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On some new contiguous relations for the Gauss hypergeometric function with applications

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ABSTRACT

Contiguous relations for hypergeometric series contain an enormous amount of hidden information. Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series. In this paper, a new set of contiguous function relations are established. Applications of such relations to hypergeometric summation formulas and the theory of Jacobi polynomials are presented.

1. Introduction and preliminaries

The major development of the theory of hypergeometric function was carried out by Gauss and published in his famous work of 1812. This work is also noted as being the real beginning of rigour in Mathematics. Some important results concerning the hypergeometric function have been developed earlier by Euler and others, but it was Gauss who made the first systematic study of the series that defines this function.

Hypergeometric series are very important in Mathematics. Almost all of the elementary functions of Mathematics are either hypergeometric, ratios of hypergeometric functions or limiting cases of a hypergeometric series.

The study of hypergeometric series was essentially started in 1812 by Gauss when he considered the infinite series

$$1+\frac{ab}{1!.c}z+\frac{a(a+1)b(b+1)}{2!.c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!.c(c+1)(c+2)}z^3+\cdots \tag{1.1}$$

as a function of a, b, c and z, where it is assumed that c cannot be zero or a negative integer, so that no zero factor appears in the denominators of the terms of the series.

By the hypergeometric series (1.1), is meant the power series

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{1.2}$$

where z is the complex variable, a, b and c are parameters which can take arbitrary real or complex values (provided that $c\neq 0,-1,-2,\ldots$), and the symbol $(\lambda)_n$ denotes the Pochhammer symbol or the shifted factorial defined as

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$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)\dots(a+n-1), & n = 1, 2, 3, \dots. \end{cases} \tag{1.3}$$

The sum of this series is called the hypergeometric function and denoted by ${}_2F_1(a, b; c; z)$, [1, Def. 2.1.5, Page 64]. One of the most important properties of the hypergeometric function is that terms of the series do not change if the numerator parameters a and b are permuted, we obtain *symmetry property*

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z). \tag{1.4}$$

Two hypergeometric functions with the same argument z are *contiguous* if their parameters a, b and c differ by integers. Gauss derived analogous relations between ${}_2F_1[a, b; c; z]$ and any two contiguous hypergeometrics in which a parameter has been changed by ± 1. Rainville [2] generalized this to cases with more parameters.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series, they can be used to evaluate a hypergeometric function that is contiguous to a hypergeometric series which can be satisfactorily evaluated. Contiguous relations are also used to make a correspondence between Lie algebras and special functions. The correspondence yields formulas of special functions [3].

Gauss [4] defined as contiguous to ${}_2F_1(a, b; c; z)$ or simply $F(a, b; c; z)$ each of the six functions obtained by increasing or decreasing one of the parameters by unity. He also proved that between F and any two of its contiguous functions, there exists a linear relation with coefficients at most linear and obtained the following fifteen interesting and useful results. Notice that, F is shorthand for ${}_2F_1(a, b; c; z)$, $F(a^{\pm})$, $F(b^{\pm})$ and $F(c^{\pm})$ represent the close neighbors of F, ${}_2F_1(a \pm 1, b; c; z)$, ${}_2F_1(a, b \pm 1; c; z)$ and ${}_2F_1(a, b; c \pm 1; z)$ respectively.

$$(a - c + 1)F = aF(a^+) - (c - 1)F(c^-) \tag{1.5}$$

$$(1 - z)F = F(a^-) - c^{-1}(c - b)zF(c^+) \tag{1.6}$$

$$(a + b - c)F = (b - c)F(b^-) + a(1 - z)F(a^+) \tag{1.7}$$

$$[2a - c + (b - a)z]F = a(1 - z)F(a^+) - (c - a)F(a^-) \tag{1.8}$$

$$[1 - a + (c - b - 1)z]F = (c - a)F(a^-) - (c - 1)(1 - z)F(c^-) \tag{1.9}$$

$$[b + (a - c)z]F = b(1 - z)F(b^+) - c^{-1}(c - a)(c - b)zF(c^+) \tag{1.10}$$

$$(b - c + 1)F = bF(b^+) - (c - 1)F(c^-) \tag{1.11}$$

$$(1 - z)F = F(b^-) - c^{-1}(c - a)zF(c^+) \tag{1.12}$$

$$(a + b - c)F = (a - c)F(a^-) + b(1 - z)F(b^+) \tag{1.13}$$

$$[2b - c + (a - b)z]F = b(1 - z)F(b^+) - (c - b)F(b^-) \tag{1.14}$$

$$[1 - b + (c - a - 1)z]F = (c - b)F(b^-) - (c - 1)(1 - z)F(c^-) \tag{1.15}$$

$$[a + (b - c)z]F = a(1 - z)F(a^+) - c^{-1}(c - a)(c - b)zF(c^+) \tag{1.16}$$

$$(a - b)F = aF(a^+) - bF(b^+) \tag{1.17}$$

$$(b - a)(1 - z)F = (c - a)F(a^-) - (c - b)F(b^-) \tag{1.18}$$

$$[c - 1 + (a + b + 1 - 2c)z]F = (c - 1)(1 - z)F(c^-) - c^{-1}(c - a)(c - b)zF(c^+). \tag{1.19}$$

In these relations, two hypergeometric series differ just in one parameter from the third hypergeometric series, and the difference is 1. A contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations.

Although the 15 Gauss relations can be proved by the expansion of the various power series in z, by direct substitution of the series (1.2) such as in (1.5)–(1.7). Some other Gauss relations can be obtained as an immediate consequence of (1.5)–(1.7) together with the symmetry property (1.4) such as (1.11)–(1.13).

In a series of four research papers, Rakha et al. [5–8] have obtained some very interesting results regarding contiguous function relations and their computations. In [5], some interesting consequences of the contiguous relations of ${}_2F_1$ were proved, while in [6], a new method of the shifted operators for computing the contiguous relations of ${}_2F_1$ are introduced. In [7], a general form of the relation between three Gauss functions has been established and an implementation with the help of the computer algebra system *Mathematica*, to verify such a relation is presented. In [8], a general formula joining three Gauss functions of the form ${}_2F_1[a_1, a_2; a_3; z]$ with arbitrary shifts is presented using shifted operators attached to the three parameters a_1, a_2 and a_3, the existence conditions of this formula is also discussed.

On the other hand, applications of the contiguous relations range from the evaluation of hypergeometric series to the derivation of the summation and transformation formulas for such series, they can be used to evaluate hypergeometric functions which are contiguous to a hypergeometric series. For this, in a series of three research papers, Lavoie et al. [9–11] have obtained a large number of very interesting results contiguous to the Gauss second, the Kummer and Bailey theorems for the series ${}_2F_1$ and the Watson, Dixon and Whipple theorems for the series ${}_3F_2$. These results have been obtained, checked and verified with the help of *Mathematica*, a general system of doing mathematics on a computer. Very recently the Kummer identity has been generalized by Vidúnas [12], by using the contiguous relation. For more details about hypergeometric series and their contiguous relations see [1,2,13–23].

In [23], several properties of coefficients of these general contiguous relations were proved and then used to propose effective ways to compute contiguous relations. Contiguous relations are also used to make a correspondence between Lie algebra and special functions; this correspondence yields formulas of special functions [3].

In [24], contiguous relations were used to establish and prove sharp inequalities between Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean.

In 1989, Takayama [25] presented an algorithm to obtain the contiguous relations of hypergeometric functions of several variables. In fact, his algorithm was based on Buchberger's algorithm on the Gröbner basis.

In 2006, Kalmykov [26] presented the reduction algorithm for Gauss hypergeometric functions ${}_2F_1$ with arbitrary values of parameters to two functions with fixed values of the parameters, which differ from the original ones by integers and has shown that the Gauss hypergeometric functions with integer/half-integer values of parameters can be divided into six types. Only three types of them are algebraically independent.

Very recently, in the context of evaluating Feynman diagrams, Bytev, et al. [27] discussed the differential–reduction algorithm, which allows one to express generalized hypergeometric functions with parameters of arbitrary values in terms of the same functions with parameters whose values differ from the original ones by integers. Also, where possible, they have compared their results with those obtained using standard techniques and have shown that the criterion of reducibility of multi-loop Feynman integrals can be reformulated in terms of criteria of reducibility of hypergeometric functions.

Recently and with the help of the Gauss contiguous relations, Cho et al. [28] obtained twenty five contiguous function relations. With the help of the symmetry property of the Gauss hypergeometric function (1.4), these relations can, in fact, be treated as only 13 contiguous relations as follows

$$F = F(a^-, b^+) + c^{-1}(b+1-a)zF(b^+, c^+) \tag{1.20}$$

$$F = (1-z)F(a^+) + c^{-1}(c-b)zF(a^+, c^+) \tag{1.21}$$

$$cF = (c-a)F(c^+) + aF(a^+, c^+) \tag{1.22}$$

$$(a-1)F = (a-b-1)F(a^-) + bF(a^-, b^+) \tag{1.23}$$

$$(a-1)F = (a-c)F(a^-) + (c-1)F(a^-, c^-) \tag{1.24}$$

$$c(1-z)F = [a-1-(c-b)z]F(c^+) + (c-a+1)F(a^-, c^+) \tag{1.25}$$

$$(c-b-1)F = (c-a-b-1)F(b^+) + a(1-z)F(a^+, b^+) \tag{1.26}$$

$$(c-a-1)F = (b-a-1)(1-z)F(a^+) + (c-b)F(a^+, b^-) \tag{1.27}$$

$$(c-a-1)F = [-a+(c-b-1)z]F(a^+) + (c-1)(1-z)F(a^+, c^-) \tag{1.28}$$

$$(a-1)(1-z)F = [a-1+(b-c)z]F(a^-) + c^{-1}(c-a+1)(c-b)zF(a^-, c^+) \tag{1.29}$$

$$(a-1)(1-z)F = (a+b-c-1)F(a^-) + (c-b)F(a^-, b^-) \tag{1.30}$$

$$(c-1)^{-1}(c-b-1)zF = (z-1)F(c^-) + F(a^-, c^-) \tag{1.31}$$

$$(c-1)^{-1}(c-a-1)(c-b-1)zF = [(c-b-1)z-a]F(c^-) + a(1-z)F(a^+, c^-). \tag{1.32}$$

In this paper, we obtain twenty four more contiguous function relations closely related to the results of Cho et al. in [28].

2. Main results

The following twenty two (forty four by applying the symmetry property (1.4)) new and interesting contiguous relations to be established are

$$[(a-b)(a-b-1)(1-z) + b(c-b-1)]F = a(a-b-1)(1-z)F(a^+) + b(c-a)F(a^-, b^+) \tag{2.1}$$

$$\begin{aligned}[a(a+2b-2c-1) - c(b-c) - (1-b)(b-c)z]F &= a(a+b-c-1)(1-z)F(a^+) \\ &\quad + (c-b)(c-a)F(a^-, b^-)\end{aligned} \tag{2.2}$$

$$[a-(a-b)z]F = a(1-z)F(a^+) + c^{-1}b(c-a)zF(b^+, c^+) \tag{2.3}$$

$$(c-a-1)[b+(a-b)z]F + a[(c-a-1)z-b]F(a^+) + b(c-1)(1-z)F(b^+, c^-) \tag{2.4}$$

$$\begin{aligned}&[a(b-1) + \{(c-a)(c-a-b) + (b-1)(b-c)\}z]F \\ &\quad = a(1-z)[(b-1)-(c-a)z]F(a^+) + c^{-1}(c-a)(c-b)(c-b-1)zF(b^-, c^+)\end{aligned} \tag{2.5}$$

$$(a-c+1)F = a(1-z)F(a^+) - (c-1)F(b^-, c^-) \tag{2.6}$$

$$[a(a-1)+(c-b)(c-3a+1)z+(b-a)(b-c)z^2]F$$
$$=a(1-z)[(a-1)-(c-b)z]F(a^+)+c^{-1}(c-a)(c-b)(c-a+1)zF(a^-,c^+) \quad (2.7)$$

$$[(c-a-1)+(a-b)z]F=a(z-1)F(a^+)+(c-1)F(a^-,c^-) \quad (2.8)$$

$$[(c-a)(c-a-b-1)+ab-b(c-b-1)z]F=(c-a)(c-a-b-1)F(a^-)+ab(1-z)^2F(a^+,b^+) \quad (2.9)$$

$$[c(b-1)+a(a-2b+1)-(b-a+1)(b-a)z]F=(c-a)(b-a-1)F(a^-)+a(c-b)F(a^+,b^-) \quad (2.10)$$

$$F=F(a^-)+c^{-1}bzF(b^+,c^+) \quad (2.11)$$

$$[b(a-1)+z\{(c-a-b)(c-a-1)-b(c-b-1)\}]F$$
$$=(c-a)[(c-a-1)z-b]F(a^-)+b(c-1)(1-z)^2F(b^+,c^-) \quad (2.12)$$

$$[(b-1)(1-z)+(a-b)z(1-z)]F=[(b-1)+(a-c)z]F(a^-)+c^{-1}(c-b)(c-b+1)zF(b^-,c^+) \quad (2.13)$$

$$[(a-1)+(b-a)z]F=(a-c)F(a^-)+(c-1)F(b^-,c^-) \quad (2.14)$$

$$[(c-a)-(b-a)z]F=(c-a)F(a^-)+ac^{-1}(c-b)zF(a^+,c^+) \quad (2.15)$$

$$[a(a-1)+(c-3a)(c-b-1)z+(a-b)(c-b-1)z^2]F$$
$$=(c-a)[(c-b-1)z-a]F(a^-)+a(c-1)(1-z)^2F(a^+,c^-) \quad (2.16)$$

$$[a+(b-a-1)z]F=aF(a^+,b^-)+c^{-1}(c-a)(b-a-1)zF(c^+) \quad (2.17)$$

$$[b(c-1)+bz(2a+b-3c+2)-(a-c)(c-a-1)z^2]F$$
$$=c^{-1}(c-a)(c-b)z[(c-a-1)z-b]F(c^+)+b(c-1)(1-z)^2F(b^+,c^-) \quad (2.18)$$

$$[(c-1)+(b-c)z]F=c^{-1}(c-a)(b-c)zF(c^+)+(c-1)F(b^-,c^-) \quad (2.19)$$

$$(c-a-1)\{(b-1)-(b-a-1)z\}F=(c-1)(b-a-1)(1-z)F(c^-)+a(c-b)F(a^+,b^-) \quad (2.20)$$

$$[(c-1)+(b-c+1)z]F=(c-1)(1-z)F(c^-)+c^{-1}b(c-a)zF(b^+,c^+) \quad (2.21)$$

$$[(c-1)(b-1)+(b-1)(2a+b-3c+1)z-(z-a)(1+a-c)z^2]F$$
$$=(c-1)[(b-1)-(c-a)z](1-z)F(c^-)+c^{-1}(c-a)(c-b)(c-b+1)zF(b^-,c^+) \quad (2.22)$$

together with the following additional four contiguous relations

$$[ab-z\{(c-b-1)(a+b-c)-a(a-c)\}]F = c^{-1}(c-a)(c-b)(c-a-b-1)zF(c^+) + ab(1-z)^2F(a^+,b^+) \quad (2.23)$$

$$(1-z)F=c^{-1}(a+b-c-1)zF(c^+)+F(a^-,b^-) \quad (2.24)$$

$$(c-b-1)(c-a-1)F=(c-1)(c-a-b-1)F(c^-)+ab(1-z)F(a^+,b^+) \quad (2.25)$$

$$[(a-1)(b-1)+(a^2+b^2+c^2+ab-2bc-2ac+c-1)]F$$
$$=(c-1)(a+b-c-1)(1-z)F(c^-)+(c-a)(c-b)F(a^-,b^-). \quad (2.26)$$

It is well known that the 15 Gauss relations ((1.5)–(1.19)) have wide applications. In order to extend these relations, in all 48 new results (including those obtained by Cho [28]) have been investigated in this paper. The importance of the discovery of such relations lies in the fact that from these new relations, there exists a strong possibility of obtaining new and useful summation and transformation formulas which may be useful in practical applications. Some of the applications in obtaining new summation formulas are given in this paper. Other applications are under investigation and will form a part of a subsequent paper in this direction.

3. Derivation

The derivations of our new contiguous relations (2.1)–(2.26) are straightforward and in this section we will obtain the proof of these relations by two different methods:

1. By algebraic manipulations, for example, if we wish to derive the result (2.1), we take the Gauss contiguous relation (1.8) and Cho et al. result (1.23) and eliminate $F(a^-)$ or by taking the Gauss result (1.17) together with the Cho et al.result (1.27) after applying the symmetry property (1.4) on it and then eliminating $F(b^+)$, we get the required result (2.1). In a similar manner, other results can be easily obtained. The scheme is outlined in (Table 1) including that of (2.1).
2. With the use of computer algebra systems *Maple* or *Mathematica* all these relations can be easily checked. For example, as in [7], the following *Mathematica* command: *Collect*[*Simplify'InducedRecurrence*[1 &, {##}&, *Hypergeometric2F1*, {{a, b, c, z}, {$a+\alpha_1, b+\beta_1, c+\gamma_1, z$}, {$a+\alpha_2, b+\beta_2, c+\gamma_2, z$}}, *Hypergeometric2F1*], _*Hypergeometric2F1, FullSimplify*] will find out such contiguous relations that relate

Table 1
Derivations by algebraic manipulations.

In Gauss relation	and Cho et al. relation	If we eliminate	We get the result
(1.8) or (1.17)	(1.23) or $(1.27)_{a\leftrightarrow b}$	$F(a^-)$ or $F(b^+)$	(2.1)
(1.8) or (1.7)	(1.30) or $(1.30)_{a\leftrightarrow b}$	$F(a^-)$ or $F(b^-)$	(2.2)
(1.17) or (1.16)	$(1.21)_{a\leftrightarrow b}$ or $(1.22)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^+)$	(2.3)
(1.17) or (1.5)	$(1.28)_{a\leftrightarrow b}$ or $(1.32)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^-)$	(2.4)
(1.16) or (1.7)	$(1.25)_{a\leftrightarrow b}$ or $(1.29)_{a\leftrightarrow b}$	$F(c^+)$ or $F(b^-)$	(2.5)
(1.7) or (1.5)	$(1.24)_{a\leftrightarrow b}$ or $(1.31)_{a\leftrightarrow b}$	$F(b^-)$ or $F(c^-)$	(2.6)
(1.8) or (1.16)	(1.29) or $(1.25)_{a\leftrightarrow b}$	$F(a^-)$ or $F(c^+)$	(2.7)
(1.8) or (1.5)	(1.24) or (1.31)	$F(a^-)$ or $F(c^-)$	(2.8)
(1.8) or (1.13)	$(1.26)_{a\leftrightarrow b}$ or (1.26)	$F(a^+)$ or $F(b^+)$	(2.9)
(1.8) or (1.18)	(1.27) or $(1.23)_{a\leftrightarrow b}$	$F(a^+)$ or $F(b^-)$	(2.10)
(1.13) or (1.6)	$(1.21)_{a\leftrightarrow b}$ or $(1.22)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^+)$	(2.11)
(1.13) or (1.9)	$(1.28)_{a\leftrightarrow b}$ or $(1.32)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^-)$	(2.12)
(1.6) or (1.18)	$(1.25)_{a\leftrightarrow b}$ or $(1.29)_{a\leftrightarrow b}$	$F(c^+)$ or $F(b^-)$	(2.13)
(1.18) or (1.9)	$(1.24)_{a\leftrightarrow b}$ or $(1.31)_{a\leftrightarrow b}$	$F(b^-)$ or $F(c^-)$	(2.14)
(1.8) or (1.6)	(1.21) or (1.22)	$F(a^+)$ or $F(c^+)$	(2.15)
(1.8) or (1.9)	(1.28) or (1.32)	$F(a^+)$ or $F(c^-)$	(2.16)
(1.16) or (1.12)	(1.27) or $(1.23)_{a\leftrightarrow b}$	$F(a^+)$ or $F(b^-)$	(2.17)
(1.10) or (1.19)	$(1.27)_{a\leftrightarrow b}$ or $(1.31)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^-)$	(2.18)
(1.12) or (1.19)	$(1.24)_{a\leftrightarrow b}$ or $(1.31)_{a\leftrightarrow b}$	$F(b^-)$ or $F(c^-)$	(2.19)
(1.5) or (1.15)	$(1.27)_{a\leftrightarrow b}$ or $(1.23)_{a\leftrightarrow b}$	$F(a^+)$ or $F(b^-)$	(2.20)
(1.11) or (1.19)	$(1.21)_{a\leftrightarrow b}$ or $(1.22)_{a\leftrightarrow b}$	$F(b^+)$ or $F(c^+)$	(2.21)
(1.19) or (1.15)	$(1.25)_{a\leftrightarrow b}$ or $(1.29)_{a\leftrightarrow b}$	$F(c^+)$ or $F(b^-)$	(2.22)
(1.16) or (1.10)	$(1.26)_{a\leftrightarrow b}$ or (1.26)	$F(a^+)$ or $F(b^+)$	(2.23)
(1.6) or (1.12)	(1.30) or $(1.30)_{a\leftrightarrow b}$	$F(a^-)$ or $F(b^-)$	(2.24)
(1.5) or (1.11)	$(1.26)_{a\leftrightarrow b}$ or $(1.26)_{a\leftrightarrow b}$	$F(a^+)$ or $F(b^+)$	(2.25)
(1.9) or (1.15)	(1.30) or $(1.30)_{a\leftrightarrow b}$	$F(a^-)$ or $F(b^-)$	(2.26)

between the three hypergeometric functions

$$ {}_2F_1[a, b; c, d, z], {}_2F_1[a+\alpha_1, b+\beta_1; c+\gamma_1; z] \quad \text{and} \quad {}_2F_1[a+\alpha_2, b+\beta_2; c+\gamma_2; z]. $$

For example in order to derive the contiguous function relation (2.1), the required shifts will be $\alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0$ and $\alpha_2 = -1, \beta_2 = 1, \gamma_2 = 0$.

4. Application

4.1. Hypergeometric summation formulas

It is well known that the classical summation theorems such as those of Gauss, Kummer and Gauss's second for the series ${}_2F_1$, viz.

Gauss's theorem [29, Section 1.3, Page 2]

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4.1} $$

provided $\Re(c-a-b) > 0$.

Kummer's theorem [29, Section 2.3, Page 9]

$$ {}_2F_1[a, b; 1+a-b; -1] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma(1+a)}. \tag{4.2} $$

Gauss's second theorem [29, Eq. 2, Page 11]

$$ {}_2F_1\left[a, b; \frac{1}{2}(a+b+1); \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{4.3} $$

play an important role in the theory of hypergeometric series. In this section, we shall establish a number of summation formulas contiguous to such well known summation theorems.

For this, in the contiguous function relation (2.11), if we multiply it by $(a-c)$, we have

$$(a-c)F = (a-c)F(a^-) + c^{-1}b(a-c)zF(b^+, c^+) \tag{4.4}$$

and by (2.15), we have

$$-(a-c)F - (b-a)zF = (c-a)F(a^-) + ac^{-1}(c-b)zF(a^+, c^+). \tag{4.5}$$

Now, if we eliminate $(a-c)F$ from (4.4) and (4.5), we get after a little simplification

$$(a-b)cF = b(a-c)F(b^+, c^+) + a(c-b)F(a^+, c^+). \tag{4.6}$$

Further, in (4.6) if we take $z = \frac{1}{2}$ and $c = \frac{1}{2}(a+b)$, we get after little simplification,

$$(a+b)\,{}_2F_1\left[a, b; \frac{1}{2}(a+b); \frac{1}{2}\right] = a\,{}_2F_1\left[a+1, b; \frac{1}{2}(a+b+2); \frac{1}{2}\right] + b\,{}_2F_1\left[a, b+1; \frac{1}{2}(a+b+2); \frac{1}{2}\right]. \tag{4.7}$$

Now, it is easy to see that the two ${}_2F_1$ on the right-hand side of (4.7) can be evaluated with the help of Gauss's second summation theorem (4.3) and after little simplification, we get the following summation formula

$${}_2F_1\left[a, b; \frac{1}{2}(a+b); \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\left[\frac{1}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)}\right]. \tag{4.8}$$

Similarly, in (4.6), if we take $z = \frac{1}{2}$ and $c = \frac{1}{2}(a+b-1)$, we get

$$(a-b)(a+b-1)\,{}_2F_1\left[a, b; \frac{1}{2}(a+b-1); \frac{1}{2}\right] = b\,(a-b+1){}_2F_1\left[a, b+1; \frac{1}{2}(a+b+1); \frac{1}{2}\right] + a(a-b-1)\,{}_2F_1\left[a+1, b; \frac{1}{2}(a+b+1); \frac{1}{2}\right]. \tag{4.9}$$

Again, it is easy to see that the two ${}_2F_1$ on the right-hand side of (4.9) can be evaluated with the help of (4.8) and after little simplification we get the following summation formula

$${}_2F_1\left[a, b; \frac{1}{2}(a+b-1); \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\right)\left[\frac{\frac{1}{2}(a+b-1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} + \frac{2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right]. \tag{4.10}$$

Similarly, in (4.6), if we take $z = \frac{1}{2}, c = \frac{1}{2}(a+b-i),\ i = 2, 3$ and 4 and proceeding as above, we get the following summation formulas

$${}_2F_1\left[a, b; \frac{1}{2}(a+b-2); \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-1\right)\left[\frac{\frac{1}{2}(3a+b-2)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)} + \frac{\frac{1}{2}(3b+a-b)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\right], \tag{4.11}$$

$${}_2F_1\left[a, b; \frac{1}{2}(a+b-3); \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{3}{2}\right)\left[\frac{A_4}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} + \frac{B_4}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right], \tag{4.12}$$

where

$$A_4 = \frac{1}{2}(b+a-3)(b+a+1) - \frac{1}{4}(b-a-3)(b-a+3)$$
$$B_4 = 2(b+a-1)$$

and for $i = 4$, we will have

$${}_2F_1\left[a, b; \frac{1}{2}(a+b-4); \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-2\right)\left[\frac{A_5}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)} + \frac{B_5}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\right], \tag{4.13}$$

where

$$A_5 = (b+a-4)^2 - \frac{1}{2}(b+a-4)(b-a-4) - \frac{1}{4}(b-a-4)^2 + 4(b+a-4) - \frac{7}{2}(b-a-4)$$
$$B_5 = (b+a-4)^2 + \frac{1}{2}(b-a-4)(b+a-4) - \frac{1}{4}(b-a-4)^2 + 8(b+a-4) - \frac{1}{2}(b-a-4) + 12.$$

Clearly the results (4.8), (4.10)–(4.13) are closely related to Gauss's second summation theorem (4.3).

Now in the known result [29, Section 2.4, Page 11]

$$ {}_2F_1\left[a, b; c; -1\right] = 2^{-a}\,{}_2F_1\left[a, c-b; c; \frac{1}{2}\right] \tag{4.14}$$

if we take $c = a - b - i$, $i = 0, 1, 2, 3$ and 4, then it is easy to see that the right-hand side of (4.14) can be evaluated by (4.8), (4.10)–(4.13) respectively and after little simplification, we get the following results

$$ {}_2F_1\left[a, b; a-b; -1\right] = 2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b)\left[\frac{1}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b\right)}\right], \tag{4.15}$$

$$ {}_2F_1\left[a, b; a-b-1; -1\right] = 2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-1)\left[\frac{(a-b-1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b\right)} + \frac{2}{\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a\right)}\right], \tag{4.16}$$

$$ \begin{aligned} {}_2F_1\left[a, b; a-b-2; -1\right] &= 2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-2) \\ &\quad\times\left[\frac{(2a-3b-4)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}\right)} + \frac{(2a-b-2)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}\right)}\right], \end{aligned} \tag{4.17}$$

$$ \begin{aligned} {}_2F_1\left[a, b; a-b-3; -1\right] &= 2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-3) \\ &\quad\times\left[\frac{2(a-b-3)(a-b-1)-b(b+3)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b-1\right)} + \frac{4(a-b-2)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b-\frac{3}{2}\right)}\right] \end{aligned} \tag{4.18}$$

and

$$ \begin{aligned} {}_2F_1\left[a, b; a-b-4; -1\right] &= 2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-4) \\ &\quad\times\left[\frac{A_5}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b-\frac{3}{2}\right)} + \frac{B_5}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b-2\right)}\right] \end{aligned} \tag{4.19}$$

where

$$A_5 = 4(a-b-4)^2 - 2(b-4)(a-b-4) - b^2 - 7b$$
$$B_5 = 4(a-b-4)^2 + 2(b+8)(a-b-4) - b^2 - b + 12.$$

Clearly the results (4.15)–(4.19) are closely related to Kummer's summation theorem (4.2).

Next, consider the contiguous functions relation (1.22)

$$ (c-a)\,{}_2F_1\left[a, b; c+1; z\right] = c\,{}_2F_1\left[a, b; c; z\right] - a\,{}_2F_1\left[a+1, b; c+1; z\right]. \tag{4.20}$$

In (4.20), if we take $z = -1$, and $c = 1 + a - b$, we get

$$ (1-b)\,{}_2F_1\left[a, b; 2+a-b; -1\right] = (1+a-b)\,{}_2F_1\left[a, b; 1+a-b; -1\right] - a\,{}_2F_1\left[a+1, b; 2+a-b; -1\right]. \tag{4.21}$$

Now, it is easy to see that the two ${}_2F_1$ on the right-hand side of (4.21) can be evaluated by the Kummer summation theorem (4.2) and after little simplification, we get the following summation formula

$$ {}_2F_1\left[a, b; 2+a-b; -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+a-b)}{2^a(1-b)}\left[\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1\right)} - \frac{1}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{3}{2}\right)}\right]. \tag{4.22}$$

Similarly, in (4.20), if we take $z = -1$ and $c = 2 + a - b$, we get

$$ (2-b)\,{}_2F_1\left[a, b; 3+a-b; -1\right] = (2+a-b)\,{}_2F_1\left[a, b; 2+a-b; -1\right] - a\,{}_2F_1\left[a+1, b; 3+a-b; -1\right]. \tag{4.23}$$

Again, it is easy to see that the two ${}_2F_1$ on the right-hand side of (4.23) can be evaluated with the help of (4.22) and after a little simplification, we get the following summation formula

$$ {}_2F_1\left[a, b; 3+a-b; -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3+a-b)}{2^a(1-b)(2-b)}\left[\frac{(1+a-b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+2\right)} - \frac{2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{3}{2}\right)}\right]. \tag{4.24}$$

Similarly, in (4.20), if we set

(i) $z = -1$, $c = 3 + a - b$, and
(ii) $z = -1$, $c = 4 + a - b$

and proceed as above, we get the following summation formulas

$$ {}_2F_1\left[a, b; 4+a-b; -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(4+a-b)}{2^a(1-b)(2-b)(3-b)} \times \left[\frac{(3b-2a-5)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{5}{2}\right)} + \frac{(2a-b+1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+2\right)}\right], \quad (4.25) $$

and

$$ {}_2F_1\left[a, b; 5+a-b; -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(5+a-b)}{2^a(1-b)(2-b)(3-b)(4-b)} \times \left[\frac{2(a-b+3)(a-b+1)-(b-1)(b-4)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+3\right)} - \frac{4(a-b+2)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+3\right)}\right]. \quad (4.26) $$

Further, in the known result (4.14), written in the form

$$ {}_2F_1\left[a, b; c; \frac{1}{2}\right] = 2^a\, {}_2F_1\left[a, c-b; c; -1\right], \quad (4.27) $$

if we set $c = \frac{1}{2}(a+b+i)$, $i = 2, 3, 4$ and 5, then it is easy to see that the right-hand side of (4.27) can be evaluated by (4.23)–(4.26) respectively and we get the following summation formulas

$$ {}_2F_1\left[a, b; \frac{1}{2}(a+b+2); \frac{1}{2}\right] = \frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)}{(a-b)}\left[\frac{1}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} - \frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)}\right], \quad (4.28) $$

$$ {}_2F_1\left[a, b; \frac{1}{2}(a+b+3); \frac{1}{2}\right] = \frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)}{[(a-b)^2-1]}\left[\frac{(b+a-1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} - \frac{4}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right], \quad (4.29) $$

$$ {}_2F_1\left[a, b; \frac{1}{2}(a+b+4); \frac{1}{2}\right] = \frac{4\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+2\right)}{((a-b)^2-2^2)(a-b)}\left[\frac{(3a+b-2)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} - \frac{(3a+b-2)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)}\right], \quad (4.30) $$

$$ {}_2F_1\left[a, b; \frac{1}{2}(a+b+5); \frac{1}{2}\right] = \frac{16\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{5}{2}\right)}{[(a-b)^2-1][(a-b)^2-3^2]}\left[\frac{A_4}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} - \frac{B_4}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right], \quad (4.31) $$

where

$$ A_4 = \frac{1}{2}(b+a+1)(a+b-3) - \frac{1}{4}(b-a+3)(b-a-3) $$
$$ B_4 = 2(b+a-1). $$

Clearly, the results (4.28)–(4.31) are closely related to Gauss's second summation theorem (4.3).

Remark 1. 1. The results (4.28)–(4.31) can also be obtained from the contiguous function relation (1.17).
2. The results (4.8), (4.10)–(4.13) and (4.28)–(4.31) contiguous to Gauss's second theorem (4.3), and the results (4.15)–(4.19), (4.22) and (4.24)–(4.26) contiguous to the Kummer summation theorem (4.2) have also been obtained by Lavoie et al. [9–11] by following a different method.

4.2. Jacobi polynomials

In this part, we shall demonstrate, how one can easily obtain some of the well known and interesting results involving Jacobi polynomials from our new contiguous function relations established in Section 2.

When a or b is a negative integer, $-n$, the hypergeometric function (1.2) becomes a polynomial, called *Jacobi polynomial of degree n* (or, hypergeometric polynomial of degree n):

$$ F(-n, b; c; z) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(c)_k} z^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)_k}{(c)_k} z^k. $$

Many important polynomials, such as the Legendre polynomial, spherical polynomial, Chebyshev polynomial, etc., are special cases of Jacobi polynomials.

The Jacobi polynomial $P_n^{(a,b)}(x)$ may be defined by [13, Eq. 1, Page 254]

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\,{}_2F_1\left(-n,\,1+a+b+n;\,1+a;\,\frac{1-x}{2}\right). \tag{4.32}$$

When $a = b = 0$, the polynomial in (4.32) becomes the Legendre polynomial. From (4.32) it follows that $P_n^{(a,b)}(x)$ is a polynomial of degree precisely n and that

$$P_n^{(a,b)}(1) = \frac{(1+a)_n}{n!}.$$

In the well known Euler transformation [13, Theorem 20, Page 60]

$${}_2F_1\,(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a,\,c-b;\,c;\,\frac{-z}{1-z}\right) \tag{4.33}$$

valid for $|z| < 1$ and $\left|\frac{z}{1-z}\right| < 1$, setting $a = -n$, $b = 1+a+b+n$, $c = 1+a$ and $z = \frac{1-x}{2}$ in (4.33) and then applying (4.33) on (4.32) yields

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\left(\frac{x+1}{2}\right)^n\,{}_2F_1\left(-n,\,-b-n;\,1+a;\,\frac{x-1}{x+1}\right). \tag{4.34}$$

Now, let $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$, then from (4.32), we will have, after some simplification

$$\mathcal{D}\left(P_n^{(a,b)}(x)\right) = \frac{1}{2}(1+a+b+n)P_{n-1}^{(a+1,b+1)}(x). \tag{4.35}$$

Differentiating both sides of (4.34), after simplification and upon application of the parameter shift $n \to n+1$, $a \to a-1$ and $b \to b-1$, we will have the following known result [13, Eq. 9, Page 264]

$$(a+b+n)P_n^{(a,b)}(x) = (b+n)P_n^{(a,b-1)}(x) + (a+n)P_n^{(a-1,b)}(x). \tag{4.36}$$

Now, we will use some of our new contiguous function relations (2.1)–(2.26) to obtain some identities involving Jacobi polynomial. For example, consider the contiguous relations (2.24)

$$(1-z)F = c^{-1}(a+b-c-1)zF(c^+) + F(a^-, b^-)$$

which can be rewritten as

$$c(1-z)\,{}_2F_1(a,b;c;z) = (a+b-c-1)z\,{}_2F_1(a,b;c+1;z) + c\,{}_2F_1(a-1,b-1;c;z). \tag{4.37}$$

Setting $a = -n+1$, $b = 1+a+b+n$, $c = 1+a$ and $z = \frac{1-x}{2}$ in (4.37), we will have, after simplification

$$\begin{aligned}(1+a)\left(\frac{1+x}{2}\right){}_2F_1&\left[-(n-1),\,1+a+(b+1)+(n-1);\,1+a;\,\frac{1-x}{2}\right]\\ &= b\left(\frac{1-x}{2}\right){}_2F_1\left[-(n-1),\,1+(a+1)+b+(n-1);\,1+(a+1);\,\frac{1-x}{2}\right]\\ &\quad+(1+a)\,{}_2F_1\left[-n,\,1+a+(b-1)+n;\,1+a;\,\frac{1-x}{2}\right].\end{aligned} \tag{4.38}$$

Applying (4.32) on (4.38), we will have, after some simplification

$$2nP_n^{(a,b-1)}(x) = (1+x)(a+n)P_{n-1}^{(a,b+1)}(x) - b(1-x)P_{n-1}^{(a+1,b)}(x). \tag{4.39}$$

Now, consider the contiguous relation (2.19), and writing it in the form

$$c[(c-1)+(b-c)z]\,{}_2F_1(a,b;c;z) = (c-a)(b-c)z\,{}_2F_1(a,b;c+1;z) + c(c-1)\,{}_2F_1(a,b-1;c-1;z). \tag{4.40}$$

Setting $a = -n+1$, $b = 1+a+b+n$, $c = 1+a$ and $z = \frac{1-x}{2}$ in (4.40), we will have, after simplification

$$\begin{aligned}(1+a)\left[a+(b+n)\left(\frac{1-x}{2}\right)\right]{}_2F_1&\left[-(n-1),\,1+a+(b+1)+(n-1);\,1+a;\,\frac{1-x}{2}\right]\\ &= (a+n)(b+n)\left(\frac{1-x}{2}\right){}_2F_1\left[-(n-1),\,1+(1+a)+b+(n-1);\,1+(a+1);\,\frac{1-x}{2}\right]\\ &\quad+a(1+a)\,{}_2F_1\left[-(n-1),\,1+(a-1)+(b+1)+(n-1);\,1+(a-1);\,\frac{1-x}{2}\right].\end{aligned} \tag{4.41}$$

Applying (4.32) on (4.41), we will have after replacing n by $n+1$, the following result

$$\left[a+(b+n+1)\left(\frac{1-x}{2}\right)\right]P_n^{(a,b+1)}(x)=(b+n+1)\left(\frac{1-x}{2}\right)P_n^{(a+1,b)}(x)+(a+n)P_n^{(a-1,b+1)}(x). \quad (4.42)$$

Using a known result [13, Eq. 17, Page 265], this can be written in the following compact form

$$aP_n^{(a,b+1)}(x)-(b+n+1)\left(\frac{1-x}{2}\right)P_{n-1}^{(a+1,b+1)}(x)=(a+n)P_n^{(a-1,b+1)}(x). \quad (4.43)$$

Now, using a known result [13, Eq. 8, Page 264],

$$2\mathcal{D}P_n^{(a,b)}(x)=(b+n)P_{n-1}^{(a+1,b)}(x)+(a+n)P_{n-1}^{(a,b+1)}(x)$$

and (4.35), we get after little simplification, the following result

$$(a+n)P_n^{(a-1,b+1)}(x)=(a+b+n+1)P_n^{(a,b+1)}(x)-(b+n+1)P_n^{(a,b)}(x) \quad (4.44)$$

from which (4.42), after some simplification, we arrive at the following known result [13, Eq. 16, Page 265]

$$(1+x)P_n^{(a,b+1)}(x)+(1-x)P_n^{(a+1,b)}(x)=2P_n^{(a,b)}(x).$$

We remark in passing that the results (4.39) and (4.44) are straightforward.

Concluding remark

Further applications of our new contiguous function relations are under investigation and will be submitted for publication soon.

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