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Models for Noncommuting Operators

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This paper develops a model theory for a pair of noncommuting operators. Using backward shift operators on a Fock space Rota's Theorem is generalized, i.e., it is shown 'that any two bounded operators on a Hilbert space are simultaneously similar to part of a pair of backward shift operators on a Fock space. These shift operators and the Fock space framework are also used to develop a dilation theory for two noncommuting operators.

1. INTRODUCTION

Rota [4] proved that any bounded operator A, on a Hilbert space $\mathscr{X}$, with spectral radius less than one, is similar to part of a backward shift operator. Another result along this line is given in [1, 3, 5]. It states that any contraction is unitarily equivalent to part of a co-isometry, i.e., if A on $\mathscr{X}$ is a contraction then $HA = (V \mid \mathscr{W})H$, where H is a unitary operator, from $\mathscr{X}$ onto $\mathscr{W}$, V is a co-isometry, and $\mathscr{W}$ is an invariant subspace for V. (An operator V on $\mathscr{X}$ is an isometry if $V^*V = I$, the identity on $\mathscr{X}$. A co-isometry is the adjoint of an isometry.) In this paper we generalize the above results to a pair of bounded operators, A, N on $\mathscr{X}$. First it is shown that A, N are simultaneously similar to part of two shift operators. Then this result is refined; if $A^*A + N^*N \leqslant I$, then it is shown that A, N are simultaneously unitarily equivalent to part of two co-isometries. We say that A, N are *simultaneously similar* to [*unitarily equivalent* to] *part of* R, T, if (1) R, T are operators on a Hilbert space $\mathscr{V}$, (2) there exists an invariant subspace $\mathscr{W}$, for both R and T, (3) there exists a similarity [unitary] transformation H mapping $\mathscr{X}$ onto $\mathscr{W}$ such that $HA = (R \mid \mathscr{W})H$ and $HN = (T \mid \mathscr{W})H$, respectively. It is emphasized that the same operator H is used to intertwine both A with $R \mid \mathscr{W}$ and N with $T \mid \mathscr{W}$. (Note: A, N is *simultaneously similar to* [*unitarily equivalent to*] R, T if (1) holds, and there exists a similarity [unitary] transformation H mapping $\mathscr{X}$ onto $\mathscr{V}$ such that $HA = RH$ and $HN = TH$, respectively.)

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Our model theory for noncommuting operators is motivated by problems arising in nonlinear systems [2]. It can also be viewed as a representation theory for an operator A perturbed by N. The models obtained are shift operators defined on a Fock space. The result is a generalization of the existing dilation theory for one operator [1, 3–5], a deeper understanding of how noncommuting operators interact, and a solution to certain problems in mathematical systems theory [2].

2. The Shift Operators S and E

In this section we introduce several different shift operators on a Fock space. These operators will be used to develop a model theory.

First some notation is established. Throughout, all spaces are Hilbert spaces, and A, N are bounded linear operators on $\mathscr{X}$. The adjoint of an operator A, is denoted by A^*, the open unit disc by D, and the (n-fold) unit polydisc by $D^n = D \times D \times \cdots \times D$. The Hardy space, $\mathscr{H}_n(\mathscr{X})$ is the space of all analytic functions, f in D^n with values in the Hilbert space $\mathscr{X}$, such that the Taylor coefficients are square summable. Each f in $\mathscr{H}_n(\mathscr{X})$ has a power series expansion given by

$$f(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i_1 \geqslant 0, \ldots, i_n \geqslant 0} f_{i_1, i_2, \ldots, i_n} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n}, \tag{2.1}$$

where the series converges uniformly in D^n, all $f_{i_1, i_2, \ldots, i_n}$ are elements in $\mathscr{X}$, and the norm is

$$\| f \|^2_{\mathscr{H}_n} \doteq \sum_{i_n \geqslant 0, \ldots, i_n \geqslant 0} \| f_{i_1, \ldots, i_n} \|^2_{\mathscr{X}}. \tag{2.2}$$

Clearly $\mathscr{H}_n(\mathscr{X})$ is a Hilbert space. The Fock space $\mathscr{F}_1(\mathscr{X})$ is the Hilbert space defined as the orthogonal direct sum of the $\mathscr{H}_n$'s:

$$\mathscr{F}_1(\mathscr{X}) \doteq \bigoplus_{n=1}^{\infty} \mathscr{H}_n(\mathscr{X}). \tag{2.3}$$

For convenience elements in $\mathscr{F}_1(\mathscr{X})$ are represented by two different notations: both $\oplus_1^\infty f_n$ and $\{f_1, f_2, ...\}$ represent the same element in $\mathscr{F}_1(\mathscr{X})$.

The *backward shift operator*, S_n mapping $\mathscr{H}_n(\mathscr{X})$ into $\mathscr{H}_n(\mathscr{X})$ is the linear operator defined by

$$S_n f(\lambda_1, ..., \lambda_n) \doteq \frac{1}{\lambda_1} [f(\lambda_1, ..., \lambda_n) - f(0, \lambda_2, \lambda_3, ..., \lambda_n)]. \tag{2.4}$$

Note the operator S_n only acts on the Taylor coefficients of λ_1 in the power series expansion of f. The *generalized backward shift operator* $S_{\mathscr{X}}$ mapping $\mathscr{F}_1(\mathscr{X})$ into $\mathscr{F}_1(\mathscr{X})$ is defined by:

$$S_{\mathscr{X}} \bigoplus_1^{\infty} f_n \doteq \bigoplus_1^{\infty} S_n f_n \qquad \left(\bigoplus_1^{\infty} f_n \in \mathscr{F}_1(\mathscr{X})\right). \tag{2.5}$$

The adjoint $S_{\mathscr{X}}$ is

$$S_{\mathscr{X}}^* \bigoplus_1^{\infty} f_n = \bigoplus_1^{\infty} \lambda_1 f_n \qquad \left(\bigoplus_1^{\infty} f_n \in \mathscr{F}_1(\mathscr{X})\right). \tag{2.6}$$

Clearly $S_{\mathscr{X}}^*$ is an isometry. Thus, $S_{\mathscr{X}}$ is a co-isometry.

The *evaluation operator*, E_n mapping $\mathscr{H}_n(\mathscr{X})$ into $\mathscr{H}_{n-1}(\mathscr{X})$ for $n \geqslant 1$ is given by

$$E_n f(\lambda_1, \lambda_2, ..., \lambda_{n-1}) \doteq f(0, \lambda_1, \lambda_2, ..., \lambda_{n-1}) \qquad (f \in \mathscr{H}_n(\mathscr{X})). \tag{2.7}$$

The E_n operator evaluates λ_1 at zero and relabels the complex variables $\lambda_i \to \lambda_{i-1}$. By convention $\mathscr{H}_0(\mathscr{X}) \doteq \mathscr{X}$. Thus $E_1 f(\lambda_1) = f(0) \in \mathscr{X}$ if $f \in \mathscr{H}_1(\mathscr{X})$. The *generalized evaluation* operator $E_{\mathscr{X}}$ mapping $\mathscr{F}_1(\mathscr{X})$ into $\mathscr{F}_1(\mathscr{X})$ is

$$\begin{aligned} E_{\mathscr{X}} \bigoplus_1^{\infty} f_n &\doteq \bigoplus_{n=1}^{\infty} E_{n+1} f_{n+1} \\ &\doteq \{f_2(0, \lambda_1), f_3(0, \lambda_1, \lambda_2), f_4(0, \lambda_1, \lambda_2, \lambda_3), f_5(0, \lambda_1, \lambda_2, \lambda_3, \lambda_4), ...\}, \end{aligned} \tag{2.8}$$

where $\bigoplus_1^{\infty} f_n = \{f_1(\lambda_1), f_2(\lambda_1, \lambda_2), ...\} \in \mathscr{F}_1(\mathscr{X})$. The adjoint of $E_{\mathscr{X}}$ is given by

$$E_{\mathscr{X}}^* \bigoplus_1^{\infty} f_n \doteq \{0, f_1(\lambda_2), f_2(\lambda_2, \lambda_3), f_4(\lambda_2, \lambda_3, \lambda_4), ...\}. \tag{2.9}$$

Clearly, $E_{\mathscr{X}}^*$ is an isometry and $E_{\mathscr{X}}$ is a co-isometry. The subscript $\mathscr{X}$ is dropped from S and E when the underlying space is understood.

The operators S and E are the models we use. It turns out that "any" pair of operators A, N are simultaneously similar to part of S and E; see Proposition 1. Furthermore, these operators have several interesting properties. The spectrum of S and E is the closed unit disc; the point spectrum of S and E is the open unit disc (Problem 67 of [3]). It is easy to verify that ran S^* is orthogonal to ran E^* (ran denotes the range). Furthermore, $\mathscr{F}_1(\mathscr{X})$ is the orthogonal direct sum of ran S^*, ran E^* and $\mathscr{X}$ (identifying $\mathscr{X}$ with the obvious subspace of $\mathscr{H}_1(\mathscr{X})$). Thus $S^*S + E^*E \leqslant I$. The dimension of $\mathscr{X}$ is called the *multiplicity* of S and E, in accordance with the usual definition for shift operators, since $\mathscr{X}$ is cyclic for the algebra generated by S^* and E^*.

Finally, we introduce an operator Φ. Let A, N be two bounded linear operators on $\mathscr{X}$ and let

$$F_i \doteq (I - \lambda_i A)^{-1} = \sum_{n=0}^{\infty} A^n \lambda_i^n. \tag{2.10}$$

Define a sequence of mappings $\Phi_i: \mathscr{X} \to \mathscr{H}_i(\mathscr{X})$ by $\Phi_1(\lambda_1) \doteq F_1$; $\Phi_2(\lambda_1, \lambda_2) \doteq F_2 N F_1$ and generally

$$\Phi_n(\lambda_1, \lambda_2, ..., \lambda_n) \doteq F_n N F_{n-1} N \cdots N F_1 = F_n N \Phi_{n-1} \qquad (n \geqslant 2). \tag{2.11}$$

In the case that

$$\sum_{i=1}^{\infty} \| \Phi_i x \|^2 \leqslant (\text{const.}) \, \|x\|^2 \qquad (\text{for all } x \in \mathscr{X}) \tag{2.12}$$

define the bounded linear operator Φ from $\mathscr{X}$ into $\mathscr{F}_1(\mathscr{X})$ by

$$\Phi x = \bigoplus_{n=1}^{\infty} \Phi_n x = \{\Phi_1 x, \Phi_2 x, \Phi_3 x, ...\}. \tag{2.13}$$

Throughout, A and N are fixed, and Φ always refers to the above transformation. Clearly $\|\Phi x\| \geqslant \|x\|$ for all $x \in \mathscr{X}$. Hence Φ (if defined) always has closed range and is a similarity transformation from $\mathscr{X}$ onto its range.

3. Simultaneous Similarity

Proposition 1. *Let A, N be operators on $\mathscr{X}$ such that* (2.12) *holds. Then A and N are simultaneously similar to part of S and E on $\mathscr{F}_1(\mathscr{X})$.*

Proof. First we show that $S\Phi = \Phi A$ and $E\Phi = \Phi N$. The former equality follows from $S_1 F_1 = F_1 A$ (see (2.10)) and

$$\begin{aligned} S\Phi &= \bigoplus_1^{\infty} S_n \Phi_n = \bigoplus_1^{\infty} F_n N F_{n-1} N \cdots N S_1 F_1 \\ &= \bigoplus_1^{\infty} F_n N F_{n-1} N \cdots N F_1 A \\ &= \bigoplus_1^{\infty} \Phi_n A = \Phi A. \end{aligned} \tag{3.1}$$

The other equality follows from $F_1(0) = I$ and

$$\begin{aligned} E\Phi &= \bigoplus_1^\infty E_{n+1}\Phi_{n+1} \\ &= \bigoplus_{n=1}^\infty E_{n+1}F_{n+1}NF_nN\cdots NF_1 \\ &= \bigoplus_1^\infty F_nNF_{n-1}N\cdots NF_1NF_1(0) = \Phi N. \end{aligned} \tag{3.2}$$

Since the ran Φ is invariant for both S and E, the proof is complete.

Clearly (2.12) does not hold for all A and N. For instance choose $A = I$ and $N = 0$. However, there always exists a $\varepsilon > 0$ such that the corresponding condition for εA and εN holds. Therefore our models S, E for A, N are perfectly general.

COROLLARY 1. *If A, N are bounded operators on $\mathscr{X}$ and $A^*A + N^*N \leqslant rI$, where $r < 1$ then A, N are simultaneously similar to part of S, E on $\mathscr{F}_1(\mathscr{X})$.*

Proof. We must verify that Φ is a bounded operator. Let P be any positive self-adjoint operator, and L be the transformation mapping positive operators into positive operators defined by $LP \doteq A^*PA + N^*PN$. Clearly $LI \leqslant rI$ and $LP \leqslant LQ$ if $P \leqslant Q$. Thus

$$L^nI = LL^{n-1}I \leqslant rL^{n-1}I \leqslant r^nI. \tag{3.3}$$

This implies that $\sum_{i=0}^n L^iI$ is an increasing sequence of positive operators bounded by $(1-r)^{-1}$. By Problem 94 of [3], this sequence has a limit

$$\begin{aligned} R = \sum_{i=0}^\infty L^iI &= I + A^*A + N^*N + A^{*2}A^2 + A^*N^*NA \\ &\quad + N^*A^*AN + N^{*2}N^2 + \cdots \end{aligned} \tag{3.4}$$

in the strong operator topology.

The expansion for $\|\Phi x\|^2$ is

$$\begin{aligned} \|\Phi x\|^2 = \|x\|^2 &+ \sum_{i=1}^\infty (A^{*i}A^ix, x) \\ &+ \sum_{i\geqslant 0, j\geqslant 0} (A^{*j}N^*A^{*i}A^iNA^jx, x) + \cdots . \end{aligned} \tag{3.5}$$

It is easy to show that (3.4) and (3.5) contain exactly the same terms, i.e., $(Rx, x) = \|\Phi x\|^2$ for all $x \in \mathscr{X}$. Since R is bounded

$$\|\Phi x\|^2 = \sum_{i=1}^{\infty} \|\Phi_i x\|^2 = (Rx, x) \leqslant M \|x\|^2 \tag{3.6}$$

and the proof is complete.

Equation (3.6) also proves

COROLLARY 2. *If* $A^*A + N^*N \leqslant rI$ *for some* $r < 1$, *then* (2.12) *holds.*

A converse to Corollary 2 is

COROLLARY 3. *Let* A, N *be bounded operators on* $\mathscr{X}$. *If* (2.12) *holds, then there exists a Hilbert norm* $\|\cdot\|_0$ *on* $\mathscr{X}$ *equivalent to* $\|\cdot\|$ *such that*

$$\|Ax\|_0^2 + \|Nx\|_0^2 \leqslant r \|x\|_0^2 \qquad (x \in \mathscr{X}) \tag{3.7}$$

for some $r < 1$.

Proof. The proof is omitted. It is almost identical to problem 122 in [3]; the other norm on $\mathscr{X}$ is defined by $\|x\|_0^2 \doteq \|\Phi x\|_{\mathscr{F}}^2$. ∎

Corollaries 2 and 3 show that (2.12) holds if and only if A, N are simultaneously similar to a pair A_0, N_0 such that for some $r < 1$ and all $x \in \mathscr{X}$,

$$\|A_0 x\|^2 + \|N_0 x\|^2 \leqslant r \|x\|^2.$$

If $N = 0$ the above reduces to the following standard result (Problem 122 in [3]): A on $\mathscr{X}$ is similar to a strict contraction if and only if

$$\sum_{i=0}^{\infty} \|A^i x\|^2 \leqslant M \|x\|^2 \qquad (x \in \mathscr{X}). \tag{3.8}$$

(T is a strict contraction if $\|T\| < 1$.) In other words, the spectral radius of A is strictly less than one if and only if (3.8) holds.

Remark. In Proposition 1 and Corollaries 2, 3 the condition (2.12) plays an important role. One can express this condition through a Lyapunov equation. We claim that (2.12) holds if and only if there exists a positive operator P such that $0 < P < \infty$ and

$$P - A^*PA - N^*PN = I \qquad (0 < P < \infty). \tag{3.9}$$

Assume (2.12). Then $P \doteq \Phi^*\Phi$ satisfies (3.9). This follows from the expansion of $\Phi^*\Phi$:

$$\Phi^*\Phi \doteq \sum_{i \geqslant 0} A^{*i}A + \sum_{j \geqslant 0, i \geqslant 0} A^{*j}N^*A^{*i}A^iNA^j + \sum_{k \geqslant 0, j \geqslant 0, i \geqslant 0} A^{*k}N^*A^{*j}N^*A^{*i}A^iNA^jNA^k + \cdots . \tag{3.10}$$

Assume P satisfies (3.9). Let $\mathscr{X}_0$ be the Hilbert space $\mathscr{X}$ equipped with the following inner product $(x, x)_0 \doteq (Px, x)$. Clearly $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms. Set $Q = A^*PA + N^*PN$. Using (3.9) and $P = I + Q$ a simple calculation gives

$$\begin{aligned} \frac{\|Ax\|_0^2 + \|Nx\|_0^2}{\|x\|_0^2} &= \frac{(A^*PA + N^*PNx, x)}{(Px, x)} \\ &= \frac{(Qx, x)}{(x, x) + (Qx, x)} \\ &= \frac{\dfrac{(Qx, x)}{(x, x)}}{1 + \dfrac{(Qx, x)}{(x, x)}} \leqslant \frac{\|Q\|}{1 + \|Q\|} < 1. \end{aligned} \tag{3.11}$$

Hence (3.7) holds. Corollary 2 gives (2.12). In many applications, obtaining a solution P to (3.9) is easier than proving that (2.12) holds. Finally, it is noted that the solution to (3.9) (if it exists) is unique.

4. Unitary Equivalence

If $N = 0$ then Proposition 1 reduces to Rota's Theorem [4]. Problem 121 of [3] is a refinement of Rota's Theorem. In our more general setting, this refinement becomes

Proposition 2. *Let A, N be bounded operators on $\mathscr{X}$, such that $A^*A + N^*N \leqslant I$, and let Φ_n for $n \geqslant 1$ be defined by (2.11). If $A^n \to 0$ strongly and $\|N\Phi_n x\|_{\mathscr{F}_n} \to 0$ for all $x \in \mathscr{X}$, as $n \to \infty$, then A, N are simultaneously unitarily equivalent to part of the shifts $S_{\mathscr{D}}$, $E_{\mathscr{D}}$ on $\mathscr{F}_1(\mathscr{D})$, for some closed linear subspace $\mathscr{D}$ of $\mathscr{X}$.*

The proof depends on the following

Lemma 1. *Let A and N be bounded operators on $\mathscr{X}$ such that*

$A^*A + N^*N \leqslant I$, and D be the positive square root of $I - A^*A - N^*N$. Let $\mathscr{D}$ be the closure of the range of D. Then

(i) *$A^{*n}A^n$ strongly converges to the positive operator A_∞^2, as $n \to \infty$.*

(ii) *For each $x \in \mathscr{X}$ the sequence $\|N\Phi_n x\|_{\mathscr{F}_n}$ is decreasing.*

(iii)

$$\sum_{n=1}^{\infty} \|D\Phi_n x\|^2 \leqslant \|x\|^2 \qquad (\textit{for all } x \in \mathscr{X})$$

so that the operator $D\Phi$ mapping $\mathscr{X}$ into $\mathscr{F}_1(\mathscr{D})$ defined by $D\Phi x = \oplus_1^\infty D\Phi_n x$, is well defined. In fact

$$\|D\Phi x\|_{\mathscr{F}_1}^2 + \|A_\infty x\|_{\mathscr{X}}^2 + \sum_1^\infty \|A_\infty N\Phi_n x\|_{\mathscr{F}_n}^2 + \lim_{n\to\infty} \|N\Phi_n x\|_{\mathscr{F}_n}^2 = \|x\|^2 \tag{4.1}$$

for all $x \in \mathscr{X}$.

Proof. Part (i) follows because A is a contraction, i.e., $A^{*n}A^n$ is a sequence of decreasing positive operators.

Consulting (2.11) gives

$$\begin{aligned}\|D\Phi_1 x\|_{\mathscr{F}_1}^2 &= \lim_{k\to\infty} \sum_{i=0}^{k} \|DA^i x\|_{\mathscr{X}}^2 \\ &= \lim_{k\to\infty} \sum_{i=0}^{k} (A^{*i}(I - A^*A - N^*N) A^i x, x) \quad (4.2)\\ &= \|x\|^2 - \lim_{k\to\infty} \|A^k x\|^2 - \|N\Phi_1 x\|_{\mathscr{F}_1}^2.\end{aligned}$$

Therefore,

$$\|D\Phi_1 x\|^2 = \|x\|^2 - \|A_\infty x\|^2 - \|N\Phi_1 x\|^2. \tag{4.3}$$

Following the same procedure on the general term $n > 1$ gives

$$\|D\Phi_n x\|^2 = \|N\Phi_{n-1} x\|^2 - \|A_\infty N\Phi_{n-1} x\|^2 - \|N\Phi_n x\|^2 \tag{4.4}$$

summing to n on (4.3), (4.4) and rearranging terms:

$$\begin{aligned}\sum_{i=1}^{n} \|D\Phi_i x\|^2 + \|A_\infty x\|^2 + \sum_{i=1}^{n-1} \|A_\infty N\Phi_i x\|^2 \\ = \|x\| - \|N\Phi_n x\|^2.\end{aligned} \tag{4.5}$$

Since the left-hand side is positive and increasing in n, the $\|N\Phi_n x\|^2$ are decreasing. Part (iii) follows by taking limits in (4.5).

Proof of Proposition 2. Let $D\Phi$ be the operator given in the lemma. Following (3.1), (3.2), it is easy to verify that

$$S_{\mathscr{D}} D\Phi = D\Phi A \qquad \text{and} \qquad E_{\mathscr{D}} D\Phi = D\Phi N. \tag{4.6}$$

The hypothesis of the Proposition and (4.1) guarantees that $D\Phi$ is an isometry. Since the ran $D\Phi$ is invariant under $S_{\mathscr{D}}$ and $E_{\mathscr{D}}$, the proof is complete.

COROLLARY 4. *If A, N are bounded operators on $\mathscr{X}$ and $A^*A + N^*N \leqslant rI$ where $r < 1$ then A nd N are simultaneously unitarily equivalent to part of S and E on $\mathscr{F}_1(\mathscr{X})$.*

Proof. We verify that the hypothesis of the proposition are satisfied. Clearly $A^n \to 0$. Equation (3.6) and Corollary 2 guarantees that $\|N\Phi_n x\|^2_{\mathscr{F}_n} \to 0$ as $n \to \infty$ for all $x \in \mathscr{X}$. Since $r < 1$ we have $\mathscr{D} = \mathscr{X}$ and the proof is complete.

By employing a trick found in [1, 5] the hypothesis $A^n \to 0$ and $N\Phi_n \to 0$ strongly as $n \to \infty$ in Proposition 2 are removed. This begins with

PROPOSITION 3. *Let A, N be operators on $\mathscr{X}$, and $A^*A + N^*N \leqslant I$. If $\|N\Phi_n x\| \to 0$ as $n \to \infty$ for all $x \in \mathscr{X}$, then A, N are simultaneously unitarily equivalent to part of a pair of co-isometries.*

Proof. Throughout the notation of Lemma 1 is used. Let $\mathscr{A}$ be the closure of the ran A_∞ and W the operator mapping $\mathscr{A}$ into $\mathscr{A}$ defined by $WA_\infty x \doteq A_\infty Ax$. It is easy to show that W is an isometry (see p. 51 of [1] or p. 39 of [5]). By Proposition (2.3), p. 6 of [5], W can be extended to a unitary operator S_0 on some larger Hilbert space $\mathscr{Y}$, i.e., $\mathscr{A}$ is a subspace of $\mathscr{Y}$ and $W = S_0 \mid \mathscr{A}$. Further, $S_0 A_\infty = A_\infty A$.

Let $\mathscr{F}_0(\mathscr{Y})$ be the following Fock space

$$\mathscr{F}_0(\mathscr{Y}) \doteq \bigoplus_{n=0}^{\infty} \mathscr{H}_n(\mathscr{Y}). \tag{4.7}$$

(Recall $\mathscr{H}_0(\mathscr{Y}) \doteq \mathscr{Y}$.) Define the co-isometry $S_{\mathscr{Y}}$ on $\mathscr{F}_0(\mathscr{Y})$ by

$$S_{\mathscr{Y}} \bigoplus_0^\infty f_n \doteq \bigoplus_0^\infty S_n f_n = \{S_0 f_0, S_1 f_1, S_2 f_2, \ldots\}, \tag{4.8}$$

where S_0 is the above unitary operator and S_n is the usual backward shift operator on $\mathscr{H}_n(\mathscr{Y})$ for $n \geqslant 1$, (see (2.4)). The co-isometry $E_{\mathscr{Y}}$ on $\mathscr{F}_0(\mathscr{Y})$ is defined by

$$E_{\mathscr{Y}} \bigoplus_0^\infty f_n \doteq \bigoplus_{n=0}^\infty E_{n+1} f_{n+1} = \{E_1 f_1, E_2 f_2, E_3 f_3, \ldots\}, \tag{4.9}$$

where E_n for $n \geqslant 1$ is the evaluation operator mapping $\mathscr{H}_n(\mathscr{Y})$ into $\mathscr{H}_{n-1}(\mathscr{Y})$, (see (2.7)).

Consider the operator Φ_0 mapping $\mathscr{X}$ into $\mathscr{F}_0(\mathscr{Y})$ defined by

$$\begin{aligned} \Phi_\infty x &\doteq A_\infty x \oplus A_\infty N\Phi x \\ &= \{A_\infty x, A_\infty N\Phi_1 x, A_\infty N\Phi_2 x,...\} \qquad (x \in \mathscr{X}). \end{aligned} \tag{4.10}$$

By following the calculations in (3.1), (3.2) with the definition of S_0 it is easy to verify that

$$S_{\mathscr{Y}}\Phi_\infty = \Phi_\infty A \qquad \text{and} \qquad E_{\mathscr{Y}}\Phi_\infty = \Phi_\infty N. \tag{4.11}$$

To complete the proof we combine the above with the proof of Proposition 2. Consider the operator $D\Phi \oplus \Phi_\infty$ mapping $\mathscr{X}$ into $\mathscr{F}_1(\mathscr{D}) \oplus \mathscr{F}_0(\mathscr{Y})$ defined by $D\Phi x \oplus \Phi_\infty x$ when $x \in \mathscr{X}$. This operator is an isometry, by (4.1) and (4.10). Clearly the operators $S_{\mathscr{D}} \oplus S_{\mathscr{Y}}$ and $E_{\mathscr{D}} \oplus E_{\mathscr{Y}}$ on $\mathscr{F}_1(\mathscr{D}) \oplus \mathscr{F}_0(\mathscr{Y})$ are co-isometries. Further (4.6), (4.11) give $(S_{\mathscr{D}} \oplus S_{\mathscr{Y}})$ $(D\Phi x \oplus \Phi_\infty x) = D\Phi Ax \oplus \Phi_\infty Ax$ and $(E_{\mathscr{D}} \oplus E_{\mathscr{Y}})$ $(D\Phi x \oplus \Phi_\infty x) = D\Phi Nx \oplus \Phi_\infty Nx$, where $x \in \mathscr{X}$. Since the range of $D\Phi \oplus \Phi_\infty$ is an invariant subspace for both $S_{\mathscr{D}} \oplus S_{\mathscr{Y}}$ and $E_{\mathscr{D}} \oplus E_{\mathscr{Y}}$ the proof is complete.

Finally we are ready to prove

PROPOSITION 4. *If A, N are operators on $\mathscr{X}$ such that $A^*A + N^*N \leqslant I$, then A, N are simultaneously unitarily equivalent to part of a pair of co-isometries.*

Proof. Since $\|N\Phi_n x\|^2$ is a decreasing sequence (see Lemma 1), there exists a positive operator P on $\mathscr{X}$ such that

$$(P^2x, x) = \|Px\|^2 = \lim_{n\to\infty} (\Phi_n^* N^* N\Phi_n x, x) = \lim_{n\to\infty} \|N\Phi_n x\|^2_{\mathscr{F}_n}$$

By the definitions (2.2) and (2.11) we have

$$\|Px\|^2 = \lim_{n\to\infty} \sum_{i_1\geqslant 0,\ldots,i_n\geqslant 0} \|NA^{i_n}NA^{i_{n-1}}N \cdots NA^{i_1}x\|^2_{\mathscr{X}} \tag{4.12}$$

and

$$\|Px\|^2_{\mathscr{X}} = \|PN\Phi_1 x\|^2_{\mathscr{F}_1} \tag{4.13}$$

for all $x \in \mathscr{X}$. Let $\mathscr{Z}$ be the closure of the range of P, and let Z be any unitary operator on the Fock space $\mathscr{F}_0(\mathscr{Z})$ such that

$$Z\{Px, 0, 0, 0,...\} = \{0, PN\Phi_1 x, 0, 0,...\}. \tag{4.14}$$

Such an operator exists by (4.13).

Let $S_{\mathscr{Z}}$ be the co-isometry on $\mathscr{F}_0(\mathscr{Z})$ defined by

$$S_{\mathscr{Z}} \bigoplus_0^\infty f_n \doteq \bigoplus_0^\infty S_n f_n \qquad \left(\bigoplus_0^\infty f_n \in \mathscr{F}_0(\mathscr{Z})\right), \tag{4.15}$$

where S_n is the backward shift operator in $\mathscr{H}_n(\mathscr{Z})$, for $n > 0$, and $S_0 \doteq I$. Define $\tilde{E}_{\mathscr{Z}}$ to be the co-isometry on $\mathscr{F}_0(Z)$ given by $\tilde{E}_{\mathscr{Z}} \doteq ZE_{\mathscr{Z}}$, where $E_{\mathscr{Z}}$ is the generalized evaluation operator, (replace $\mathscr{Y}$ by $\mathscr{Z}$ in (4.9)). Consider the operator $\Phi_{\mathscr{Z}}$ mapping $\mathscr{X}$ into $\mathscr{F}_0(\mathscr{Z})$ defined by

$$\Phi_{\mathscr{Z}} x \doteq \{0, PN\Phi_1 x, 0, 0, \ldots\} \qquad (x \in \mathscr{X}). \tag{4.16}$$

From (4.13)

$$\|\Phi_{\mathscr{Z}} x\|^2 = \lim_{n \to \infty} \|N\Phi_n x\|^2 \qquad (x \in \mathscr{X}). \tag{4.17}$$

Using (4.14), (2.4), (2.7) a simple calculation verifies that

$$S_{\mathscr{Z}} \Phi_{\mathscr{Z}} = \Phi_{\mathscr{Z}} A \qquad \text{and} \qquad \tilde{E}_{\mathscr{Z}} \Phi_{\mathscr{Z}} = \Phi_{\mathscr{Z}} N. \tag{4.18}$$

At this point the proof is exactly the same as Proposition 3 except one uses the co-isometries $S_{\mathscr{D}} \oplus S_{\mathscr{Y}} \oplus S_{\mathscr{Z}}$ and $E_{\mathscr{D}} \oplus E_{\mathscr{Y}} \oplus \tilde{E}_{\mathscr{Z}}$ on $\mathscr{F}_1(\mathscr{D}) \oplus \mathscr{F}_0(\mathscr{Y}) \oplus \mathscr{F}_0(\mathscr{Z})$, along with the isometry mapping $\mathscr{X}$ into $\mathscr{F}_1(\mathscr{D}) \oplus \mathscr{F}_0(\mathscr{Y}) \oplus \mathscr{F}_0(\mathscr{Z})$ defined by $D\Phi x \oplus \Phi_\infty x \oplus \Phi_{\mathscr{Z}} x$ where $x \in \mathscr{X}$. Note (4.1), (4.10) and (4.17) guarantee that the last operator preserves the norm. The intertwining property follows from (4.6), (4.11), and (4.18).

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