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Global well-posedness of the critical Burgers equation in critical Besov spaces

Changxing Miao^{a,*}, Gang Wu^b

^a Institute of Applied Physics and Computational Mathematics, PO Box 8009, Beijing 100088, PR China

^b The Graduate School of China Academy of Engineering Physics, PO Box 2101, Beijing 100088, PR China

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ABSTRACT

We make use of the method of modulus of continuity [A. Kiselev, F. Nazarov, R. Shterenberg, Blow up and regularity for fractal Burgers equation, *Dyn. Partial Differ. Equ.* 5 (2008) 211–240] and Fourier localization technique [H. Abidi, T. Hmidi, On the global well-posedness of the critical quasi-geostrophic equation, *SIAM J. Math. Anal.* 40 (1) (2008) 167–185] [H. Abidi, T. Hmidi, On the global well-posedness of the critical quasi-geostrophic equation, *SIAM J. Math. Anal.* 40 (1) (2008) 167–185] to prove the global well-posedness of the critical Burgers equation $\partial_t u + u\partial_x u + \Lambda u = 0$ in critical Besov spaces $\dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R})$ with $p \in [1, \infty)$, where $\Lambda = \sqrt{-\Delta}$.

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1. Introduction

We consider the Burgers equation with fractional dissipation in \mathbb{R} ,

$$\begin{cases} \partial_t u + u\partial_x u + \Lambda^\alpha u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $0 \leq \alpha \leq 2$ and the operator Λ^α is defined by Fourier transform

$$\mathcal{F}(\Lambda^\alpha u)(\xi) = |\xi|^\alpha \mathcal{F}u(\xi).$$

* Corresponding author.

E-mail addresses: miao_changxing@iapcm.ac.cn (C. Miao), wugangmaths@yahoo.com.cn (G. Wu).

The Burgers equation (1.1) with $\alpha = 0$ and $\alpha = 2$ has received an extensive amount of attention since the studies by Burgers in the 1940s. If $\alpha = 0$, the equation is perhaps the most basic example of a PDE evolution leading to shocks; if $\alpha = 2$, it provides an accessible model for studying the interaction between nonlinear and dissipative phenomena. Recently, in [14] for the periodic case authors give a complete study for general $\alpha \in [0, 2]$, see also [3,10,13,16]. In particular, for $\alpha = 1$, with help of the method of modulus of continuity they proved the global well-posedness of the equation in the critical Hilbert space $H^{\frac{1}{2}}(\mathbb{T}^1)$.

In this paper, we study the following critical case,

$$\begin{cases} \partial_t u + u \partial_x u + \Lambda u = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.2}$$

We use similar arguments as in [1]. Making use of Fourier localization technique and the method of modulus of continuity [14], we prove the global well-posedness of the critical Burgers equation (1.2) in critical Besov spaces $\dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R})$ with $p \in [1, \infty)$.

It is well known that $\dot{B}_{p,1}^{\frac{1}{p}}$ is the critical space under the scaling invariance. That is, if $u(x, t)$ is a solution of (1.2), then $u_\lambda(x, t) = u(\lambda x, \lambda t)$ is also a solution of the same equation and $\|u_\lambda(\cdot, t)\|_{\dot{B}_{p,1}^{\frac{1}{p}}} \approx \|u(\cdot, \lambda t)\|_{\dot{B}_{p,1}^{\frac{1}{p}}}$.

Now we give out our main results. The first main result is the following:

Theorem 1.1. *Let $u_0 \in \dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R})$ with $p \in [1, \infty)$, then the critical Burgers equation (1.2) has a unique global solution u such that*

$$u \in \mathcal{C}(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{1}{p}}) \cap L^1_{loc}(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{1}{p}+1}).$$

Remark 1.1. Because of the restriction of the smooth index s stemming from the *a priori* estimate for the transport–diffusion equation (see Theorem 1.2), we cannot get the result for the limit case $p = \infty$.

Remark 1.2. In [2], N. Alibaud constructed a unique weak solution to (1.1) for any $\alpha \leq 1$, but the author did not shown whether the solution is smooth. Our Theorem 1.1 ensures that the solution is smooth for $\alpha = 1$. In fact, since regularity is a local property, making use of the smoothing effect (4.4) and the embedding $\dot{B}_{p,1}^{\frac{1}{p}} \hookrightarrow L^\infty$, we can get that for any $t > 0$, $u(\cdot, t) \in C^\infty(\mathbb{R})$. Moreover, from Eq. (1.2), we find that

$$u(x, t) \in C^\infty(\mathbb{R} \times \mathbb{R}^+),$$

which implies that the solution is smooth in space and time.

Remark 1.3. The corresponding question for the quasi-geostrophic equation has been a focus of significant effort (see e.g. [1,5,6,9,15,20]) and the critical Q-G equation has been recently resolved in [15] for periodic case. Based on [15], Abidi and Hmidi in [1] and Dong and Du in [9] give the corresponding result for Cauchy problem of the critical Q-G equation in the framework of Besov space and Sobolev space, respectively. After the present paper is completed, Prof. J. Wu and H. Dong informed us that the authors in [10] gave the global well-posedness for the critical fractal Burgers equation in inhomogeneous space $H^{\frac{1}{2}}$ by similar argument in [9].

In order to prove this theorem, we first prove the local well-posedness which is the major part of this paper. Next we make use of the modulus of continuity [14] to get the global well-posedness. We mention that the property allowing us to remove the periodicity is the spatial decay of the solution.

The key to prove the local well-posedness is an optimal *a priori* estimate for the following transport–diffusion equation in \mathbb{R}^N :

$$\begin{cases} \partial_t u + v \cdot \nabla u + \nu \Delta^\alpha u = f, \\ u(x, 0) = u_0(x), \end{cases} \tag{TD}_{\nu, \alpha}$$

where v is a given vector field which needs not to be divergence free, u_0 is the initial data, f is a given external force term, $\nu \geq 0$ is a constant, $0 \leq \alpha \leq 2$. Our second main result is the following:

Theorem 1.2. *Let $1 \leq \rho_1 \leq \rho \leq \infty$, $1 \leq p \leq p_1 \leq \infty$ and $1 \leq r \leq \infty$. Let $s \in \mathbb{R}$ satisfy the following*

$$\begin{aligned} s &< 1 + \frac{N}{p_1} \quad \left(\text{or } s \leq 1 + \frac{N}{p_1} \text{ if } r = 1 \right), \\ s &> -N \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \left(\text{or } s > -1 - N \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \text{ if } \operatorname{div} v = 0 \right). \end{aligned}$$

There exists a constant $C > 0$ depending only on N, α, s, p, p_1 and r , such that for any smooth solution u of $(\text{TD})_{\nu, \alpha}$ with $\nu \geq 0$, we have the following *a priori* estimate:

$$\nu^{\frac{1}{\rho}} \|u\|_{L_T^\rho \dot{B}_{p,r}^{s+\frac{\alpha}{\rho}}} \leq C e^{CZ(T)} \left(\|u_0\|_{\dot{B}_{p,r}^s} + \nu^{\frac{1}{\rho_1}-1} \|f\|_{L_T^{\rho_1} \dot{B}_{p,r}^{s-\alpha+\frac{\alpha}{\rho_1}}} \right) \tag{1.3}$$

with $Z(T) := \int_0^T \|\nabla v(t)\|_{\dot{B}_{p_1, \infty}^{\frac{N}{p_1}} \cap L^\infty} dt$.

Besides if $u = v$, then for all $s > 0$ ($s > -1$ if $\operatorname{div} v = 0$), the estimate (1.3) holds with $Z(T) = \int_0^T \|\nabla v(t)\|_{L^\infty} dt$.

Remark 1.4. When $\alpha = 2$, the above *a priori* estimate has been proved by R. Danchin in [7], where the method used has been introduced by T. Hmidi in [11]. In this paper, we extend Danchin’s results to the general case $\alpha \in [0, 2]$. The proof’s key is the use of Lagrangian coordinates transformation together with an important commutator estimate.

The rest of this paper is arranged as follows:

In Section 2, we recall some definitions and properties about homogeneous Besov spaces, and we will also list some useful lemmas. In Section 3, we prove Theorem 1.2. In Section 4, we prove the local well-posedness. In Section 5, we give the blow-up criterion. In Section 6, we complete the proof of the global well-posedness.

Notation. Throughout the paper, C stands for a constant which may be different in each occurrence. We shall sometimes use the notation $A \lesssim B$ instead of $A \leq CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

Let us first recall the Littlewood–Paley theory. Let χ and φ be a couple of smooth radial functions valued in $[0, 1]$ such that χ is supported in the ball $\{ \xi \in \mathbb{R}^N \mid |\xi| \leq \frac{4}{3} \}$, φ is supported in the shell $\{ \xi \in \mathbb{R}^N \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}$ and

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N,$$

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Denoting $\varphi_q(\xi) = \varphi(2^{-q}\xi)$ and $h_q = \mathcal{F}^{-1}\varphi_q$, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_q u := \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x - y) \, dy, \quad \forall q \in \mathbb{Z}.$$

We can also define the following low-frequency cut-off:

$$\dot{S}_q u := \sum_{j \leq q-1} \dot{\Delta}_j u.$$

Definition 2.1. Let S'_h be the space of temperate distributions u such that

$$\lim_{q \rightarrow -\infty} \dot{S}_q u = 0, \quad \text{in } S'.$$

The formal equality

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u$$

holds in S'_h and is called the *homogeneous Littlewood–Paley decomposition*. It has nice properties of quasi-orthogonality:

$$\dot{\Delta}_{q'} \dot{\Delta}_q u \equiv 0 \quad \text{if } |q' - q| \geq 2 \quad \text{and} \quad \dot{\Delta}_{q'} (\dot{S}_{q-1} u \dot{\Delta}_q v) \equiv 0 \quad \text{if } |q' - q| \geq 5. \tag{2.1}$$

Let us now define the homogeneous Besov spaces:

Definition 2.2. For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $u \in S'_h$, we set

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty,$$

and

$$\|u\|_{\dot{B}^s_{p,\infty}} := \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p}.$$

We then define the *homogeneous Besov spaces* as

$$\dot{B}^s_{p,r} := \{u \in S'_h \mid \|u\|_{\dot{B}^s_{p,r}} < \infty\}.$$

The above definition does not depend on the choice of the couple (χ, φ) . Remark that if $s < \frac{N}{p}$ or $s = \frac{N}{p}$ and $r = 1$, then $\dot{B}_{p,r}^s$ is a Banach space.

We now recall some basic properties of the homogeneous Besov spaces.

Proposition 2.1. *The following properties hold true (cf. [17,18]):*

1. *Generalized derivatives: Let $\sigma \in \mathbb{R}$, then the operator Λ^σ is an isomorphism from $\dot{B}_{p,r}^s$ to $\dot{B}_{p,r}^{s-\sigma}$.*
2. *Sobolev embedding: If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $\dot{B}_{p_1,r_1}^s \hookrightarrow \dot{B}_{p_2,r_2}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}$.*
3. *If $(p, r) \in [1, \infty]^2$ and $s > 0$, there exists a positive constant $C = C(N, s)$ such that*

$$\|uv\|_{\dot{B}_{p,r}^s} \leq C(\|u\|_{L^\infty}\|v\|_{\dot{B}_{p,r}^s} + \|v\|_{L^\infty}\|u\|_{\dot{B}_{p,r}^s}).$$

In our next study we require two kinds of coupled space–time Besov spaces. The first one is defined by the following manner: for $T > 0$ and $\rho \in [1, \infty]$, we denote by $L_T^\rho \dot{B}_{p,r}^s$ the set of all tempered distribution u satisfying

$$\|u\|_{L_T^\rho \dot{B}_{p,r}^s} := \left\| \left(\sum_{q \in \mathbb{Z}} 2^{qs\rho} \|\dot{\Delta}_q u\|_{L^p}^r \right)^{\frac{1}{r}} \right\|_{L_T^\rho} < \infty.$$

The second mixed space is $\tilde{L}_T^\rho \dot{B}_{p,r}^s$ which is the set of all tempered distribution u satisfying

$$\|u\|_{\tilde{L}_T^\rho \dot{B}_{p,r}^s} := \left(\sum_{q \in \mathbb{Z}} 2^{qs\rho} \|\dot{\Delta}_q u\|_{L^p}^r \right)^{\frac{1}{r}} < \infty.$$

Let us remark that, by virtue of the Minkowski inequality, we have

$$\|u\|_{\tilde{L}_T^\rho \dot{B}_{p,r}^s} \leq \|u\|_{L_T^\rho \dot{B}_{p,r}^s} \quad \text{if } \rho \leq r,$$

and

$$\|u\|_{L_T^\rho \dot{B}_{p,r}^s} \leq \|u\|_{\tilde{L}_T^\rho \dot{B}_{p,r}^s} \quad \text{if } \rho \geq r.$$

Now we give some useful lemmas.

Lemma 2.1. (Cf. [12,19].) *Let ϕ be a smooth function supported in the shell $\{\xi \in \mathbb{R}^N \mid R_1 \leq |\xi| \leq R_2, 0 < R_1 < R_2\}$. There exist two positive constants κ and C depending only on ϕ such that for all $1 \leq p \leq \infty$, $\tau \geq 0$ and $\lambda > 0$, we have*

$$\|\phi(\lambda^{-1}D)e^{-\tau\Lambda^\alpha}u\|_{L^p} \leq Ce^{-\kappa\tau\lambda^\alpha} \|\phi(\lambda^{-1}D)u\|_{L^p}.$$

Lemma 2.2. (Cf. [7].) *Let v be a smooth vector field. Let ψ_t be the solution to*

$$\psi_t(x) = x + \int_0^t v(\tau, \psi_\tau(x)) \, d\tau.$$

Then for all $t \in \mathbb{R}^+$, the flow ψ_t is a C^1 diffeomorphism over \mathbb{R}^N and one has

$$\begin{aligned} \|\nabla\psi_t^{\pm 1}\|_{L^\infty} &\leq e^{V(t)}, \\ \|\nabla\psi_t^{\pm 1} - \text{Id}\|_{L^\infty} &\leq e^{V(t)} - 1, \\ \|\nabla^2\psi_t^{\pm 1}\|_{L^\infty} &\leq e^{V(t)} \int_0^t \|\nabla^2 v(\tau)\|_{L^\infty} e^{V(\tau)} d\tau, \end{aligned}$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$.

Lemma 2.3. (Cf. [4].) Let v be a given vector field belonging to $L^1_{loc}(\mathbb{R}^+; \text{Lip})$. For $q \in \mathbb{Z}$ we set $u_q := \dot{\Delta}_q u$ and denote by ψ_q the flow of the regularized vector field $\dot{S}_{q-1} v$. Then for $u \in \dot{B}^{\alpha,p}_\infty$ with $\alpha \in [0, 2)$ and $p \in [1, \infty]$ we have

$$\|\Lambda^\alpha(u_q \circ \psi_q) - (\Lambda^\alpha u_q) \circ \psi_q\|_{L^p} \leq C e^{CV(t)} V^{1-\frac{\alpha}{2}}(t) 2^{q\alpha} \|u_q\|_{L^p},$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$ and $C = C(\alpha, p) > 0$ is a constant.

Lemma 2.4. Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq p_1 \leq \infty$, $p_2 := (1/p - 1/p_1)^{-1}$. Let $R_q := (\dot{S}_{q-1} v - v) \cdot \nabla \dot{\Delta}_q u - [\dot{\Delta}_q, v \cdot \nabla]u$. There exists a constant $C = C(N, \sigma)$ such that

$$\begin{aligned} 2^{q\sigma} \|R_q\|_{L^p} &\leq C \left(\sum_{|q'-q| \leq 4} \|\dot{S}_{q'-1} \nabla v\|_{L^\infty} 2^{q'\sigma} \|\dot{\Delta}_{q'} u\|_{L^p} \right. \\ &\quad + \sum_{q' \geq q-3} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} 2^{q\sigma} \|\dot{\Delta}_q u\|_{L^p} \\ &\quad + \sum_{\substack{|q'-q| \leq 4 \\ q'' \leq q'-2}} 2^{(q-q')(\sigma-1-\frac{N}{p_1})} 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q''\sigma} \|\dot{\Delta}_{q''} u\|_{L^p} \\ &\quad + \sum_{\substack{q' \geq q-3 \\ |q''-q'| \leq 1}} 2^{(q-q')(\sigma+N \min(\frac{1}{p_1}, \frac{1}{p'})} \\ &\quad \times 2^{q' \frac{N}{p_1}} (2^{q-q'} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} + \|\dot{\Delta}_{q'} \text{div } v\|_{L^{p_1}}) 2^{q''\sigma} \|\dot{\Delta}_{q''} u\|_{L^p} \Big), \end{aligned}$$

and the third term in the right-hand side may be replaced by

$$C \sum_{|q'-q| \leq 4} 2^{q'(\sigma-1)} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} \|\dot{S}_{q'-1} \nabla u\|_{L^{p_2}}.$$

Besides if $u = v$, the following estimate holds true:

$$\begin{aligned} 2^{q\sigma} \|R_q\|_{L^p} &\leq C \left(\sum_{|q'-q| \leq 4} \|\dot{S}_{q'-1} \nabla u\|_{L^\infty} 2^{q'\sigma} \|\dot{\Delta}_{q'} u\|_{L^p} + \sum_{q' \geq q-3} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla u\|_{L^\infty} 2^{q\sigma} \|\dot{\Delta}_q u\|_{L^p} \right. \\ &\quad \left. + \sum_{\substack{q' \geq q-3 \\ |q''-q'| \leq 1}} 2^{(q-q')\sigma} (2^{q-q'} \|\dot{\Delta}_{q'} \nabla u\|_{L^\infty} + \|\dot{\Delta}_{q'} \text{div } u\|_{L^\infty}) 2^{q''\sigma} \|\dot{\Delta}_{q''} u\|_{L^p} \right). \end{aligned}$$

R. Danchin in [8] gave the proof for the nonhomogeneous case. For the convenience of the reader, we will give the proof for the homogeneous case in Appendix A.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Here we only prove the case $\alpha \in [0, 2)$ (for the case $\alpha = 2$, see [7]).

Let $u_q := \dot{\Delta}_q u$ and $f_q := \dot{\Delta}_q f$. Applying $\dot{\Delta}_q$ to (TD) $_{v,\alpha}$ yields

$$\partial_t u_q + \dot{S}_{q-1} v \cdot \nabla u_q + \nu \Lambda^\alpha u_q = f_q + R_q$$

with $R_q := (\dot{S}_{q-1} v - \nu) \cdot \nabla u_q - [\dot{\Delta}_q, \nu \cdot \nabla] u$.

Let ψ_q be the flow of the regularized vector field $\dot{S}_{q-1} v$. Denote $\bar{u}_q := u_q \circ \psi_q$, $\bar{f}_q := f_q \circ \psi_q$ and $\bar{R}_q := R_q \circ \psi_q$. Then we have

$$\partial_t \bar{u}_q + \nu \Lambda^\alpha \bar{u}_q = \bar{f}_q + \bar{R}_q + \nu G_q \tag{3.1}$$

with $G_q := \Lambda^\alpha (u_q \circ \psi_q) - (\Lambda^\alpha u_q) \circ \psi_q$.

Applying $\dot{\Delta}_j$ to (3.1) and using Lemma 2.1, we get

$$\begin{aligned} \|\dot{\Delta}_j \bar{u}_q(t)\|_{L^p} &\lesssim e^{-\kappa \nu t 2^{j\alpha}} \|\dot{\Delta}_j u_{0,q}\|_{L^p} + \int_0^t e^{-\kappa \nu (t-\tau) 2^{j\alpha}} \\ &\quad \times (\|\dot{\Delta}_j \bar{f}_q\|_{L^p} + \|\dot{\Delta}_j \bar{R}_q\|_{L^p} + \nu \|\dot{\Delta}_j G_q\|_{L^p}) \, d\tau. \end{aligned} \tag{3.2}$$

Now from Lemma 2.3 we have

$$\|\dot{\Delta}_j G_q(t)\|_{L^p} \leq C e^{CV(t)} V^{1-\frac{\alpha}{2}}(t) 2^{q\alpha} \|u_q\|_{L^p}. \tag{3.3}$$

According to Bernstein lemma and Lemma 2.2, we can get

$$\begin{aligned} \|\dot{\Delta}_j \bar{f}_q(t)\|_{L^p} &\lesssim 2^{-j} \|\nabla \dot{\Delta}_j \bar{f}_q\|_{L^p} \\ &\lesssim 2^{-j} \|(\nabla f_q) \circ \psi_q\|_{L^p} \|\nabla \psi_q\|_{L^\infty} \\ &\lesssim 2^{-j} \|\nabla f_q\|_{L^p} \|J_{\psi_q^{-1}}\|_{L^\infty}^{\frac{1}{p}} \|\nabla \psi_q\|_{L^\infty} \\ &\lesssim e^{CV(t)} 2^{q-j} \|f_q\|_{L^p}. \end{aligned} \tag{3.4}$$

Arguing similarly as in deriving (3.4), we obtain

$$\|\dot{\Delta}_j \bar{R}_q(t)\|_{L^p} \lesssim e^{CV(t)} 2^{q-j} \|R_q\|_{L^p}.$$

According to Lemma 2.4, we get

$$\|\dot{\Delta}_j \bar{R}_q(t)\|_{L^p} \lesssim e^{CV(t)} 2^{q-j} c_q(t) 2^{-qs} Z'(t) \|u(t)\|_{\dot{B}_{p,r}^s} \tag{3.5}$$

with $\|c_q(t)\|_{\ell^r} = 1$.

Plugging (3.3), (3.4) and (3.5) into (3.2), taking the L^p norm over $[0, t]$ and multiplying both sides by $\nu^{\frac{1}{p}} 2^{q(s+\frac{\alpha}{p})}$, we obtain

$$\begin{aligned}
 v^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} \|\dot{\Delta}_j \bar{u}_q\|_{L_t^\rho L^p} &\lesssim 2^{\frac{(q-j)\alpha}{\rho}} (1 - e^{-\kappa v \rho t 2^{j\alpha}})^{\frac{1}{\rho}} 2^{qs} \|\dot{\Delta}_j u_{0,q}\|_{L^p} \\
 &+ v^{-\frac{1}{\rho_1}} 2^{(q-j)(1+\frac{\alpha}{\rho}+\frac{\alpha}{\rho_1})} e^{CV(t)} 2^{q(s-\frac{\alpha}{\rho_1})} \|f_q\|_{L_t^{\rho_1} L^p} \\
 &+ v^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} 2^{(q-j)\alpha} e^{CV(t)} v^{1-\frac{\alpha}{2}}(t) \|u_q\|_{L_t^\rho L^p} \\
 &+ 2^{(q-j)(1+\frac{\alpha}{\rho})} \int_0^t c_q(\tau) Z'(\tau) e^{CV(\tau)} \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau. \tag{3.6}
 \end{aligned}$$

Let $M_0 \in \mathbb{Z}$ to be fixed hereafter. Decomposing

$$u_q = \dot{S}_{q-M_0} \bar{u}_q \circ \psi_q^{-1} + \sum_{j \geq q-M_0} \dot{\Delta}_j \bar{u}_q \circ \psi_q^{-1},$$

we have for all $t \in [0, T]$,

$$\|u_q\|_{L_t^\rho L^p} \leq e^{CV(t)} \left(\|\dot{S}_{q-M_0} \bar{u}_q\|_{L_t^\rho L^p} + \sum_{j \geq q-M_0} \|\dot{\Delta}_j \bar{u}_q\|_{L_t^\rho L^p} \right). \tag{3.7}$$

By Lemma A.1 in [7], we have

$$\|\dot{S}_{q-M_0} \bar{u}_q\|_{L^p} \lesssim \|J_{\psi_q^{-1}}\|_{L^\infty}^{\frac{1}{p}} (2^{-q} \|\nabla J_{\psi_q^{-1}}\|_{L^\infty} \|J_{\psi_q}\|_{L^\infty} + 2^{-M_0} \|\nabla \psi_q^{-1}\|_{L^\infty}) \|u_q\|_{L^p}.$$

This together with Lemma 2.2 and Bernstein lemma leads to

$$\|\dot{S}_{q-M_0} \bar{u}_q\|_{L_t^\rho L^p} \lesssim e^{CV(t)} (e^{CV(t)} - 1 + 2^{-M_0}) \|u_q\|_{L_t^\rho L^p}. \tag{3.8}$$

As $\dot{\Delta}_j u_{0,q} = 0$ for $|j - q| > 1$, from (3.6) we get

$$\begin{aligned}
 &\sum_{j \geq q-M_0} v^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} \|\dot{\Delta}_j \bar{u}_q\|_{L_t^\rho L^p} \\
 &\lesssim (1 - e^{-\kappa v \rho t 2^{q\alpha}})^{\frac{1}{\rho}} 2^{qs} \|u_{0,q}\|_{L^p} + v^{-\frac{1}{\rho_1}} 2^{M_0(1+\alpha)} e^{CV(t)} 2^{q(s-\frac{\alpha}{\rho_1})} \|f_q\|_{L_t^{\rho_1} L^p} \\
 &+ v^{\frac{1}{\rho}} 2^{M_0\alpha} e^{CV(t)} v^{1-\frac{\alpha}{2}}(t) 2^{q(s+\frac{\alpha}{\rho})} \|u_q\|_{L_t^\rho L^p} \\
 &+ 2^{M_0(1+\alpha)} \int_0^t c_q(\tau) Z'(\tau) e^{CV(\tau)} \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau. \tag{3.9}
 \end{aligned}$$

Plugging (3.8) and (3.9) into (3.7) yields that

$$\begin{aligned}
 v^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} \|u_q\|_{L_t^\rho L^p} &\leq C (1 - e^{-\kappa v \rho t 2^{q\alpha}})^{\frac{1}{\rho}} 2^{qs} \|u_{0,q}\|_{L^p} \\
 &+ C e^{CV(t)} \left(v^{-\frac{1}{\rho_1}} 2^{M_0(1+\alpha)} 2^{q(s-\frac{\alpha}{\rho_1})} \|f_q\|_{L_t^{\rho_1} L^p} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ (2^{-M_0} + 2^{M_0\alpha} \nu^{1-\frac{\alpha}{\rho}}(t)) \nu^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} \|u_q\|_{L_t^\rho L^p} \\
 &+ 2^{M_0(1+\alpha)} \int_0^t c_q(\tau) Z'(\tau) \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau.
 \end{aligned}$$

Choose M_0 to be the unique integer such that $2C2^{-M_0} \in (\frac{1}{8}, \frac{1}{4}]$ and T_1 to be the largest real number such that

$$T_1 \leq T \quad \text{and} \quad CV(T_1) \leq C_0 \quad \text{with} \quad C_0 = \min\left(\ln 2, \left(\frac{2^{-M_0\alpha}}{8C^{\frac{\alpha}{2}}}\right)^{\frac{2}{2-\alpha}}\right).$$

Thus for $t \in [0, T_1]$, there exists a constant C_1 such that

$$\begin{aligned}
 \nu^{\frac{1}{\rho}} 2^{q(s+\frac{\alpha}{\rho})} \|u_q\|_{L_t^\rho L^p} &\leq C_1 \left((1 - e^{-\kappa\nu\rho t 2^{q\alpha}})^{\frac{1}{\rho}} 2^{q\alpha} \|u_{0,q}\|_{L^p} \right. \\
 &\quad \left. + \nu^{-\frac{1}{\rho_1}} 2^{q(s-\frac{\alpha}{\rho_1})} \|f_q\|_{L_t^{\rho_1} L^p} + \int_0^t c_q(\tau) Z'(\tau) \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau \right).
 \end{aligned}$$

Taking ℓ^r norm yields

$$\nu^{\frac{1}{\rho}} \|u\|_{\tilde{L}_t^\rho \dot{B}_{p,r}^{s+\frac{\alpha}{\rho}}} \leq C_1 \left(\|u_0\|_{\dot{B}_{p,r}^s} + \nu^{-\frac{1}{\rho_1}} \|f\|_{\tilde{L}_t^{\rho_1} \dot{B}_{p,r}^{s-\frac{\alpha}{\rho_1}}} + \int_0^t Z'(\tau) \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau \right). \tag{3.10}$$

Splitting $[0, T]$ into m subintervals like as $[0, T_1], [T_1, T_2]$ and so on, such that

$$C \int_{T_k}^{T_{k+1}} \|\nabla v(t)\|_{L^\infty} dt \approx C_0.$$

Arguing similarly as in deriving (3.10), we get for all $t \in [T_k, T_{k+1}]$,

$$\nu^{\frac{1}{\rho}} \|u\|_{\tilde{L}_{[T_k,t]}^\rho \dot{B}_{p,r}^{s+\frac{\alpha}{\rho}}} \leq C_1 \left(\|u(T_k)\|_{\dot{B}_{p,r}^s} + \nu^{-\frac{1}{\rho_1}} \|f\|_{\tilde{L}_{[T_k,t]}^{\rho_1} \dot{B}_{p,r}^{s-\frac{\alpha}{\rho_1}}} + \int_{T_k}^t Z'(\tau) \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau \right).$$

By a standard induction argument, it can be shown that

$$\nu^{\frac{1}{\rho}} \|u\|_{\tilde{L}_t^\rho \dot{B}_{p,r}^{s+\frac{\alpha}{\rho}}} \leq C_1^{k+1} \left(\|u_0\|_{\dot{B}_{p,r}^s} + \nu^{-\frac{1}{\rho_1}} \|f\|_{\tilde{L}_t^{\rho_1} \dot{B}_{p,r}^{s-\frac{\alpha}{\rho_1}}} + \int_0^t Z'(\tau) \|u(\tau)\|_{\dot{B}_{p,r}^s} d\tau \right).$$

Since the number of such subintervals is $m \approx CV(T)C_0^{-1}$, one can readily conclude that up to a change of C ,

$$\nu^{\frac{1}{\rho}} \|u\|_{\tilde{L}_T^\rho \dot{B}_{p,r}^{s+\frac{\alpha}{\rho}}} \leq C e^{CV(T)} \left(\|u_0\|_{\dot{B}_{p,r}^s} + \nu^{-\frac{1}{\rho_1}} \|f\|_{\tilde{L}_T^{\rho_1} \dot{B}_{p,r}^{s-\frac{\alpha}{\rho_1}}} + \int_0^T Z'(t) \|u(t)\|_{\dot{B}_{p,r}^s} dt \right). \tag{3.11}$$

Of course, the above inequality is valid for all $\rho \in [\rho_1, \infty]$. Choosing first $\rho = \infty$ in (3.11) and applying Gronwall lemma leads to

$$\|u\|_{\tilde{L}_T^\infty \dot{B}_{p,r}^s} \leq C e^{CZ(T)} (\|u_0\|_{\dot{B}_{p,r}^s} + \nu^{\frac{1}{\rho_1}-1} \|f\|_{\tilde{L}_T^{\rho_1} \dot{B}_{p,r}^{s-\alpha+\frac{\alpha}{\rho_1}}}). \tag{3.12}$$

Now plugging (3.12) into (3.11) yields the desired estimate for general ρ . \square

4. Local well-posedness

In this section, we prove the following result:

Proposition 4.1. *Let $u_0 \in \dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R})$ with $p \in [1, \infty)$, then there exists $T > 0$ such that Eq. (1.2) has a unique solution u such that*

$$u \in \tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}} \cap L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}.$$

Besides for all $\beta \in \mathbb{R}^+$, we have $t^\beta u \in \tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+\beta}$.

Proof. We prove this proposition by making use of an iterative method.

Step 1. Approximation solution.

Let $u^0 := e^{-tA}u_0(x)$ and let u^{n+1} be the solution of the linear equation

$$\begin{cases} \partial_t u^{n+1} + u^n \partial_x u^{n+1} + Au^{n+1} = 0, \\ u^{n+1}(x, 0) = u_0(x). \end{cases}$$

Obviously $u^0 \in L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{1}{p}+1})$, thus according to Theorem 1.2, we have $\forall n \in \mathbb{N}$,

$$u^n \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{1}{p}}) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{1}{p}+1}).$$

Step 2. Uniform bounds.

Now we intend to obtain uniform bounds, with respect to the parameter n , for some $T > 0$ independent of n .

By Lebesgue theorem, there exist $T > 0$ and an absolute constant $\varepsilon_0 \in (0, \min\{\frac{1}{4C^2}, \frac{C_0}{2}\})$ such that

$$\sum_{q \in \mathbb{Z}} (1 - e^{-\kappa T 2^q})^{\frac{1}{2}} 2^{\frac{q}{p}} \|\dot{\Delta}_q u_0\|_{L^p} \leq \varepsilon_0. \tag{4.1}$$

By making use of Lemma 2.4 and similar arguments as in the proof of Theorem 1.2, for all $T > 0$ such that

$$\int_0^T \|u^n(\tau)\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}} d\tau \leq CC_0$$

and

$$\|u^n\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}} \leq 2C\varepsilon_0,$$

we have

$$\begin{aligned} & \|u^{n+1}\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}} + \|u^{n+1}\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}} \\ & \leq C \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa t 2^q})^{\frac{1}{2}} 2^{\frac{q}{p}} \|\dot{\Delta}_q u_0\|_{L^p} + C \|u^n\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}} \|u^{n+1}\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}}. \end{aligned}$$

Therefore, from the inequality $C \|u^n\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}} \leq 2C^2\varepsilon_0 \leq \frac{1}{2}$, we find that

$$\|u^{n+1}\|_{\tilde{L}_T^2 \dot{B}_{p,1}^{\frac{1}{p}+\frac{1}{2}}} + \|u^{n+1}\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}} \leq 2C\varepsilon_0 \leq CC_0. \tag{4.2}$$

On the other hand, by Theorem 1.2 and the Sobolev embedding $\dot{B}_{p,1}^{\frac{1}{p}} \hookrightarrow L^\infty$, we have

$$\|u^{n+1}\|_{\tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq C e^{C \int_0^T \|u^n(\tau)\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}} d\tau} \|u_0\|_{\dot{B}_{p,1}^{\frac{1}{p}}} \leq C \|u_0\|_{\dot{B}_{p,1}^{\frac{1}{p}}}.$$

Combining the above results, we have proved that the sequence $(u^n)_{n \in \mathbb{N}}$ is uniformly bounded in $\tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}} \cap L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}$.

Step 3. Strong convergence.

We first prove that $(u^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}$.

Let $(n, m) \in \mathbb{N}^2$, $n > m$ and $u^{n,m} := u^n - u^m$. One easily verifies that

$$\begin{cases} \partial_t u^{n+1,m+1} + u^n \partial_x u^{n+1,m+1} + \Lambda u^{n+1,m+1} = -u^{n,m} \partial_x u^{m+1}, \\ u^{n+1,m+1}(x, 0) = 0. \end{cases}$$

According to Theorem 1.2, we have

$$\|u^{n+1,m+1}\|_{\tilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq C e^{C \|u^n\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \int_0^T \|u^{n,m} \partial_x u^{m+1}(\tau)\|_{\dot{B}_{p,1}^{\frac{1}{p}}} d\tau. \tag{4.3}$$

By Proposition 2.1 and the embedding $\dot{B}_{p,1}^{\frac{1}{p}} \hookrightarrow L^\infty$, we have

$$\|u^{n,m} \partial_x u^{m+1}\|_{\dot{B}_{p,1}^{\frac{1}{p}}} \lesssim \|u^{n,m}\|_{\dot{B}_{p,1}^{\frac{1}{p}}} \|u^{m+1}\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}}.$$

Substituting this into (4.3) yields

$$\|u^{n+1,m+1}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq C \|u^{n,m}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} e^{C\|u^n\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \int_0^T \|u^{m+1}(\tau)\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}} d\tau.$$

By (4.1), we can choose ε_0 small enough such that

$$\|u^{n+1,m+1}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq \epsilon \|u^{n,m}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}}$$

with $\epsilon < 1$. Now we can get by induction

$$\|u^{n+1,m+1}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq \epsilon^{m+1} \|u^{n,0}\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq C\epsilon^{m+1} \|u_0\|_{\dot{B}_{p,1}^{\frac{1}{p}}}.$$

This implies that $(u^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}$. Thus there exists $u \in \widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}$ such that u^n converges strongly to u in $\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}$. Fatou lemma and (4.2) ensure that $u \in L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}$. Thus by passing to the limit into the approximation equation, we can get a solution to (1.2) in $\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}} \cap L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}$.

Step 4. Uniqueness.

Let u_1 and u_2 be two solutions of Eq. (1.2) with the same initial data and belonging to the space $\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}} \cap L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}$. Let $u_{1,2} := u_1 - u_2$, then we have

$$\begin{cases} \partial_t u_{1,2} + u_1 \partial_x u_{1,2} + \Lambda u_{1,2} = -u_{1,2} \partial_x u_2, \\ u_{1,2}(x, 0) = 0. \end{cases}$$

By similar arguments as in Step 3, we have

$$\|u_{1,2}\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \leq C e^{C\|u_1\|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \int_0^t \|u_{1,2}\|_{\widetilde{L}_\tau^\infty \dot{B}_{p,1}^{\frac{1}{p}}} \|u_2(\tau)\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}} d\tau.$$

Gronwall’s inequality ensures that $u_1 = u_2, \forall t \in [0, T]$.

Step 5. Smoothing effect.

We will prove that for all $\beta \in \mathbb{R}^+$, we have

$$\|t^\beta u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+\beta}} \leq C_\beta e^{C(\beta+1)\|u\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \|u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}}. \tag{4.4}$$

It is obvious that

$$\begin{cases} \partial_t(t^\beta u) + u \partial_x(t^\beta u) + \Lambda(t^\beta u) = \beta t^{\beta-1} u, \\ (t^\beta u)(x, 0) = 0. \end{cases}$$

When $\beta = 1$, by Theorem 1.2, we have

$$\|tu(t)\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+1}} \leq C e^{C\|u\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \|u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}}.$$

Suppose (4.4) is true for n , we will prove it for $n + 1$. Applying Theorem 1.2 to the equation of $t^{n+1}u$ yields that

$$\begin{aligned} \|t^{n+1}u(t)\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+n+1}} &\leq C(n+1)e^{C\|u\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \|t^n u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+n}} \\ &\leq C_n e^{C(n+2)\|u\|_{L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}}} \|u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}}}. \end{aligned}$$

For general $\beta \in \mathbb{R}^+$, obviously $[\beta] \leq \beta \leq [\beta] + 1$. Thus by the following interpolation

$$\|t^\beta u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}+\beta}} \lesssim \|t^{[\beta]}u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{[\beta]+1-\beta}} \|t^{[\beta]+1}u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{\beta-[\beta]}},$$

we can get the estimate for general $\beta \in \mathbb{R}^+$. \square

5. Blow-up criterion

In this section, we prove the following blow-up criterion:

Proposition 5.1. *Let T^* be the maximum local existence time of u in $\widetilde{L}_T^\infty \dot{B}_{p,1}^{\frac{1}{p}} \cap L_T^1 \dot{B}_{p,1}^{\frac{1}{p}+1}$. If $T^* < \infty$, then*

$$\int_0^{T^*} \|\partial_x u(t)\|_{L^\infty} dt = \infty.$$

Proof. Suppose $\int_0^{T^*} \|\partial_x u(t)\|_{L^\infty} dt$ be finite, then by Theorem 1.2, we have

$$\forall t \in [0, T^*), \quad \|u(t)\|_{\dot{B}_{p,1}^{\frac{1}{p}}} \leq M_{T^*} := C e^{C \int_0^{T^*} \|\partial_x u(t)\|_{L^\infty} dt} \|u_0\|_{\dot{B}_{p,1}^{\frac{1}{p}}} < \infty. \tag{5.1}$$

Let $\widetilde{T} > 0$ such that

$$\sum_{q \in \mathbb{Z}} (1 - e^{-\kappa \widetilde{T} 2^q})^{\frac{1}{2}} M_{T^*} \leq \varepsilon_0, \tag{5.2}$$

where ε_0 is the absolute constant emerged in the proof of Proposition 4.1. Now (5.1) and (5.2) imply that

$$\forall t \in [0, T^*), \quad \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa \widetilde{T} 2^q})^{\frac{1}{2}} 2^{\frac{q}{p}} \|\Delta_q u(t)\|_{L^p} \leq \varepsilon_0.$$

This together with the local existence theory ensures that, there exists a solution $\tilde{u}(t)$ on $[0, \tilde{T})$ to (1.2) with the initial datum $u(T^* - \tilde{T}/2)$. By uniqueness, $\tilde{u}(t) = u(t + T^* - \tilde{T}/2)$ on $[0, \tilde{T}/2)$ so that \tilde{u} extends the solution u beyond T^* . \square

6. Global well-posedness

In this section, making use of the method of modulus of continuity [14], with help of similar arguments as in [1], we give the proof of the global well-posedness.

Let T^* be the maximal existence time of the solution u to (1.2) in the space $\tilde{L}^\infty([0, T^*]; \dot{B}_{p,1}^{\frac{1}{p}}) \cap L^1_{loc}([0, T^*]; \dot{B}_{p,1}^{\frac{1}{p}+1})$. From Proposition 4.1, there exists $T_0 > 0$ such that

$$\forall t \in [0, T_0], \quad t \|\partial_x u(t)\|_{L^\infty} \leq C \|u^0\|_{\dot{B}_{p,1}^{\frac{1}{p}}}.$$

Let λ be a positive real number that will be fixed later and $T_1 \in (0, T_0)$. We define the set

$$I := \{T \in [T_1, T^*]; \forall t \in [T_1, T], \forall x \neq y \in \mathbb{R}, |u(x, t) - u(y, t)| < \omega_\lambda(|x - y|)\},$$

where $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is strictly increasing, concave, $\omega(0) = 0$, $\omega'(0) < +\infty$, $\lim_{\xi \rightarrow 0^+} \omega''(\xi) = -\infty$ and

$$\omega_\lambda(|x - y|) = \omega(\lambda|x - y|).$$

The function ω is a modulus of continuity chosen as in [14].

We first prove that T_1 belongs to I under suitable conditions over λ . Let C_0 be a large positive number such that

$$2\|u_0\|_{L^\infty} < \omega(C_0) < 3\|u_0\|_{L^\infty}. \tag{6.1}$$

Since ω is strictly increasing, then by maximum principle (refer [2]) we have

$$\lambda|x - y| \geq C_0 \implies |u(x, T_1) - u(y, T_1)| \leq 2\|u_0\|_{L^\infty} < \omega_\lambda(|x - y|).$$

On the other hand we have from Mean Value Theorem

$$|u(x, T_1) - u(y, T_1)| \leq |x - y| \|\partial_x u(T_1)\|_{L^\infty}.$$

Let $0 < \delta_0 < C_0$. Then by the concavity of ω we have

$$\lambda|x - y| \leq \delta_0 \implies \omega_\lambda(|x - y|) \geq \frac{\omega(\delta_0)}{\delta_0} \lambda|x - y|.$$

If we choose λ so that

$$\lambda > \frac{\delta_0}{\omega(\delta_0)} \|\partial_x u(T_1)\|_{L^\infty},$$

then we get

$$0 < \lambda|x - y| \leq \delta_0 \implies |u(x, T_1) - u(y, T_1)| < \omega_\lambda(|x - y|).$$

Let us now consider the case $\delta_0 \leq \lambda|x - y| \leq C_0$. By Mean Value Theorem and the increasing property of ω , we can get

$$|u(x, T_1) - u(y, T_1)| \leq \frac{C_0}{\lambda} \|\partial_x u(T_1)\|_{L^\infty} \quad \text{and} \quad \omega(\delta_0) \leq \omega_\lambda(|x - y|).$$

Choosing λ such that

$$\lambda > \frac{C_0}{\omega(\delta_0)} \|\partial_x u(T_1)\|_{L^\infty},$$

thus we get

$$\delta_0 \leq \lambda|x - y| \leq C_0 \quad \Rightarrow \quad |u(x, T_1) - u(y, T_1)| < \omega_\lambda(|x - y|).$$

All the preceding conditions over λ can be obtained if we take

$$\lambda = \frac{\omega^{-1}(3\|u_0\|_{L^\infty})}{2\|u_0\|_{L^\infty}} \|\partial_x u(T_1)\|_{L^\infty}. \tag{6.2}$$

From the construction, the set I is an interval of the form $[T_1, T_*)$. We have three possibilities which will be discussed separately.

Case 1. The first possibility is $T_* = T^*$. In this case we necessarily have $T^* = \infty$ because the Lipschitz norm of u does not blow up.

Case 2. The second possibility is $T_* \in I$ and we will show that is not possible.

Let C_0 satisfy (6.1), then for all $t \in [T_1, T^*)$, we have

$$\lambda|x - y| \geq C_0 \quad \Rightarrow \quad |u(x, t) - u(y, t)| < \omega_\lambda(|x - y|).$$

Since $\partial_x u(t)$ belongs to $\mathcal{C}((0, T^*]; \dot{B}_{p,1}^{\frac{1}{p}})$, then for $\varepsilon > 0$ there exist $\eta_0, R > 0$ such that $\forall t \in [T_*, T_* + \eta_0]$,

$$\|\partial_x u(t)\|_{L^\infty} \leq \|\partial_x u(T_*)\|_{L^\infty} + \frac{\varepsilon}{2} \quad \text{and} \quad \|\partial_x u(T_*)\|_{L^\infty(B_{(0,R)}^c)} \leq \frac{\varepsilon}{2},$$

where $B_{(0,R)}$ is the ball of radius R and with center the origin. Hence for $\lambda|x - y| \leq C_0$ and x or $y \in B_{(0,R+\frac{C_0}{\lambda})}^c$, we have for $\forall t \in [T_*, T_* + \eta_0]$

$$|u(x, t) - u(y, t)| \leq |x - y| \|\partial_x u(t)\|_{L^\infty(B_{(0,R)}^c)} \leq \varepsilon|x - y|.$$

On the other hand we have from the concavity of ω

$$\lambda|x - y| \leq C_0 \quad \Rightarrow \quad \frac{\omega(C_0)}{C_0} \lambda|x - y| \leq \omega_\lambda(|x - y|).$$

Thus if we take ε sufficiently small such that

$$\varepsilon < \frac{\omega(C_0)}{C_0} \lambda,$$

then we find that

$$\lambda|x - y| \leq C_0; \quad x \text{ or } y \in B^c_{(0, R + \frac{C_0}{\lambda})} \Rightarrow |u(x, t) - u(y, t)| < \omega_\lambda(|x - y|).$$

It remains to study the case where $x, y \in B_{(0, R + \frac{C_0}{\lambda})}$. Since $\|\partial_x^2 u(T_*)\|_{L^\infty}$ is finite (see Proposition 4.1) then we get for each $x \in \mathbb{R}$

$$|\partial_x u(x, T_*)| < \lambda \omega'(0).$$

From the continuity of $x \rightarrow |\partial_x u(x, T_*)|$ we obtain

$$\|\partial_x u(T_*)\|_{L^\infty(B_{(0, R + \frac{C_0}{\lambda})})} < \lambda \omega'(0).$$

Let $\delta_1 \ll 1$. By the continuity in time of the quantity $\|\partial_x u(t)\|_{L^\infty}$, there exists $\eta_1 > 0$ such that $\forall t \in [T_*, T_* + \eta_1]$

$$\|\partial_x u(t)\|_{L^\infty(B_{(0, R + \frac{C_0}{\lambda})})} < \lambda \frac{\omega(\delta_1)}{\delta_1}.$$

Therefore for $\lambda|x - y| \leq \delta_1$ and $x \neq y$ belonging together to $B_{(0, R + \frac{C_0}{\lambda})}$, we have for all $t \in [T_*, T_* + \eta_1]$

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq |x - y| \|\partial_x u(t)\|_{L^\infty(B_{(0, R + \frac{C_0}{\lambda})})} \\ &< \lambda|x - y| \frac{\omega(\delta_1)}{\delta_1} \\ &\leq \omega_\lambda(|x - y|). \end{aligned}$$

Now for the other case since

$$\forall x, y \in B_{(0, R + \frac{C_0}{\lambda})}, \quad \delta_1 \leq \lambda|x - y|; \quad |u(x, T_*) - u(y, T_*)| < \omega_\lambda(|x - y|),$$

then we get from a standard compactness argument the existence of $\eta_2 > 0$ such that for all $t \in [T_*, T_* + \eta_2]$

$$\forall x, y \in B_{(0, R + \frac{C_0}{\lambda})}, \quad \delta_1 \leq \lambda|x - y|; \quad |u(x, t) - u(y, t)| < \omega_\lambda(|x - y|).$$

Taking $\eta = \min(\eta_0, \eta_1, \eta_2)$, we obtain that $T_* + \eta \in I$ which contradicts the fact that T_* is maximal.

Case 3. The last possibility is that T_* does not belong to I . By the continuity in time of u , there exist $x \neq y$ such that

$$u(x, T_*) - u(y, T_*) = \omega_\lambda(\xi), \quad \text{with } \xi = |x - y|.$$

We will show that this scenario cannot occur and more precisely:

$$f'(T_*) < 0 \quad \text{where } f(t) := u(x, t) - u(y, t).$$

This is impossible since $f(t) \leq f(T_*)$, $\forall t \in [0, T_*]$.

The proof is the same as [14] and for the convenience of the reader we sketch out the proof. From the regularity of the solution we see that the equation can be defined in the classical manner and

$$f'(T_*) = u(y, T_*)\partial_x u(y, T_*) - u(x, T_*)\partial_x u(x, T_*) + \Lambda u(y, T_*) - \Lambda u(x, T_*).$$

From [14] we have

$$u(y, T_*)\partial_x u(y, T_*) - u(x, T_*)\partial_x u(x, T_*) \leq \omega_\lambda(\xi)\omega'_\lambda(\xi).$$

Again from [14]

$$\begin{aligned} \Lambda u(y, T_*) - \Lambda u(x, T_*) &\leq \frac{1}{\pi} \int_0^{\frac{\xi}{2}} \frac{\omega_\lambda(\xi + 2\eta) + \omega_\lambda(\xi - 2\eta) - 2\omega_\lambda(\xi)}{\eta^2} d\eta \\ &\quad + \frac{1}{\pi} \int_{\frac{\xi}{2}}^\infty \frac{\omega_\lambda(2\eta + \xi) - \omega_\lambda(2\eta - \xi) - 2\omega_\lambda(\xi)}{\eta^2} d\eta \\ &\leq \lambda J(\lambda\xi), \end{aligned}$$

where

$$\begin{aligned} J(\xi) &= \frac{1}{\pi} \int_0^{\frac{\xi}{2}} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta \\ &\quad + \frac{1}{\pi} \int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^2} d\eta. \end{aligned}$$

Thus we get

$$f'(T_*) \leq \lambda(\omega\omega' + J)(\lambda\xi).$$

Now, we choose the same function as in [14]

$$\omega(\xi) = \begin{cases} \frac{\xi}{1+4\pi\sqrt{\xi_0\xi}}, & \text{if } 0 \leq \xi \leq \xi_0, \\ C_{\xi_0} \log \xi, & \text{if } \xi \geq \xi_0, \end{cases}$$

here ξ_0 is a sufficiently large number and C_{ξ_0} is chosen to provide continuity of ω . It is shown in [14],

$$\forall \xi \neq 0, \quad \omega(\xi)\omega'(\xi) + J(\xi) < 0.$$

Thus we can get that $f'(T_*) < 0$.

Combining the above discussion, we conclude that $T^* = \infty$ and

$$\forall t \in [T_1, \infty), \quad \|\partial_x u\|_{L^\infty} \leq \lambda \omega'(0) = \lambda.$$

The value of λ is given by (6.2).

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Appendix A. Commutator estimate

In this appendix, we give the proof of Lemma 2.4. By Bony’s decomposition, we have

$$\begin{aligned} R_q &= (\dot{S}_{q-1} v - v) \cdot \nabla \dot{\Delta}_q u - [\dot{\Delta}_q, v \cdot \nabla] u \\ &= [\dot{T}_{v^j}, \dot{\Delta}_q] \partial_j u + \dot{T}_{\partial_j \dot{\Delta}_q u} v^j - \dot{\Delta}_q \dot{T}_{\partial_j u} v^j \\ &\quad + \{ \partial_j \dot{R}(v^j, \dot{\Delta}_q u) - \partial_j \dot{\Delta}_q \dot{R}(v^j, u) \} \\ &\quad + \{ \dot{\Delta}_q \dot{R}(\operatorname{div} v, u) - \dot{R}(\operatorname{div} v, \dot{\Delta}_q u) \} \\ &\quad + (\dot{S}_{q-1} v - v) \cdot \nabla \dot{\Delta}_q u \\ &=: R_q^1 + R_q^2 + R_q^3 + R_q^4 + R_q^5 + R_q^6. \end{aligned} \tag{A.1}$$

Above, the summation convention over repeated indices has been used. The notation \dot{T} stands for homogeneous Bony’s paraproduct which is defined by

$$\dot{T}_f g := \sum_{q' \in \mathbb{Z}} \dot{S}_{q'-1} f \dot{\Delta}_{q'} g,$$

and \dot{R} stands for the remainder operator defined by

$$\dot{R}(f, g) := \sum_{q' \in \mathbb{Z}} \dot{\Delta}_{q'} f (\dot{\Delta}_{q'-1} g + \dot{\Delta}_{q'} g + \dot{\Delta}_{q'+1} g).$$

Note that

$$\|\dot{\Delta}_{q'} \nabla v\|_{L^a} \approx 2^{q'} \|\dot{\Delta}_{q'} v\|_{L^a}, \quad \forall a \in [1, \infty], \quad q' \in \mathbb{Z}. \tag{A.2}$$

Now let us estimate each term in (A.1).

Bounds for $2^{q\sigma} \|R_q^1\|_{L^p}$. By (2.1) and the definition of $\dot{\Delta}_q$, we have

$$\begin{aligned} R_q^1 &= \sum_{|q'-q| \leq 4} [\dot{S}_{q'-1} v^j, \dot{\Delta}_q] \partial_j \dot{\Delta}_{q'} u \\ &= \sum_{|q'-q| \leq 4} \int_{\mathbb{R}^N} h(y) [\dot{S}_{q'-1} v^j(x) - \dot{S}_{q'-1} v^j(x - 2^{-q} y)] \partial_j \dot{\Delta}_{q'} u(x - 2^{-q} y) dy. \end{aligned} \tag{A.3}$$

Applying Mean Value Theorem and Young's inequality to (A.3) yields

$$2^{q\sigma} \|R_q^1\|_{L^p} \leq C \sum_{|q'-q|\leq 4} \|\dot{S}_{q'-1} \nabla v\|_{L^\infty} 2^{q'\sigma} \|\dot{\Delta}_{q'} u\|_{L^p}. \tag{A.4}$$

Bounds for $2^{q\sigma} \|R_q^2\|_{L^p}$. According to (2.1), we have

$$R_q^2 = \sum_{q' \geq q-3} \dot{S}_{q'-1} \partial_j \dot{\Delta}_q u \dot{\Delta}_{q'} v^j.$$

By (A.2), we can get

$$\begin{aligned} 2^{q\sigma} \|R_q^2\|_{L^p} &\leq C \sum_{q' \geq q-3} 2^{q\sigma} \|\dot{\Delta}_{q'} v^j\|_{L^\infty} \|\dot{S}_{q'-1} \partial_j \dot{\Delta}_q u\|_{L^p} \\ &\leq C \sum_{q' \geq q-3} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} 2^{q\sigma} \|\dot{\Delta}_q u\|_{L^p}. \end{aligned} \tag{A.5}$$

Bounds for $2^{q\sigma} \|R_q^3\|_{L^p}$. Again from (2.1), we have

$$R_q^3 = - \sum_{|q'-q|\leq 4} \dot{\Delta}_q (\dot{S}_{q'-1} \partial_j u \dot{\Delta}_{q'} v^j) = - \sum_{\substack{|q'-q|\leq 4 \\ q'' \leq q'-2}} \dot{\Delta}_q (\dot{\Delta}_{q''} \partial_j u \dot{\Delta}_{q'} v^j). \tag{A.6}$$

Therefore, denoting $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$ and taking advantage of (A.2), we can obtain

$$\begin{aligned} 2^{q\sigma} \|R_q^3\|_{L^p} &\leq C \sum_{\substack{|q'-q|\leq 4 \\ q'' \leq q'-2}} 2^{q\sigma} \|\dot{\Delta}_{q'} v^j\|_{L^{p_1}} \|\dot{\Delta}_{q''} \partial_j u\|_{L^{p_2}} \\ &\leq C \sum_{\substack{|q'-q|\leq 4 \\ q'' \leq q'-2}} 2^{(q-q'')(\sigma-1-\frac{N}{p_1})} 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q''\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}. \end{aligned} \tag{A.7}$$

Note that, starting from the first equality of (A.6), one can alternately get

$$\begin{aligned} 2^{q\sigma} \|R_q^3\|_{L^p} &\leq C \sum_{|q'-q|\leq 4} 2^{q\sigma} \|\dot{\Delta}_{q'} v^j\|_{L^{p_1}} \|\dot{S}_{q'-1} \partial_j u\|_{L^{p_2}} \\ &\leq C \sum_{|q'-q|\leq 4} 2^{q'(\sigma-1)} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} \|\dot{S}_{q'-1} \partial_j u\|_{L^{p_2}}. \end{aligned} \tag{A.8}$$

Bounds for $2^{q\sigma} \|R_q^4\|_{L^p}$.

$$\begin{aligned} R_q^4 &= \sum_{\substack{|q'-q|\leq 2 \\ |q''-q'|\leq 1}} \partial_j (\dot{\Delta}_{q'} v^j \dot{\Delta}_q \dot{\Delta}_{q''} u) - \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} \partial_j \dot{\Delta}_q (\dot{\Delta}_{q'} v^j \dot{\Delta}_{q''} u) \\ &=: R_q^{4,1} + R_q^{4,2}. \end{aligned}$$

By (A.2), we can get

$$\begin{aligned}
 2^{q\sigma} \|R_q^{4,1}\|_{L^p} &\leq C \sum_{\substack{|q'-q|\leq 2 \\ |q''-q'|\leq 1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p} \\
 &\leq C \sum_{\substack{|q'-q|\leq 2 \\ |q''-q'|\leq 1}} 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}.
 \end{aligned} \tag{A.9}$$

For $R_q^{4,2}$, we proceed differently according to the value of $\frac{1}{p} + \frac{1}{p_1}$. If $\frac{1}{p} + \frac{1}{p_1} \leq 1$, we denote $\frac{1}{p_3} := \frac{1}{p} + \frac{1}{p_1}$ and have

$$\begin{aligned}
 2^{q\sigma} \|R_q^{4,2}\|_{L^p} &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{q(1+\sigma)} 2^{q(\frac{N}{p_3} - \frac{N}{p})} \|\dot{\Delta}_{q'} v \dot{\Delta}_{q''} u\|_{L^{p_3}} \\
 &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{q(1+\sigma)} 2^{q \frac{N}{p_1}} \|\dot{\Delta}_{q'} v\|_{L^{p_1}} \|\dot{\Delta}_{q''} u\|_{L^p} \\
 &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{(q-q')(1+\sigma + \frac{N}{p_1})} 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}.
 \end{aligned} \tag{A.10}$$

If $\frac{1}{p} + \frac{1}{p_1} > 1$, taking $p_1 = p'$ in the above computations yields

$$\begin{aligned}
 2^{q\sigma} \|R_q^{4,2}\|_{L^p} &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{q(1+\sigma)} 2^{q \frac{N}{p'}} \|\dot{\Delta}_{q'} v \dot{\Delta}_{q''} u\|_{L^1} \\
 &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{q(1+\sigma)} 2^{q \frac{N}{p'}} \|\dot{\Delta}_{q'} v\|_{L^{p'}} \|\dot{\Delta}_{q''} u\|_{L^p} \\
 &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{(q-q')(1+\sigma + \frac{N}{p'})} 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}.
 \end{aligned} \tag{A.11}$$

Putting (A.9), (A.10) and (A.11) together, we obtain

$$\begin{aligned}
 2^{q\sigma} \|R_q^4\|_{L^p} &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{(q-q')(1+\sigma + N \min(\frac{1}{p_1} + \frac{1}{p'}))} \\
 &\quad \times 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \nabla v\|_{L^{p_1}} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}.
 \end{aligned} \tag{A.12}$$

Bounds for $2^{q\sigma} \|R_q^5\|_{L^p}$. Similar computations yield

$$\begin{aligned}
 2^{q\sigma} \|R_q^5\|_{L^p} &\leq C \sum_{\substack{q' \geq q-3 \\ |q''-q'|\leq 1}} 2^{(q-q')(\sigma + N \min(\frac{1}{p_1} + \frac{1}{p'}))} \\
 &\quad \times 2^{q' \frac{N}{p_1}} \|\dot{\Delta}_{q'} \operatorname{div} v\|_{L^{p_1}} 2^{q'\sigma} \|\dot{\Delta}_{q''} u\|_{L^p}.
 \end{aligned} \tag{A.13}$$

Bounds for $2^{q\sigma} \|R_q^6\|_{L^p}$.

$$R_q^6 = - \sum_{q' \geq q-1} \dot{\Delta}_{q'} v \cdot \nabla \dot{\Delta}_q u,$$

thus by Bernstein lemma, we have

$$2^{q\sigma} \|R_q^6\|_{L^p} \leq C \sum_{q' \geq q-1} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} 2^{q\sigma} \|\dot{\Delta}_q u\|_{L^p}. \quad (\text{A.14})$$

Combining inequalities (A.4), (A.5), (A.7) or (A.8), (A.12), (A.13) and (A.14), we end up with the desired estimate for R_q .

Straightforward modifications in the estimates for R_q^3 , R_q^4 and R_q^5 leads to the desired estimate in the special case where $u = v$.

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