On Umbral Calculus I

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In this paper we establish weaker characteristic conditions of the operator $t_e$ and give some new results of invariant operators, basic sequences, and expansion theorem. © 2000 Academic Press

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1. INTRODUCTION

In this paper, we will follow Roman [1] and investigate the characteristic conditions of the operators $t_e$ and $\sigma_y$ and establish some new results of invariant operators, basic sequences, and expansion theorem.

For the sake of simplicity, we use the same notions and notations as [1]. Roman [1] gave the characteristic conditions of the operator $t_e$: Let $\tau$ be a linear operator on polynomials. Then $\tau = t_e$ for some admissible sequence $e$ if and only if

$$\tau x^n \neq 0 \text{ for all } n > 0$$

and

$$\tau \sigma_y = y \sigma_y \tau \quad (1.1)$$

for all constants $y$.

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Roman [1] established the characteristic condition of basic sequences for \((t_{e}, d, L)\)

\[
p_{n}(xy) = \sum_{k=0}^{n} \frac{d_{n}}{d_{k}d_{n-k}} p_{k}(x)y^{k}\langle L | p_{n-k}(xy) \rangle \quad (n = 0, 1, \ldots) \tag{1.2}
\]

for all \(x\) and constants \(y\).

In Section 2, we will point out that condition (1.1) for all constants \(y\) can be weakened as there exist a constant \(y_0\) which is not equal to 0 and a root of unity such that (1.1) holds.

We will also weaken condition (1.2) and prove that if (1.2) holds for one constant \(y_0\) satisfying that \(y_0\) is not equal to 0 and for a root of unity then \(p_{n}(x)\) is basic for \((t_{e}, d, L)\).

In Section 3 we will investigate invariant operators. We will prove that if an operator \(\tau\) can commute with one of the operators \(\sigma_{a}\) \((a \neq 0\) and a root of unity\), i.e.,

\[
\tau \sigma_{a} = \sigma_{a} \tau,
\]

then \(\tau\) is an invariant operator. We also point out that the set of all invariant operators is a ring which is an algebra isomorphism with the ring of sequences of constants.

In Section 4, we will establish a reverse result of the expansion theorem.

### 2. BASIC RESULTS

The following theorem considerably weakens the conditions of Theorem 2.4 of [1].

**Theorem 2.1.** Let \(\tau\) be a linear operator on polynomials. Then \(\tau = t_{e}\) for some admissible sequence \(e\) if and only if there exist a constant \(y_0\) which is not equal to 0 and a root of unity such that

\[
\tau x^n \neq 0 \quad \text{for all } n > 0
\]

and

\[
\tau \sigma_{y_0} = y_0 \sigma_{y_0} \tau.
\]

**Proof.** According to Theorem 2.4 of [1] we only need to prove the “if” part. Set

\[
\tau x^n = s_{n}(x),
\]
where \( s_m(x) \) is a polynomial of degree \( \leq m \). It is easy to prove that
\[
\deg(s_m(x)) \leq n - 1.
\]
Hence we can set
\[
\tau x^n = s_n(x),
\]
where \( s_n(x) \) is a polynomial of degree \( \leq n \). And if \( n \geq 1 \),
\[
s_n(xy) = y_0^{n-1}s_n(x).
\] (2.1)
Assume that
\[
s_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
\] (2.2)
Then from (2.1) and (2.2), comparing the coefficients of \( x^k \) \( (k = 0, 1, \ldots, n) \), we obtain
\[
a_k = 0 \quad (k = 0, 1, \ldots, n - 2, n).
\]
Then we have
\[
s_n(x) = a_{n-1}x^{n-1} = s_n(1)x^{n-1} \quad (n \geq 1).
\]
Hence
\[
\tau x^n = s_n(1)x^{n-1} \quad (n \geq 1).
\]
Since
\[
\tau 1 = \tau y_1 = y_0\alpha y_1 = y_0\tau 1,
\]
we have
\[
\tau 1 = s_0(x) = 0. \quad (2.3)
\]
Since \( s_n(1) = a_{n-1} \neq 0 \) \( (n \geq 1) \), we may set
\[
c_n = c_0s_1(1) \cdots s_n(1) \quad (c_0 \neq 0; n = 0, 1, \ldots).
\]
Then
\[
\tau x^n = s_n(1)x^{n-1} = c_n c_{n-1} x^n = tc^n \quad (n = 1, 2, \ldots). \quad (2.4)
\]
Therefore, from (2.3) and (2.4) it follows that
\[
\tau = t_c.
\]
THEOREM 2.2. Let \( p_n(x) \) be a polynomial sequence. Then \( p_n(x) \) is basic for \((t_c, d, L)\) for some admissible sequence \( c \) if and only if there exist a constant \( y_0 \) which is not equal to 0 and a root of unity such that

\[
p_n(xy_0) = \sum_{k=0}^{n} \frac{d_k}{d_{n-k}} p_k(x) y_0^{k} \langle L | p_{n-k}(xy_0) \rangle \quad (n = 0, 1, \ldots) \quad (2.5)
\]

hold for all \( x \).

Proof. In view of Theorem 2.5 of [1], we only need to prove the “if” part. Now, assume (2.5) is true. Using the proof of [1, Theorem 2.5], but applying our Theorem 2.1 instead of Theorem 2.4 of [1], we can prove

\[
L = t_c \quad \text{for some admissible} \ c
\]

and

\[
t_c p_n(x) = \frac{d_n}{d_{n-1}} p_{n-1}(x).
\]

Since \( p_n(x) \) is a polynomial sequence, from (2.5) it follows that

\[
\langle L | p_0(xy_0) \rangle = \langle L | p_0(x) \rangle \neq 0.
\]

Applying the functional \( L \) to the two sides of (2.5) gives

\[
\langle L | p_n(xy_0) \rangle = \sum_{k=0}^{n} \frac{d_k}{d_{n-k}} \langle L | p_k(x) \rangle y_0^k \langle L | p_{n-k}(xy_0) \rangle \quad (n = 0, 1, \ldots)
\]

Setting \( n = 0 \) in the above gives

\[
\langle L | p_0(xy_0) \rangle = \frac{1}{d_0} \langle L | p_0(x) \rangle \langle L | p_0(xy_0) \rangle,
\]

from which it follows that

\[
\langle L | p_0(x) \rangle = d_0.
\]

Similarly, setting \( n = 1 \), we get

\[
\langle L | p_1(x) \rangle = 0.
\]

By induction, we can easily deduce

\[
\langle L | p_n(x) \rangle = 0 \quad (n = 1, 2, \ldots).
\]

This shows that \( p_n(x) \) is basic for \((t_c, d, L)\). \( \square \)
Denote by $\varepsilon_1$ the evaluation functional at $x = 1$, i.e., $\langle \varepsilon_1 \mid p(x) \rangle = p(1)$ for every polynomial $p(x)$.

**Corollary 2.2.1.** A polynomial sequence $p_n(x)$ is basic for $(t_\varepsilon, d, \varepsilon_1)$ if and only if there exist a constant $y_0$ which is not equal to $0$ and a root of unity such that

$$
p_n(xy_0) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} p_k(x) y_0^k p_{n-k}(y_0) \quad (n = 0, 1, \ldots).
$$

Let $p_n(x)$ be a basic sequence for $(t_\varepsilon, d, L)$. If a polynomial sequence $s_n(x)$ satisfies

$$
s_n(xy) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} s_k(x) y^k \langle L \mid p_{n-k}(xy) \rangle \quad (n = 0, 1, \ldots) \quad (2.6)
$$

for all $x$ and $y$, then we say $s_n(x)$ is an associated sequence for $(t_\varepsilon, d, L)$.

On the associated sequence $s_n(x)$, we have the following theorem, the proof of which is omitted.

**Theorem 2.3.** Let $p_n(x)$ be the basic sequence for $(t_\varepsilon, d, L)$, then $s_n(x)$ is an associated sequence for $(t_\varepsilon, d, L)$ if and only if there exists an invertible invariant operator $T$ such that

$$
Tp_n(x) = s_n(x) \quad (n = 0, 1, \ldots). \quad (2.7)
$$

### 3. On the Invariant Operators

The following theorem establishes a characteristic condition of an invariant operator $\tau$.

**Theorem 3.1.** A linear operator $\tau$ is invariant if and only if there exist a constant $y_0$ which is not equal to $0$ and a root of unity such that

$$
\tau \sigma_{y_0} = \sigma_{y_0} \tau.
$$

**Proof.** Obviously, we only need to prove the “if” part. Assume $y_0 \neq 0$ and $y_0$ is not a root of unity. Set $\tau x^n = s_m(x)$, a polynomial of degree $\leq m$. Then we have

$$
s_m(y_0 x) = \sigma_{y_0} s_m(x) = \sigma_{y_0} \tau x^n = \tau \sigma_{y_0} x^n = y_0^n \tau x^n = y_0^n s_m(x).
$$

Let

$$
s_m(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.
$$
It is easy to obtain
\[ a_i = 0 \quad (i = 0, 1, \ldots, m; i \neq n). \]

Hence
\[ \tau x^n = s_n(x) = a_n x^n = s_n(1) x^n. \]

Therefore, for any constant \( y \)
\[
\alpha y \tau x^n = \alpha y s_n(1) x^n = s_n(1) \alpha y x^n = y^n s_n(1) x^n
\]
\[ = y^n \tau x^n = \tau (y^n x^n) = \tau y x^n \quad (n = 0, 1, \ldots). \]

By linearity, we have for any constant \( y \),
\[ \alpha y \tau = \tau \alpha y. \]

Let \( \mathbb{C} \) be the set of all sequences of constants. Let \( C = \{c_0, c_1, c_2, \ldots\} \)
\( \in \mathbb{C}, D = \{d_0, d_1, d_2, \ldots\} \in \mathbb{C} \). Define the sum and product of \( C \) and \( D \) by
\[ C + D = \{c_0 + d_0, c_1 + d_1, c_2 + d_2, \ldots\} \]
and
\[ CD = \{c_0 d_0, c_1 d_1, c_2 d_2, \ldots\}. \]

Define the zero element \( 0 \), identity element \( I \), and negative element \( -C \) as
\[ 0 = \{0, 0, \ldots\}, \quad I = \{1, 1, \ldots\}, \]
and
\[ -C = \{-c_0, -c_1, -c_2, \ldots\}, \]
respectively.

If \( c_i \neq 0 \ (i = 0, 1, \ldots) \), define its reverse element
\[ C^{-1} = \{c_0^{-1}, c_1^{-1}, c_2^{-1}, \ldots\}. \]

From the above it is clear that the set \( \mathbb{C} \) is a ring.

Denote by \( \mathcal{T} \) the set of all invariant operators. From Theorem 3.1 we can deduce

**Theorem 3.2.** The set \( \mathcal{T} \) of all invariant operators is a ring. And the mapping
\[ \phi: \tau \in \mathcal{T} \mapsto A = \{a_0, a_1, \ldots\}, \]
where \( \tau \) satisfies \( \tau x^n = a_n x^n \ (n = 0, 1, \ldots) \), is an algebra isomorphism from the ring \( \mathcal{T} \) onto the ring \( \mathbb{C} \) of all sequences of constants.
4. ON THE REVERSE RESULT OF THE EXPANSION THEOREM

Let $L$ be a linear operator on polynomials. If $L$ satisfies the conditions

(i) for every polynomial $p_n(x)$ of degree $n$ exactly

$$\deg\{Lp_n(x)\} \leq n - 1,$$

(ii) $L1 = 0$,

then we say the operator $L$ has the property of decreasing degree, denoted by $L \in (D)$ (see [6]).

Roman [1] established the following expansion theorem and its corollary:

**Theorem R.** Let $\tau$ be invariant and suppose that $p_n(x)$ is basic for $(t_c, d, L)$. Then

$$\tau = \sum_{k=0}^{\infty} \frac{1}{d_k} \langle \epsilon_1 | \tau p_k(x) \rangle x^k t_c^k,$$

where $\epsilon_1$ is evaluation at $x = 1$.

**Corollary.** If $p_n(x)$ is basic for $(t_c, d, \epsilon_1)$ then

$$\sigma_y = \sum_{k=0}^{\infty} \frac{p_k(y)}{d_k} x^k t_c^k.$$

In the following, we will establish a partial reverse result of the expansion theorem of the invariant operator $\sigma_y$.

**Theorem 4.1.** Let $L \in (D)$. If for all $y$ the operator $\sigma_y$ can be expressed by a formal power series on the operator $L$,

$$\sigma_y = \sum_{k=0}^{\infty} \frac{a_k(\gamma)}{d_k} b_k(x) L^{k},$$

where $a_k(y), b_k(x)$ ($k = 0, 1, \ldots$) are sequences of functions on variables $y$ and $x$ with $b_k(1) = 1$, then we have

(i) $b_k(x) = x^k$ and $a_k(y)$ are polynomials of degree $k$ on variable $y$.

(ii) $L = t_c$ for some $c$.

(iii) $a_k(y)$ is basic for $(t_c, d, \epsilon_1)$.

In order to prove Theorem 4.1 we need some lemmas.
**Lemma 4.1.1.** Let $p_n(x)$ be basic for $(t_e, d, e)$. If there exists a constant $y_0$ and a sequence $a_k$ of constants satisfying
\[
p_n(xy_0) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} a_k x^k p_{n-k}(x) \quad (n = 0, 1, \ldots),
\] (4.2)
then
\[
a_k = p_k(y_0) \quad (k = 0, 1, \ldots). \tag{4.3}
\]

*Proof.* Since $p_0(x) = \langle \epsilon_1 | p_0(x) \rangle = d_0$, then $p_0(x) \neq 0$. Setting $n = 0$ in (4.2) yields
\[a_0 = d_0 = p_0(y_0).
\]
Now assume
\[a_k = p_k(y_0) \quad (k = 0, 1, \ldots, n - 1).
\]
Then from (4.2) we have
\[
p_n(xy_0) = \sum_{k=0}^{n-1} \frac{d_n}{d_k d_{n-k}} p_k(y_0) x^k p_{n-k}(x) + \frac{a_n}{d_0} x^n p_0(x). \tag{4.4}
\]
On the other hand, applying Theorem 2.5 of 1 yields
\[
p_n(xy_0) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} p_k(y_0) x^k p_{n-k}(x). \tag{4.5}
\]
Comparing (4.4) and (4.5) yields (4.3) for $k = n$. \qed

From the above lemma it is easy to obtain

**Lemma 4.1.2.** Let $p_n(x)$ be basic for $(t_e, d, e)$. If a polynomial sequence $a_k(x)$ satisfies
\[
p_n(xy) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} a_k(x) x^k p_{n-k}(x) \quad (n = 0, 1, \ldots),
\]
then
\[
a_k(y) = p_k(y) \quad (k = 0, 1, \ldots).
\]

*Proof of Theorem 4.1.* First, we prove $a_k(y)$ is a polynomial of degree $k$ on variable $y$. Applying the two sides of (4.1) to 1 yields
\[
1 = \alpha_1 = \frac{a_0(y)}{d_0} b_0(x).
\]
from which it follows that
\[ a_0(y) = d_0 \neq 0, \quad b_0(x) = 1. \]

Now applying the two sides of (4.1) to \( x \), we have
\[ xy = \sigma_y x = \frac{a_0(y)}{d_0} b_0(x) x + \frac{a_1(y)}{d_1} b_1(x) Lx = x + \frac{a_1(y)}{d_1} b_1(x) Lx. \]

From the above, we obtain \( Lx = \text{const.} \neq 0 \) and
\[ a_1(y) b_1(x) Lx = d_1 x(y - 1). \]

Hence, observing \( b_1(1) = 1 \), we must have
\[ b_1(x) = x \quad (4.6) \]

and \( a_1(y) \) is a polynomial of degree 1 on variable \( y \). Assume \( a_k(y) \) \((k = 0, 1, \ldots, n - 1)\) are polynomials of degree \( k \) on variable \( y \); then from (4.1)
\[ (xy)^n = \sum_{k=0}^{n-1} \frac{a_k(y)}{d_k} b_k(x) L^k x^n + \frac{a_n(y)}{d_n} b_n(x) L^n x^n. \quad (4.7) \]

Observing that
\[ \sum_{k=0}^{n-1} \frac{a_k(y)}{d_k} b_k(x) L^k x^n \]
is a polynomial of degree \( n - 1 \) on variable \( y \), and comparing the coefficient of \( y^n \) on the two sides of (4.7), we obtain \( a_n(y) \) is polynomial of degree \( n \). From (4.7) we also obtain
\[ Lx^n \neq 0 \quad (n = 1, 2, \ldots). \quad (4.8) \]

Now we turn to prove
\[ L \sigma_x = a \sigma_x L \text{ for all } a. \quad (4.9) \]

Since \( \sigma_x \sigma_u = \sigma_u \sigma_x \),
\[ \sigma_x \sigma_u x^n = \sum_{k=0}^{n-1} \frac{a_k(y)}{d_k} b_k(x) L^k \sigma_x x^n + \frac{a_n(y)}{d_n} b_n(x) L^n \sigma_x x^n \]
\[ = \sigma_u \sigma_x x^n = \sum_{k=0}^{n} \frac{a_k(y)}{d_k} \sigma_u b_k(x) L^k x^n. \]
Comparing the coefficient of $y^n$ yields
\[ b_n(x)L^n\sigma_y x^n = \sigma_y b_n(x)L^nx^n. \]

Therefore, we have
\[ \sum_{k=0}^{n-1} \frac{a_k(y)}{d_k} b_k(x)L^k\sigma_y x^n = \sum_{k=0}^{n-1} \frac{a_k(y)}{d_k} \sigma_y b_k(x)L^k x^n. \]

Using a similar method, we obtain
\[ b_1(x)L\sigma_y x^n = \sigma_y b_1(x)Lx^n. \]

Observing that (4.6), we have for all $a$
\[ xL\sigma_y x^n = a\sigma_y Lx^n \quad (n = 0, 1, \ldots). \]

From the above, (4.9) follows. From (4.8) and (4.9), using Theorem 2.1, we have for some admissible sequence $c$
\[ L = t_c. \]

Then
\[ \sigma_y = \sum_{k=0}^{\infty} \frac{a_k(y)}{d_k} b_k(x)t_c^k. \quad (4.10) \]

Now, by induction, it is easy to prove $b_k(x) = x^k$ ($k = 0, 1, \ldots$). And we can rewrite (4.10) as
\[ \sigma_y = \sum_{k=0}^{\infty} \frac{a_k(y)}{d_k} x^k t_c^k. \quad (4.11) \]

Now assume $p_n(x)$ is basic for $(t_c, d, e_1)$. From (4.11), we have
\[ p_n(xy) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} a_k(y)x^k p_{n-k}(x). \]

Applying Lemma 2.4.2 yields
\[ a_k(y) = p_k(y) \quad (k = 0, 1, \ldots, n). \]

As an application of Theorem 4.1, we have

**Corollary 4.1.1 [1]**. A polynomial sequence $p_n(x)$ is basic for $(t_c, d, e_1)$ for some $c$ if
\[ p_n(xy) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} p_k(y)x^k p_{n-k}(x) \quad (n = 0, 1, \ldots) \quad (4.12) \]

hold for all $x$ and $y$. 


5. OPEN PROBLEM

If we observe the theory of basic sequences or polynomial sequences of binomial type and Sheffer sequences in [2–5], the theorems on basic sequences and associated sequences for \((t_{c}, d, L)\) relative to invariant operators are almost parallel to those of basic sequences and Sheffer sequences for a delta operator \(Q\) relative to shift-invariant operators.

In [6], we established the following reverse result of the expansion theorem on shift operator \(E^{y_{0}} (y_{0} \neq 0)\).

**Theorem S** [6]. Suppose \(L \in (D)\) and \(y_{0} \neq 0\). If a shift operator \(E^{y_{0}}\) can be expressed by a formal power series on a linear operator \(L\),

\[ E^{y_{0}} = \sum_{k \geq 0} \frac{a_{k}}{k!} L^{k}, \]

where \(a_{k} (k = 0, 1, \ldots)\) is a sequence of constants, then \(L\) is a delta operator with a basic set \(p_{k}(x)\) and

\[ a_{k} = p_{k}(y_{0}). \]

Now we propose an open problem. If for one \(y = y_{0} (y_{0} \neq 0\) and \(y_{0}\) not a root of unity) (4.1) holds, which assertions can we deduce? In detail, we state the problem as follows:

**Problem.** Let \(L \in (D)\). If the operator \(\sigma_{y_{0}} (y_{0} \neq 0\) and \(y_{0}\) is not a root of unity) can be expressed by a formal power series on the operator \(L\),

\[ \sigma_{y_{0}} = \sum_{k=0}^{\infty} \frac{a_{k}}{d_{k}} x^{k} L^{k}, \]

where \(a_{k} (k = 0, 1, \ldots)\) is a sequence of constants, then

(i) \(L = t_{c}\) for some \(c\).

(ii) \(a_{k} = p_{k}(y_{0}) (k = 0, 1, \ldots)\), where \(p_{k}(x)\) is basic for \((t_{c}, d, \epsilon_{1})\).

The author guesses that this problem is positive.

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REFERENCES