarly as (3.7) we have:
\[
\pi \left( q_\epsilon D_\epsilon + \sum_{i \in L_\epsilon} |d_i| \right) |\log \epsilon| (1 - o(1)) \leq J_{B_R}(u_\epsilon, A_\epsilon) \leq C h_{ex}^2 \leq C |\log \epsilon| h_{ex}.
\]
(4.3)

Therefore,
\[ \sum_{i \in L_\varepsilon} |d_i| \leq C h_{\text{ex}}, \quad (4.4) \]

By definition of \( q_\varepsilon \),
\[ q_\varepsilon D_\varepsilon \leq C. \quad (4.5) \]

Note that \((u_\varepsilon, A_\varepsilon)\) is a solution of the Ginzburg–Landau equations (1.2), (1.3). The second one leads to
\[ \frac{1}{2} \| h_\varepsilon - h_{\text{ex}} \|_{H^1(B_R)}^2 \leq J(u_\varepsilon, A_\varepsilon) \leq C h_{\text{ex}}^2 \] which with (4.4), (4.5) imply that there exist \( h_\infty \in H^1_1(B_R, \mathbb{R}) \) and a Radon measure \( \mu_\infty \) such that up an extraction of \( \varepsilon_n \) from \( \varepsilon \),
\[ \frac{h_{\varepsilon_n}}{h_{\text{ex}}} \rightharpoonup h_\infty \quad \text{weakly in } H^1_1(B_R), \quad (4.7) \]

and
\[ \mu_{\varepsilon_n} \rightarrow \mu_\infty \quad \text{in the sense of measures}. \quad (4.8) \]

We have also,
\[ \mu_\infty = -\Delta h_\infty + h_\infty. \quad (4.9) \]

4.2. Properties of \((h_\infty, \mu_\infty)\)

From (4.7) and (4.9), we can mention that \( \mu_\infty \in H^{-1} \), so in particular there is no concentration of the vorticity on isolated points. Moreover, referring to [8], Lemma 4.1, we have \( |\nabla h_\infty| \in W^{1,p}(B_R) \), \( 1 \leq p < +\infty \). In particular, \( h_\infty \in W^{1,p}_p(B_R) \) and by Sobolev injection, we conclude that \( h_\infty \in C^{0,\alpha}(B_R) \) for \( 0 \leq \alpha < 1 \).

(i) \( h_\infty \) is radial

First, for any \( x \in B_R \), we take \( x = re^{i\theta} \) where \( 0 \leq r < R \) and \( 0 \leq \theta \leq 2\pi \). Let \( \varepsilon_n \to 0 \) and \( k_n \) an integer such that
\[ \frac{2\pi k_n}{q_{\varepsilon_n}} \to \theta \quad \text{as } n \to +\infty. \]

We set \( R_n \) to be the rotation of center \( O \) and of angle \( \frac{2\pi k_n}{q_{\varepsilon_n}} \). Taking the curl in (2.11) we get for any \( n \):
\[ h_{\varepsilon_n} \circ R_n = h_{\varepsilon_n}. \quad (4.10) \]

Having \( \{\frac{h_{\varepsilon_n}}{h_{\text{ex}}}\}_n \) is bounded in \( H^1(B_R) \), there exists a subsequence still denoted \( n \) such that \( \{\frac{h_{\varepsilon_n}}{h_{\text{ex}}}\}_n \) and \( \{\frac{h_{\varepsilon_n} \circ R_n}{h_{\text{ex}}}\}_n \) converge weakly in \( H^1 \) to the same limit, which thanks to (4.7) is \( h_\infty \). In addition, by change of variables we obtain for any \( \Phi \in C_0^\infty(B_R) \):
\[ \int_{B_R} h_{\varepsilon_n} \circ R_n \Phi = \int_{B_R} h_{\varepsilon_n} \Phi \circ R_n^{-1}, \quad (4.11) \]

where \( R_n^{-1} \) is the rotation of center \( O \) of angle \( \frac{2\pi k_n}{q_{\varepsilon_n}} \). Inserting (4.10) in (4.11) we have:
\[ \int_{B_R} h_{\varepsilon_n} \Phi = \int_{B_R} h_{\varepsilon_n} \Phi \circ R_n^{-1}. \quad (4.12) \]

But, as \( n \to +\infty \)
\[ \Phi \circ R_n^{-1} \rightarrow \Phi \circ R_{-\theta} \quad \text{in } C^k(B_R) \quad \forall k, \quad (4.13) \]

where \( R_{-\theta} \) is the rotation of center \( O \) and of angle \(-\theta\). Thus, we pass to the limit in (4.12) and we use (4.13) to find:
\[ \int_{B_R} h_\infty \Phi = \int_{B_R} h_\infty (\Phi \circ R_{-\theta}). \quad (4.14) \]
Again by change of variables

$$\int_{B_R} h_\infty (\Phi \circ R_\theta) = \int_{B_R} (h_\infty \circ R_\theta) \Phi. \quad (4.15)$$

Comparing (4.14) to (4.16), we get for any $\Phi \in C_0^\infty (B_R)$

$$\int_{B_R} h_\infty \Phi = \int_{B_R} (h_\infty \circ R_\theta) \Phi. \quad (4.16)$$

We deduce that $h_\infty = h_\infty \circ R_\theta$ independently of $\theta \in [0, 2\pi]$. Step 1 is then proved.

(ii) $\mu_\infty$ is supported on a finite number of concentric circles of center $O$ and of positive radii such that the mass of $\mu_\infty$ on each one belongs to $2\pi \beta \mathbb{Z}$.

The balls $(B_i^j (a_i^j, r_i))_{i \in T_\varepsilon, 1 \leq j \leq q_\varepsilon}$ defined in $\overline{B_R \setminus B_{r_\varepsilon}}$ by Proposition 2.2 depend on $\varepsilon$, hence we can write:

$$d_i(\varepsilon) = d_i \quad \text{and} \quad a_i^j(\varepsilon) = a_i^j \quad \text{for } i \in L_\varepsilon, 1 \leq j \leq q_\varepsilon,$$

where $d_i = \deg(\frac{\mu_\varepsilon}{\mu_\varepsilon}, \partial B_i^j (a_i^j, r_i))$. (6.4) gives us $\Sigma_{i \in T_\varepsilon} |d_i| \leq C$, thus the cardinal of $\{i \in T_\varepsilon, d_i(\varepsilon) \neq 0\}$ is bounded independently of $\varepsilon$. First, if $d_i(\varepsilon) = 0$ then $D_\varepsilon = 0$, so the measure $\mu_\varepsilon$ defined by (2.18) is written as

$$\mu_\varepsilon = \frac{2\pi \Sigma_{i \in L_\varepsilon} d_i \delta_{\omega}}{h_{\text{ex}}}. $$

Using $r_\varepsilon \rightarrow 0$ together with the fact that the limit measure $\mu_\infty$ is not concentrated on isolated points we find that $\mu_\infty = 0$.

Second, if there exist points with nonzero degrees then without loss of generality there exists $m \in \mathbb{N}^*$ such that these points are denoted $\{a_i^j(\varepsilon), 1 \leq i \leq m, 1 \leq j \leq q_\varepsilon\}$. Then, up to extraction from $\varepsilon \rightarrow 0$:

$$d_i(\varepsilon_n) \rightarrow p_i, \quad \text{and} \quad a_i^j(\varepsilon_n) \rightarrow b_i^j \quad \text{as } n \rightarrow +\infty, \quad (4.17)$$

where $p_i \in \mathbb{Z}$ and $b_i^j$ is contained strictly in the sector $S_{r_i, b_i^j}$. To simplify, we take for $1 \leq k \leq m$, $b_k^j = r_k e^{i \theta_k}$ where $0 < r_1 < \cdots < r_m < R$. Note that $r_k$ is constant and does not depend on $\varepsilon$. The extended points of $(b_k^j)_{1 \leq k \leq m}$ by $S$-periodicity to $B_R \setminus B_{r_k}$ are $(b_k^j = (r_k^{-\frac{m+2n(i-j)-1}{2\varepsilon}} e^{i \theta_k}), 1 \leq k \leq m, 1 \leq j \leq q_\varepsilon)$. Let $\Gamma_k(r_k)$ be the circle of center $0$ and of radius $r_k$. It is clear that $V_1 \leq k \leq m$,

$$\sum_{j=1}^{q_{\varepsilon_n}} \frac{\delta_{b_i^j(\varepsilon_n)}}{q_{\varepsilon_n}} \rightarrow \frac{1}{2\pi r_k} \text{e}^{i \theta_k}, \quad \text{in the sense of measures.} \quad (4.18)$$

Consequently, using $d_k(\varepsilon_n) \rightarrow p_k$ together with $\beta h_{\text{ex}} \simeq q_{\varepsilon_n}$

$$2\pi \sum_{k=1}^{m} d_k(\varepsilon_n) \frac{\Sigma_{j=1}^{q_{\varepsilon_n}} \delta_{b_i^j(\varepsilon_n)}}{h_{\text{ex}}} \rightarrow \sum_{k=1}^{m} \beta \frac{p_k}{r_k} \Sigma_{j=1}^{q_{\varepsilon_n}} \delta_{\Gamma_k(r_k)} \quad \text{in the sense of measures,} \quad (4.19)$$

which with $\Sigma_{k \in L_\varepsilon} 2\pi d_k(\varepsilon_n) \rightarrow 0$ yields that $\mu_\infty = \beta \sum_{k=1}^{m} \frac{p_k}{r_k} \delta_{\Gamma_k(r_k)}$.

Set the space:

$$W := \left\{ f \in H^1 (B_R, \mathbb{R}), \quad f \text{ is radial, } \mu = -\Delta f + f \text{ is supported on a finite number of concentric circles of center } 0 \text{ and the mass of } \mu \text{ on each one belongs to } 2\pi \beta \mathbb{Z} \right\}. \quad (4.20)$$

It is clear that $h_\infty \in W$. Now, splitting the energy $J_{B_R}$ between the contribution inside the vortex-balls $(\bigcup_{1 \leq j \leq q_\varepsilon} i \in T_\varepsilon) B_i^j \bigcup \bigcup_{i \in L_\varepsilon} B_i^j)$ and the contribution outside, we get:
\[ J_{B_R}(u_\varepsilon, A_\varepsilon) \geq \pi \left( q_\varepsilon D_\varepsilon + \sum_{i \in L_\varepsilon} |d_i| \right) |\log \varepsilon| (1 - o(1)) + \frac{1}{2} \int_{B_R \setminus \bigcup_{i \in L_\varepsilon} B_i} |\nabla h_\varepsilon|^2 \\
+ \frac{1}{2} \int_{B_R \setminus \bigcup_{i \in L_\varepsilon} B_i} |h_\varepsilon - h_{\text{ex}}|^2 - o(1). \] (4.21)

We divide (4.21) by \( h_{\text{ex}}^2 \) and we proceed similarly as in [7], Lemma 3.2 to obtain:

\[ \liminf_{n \to +\infty} \frac{J_{B_R}(u_{\varepsilon n}, A_{\varepsilon n})}{h_{\text{ex}}^2} \geq \frac{\lambda}{2} \int_{B_R} |-\Delta h_\infty + h_\infty|^2 + \frac{1}{2} \int_{B_R} |\nabla h_\infty|^2 + \frac{1}{2} \int_{B_R} |h_\infty - 1|^2 = E(h_\infty). \] (4.22)

### 4.3. Upper bound of the energy

First, any \( f \in W \) is solution of:

\[
\left\{ \begin{array}{ll}
-\Delta f + f = \mu & \text{in } B_R, \\
f = 1 & \text{on } \partial B_R.
\end{array} \right.
\] (4.23)

Then \( \forall x \in B_R, (f - 1)(x) = \int_{B_R} G(x, y) \mu(y) - \delta_y \) where \( G \) is the Green potential, solution of:

\[
\left\{ \begin{array}{ll}
-\Delta_x G(x, y) + G(x, y) = \delta_y & \text{in } B_R, \\
G(x, y) = 0, & \text{in } \partial B_R.
\end{array} \right.
\] (4.24)

As (3.20), it is easy that

\[ E(f) = H(\mu) = \int_{B_R} |\mu| + \frac{1}{2} \int_{B_R B_R} G(x, y) \mu(x) \mu(y) + \frac{1}{2} \int_{B_R B_R} G(x, y) \mu(x - 1) \mu(y - 1). \] (4.25)

**Proposition 4.1.** Consider \( h_{\text{ex}} \leq C |\log \varepsilon| \). Let \( (u_\varepsilon, A_\varepsilon) \) be a minimizer of the energy \( J \) over the space \( G_\varepsilon \), then for any Radon measure \( \mu \) invariant by rotation and concentrated on a finite number of concentric circles of center \( O \) and of positive radii such that its mass on each one belongs to \( 2 \pi \beta \mathbb{N} \), we have:

\[ \limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \leq H(\mu). \] (4.26)

Thanks to (4.25), Proposition 1 can be stated differently:

**Corollary 4.1.** If \( \lambda > 0 \), then for any \( f \in W \) such that \( -\Delta f + f \) is positive

\[ \limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \leq E(f). \] (4.27)

**Proof of Proposition 4.1.** Suppose that the assumptions of Proposition 4.1 hold, then without loss of generality we assume that the measure \( \mu \) is supported on \( m \) concentric circles denoted by \( (\Gamma_i)_{1 \leq i \leq m} \), of center \( 0 \) and of positive radii. The mass of \( \mu \) on each circle belongs to \( 2\pi \beta \mathbb{N} \), hence there exist \( (r_i)_{1 \leq i \leq m} \) with \( 0 < r_1 < r_2 < \cdots < r_m < R \) where \( r_i \) is taken to be the radius of the circle \( \Gamma_i \), and \( (m_i)_{1 \leq i \leq m} \) where \( m_i \in \mathbb{N} \) such that \( \int_{\Gamma_i} \mu = 2\pi \beta m_i \). Then, \( \int_{B_R} \mu = \int_{\bigcup_{i=1}^m r_i \Gamma_i} \mu = 2\pi \beta \sum_{i=1}^m m_i \), so the measure \( \mu \) is given as

\[ \mu = \beta \sum_{i=1}^m \frac{m_i}{r_i} \delta_{\Gamma_i}, \] (4.28)

where \( \delta_{\Gamma_i} \) is the measure of arclength along \( \Gamma_i \).
The upper bound (4.26) is obtained by a construction of a test configuration \((v_\varepsilon, B_\varepsilon)\) in the space \(G_\varepsilon\). For this, we need to describe the vortices of \((v_\varepsilon, B_\varepsilon)\). We decompose the proof of Proposition 4.1 into three steps.

**Step 1.** We consider the sequence \(q_\varepsilon\) defined by (2.9). Let \(S_j\) be the sector:

\[
S_j = \left\{ r e^{i\alpha}, 0 \leq r < R, \theta \in \left[ \frac{2\pi (j - 1)}{q_\varepsilon}, \frac{2\pi j}{q_\varepsilon} \right], 1 \leq j \leq q_\varepsilon \right\}.
\]

First, we place in the sector \(S_j\) the points \((a^k_j)_{1 \leq k \leq m} = (r_k e^{i\frac{2\pi j k}{q_\varepsilon}})_{1 \leq k \leq m}\) where \(\{r_1, \ldots, r_m\}\) are the radii of the circles where the measure \(\mu\) concentrates. In particular, the extended points of \((a^k_j)_{1 \leq k \leq m}\) to the sector \(S_j, 1 \leq j \leq q_\varepsilon\), are denoted \((a^k_j)_{1 \leq k \leq m} = (r_k e^{i\frac{2\pi (j - 1) k}{q_\varepsilon}})_{1 \leq k \leq m}\). We define for \(0 \leq j \leq q_\varepsilon, \Sigma_j = \{r_k e^{i\frac{2\pi j k}{q_\varepsilon}}, 0 \leq r < R\}.\) Remark that the boundary of \(S_j\) is \(\partial S_j = (\partial B_R \cap S_j) \cup \Sigma_{j - 1} \cup \Sigma_j\). From now on, we mean by the \(S\)-periodicity of a given function \(T\) if \(T(x e^{i\frac{2\pi j}{q_\varepsilon}}) = T(x)\). Now, we define the measure:

\[
\mu_\varepsilon = \frac{2\pi}{h_\varepsilon} \sum_{k=1}^{m} \left( m_k \sum_{i=1}^{q_k} \delta_{a^k_i} \right).
\]

Let \(1 \leq k \leq m\) be fixed, then as \(\varepsilon \to 0, \frac{\sum_{i=1}^{q_k} \delta_{a^k_i}}{q_\varepsilon} \to \frac{1}{2\pi r_k} \delta_{\Gamma_k}\) in the sense of measures. Using the fact that \(\beta h_\varepsilon \simeq q_\varepsilon\) as \(\varepsilon \to 0\)

\[
2\pi m_k \sum_{i=1}^{q_k} \delta_{a^k_i} \to \beta \frac{m_k}{r_k} \delta_{\Gamma_k}.
\]

It follows that as \(\varepsilon \to 0,\)

\[
\mu_\varepsilon \to \mu = \beta \sum_{k=1}^{m} \frac{m_k}{r_k} \delta_{\Gamma_k}, \quad \text{in the sense of measures.}
\]

Thanks to [7], Proposition 2.2, we can state:

\[
\limsup_{\varepsilon \to 0} \frac{1}{2} \int_{B_R \setminus B_R} G(x, y) d(\mu_\varepsilon - 1)(x) d(\mu_\varepsilon - 1)(y) \leq \frac{\lambda}{2} \mu(B_R) + \int_{B_R} \int_{B_R} G(x, y) d(\mu - 1)(x) d(\mu - 1)(y).
\]

**Step 2.** Now, we construct a test configuration \((v_\varepsilon, B_\varepsilon)\) to be in \(G_\varepsilon\). First, we construct \(h_\varepsilon\) to be a \(S\)-periodic function. Indeed, let \(h_\varepsilon\) be the unique solution of:

\[
\begin{align*}
-\Delta h_\varepsilon + h_\varepsilon &= \sum_{k=1}^{m} 2\pi m_k \delta_{a^k_i} \quad \text{in} \ S_1, \\
{\partial h_\varepsilon \over \partial \nu} &= h_\varepsilon \quad \text{on} \ \Sigma_1 \cap \partial B_R, \\
{\partial h_\varepsilon \over \partial \nu} &= 0 \quad \text{on} \ \Sigma_0 \cup \Sigma_1.
\end{align*}
\]

Because, we have set \(\frac{\partial h_\varepsilon}{\partial \nu} = 0\) on \(\Sigma_0 \cup \Sigma_1\), and thanks to the fact that \(h_\varepsilon\) has the symmetry of the sector \(S_1\), the extended \(h_\varepsilon\) by \(S\)-periodicity to the ball \(B_R\) verifies:

\[
\begin{align*}
-\Delta h_\varepsilon + h_\varepsilon &= h_\varepsilon \mu_\varepsilon \quad \text{in} \ B_R, \\
h_\varepsilon &= h_\varepsilon \quad \text{on} \ \partial B_R.
\end{align*}
\]

In particular, we obtain \(h_\varepsilon(x e^{i\frac{2\pi j}{q_\varepsilon}}) = h_\varepsilon(x)\). \(h_\varepsilon\) is taken as the magnetic field. Having defined \(h_\varepsilon\) on \(B_R\), we let \(B_\varepsilon\) be a solution of \(\text{curl } B_\varepsilon = h_\varepsilon\). \(B_\varepsilon\) is taken to be the magnetic potential. Furthermore, we define the function \(\phi_\varepsilon\) only modulo \(2\pi\) where \(\rho_\varepsilon \neq 0\). Set \(x_0 \in B_R \setminus \bigcup_{j=1}^{q_\varepsilon} B(a^k_j, \varepsilon)\) and the function,

\[
\phi(x) = \int_{(x_0, x)} e^{-i\frac{2\pi j}{q_\varepsilon}} B_\varepsilon \cdot \tau - \nabla h_\varepsilon \cdot \nu,
\]

(4.34)
where \((x_0, x)\) is any curve joining \(x_0\) to \(x\) in \(B_R \setminus \bigcup_{1 \leq k \leq m, 1 \leq j \leq q_k} (B(a_k^j, \varepsilon))\). Let us then choose \(\rho_\varepsilon\) such that

\[0 \leq \rho_\varepsilon \leq 1, \rho_\varepsilon = 0 \text{ in } \bigcup_{1 \leq k \leq m} (B(a_k^1, \varepsilon)), \rho_\varepsilon = 1 \text{ in } S \setminus \bigcup_{1 \leq k \leq m} B(a_k^1, 2\varepsilon)), \text{and } \rho_\varepsilon = \frac{|x-a_k^1|}{\varepsilon} - 1 \text{ otherwise.}\]

We may extend \(\rho_\varepsilon\) by \(S\)-periodicity to \(B_R\), so we get \(\rho_\varepsilon(xe^{i\theta}) = \rho_\varepsilon(x)\). Let us take \(v_\varepsilon = \rho_\varepsilon e^{ih_\varepsilon}\), then after a simple exercise the test configuration \((v_\varepsilon, B_\varepsilon)\) belongs to the space \(G_\varepsilon\).

**Step 3.** From (4.33), \(h_\varepsilon\) satisfies in particular

\[(h_\varepsilon - h_{\text{ex}})(y) = h_{\text{ex}} \int_{B_R} G(y, x) d(\mu_{\varepsilon} - 1)(x), \quad \forall y \in B_R.\] (4.35)

Now, multiplying \(-\Delta h_\varepsilon + h_\varepsilon - h_{\text{ex}} = h_{\text{ex}}(\mu_{\varepsilon} - 1)\) by \((h_\varepsilon - h_{\text{ex}})\), integrating on \(B_R\), and using (4.35) it follows that

\[
\int_{B_R} |\nabla h_\varepsilon|^2 + \int_{B_R} |h_\varepsilon - h_{\text{ex}}|^2 = \int_{B_R} (-\Delta h_\varepsilon + h_\varepsilon - h_{\text{ex}})(h_\varepsilon - h_{\text{ex}})
= h^2_{\varepsilon} \int_{B_R} G(y, x) d(\mu_{\varepsilon} - 1)(x) d(\mu_{\varepsilon} - 1)(y).
\]

We use (4.32) to have

\[
\limsup_{\varepsilon \to 0} \frac{\frac{1}{2} \int_{B_R} |\nabla h_\varepsilon|^2 + \frac{1}{2} \int_{B_R} |h_\varepsilon - h_{\text{ex}}|^2}{h^2_{\varepsilon}} \leq \frac{\lambda}{2} \int_{B_R} |\mu| + \frac{1}{2} \int_{B_R \times B_R} G(x, y) d(\mu_{\varepsilon} - 1)(y) d(\mu_{\varepsilon} - 1)(x) = I(\mu).
\]

In addition, thanks to the fact that there are \((mq_\varepsilon)\) points \((a_k^j)_{1 \leq i \leq q_k, 1 \leq k \leq m}\) in \(B_R\) and by definition of \(\rho_\varepsilon\):

\[
\limsup_{\varepsilon \to 0} \frac{\frac{1}{2} \int_{B_R} |\nabla \rho_\varepsilon|^2 + \frac{1}{4q_\varepsilon} \int_{B_R} (1 - \rho_\varepsilon^2)^2}{h^2_{\varepsilon}} = 0.
\] (4.36)

By construction of \(\phi_\varepsilon\), we have \(\rho_\varepsilon |\nabla \phi_\varepsilon - B_\varepsilon| \leq |\nabla h_\varepsilon|\), so we can find:

\[
\limsup_{\varepsilon \to 0} \frac{J_{B_R}(v_\varepsilon, B_\varepsilon)}{h^2_{\varepsilon}} \leq \limsup_{\varepsilon \to 0} \left( \frac{\frac{1}{2} \int_{B_R} |\nabla h_\varepsilon|^2 + \frac{1}{2} \int_{B_R} |h_\varepsilon - h_{\text{ex}}|^2}{h^2_{\varepsilon}} \right)
+ \limsup_{\varepsilon \to 0} \left( \frac{\frac{1}{2} \int_{B_R} |\nabla \rho_\varepsilon|^2 + \frac{1}{4q_\varepsilon} \int_{B_R} (1 - \rho_\varepsilon^2)^2}{h^2_{\varepsilon}} \right) \leq H(\mu).
\] (4.37)

**New formulation of the functional \(E\).** Let \(f \in W\), then in particular the measure \((-\Delta f + f)\) is concentrated on a finite number of concentric circles of center \(O\) and of positive radii.

(i) **The case \(-\Delta f + f = 0\)**

In this case,

\[
\begin{cases}
-\Delta f + f = 0 & \text{in } B_R, \\
f = 1 & \text{on } \partial B_R.
\end{cases}
\] (4.38)

Thanks to (4.38),

\[
E(f) = \frac{1}{2} \int_{B_R} |\nabla f|^2 + \frac{1}{2} \int_{B_R} |f - 1|^2 = \frac{1}{2} \int_{B_R} (-\Delta f + f - 1)(f - 1)
= -\frac{1}{2} \int_{B_R} (f - 1) = \frac{\pi R^2}{2} - \frac{1}{2} \int_{B_R} f.
\] (4.39)

We need to calculate \(\int_{B_R} f\). Having \(f\) is radial, there exists so \(g : [0, R] \to \mathbb{R}\) such that \(f(re^{i\theta}) = g(r)\) for any \(\theta \in [0, 2\pi]\). (4.38) becomes:

\[
-g'' - \frac{g'}{r} + g = 0 \quad \text{in } [0, R] \quad \text{and} \quad g(R) = 1.
\] (4.40)
Recall that
\[ g(r) = \frac{I_0(r)}{I_0(R)} \text{ in } [0, R] \] (4.41)
where \( I_0 \) is the modified Bessel function of the first kind defined as
\[ I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2} \], \( x \geq 0 \) .
(4.42)
Note that \( I_0 \) is increasing and \( I_0(0) = 1 \). We denote by \( I_1 \) the derivative of \( I_0 \), it is then nonnegative. In particular \( g'(R) = \frac{I_1(R)}{I_0(R)} \) and so having \( f = \Delta f \text{ in } B_R \) gives us:
\[
\int_{B_R} f = \int_{B_R} \Delta f = \int_{\partial B_R} \frac{\partial f}{\partial v} = 2\pi R g'(R) = 2\pi R \frac{I_1(R)}{I_0(R)} .
\]
Inserting this in (4.39) yields \( E(f) = \pi \left( R^2 - R g'(R) \right) \). For simplification, we take:
\[
J_0 = \pi \left( R^2 - R \frac{I_1(R)}{I_0(R)} \right) .
\] (4.43)
(ii) The case \(-\Delta f + f \neq 0\)
Here, let \( f \) be in \( W \) such that \(-\Delta f + f = \mu \) is a positive measure concentrated exactly on one circle of center \( O \) and of radius \( r \in [0, R[ \) and its mass is equal to \( 2\pi \beta \). Therefore,
\[
E(f) = \frac{\pi R^2}{2} + \frac{\lambda}{2} \int_{B_R} \frac{|-\Delta f + f|}{2} + \frac{1}{2} \int_{B_R} (-\Delta f + f)(f - 1) - \frac{1}{2} \int_{B_R} f ,
\]
(4.44)
since \( f = 1 \) on \( \partial B_R \). However, as (4.28) we have:
\[
\mu = -\Delta f + f = \frac{\beta}{r} \delta_{\Gamma(r)} \text{ in } B_R .
\]
We can then write
\[
-g''(r) - \frac{g'(r)}{r} + g(r) = \frac{\beta}{r} \delta_{\Gamma(r)} \text{ in } [0, R[ , \quad g(R) = 1 .
\] (4.45)
Note that \( g \) is written on the interval \([r, R] \) as a combination of \( I_0 \) and \( K_0 \) where \( K_0 \) is the modified Bessel function of the second kind given as
\[
K_0(x) = -\left( \log \left( \frac{x}{2} \right) + \gamma \right) I_0(x) + \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2} 2^{2n} \Phi(n) ,
\] (4.46)
such that \( \Phi(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \) for \( n \neq 0 \), \( \Phi(0) = 0 \), and \( \gamma = \lim_{n \to +\infty} (\Phi(n) - \log n) \). If we denote by \( g'_l \) (resp. \( g'_r \)) the left (resp. right) derivative of \( g \), we get:
\[
\frac{\beta}{r} = g'_l(r) - g'_r(r) .
\] (4.47)
As a consequence,
\[
\int_{B_R} |-\Delta f + f| = 2\pi \beta \quad \text{and} \quad \int_{B_R} (-\Delta f + f)(f - 1) = 2\pi \beta (g - 1)(r) .
\] (4.48)
Moreover \( \int_{B_R} f = 2\pi (R g'(R) + \beta) \). Consequently, combining all the above in (4.44) yields:
\[
\frac{E(f)}{\pi} = \frac{R^2}{2} - R g'(R) + \beta (g(r) - (2 - \lambda)) .
\] (4.49)
Let us define
\[ \forall x \in [0, R], \quad X(x) := I_0(R) K_0(x) - K_0(R) I_0(x). \]
Let \( K_1 \) be the derivative of \(-K_0\) and set \( a(R) = R[I_0(R)K_1(R) + K_0(R)I_1(R)] \), then a simple calculation of \( g'(R) \) and \( g(r) \) gives us:
\[ E(f) = J_0 + \pi \beta F(r), \quad (4.50) \]
where
\[ F(r) = \lambda - \left( 2 - 2 \frac{I_0(r)}{I_0(R)} \right) + \frac{\beta}{a(R)} \frac{I_0(r)X(r)}{I_0(R)}, \quad r \in ]0, R[. \quad (4.51) \]
Now, inserting (4.50) in Corollary 4.1, we get for any \( r \in ]0, R[ \),
\[ \limsup_{n \to \infty} J(u_{\epsilon n}, A_{\epsilon n}) h_\infty \leq J_0 + \pi \beta F(r_0), \quad (4.52) \]
where \((u_\epsilon, A_\epsilon)\) is a minimizer of \( J \) over \( G_\epsilon \). Our interest is to minimize the right-hand side of (4.52). This will be the subject of the following proposition:

**Proposition 4.2.** If \( \beta < \frac{2R I_1(R)}{I_0(R)} \), then the minimum of \( F \) over \( ]0, R[ \) is achieved by a unique \( r_0 \in ]0, R[ \). In addition \( F(r_0) < 0 \) if \( \lambda < 2 - 2 \frac{I_0(R)}{I_0(R)} \).

**Corollary 4.2.** Under \( \beta < \frac{2R I_1(R)}{I_0(R)} \) and (4.53), we have \( \mu_\infty \neq 0 \).

**Proof.** We argue by contradiction and we suppose that \( \mu_\infty = -\Delta h_\infty + h_\infty = 0 \). Let \((u_\epsilon, A_\epsilon)\) be a minimizer of \( J \) over \( G_\epsilon \), then from (4.22),
\[ \liminf_{n \to +\infty} J_{B_R}(u_{\epsilon n}, A_{\epsilon n}) / h_{\epsilon n}^2 \geq \frac{1}{2} \int_{B_R} |\nabla h_\infty|^2 + \frac{1}{2} \int_{B_R} |h_\infty - 1|^2. \quad (4.54) \]
We know \( \frac{1}{2} \int_{B_R} |\nabla h_\infty|^2 + \frac{1}{2} \int_{B_R} |h_\infty - 1|^2 = J_0 \). It is then clear that
\[ \liminf_{\epsilon \to 0} J_{B_R}(u_\epsilon, A_\epsilon) / h_{\epsilon n}^2 \geq J_0. \quad (4.55) \]
However, under \( \beta < \frac{2R I_1(R)}{I_0(R)} \) and (4.53) we have from Proposition 4.2,
\[ \limsup_{n \to \infty} J(u_{\epsilon n}, A_{\epsilon n}) / h_{\epsilon n}^2 \leq J_0 + \pi \beta F(r_0) < J_0. \quad (4.56) \]
This contradicts (4.55). \( \square \)

**Remark 4.1.** The parameters \( \beta_0(R) \) and \( \lambda_0(R) \) given in Theorem 2 are then taken to be equal respectively to \( \frac{2RI_1(R)}{I_0(R)} \) and \( 2 - \frac{2}{I_0(R)} \). Note that \( \lambda_0(R) > 0 \) because \( I_0(R) > 1 \).

### 4.4. One circle of vorticity

Here assume that \( \beta < \frac{2RI_1(R)}{I_0(R)} \) and \( \lambda < 2 - \frac{2}{I_0(R)} \). In the case of the vortex’s concentration exactly along one circle, the limit measure \( \mu_\infty \) can be written as
\[ \mu_\infty = \beta d \delta_{\Gamma}, \quad (4.57) \]
where \( d \in \mathbb{Z}^+ \) and \( \Gamma \) is the circle of center \( O \) and of radius \( r \) with \( 0 < r < R \). The mass of \( \mu_\infty \) on \( \Gamma \) is then \( 2\pi \beta d \).
Lemma 4.1. The $d$ defined by (4.57) is in $\mathbb{N}^*$. 

Proof. Let $h_0$ be the solution of $-\Delta h_0 + h_0 = 0$ in $B_R$ such that $h_0 = 1$ on $\partial B_R$. Thanks to the convergence of $\mu_{\varepsilon_n}$ to $\mu_{\infty}$, we can write:

$$
\int_{B_R} (h_0 - 1)\mu_{\infty} = \lim_{n \to +\infty} \left( \frac{2\pi}{h_{\text{ex}}} \sum_{i \in L_{\varepsilon_n}} d_i (h_0 - 1)(a_i) + \frac{2\pi q_{\varepsilon_n} \sum_{i \in T_{\varepsilon_n}} d_i (\varepsilon_n)(h_0 - 1)(a_i)}{h_{\text{ex}}} \right). \tag{4.58}
$$

Using (4.4) and (4.5) in (4.21) we get from (4.58):

$$
\int_{B_R} (h_0 - 1)\mu_{\infty} \leq -\liminf_{n \to +\infty} \frac{|\log \varepsilon_n|}{2h_{\text{ex}}} \int_{B_R} |\mu_{\varepsilon_n}| = -\frac{\lambda}{2} \int_{B_R} |\mu_{\infty}| < 0. \tag{4.59}
$$

But, combining (4.57) with the fact that $h_0$ is radial yields:

$$
\int_{B_R} (h_0 - 1)\mu_{\infty} = 2\pi d(h_0 - 1)(r). \tag{4.60}
$$

Comparing (4.59) to (4.60) leads to $d(h_0 - 1)(r) \leq 0$, so $d \in \mathbb{N}^*$ since $0 < h_0 < 1$. \hfill \square

Lemma 4.2. Let $(u_{\varepsilon}, A_{\varepsilon})$ be a minimizer of $J$ over the space $G_{\varepsilon}$. If $\mu_{\infty} = \frac{\beta}{\pi} \delta_{\Gamma}$, then $r = r_0$. Moreover,

$$
\lim_{n \to +\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{\text{ex}}^2} = J_0 + \beta \pi \lambda - \beta \pi \left( 2 - \frac{2I_0(r_0)}{I_0(R)} \right) + \beta^2 \pi \frac{I_0(r_0)X(r_0)}{a(R)I_0(R)}. \tag{4.61}
$$

Proof. Similar to (4.50), we can write from $\mu_{\infty} = -\Delta h_{\infty} + h_{\infty} = \frac{\beta}{\pi} \delta_{\Gamma}$,

$$
E(h_{\infty}) = J_0 + \pi \beta F(r),
$$

where $\Gamma$ is the circle of radius $r$ and of center $O$. Combining (4.22) together with (4.56) we get $r = r_0$, since $r_0$ is the unique minimum of the functional $F$ over $]0, R[$. Finally, using (4.51) we obtain:

$$
\lim_{n \to +\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{\text{ex}}^2} = J_0 + \pi \beta F(r_0). \hfill \square
$$

As a consequence of all the above, Theorem 2 is proved.

4.5. Proof of Proposition 4.2

First, we state some properties of the Bessel functions $I_i$ and $K_i$ where $0 \leq i \leq 1$, which will be very useful for the rest.

Step 1. Some properties. Note that $K_0$ decreases, is positive and tends to $+\infty$ as $x \to 0$, then its derivative $-K_1$ is positive and thanks to (4.46), $K_1$ tends to $+\infty$ as $x \to 0$. In addition we have:

Lemma 4.3. $I_1$ is increasing on $[0, +\infty[$ and $K_1$ is decreasing on $]0, +\infty[$. Moreover for any $x > 0$,

$$
I_0 - \frac{2}{x} I_1 \geq 0 \quad \text{and} \quad K_0 + \frac{2}{x} K_1 \geq 0. \tag{4.61}
$$

In addition, for any $0 \leq i \leq 1$ and $x > 0$,

$$
I_i(x) \simeq \frac{e^x}{\sqrt{2\pi x}} \quad \text{when } x \text{ is large enough}, \tag{4.62}
$$

$$
K_i(x) \simeq \frac{e^{-x}}{\sqrt{2\pi x}} \quad \text{when } x \text{ is large enough}. \tag{4.63}
$$

Finally,

$$
I_0 \geq I_1 \quad \text{and} \quad K_0 \leq K_1. \tag{4.64}
$$
Proof. First referring to the expressions of \( I_0 \) and \( K_0 \) given respectively by (4.42) and (4.46), the assertion (4.61) is immediate. Second, the assertions (4.62) and (4.63) are well known in \([9]\). To prove (4.64) let \( N_1(x) = (I_0(x))^2 - (I_1(x))^2, x \geq 0 \). Having \( I_0'' + \frac{I_0'}{x} = I_0 \), hence \( I_1' = I_0 - \frac{I_1}{x} \). Using the fact that \( I_0 \geq \frac{2I_1}{x} \) yields \( I_1' \geq 0 \), so \( I_1 \) is increasing. A derivation of \( N_1 \) gives us:

\[
N'_1(x) = 2I_0I_1 - 2I_1 \left( I_0 - \frac{I_1}{x} \right) = 2 \left( \frac{I_1}{x} \right)^2 > 0.
\]

In particular, we deduce that \( N_1(x) \geq N_1(0) = 1 \), which proves \( I_0 \geq I_1 \) in \([0, +\infty[\). Now, let us take for \( x > 0 \), \( N_2(x) = (K_1(x))^2 - (K_0(x))^2 \). Using the same argument, \( N_2(x) \) is decreasing and tends to 0 as \( x \to +\infty \), then \( N_2(x) \geq 0 \) for any \( x \in [0, +\infty[ \). □

Step 2. One critical point of the functional \( F \) in \([0, R]\). Let \( Y \) be the derivative of \((-X)\), so that \( Y(x) = I_0(R)K_1(x) + K_0(R)I_1(x) \) for \( x \in [0, R]\). Note that \( Y(x) > 0 \) for any \( x \in [0, R]\) and in particular \( Y(R) = a(R) \). The derivative of \( F \) is:

\[
F'(r) = 2\beta \frac{I_1(r)}{I_0(r)} + \frac{\beta^2}{a(r)}I_0(r) \left( I_1(r)X(r) - I_0(r)Y(r) \right) \quad \forall r \in [0, R[.
\]

Let us define:

\[
T(x) = I_0(x)Y(x) - I_1(x)X(x) \quad \forall x \in [0, R[.
\]  

(4.65)

We know from Lemma 4.3 that \( K_1 \geq K_0 \), then it is immediate that \( Y \geq X \) in \([0, R]\), and so from (4.65) \( T \) is positive in \([0, R]\), since \( I_0 \geq I_1 \). We replace \((I_0Y - I_1X)\) with \( T \) in \( F'(r) \) to get \( \forall r \in [0, R[\),

\[
F'(r) = \frac{\beta}{I_0(r)} \left( 2I_1(r) - \frac{\beta}{a(r)}T(r) \right).
\]  

(4.66)

Letting \( F'(r) = 0 \), we get \( \frac{\beta}{a(r)} = 2\frac{I_1(r)}{T(r)} \). Hence, if we take the function,

\[
G(x) = 2 \frac{I_1(x)}{T(x)} , \quad x \in [0, R],
\]  

(4.67)

it follows that any critical point \( r \) in \([0, R]\) of the functional \( F \) satisfies the following identity \( \frac{\beta}{a(r)} = G(r) \). Consequently, the critical points of \( x \to F(x) \) in the plane \((x, y)\) are the intersection between the graph of \( x \to G(x) \) and the horizontal line of equation \( y = \frac{\beta}{a(r)} \). To determine such intersection we need to know the sense of variation of the function \( G \). Note that when it is not necessary we omit the variable \( x \). The derivatives of \( Y \) and \( T \) are:

\[
Y' = -X - \frac{Y}{x}, \quad T' = 2I_1Y - 2I_0X - \frac{T}{x}, \quad x \in [0, R[.
\]

The functions \( Y \) and \( X \) are respectively positive and nonnegative on \([0, R]\), hence \( Y' < 0 \), i.e. \( Y \) is decreasing on \([0, R]\). Using the above derivatives, we have for \( x \in [0, R]\):

\[
G'(x) = 2 \frac{2}{T^2} \left( I_0T + 2I_1(I_0X - I_1Y) \right).
\]  

(4.68)

We replace \( T \) by the right-hand side of (4.65) in (4.68) to find:

\[
\frac{T^2}{2} G' = \left( I_0^2 - 2I_1^2 \right)Y + I_0I_1 X.
\]  

(4.69)

In view of the fact that \( X(R) = 0, Y(R) = \frac{a(R)}{R} \) and \( T(R) = \frac{a(R)I_0(R)}{R} \), hence again from (4.69)

\[
\frac{a(R)I_0(R)^2}{2R} G'(R) = \left( I_0(R)^2 - 2(I_1(R))^2 \right).
\]

The sign of \( G'(R) \) depends on the sign of the quantity \((I_0(R) - \sqrt{2}I_1(R))\). Let us take for \( x \in [0, +\infty[\), \( Z(x) = (I_0(x))^2 - 2(I_1(x))^2 \). It is clear from Lemma 4.3 that \( Z \) is decreasing on \([0, +\infty[\) and \( Z(x) \) tends to \(-\infty\) as \( x \to +\infty \). This implies that there exists a unique \( 0 < R^* < +\infty \) such that \( I_0(R^*) = \sqrt{2}I_1(R^*) \). Note that \( R^* \simeq 2\),
and so from (4.69), $G'(R^*) > 0$. To determine the cardinal of the set $\{ r \in ]0, R[ , \frac{\beta}{a(R)} = G(r) \}$, we need to know the sense of variation of the function $G$ which depends on $R$. We start with:

**Case 1.** $I_0(R) \geq \sqrt{2}I_1(R) (\Leftrightarrow R \leq R^*)$. In this case, we have $G'(R) \geq 0$. Inserting the fact that the function $Z$ is decreasing on $[0, +\infty [ \text{ in } (4.69)$ to have:

$$\frac{T^2(x)}{2} G'(x) \geq \left( (I_0(R))^2 - 2(I_1(R))^2 \right) Y(x). \quad (4.70)$$

Thanks to $I_0(R) \geq \sqrt{2}I_1(R)$, $G$ is increasing on $]0, R[$, so $G(x) < G(R)$. Remember that any critical point $r$ of $F$ satisfies $\frac{\beta}{a(R)} = G(r)$, so the intersection between the graph of $x \rightarrow G(x)$ and the horizontal line of equation $y = \frac{\beta}{a(R)}$ is restricted to one point (even without a condition on $\beta$). Consequently, there is a unique critical point of $F$ in $]0, R[.$

**Case 2.** $I_0(R) < \sqrt{2}I_1(R) (\Leftrightarrow R > R^*)$. First, it is clear that $G'(r) > 0$ for any $r \in ]0, R^* ]$. But, unfortunately we have no idea on the sign of $G'$ on the interval $[R^*, R]$. Then, from now on we will be concerned with the study of the behavior of $G$ on the interval $[R^*, R]$. Knowing $R > R^*$, we have $G'(R) < 0$, then combining this with the fact that $G'(R_+) > 0$, there exists at least $r_+$ with $R^* < r_+ < R$ such that $G'(r_+ ) = 0$. We will prove that $r_+$ is the unique point in $[R^*, R]$ where the function $G'$ vanishes. Indeed with a simple calculation, the second derivative of the function $G$ at $r$ in $[R^*, R]$ is:

$$\frac{T^4 G''(r)}{2} = I_1 T^3 + I_0 T^2 \left( 2I_1 Y - 2I_0 X - \frac{T}{r} \right) + 2 \left( I_0 - \frac{I_1}{r} \right)(I_0 X - I_1 Y) T^2 - 2I_1 T^3$$

$$- 2I_1 T^2 Y \left( I_0 - \frac{I_1}{r} \right) + 2I_1^2 T^2 \left( X + \frac{Y}{r} \right) - 2TT' (I_0 T + 2I_1 (I_0 X - I_1 Y)) \quad (4.71)$$

We know that $G'(r_+) = 0$ and $r_+ \in [R^*, R]$, so the set of the critical points of $G$ in $[R^*, R]$ is not empty. For this let $r$ be a arbitrary critical point of $G$ in $[R^*, R]$, then in particular $G'(r) = 0$, so thanks to (4.68) we obtain:

$$\frac{r I_0 T^2 G''(r)}{4I_1 Y} = 2I_1 I_0 + 3r(I_1)^2 - 3r(I_0)^2. \quad (4.72)$$

It is easy that the right-hand side of (4.72) is decreasing on the interval $[R^*, R]$, then by the definition of $R^*$ which is such that $I_0(R^*) = \sqrt{2}I_1(R^*)$ we find for any $x \in [R^*, R]$: $2I_1(x)I_0(x) + 3x(I_1(x))^2 - 3x(I_0(x))^2 \leq 0.$

We conclude from (4.72) that any critical point $r$ of the function $G$ in $[R^*, R]$ satisfies $G''(r) \leq 0$, so by continuity of $G$ such $r$ is unique. But, knowing that $G'(r_+) = 0$, hence $r_+$ is the unique maximum of $G$. In this case, $G$ is increasing on $]0, r_+[ \text{ and is decreasing on } ]r_+, R[.$ Now, we assume that the parameter $\beta$ satisfies:

$$\beta < \frac{2R I_1(R)}{I_0(R)} \quad (4.73)$$

This means that $\frac{\beta}{a(R)} < G(R)$ and the set $\{ r \in ]0, R[ , G(r) = \frac{\beta}{a(R)} \}$ is then restricted to one point, so there is a unique critical point of $F$ in $]0, R[.$

**Step 3. The nature of the critical point of $F$.** Using (4.66) and the fact that $T(R) = \frac{a(R)I_0(R)}{R}$, we get under (4.73),

$$F'(R) = \frac{\beta}{I_0(R)} \left( 2I_1(R) - \frac{\beta}{a(R)} T(R) \right) = \frac{\beta}{I_0(R)} \left( 2I_1(R) - \frac{\beta I_0(R)}{R} \right) > 0.$$ 

But, by definition of the Bessel functions, $F'(x) \rightarrow -\infty$ as $x \rightarrow 0$. The unique critical point of the functional $F$ in the interval $]0, R[$ is then a minimizer. In particular it is in $]0, R[$ and we denote it by $r_0$. Recall that $F(R) = \lambda - \frac{2}{I_0(R)}$, since $X(R) = 0$. Choosing $\lambda < 2 - \frac{2}{I_0(R)}$ yields $F(R) < 0$ and so $F(r_0) < 0$. The proof of Proposition 4.2 is completed.
References