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SYMBOLIC ANALYSIS OF REDUCED FORMS OF THE NAVIER–STOKES EQUATIONS

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Abstract—A unified development of symbolic analysis is presented. Symbolic analysis is used to identify reduced forms of the linearized steady Navier–Stokes equations which permit computational solutions to be obtained in a *single* spatial march in the dominant flow direction. In particular it is demonstrated that the "parabolized" form of the Navier–Stokes equations, although not parabolic, is well-posed as an initial-value problem in space, provided the solution is restricted to functions with compact support. The effectiveness of symbolic analysis for determining the well-posedness of complex systems of equations, such as the Navier–Stokes equations, is clearly demonstrated.

1. INTRODUCTION

Traditionally partial differential equations (PDEs) are classified [1] by a determination of the existence of surfaces (directions in two dimensions) for which the Cauchy problem is not well-posed. For scalar second order PDEs the nature of the corresponding characteristic polynomial provides precise definition of elliptic, parabolic and hyperbolic PDEs.

For systems of equations such as govern compressible, viscous flow [2], a characteristic analysis is often less useful. First, the direct extraction of the characteristic form may lead to a singular matrix. Second where this degenerate behaviour can be avoided the roots of the characteristic polynomial may well not correspond to the traditional categories. Instead a mixture of real and complex roots will imply that the system of PDEs is of "mixed" type.

This is a particular problem for reduced forms of the Navier–Stokes equations which are developed with the intention of obtaining a well-posed initial-value problem in space [3, 4]. When this is achieved a very economical computational algorithm can be constructed to provide the solution in a *single* spatial march in the dominant flow direction.

What is required is an *a priori* analysis which will identify systems of PDEs which admit well-posed initial-value solutions in a particular spatial direction. The analysis should also clearly identify such time-like spatial directions. The traditional characteristic analysis, when applied to systems of equations of the complexity of the Navier–Stokes equations, is inadequate for these purposes. However by taking the Fourier transform of the governing equations and examining the behaviour of the symbol, such an *a priori* analysis is indeed available. It is suggested that by considering nonlinear equations locally, effectively freezing the nonliner coefficients and thereby enabling Fourier transforms to be obtained, the behaviour of singularities of the resulting symbol allow the existence of initial value solutions of the nonlinear equations to be inferred, given appropriate bounds on the nonlinear coefficients.

Symbolic analysis has been used previously by, for instance, Brandt and Dinar [5], who used it to determine whether equation sets are elliptic, and therefore well-posed as boundary-value problems, and hence able to be solved using relaxation type multigrid approaches. Gustafsson and Sundstrom [6] use symbolic analysis, in addition to other techniques, to determine the behaviour of solutions to the Navier–Stokes equations. A particularly good exposition of the use of symbolic analysis in determining the behaviour of PDEs is to be found in Schecter [7]. The application of symbolic analysis to determine whether reduced forms of the Navier–Stokes equations may be solved computationally in a single spatial march is believed by the authors to be novel. In particular the proof that equation set C in Section 4 is well-posed as an initial-value problem in space, provided the solution set is restricted to having compact support, has not been obtained before.

This paper will briefly describe symbolic analysis, and make use of it in investigating the behaviour of solutions of reduced Navier–Stokes equations. In Section 2 the method of obtaining the symbol for the general scalar second order PDE is given. We describe a constraint on the symbol to guarantee the equation possesses a unique initial-value solution.

In Section 3 the symbolic analysis will be extended to higher order and systems of equations, for which the traditional characteristic analysis is less useful. In Section 4 the behaviour of the symbol of various reduced forms of the Navier–Stokes equations will be considered, to illustrate how the analysis is used in practice to determine whether stable computational solutions can be obtained in a single spatial march.

2. CLASSIFICATION OF SCALAR SECOND ORDER PDEs

PDEs may be classified as being of elliptic, parabolic or hyperbolic type. From the current point of view such a classification is of interest because it separates elliptic equations, which are well-posed as boundary-value problems, from hyperbolic and parabolic equations, that are well-posed as initial-value problems. Boundary-value problems must be solved numerically using an iterative process, whereas initial-value problems may be solved using a single march algorithm, in the time-like direction. However such a classification is complete only for second order equations, or equivalent systems; for higher order systems it is not complete. In fact determining whether such a general system is elliptic or nonelliptic can be difficult. In this section the method of obtaining the symbol and using it to determine if the equation is well-posed as an initial-value problem will be briefly described, using the general second order PDE.

The general second order equation in two independent variables (x, y) and one dependent variable $u(x, y)$ may be written as,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G(x, y) = 0. \tag{1}$$

If equation (1) is nonlinear then we linearize it by freezing the values of the coefficients at their local values, thus we may consider them to be constants. To extend local results obtained in this manner to global results we must be able to impose global constraints on the behaviour of the coefficients.

It is well-known that if the discriminant, $B^2 - 4AC$, is less than zero such an equation is elliptic, if equal to zero it is parabolic, if greater than zero it is hyperbolic. Thus the classification depends only on the highest order derivatives, the principal terms, and is independent of the coordinate system [1].

Now suppose we wish to see whether a solution exists for equation (1) in the domain, $0 \leqslant \lambda \leqslant L$, $-\infty \leqslant \eta \leqslant \infty$, with initial data prescribed on $\lambda = 0$, where λ and η are linearly independent functions of x and y. Firstly we must transform equation (1) into the new coordinate system, giving

$$au_{\lambda\lambda} + bu_{\lambda\eta} + cu_{\eta\eta} + du_\lambda + eu_\eta + fu + g(\lambda, \eta) = 0, \tag{2}$$

where $b^2 - 4ac$ has the same sign as $B^2 - 4AC$ [1]. The initial data will be of the form

$$u_{\lambda k}(0, \eta) = h_k(\eta) \quad k = 0, r \quad r < 2, \tag{3}$$

where $u_{\lambda k}$ is the kth derivative of u in the λ direction. The transformation is necessary to identify increasing λ with a time-like direction. The determination of such a time-like direction may impose some restriction on the coefficients A to F.

To determine if solutions exist for equations such as equation (2), we need only consider equations of the form [7],

$$au_{\lambda\lambda} + bu_{\lambda\eta} + cu_{\eta\eta} + du_\lambda + eu_\eta + fu = 0, \tag{4}$$

with initial data,

$$u_{;k}(0, \eta) = 0, \quad k = 0, r-1, \quad u_{;k}(0, \eta) = p(\eta), \quad k = r \quad r < 2. \tag{5}$$

We will relate the existence of solutions to the roots of the symbol of the equation, denoted σ, which we define in the following way. We associate with $u(\lambda, \eta) \in C^s$ it's Fourier transform $\hat{u}(\Lambda, \Gamma) \in C^s(\Lambda, \Gamma)$, where we use C^s to denote the set of s times continuously differentiable functions. The reciprocal formula allows us to write u in terms of $\hat{u}$ as

$$u(\lambda, \eta) = (2\pi)^{-1} \int e^{i(\lambda\Lambda + \eta\Gamma)} \hat{u}(\Lambda, \Gamma) \, d\Lambda \, d\Gamma,$$

where the above integral will converge absolutely for Λ and $\Gamma \in R$, the space of real numbers, if $s \geqslant n + 1$ [8]. When substituted into equation (4) the result is

$$(2\pi)^{-1} \int e^{i(\lambda\Lambda + \eta\Gamma)}(-a\Lambda^2 - b\Lambda\Gamma - c\Gamma^2 + di\Lambda + ei\Gamma + f)\hat{u}(\Lambda, \Gamma) \, d\Lambda \, d\Gamma = 0,$$

and the symbol of equation (4), $\sigma(\Lambda, \Gamma)$, is obtained from the kernel of the above expression, i.e.

$$\sigma = -a\Lambda^2 - b\Lambda\Gamma - c\Gamma^2 + di\Lambda + ei\Gamma + f. \tag{6}$$

Similarly the principal symbol, σ_p, of equation (4) is obtained when all but the highest order terms are dropped from equation (6), thus

$$\sigma_p = -a\Lambda^2 - b\Lambda\Gamma - c\Gamma^2,$$

and it is readily seen that by considering the zeros of the principal symbol we may determine if the equation is elliptic, parabolic or hyperbolic.

We now state without proof the constraints on the symbol for equation (4) with initial data equation (5) to be well-posed as an initial-value problem in space. For equation (4) with appropriate initial data on $\lambda = 0$ a unique solution will exist in the domain $0 \leqslant \lambda \leqslant L < \infty$, $-\infty < \Gamma < \infty$, provided [7],

(1) the roots Λ of $\sigma(\Lambda, \Gamma) = 0$ are bounded below in the complex plane;
(2) the inital data is sufficiently smooth.

By a solution we mean a function $u(\lambda, \eta)$ which satisfies equation (4) and the initial data, which is at least twice differentiable and which is continuously dependent on the initial data. If this is the case then we say that equation (4) with the appropriate initial data is well-posed as an initial-value problem in the given domain.

The classification given above is for a scalar second order PDE. In general one can expect to encounter higher order and systems of PDEs, and so it is natural to wish to be able to classify such systems in a similar manner to that given for the second order scalar PDE. In the next section this extension is provided.

3. EXTENSION TO HIGHER ORDER AND SYSTEMS OF EQUATIONS

If we attempt to use the method of characteristics to classify higher order equations, or equivalent systems, by dividing them up into elliptic, parabolic and hyperbolic types, then the classification would be as follows. The equation system is elliptic if the principal symbol has no real roots, it is hyperbolic if it has as many real roots as it's order, and it is parabolic if it has only one real root. This immediately presents us with a number of problems. Firstly, if we have a system of equations, what is meant by the principal symbol? Secondly, to effectively classify the system we must in general find all the roots of what is likely to be a high order polynomial. Finally, the classification is evidently not complete for systems with a principal symbol of order higher than two. In addition, even if we can identify a system as being parabolic, initial-value solutions will exist only if the initial data is specified on a space-like surface, and the solution extended in a positive time-like direction. For these reasons it is suggested that the use of a characteristics analysis to divide equations or systems into elliptic, parabolic and hyperbolic types is not a satisfactory way to identify those that are well-posed as initial-value problems.

In the present section we define what is meant by the symbol of a system of equations, and extend the classification given in the previous section that determines the existence of initial-value solutions, to include such systems.

The relation of the behaviour of initial-value solutions to the roots of the symbol, given in the previous section, is extended to higher order systems of equations in the following way. If we have a system of N mth order differential equations in N unknowns $u(\lambda, \eta_1 \ldots \eta_n)$, and if we define the differentiation operators $D = (D_{\eta 1} \ldots D_{\eta n})$, the system may be written as,

$$P(\cdot_{\lambda}, D)u(\lambda, \eta) = G(\lambda, \eta), \tag{7}$$

where P is an $N \times N$ matrix whose elements are mth degree polynomials in each of their $n + 1$ arguments. On the surface $\lambda = 0$ we prescribe initial data,

$$u_{\lambda k}(0, \eta) = H_k(\eta), \quad k = 0 \ldots r, \quad r < m. \tag{8}$$

Once again we may reduce the problem of showing the above system has a solution to the standard problem of showing that the system,

$$\begin{aligned} P(\cdot_{\lambda}, D)u(\lambda, \eta) &= 0, \\ u_{\lambda k}(0, \eta) &= 0, \quad k = 0 \ldots r - 1, \\ u_{\lambda k}(0, \eta) &= p(\eta), \quad k = r, \quad r < m, \end{aligned} \tag{9}$$

has a solution.

To show when this is the case we write u in terms of it's Fourier transform, giving,

$$u(\lambda, \eta) = (2\pi)^{-(n+1)/2} \int e^{i(\eta \cdot \epsilon + \lambda \Lambda)} \hat{u}(\Lambda, \epsilon) \, d\epsilon \, d\Lambda, \tag{10}$$

where

$$\eta \cdot \epsilon = (\eta_1 \epsilon_1 + \cdots + \eta_n \epsilon_n). \tag{11}$$

Substituting this into equation (9) gives

$$(2\pi)^{-(n+1)/2} \int e^{i(\eta \cdot \epsilon + \lambda \Lambda)} P(i\Lambda, i\epsilon) \hat{u}(\Lambda, \epsilon) \, d\epsilon \, d\Lambda = 0, \tag{12}$$

The symbol of equation (9) is obtained from the kernel of the above expression, and is $P(i\Lambda, i\epsilon)$.

System equation (9), will possess solutions u of class C_0^m provided that all the zeros Λ of $\det P(i\epsilon, i\Lambda)$ are bounded below in the complex plane, and that r is one less than the number of zeros including multiplicities, and that the initial data is of class C_0^{m+n+1}; in which case equation (9) is well-posed as an initial-value problem [7].

4. PRACTICAL APPLICATION OF SYMBOLIC ANALYSIS

Our primary use of symbolic analysis is to identify systems of equations that will lead to stable computational solutions in a single spatial march [9]. For this to be the case it is necessary and sufficient, when the solutions Λ of $\det P(i\Lambda, i\epsilon) = 0$ are continuous functions of ϵ, that $im\,\Lambda$ is bounded below in the complex plane.

4.1. *Full Navier–Stokes equations*

We consider the linearized, nondimensional Navier–Stokes equations for incompressible flow in two-dimensional cartesian coordinates, (x, y), with corresponding velocity components (u, v), with p denoting the pressure and Re is the Reynolds number,

$$\bar{u}u_x + \bar{v}u_y = -p_x + \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}), \tag{13}$$

$$\bar{u}v_x + \bar{v}v_y = -p_y + \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}), \tag{14}$$

$$u_x + v_y = 0, \tag{15}$$

which we will call equation set A. The overbar indicates we are using frozen local values for these components.

The dependent variables u, v, p are written in terms of their Fourier transforms $\hat{u}, \hat{v}, \hat{p}$, as

$$u(x, y) = (2\pi)^{-1} \int \hat{u} e^{i(Xx + Yy)} \, dX \, dY, \tag{16}$$

with equivalent expressions for v and p. Substitution for u, v, p in the linearized equation set A gives

$$\begin{bmatrix} \alpha & 0 & iX & \hat{u} \\ 0 & \alpha & iY & \hat{v} \\ iX & iY & 0 & \hat{p} \end{bmatrix} = 0, \tag{17}$$

where

$$\alpha = i\bar{u}X + i\bar{v}Y + \frac{X^2 + Y^2}{\mathrm{Re}}.$$

The symbol of $A, \sigma(A)$, is the matrix in the above system. The principal symbol, $\sigma_p(A)$ is the matrix,

$$\begin{bmatrix} \dfrac{X^2 + Y^2}{\mathrm{Re}} & 0 & 0 \\ 0 & \dfrac{X^2 + Y^2}{\mathrm{Re}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Equation system A will be elliptic if no nonzero real roots (X, Y) exist for $\det \sigma_p(A) = 0$. It is apparent that $\det \sigma_p(A) = 0$ for all (X, Y) since the matrix A_p is degenerate, and therefore A is not elliptic. The source of the degeneracy is the lack of second derivatives in equation (15).

If we now consider the complete symbol we obtain,

$$\det \sigma(A) = \left(i\bar{u}X + i\bar{v}Y + \frac{X^2 + Y^2}{\mathrm{Re}} \right)(X^2 + Y^2). \tag{19}$$

Equations of the above sort for which the real roots of $\det \sigma(\epsilon) = \mathit{constant}$ can be contained in a finite sphere in ϵ space are members of the set of hypoelliptic equations [7]. Elliptic and parabolic equations are members of the set of hypoelliptic equations. Since $\det \sigma(A)$ has roots of the form $X = \pm iY$ clearly equation set A will not have initial-value solutions, and thus it is, as is well-known, not parabolic, although it is not possible from the above analysis to determine if it is well-posed as a boundary value problem, i.e. whether it is elliptic. We relate the apparent nonellipticity to the fact demonstrated by Gustafsson and Sundstrom [6] that the unsteady Navier–Stokes equations are not of parabolic type, but are defined as incompletely parabolic.

It is possible to obtain a genuinely elliptic equation set from the set A by replacing equation (15) with a Poisson equation for the pressure, which is constructed in the following way. Equation (13) is differentiated with respect to x, equation (14) with respect to y and the resulting equations added to create a Poisson equation for p. Using equation (15) to remove terms the following equation set D is obtained,

$$\bar{u}u_x + \bar{v}u_y = -p_x + \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}), \tag{20}$$

$$\bar{u}v_x + \bar{v}v_y = -p_y + \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}), \tag{21}$$

$$p_{xx} + p_{yy} = -u_x u_x - 2v_x u_y - v_y v_y, \tag{22}$$

where the freezing of the nonlinear terms has been carried out after the manipulation is completed.

The equation set above has as its principal symbol,

$$\begin{bmatrix} \dfrac{X^2+Y^2}{\mathrm{Re}} & 0 & 0 \\ 0 & \dfrac{X^2+Y^2}{\mathrm{Re}} & 0 \\ 0 & 0 & -(X^2+Y^2) \end{bmatrix}, \tag{23}$$

with roots obtained from,

$$\det \sigma_p = -(X^2+Y^2)^3 = 0. \tag{24}$$

The roots of equation (24) are the purely imaginary terms $X = \pm iY$, indicating that the equation system (20)–(22) is elliptic.

4.2. Reduced forms of the Navier–Stokes equations

For many flow problems with a dominant flow direction it can be shown that second derivatives in the flow direction in equations (13) and (14) are always small and can be neglected [9]. This leads to the following system of equations:

$$\bar{u}u_x + \bar{v}u_y = -p_x + \frac{1}{\mathrm{Re}} u_{yy}, \tag{25}$$

$$\bar{u}v_x + \bar{v}v_y = -p_y + \frac{1}{\mathrm{Re}} v_{yy}, \tag{26}$$

$$u_x + v_y = 0, \tag{27}$$

which we will call equation set B. Once again the principal symbol is degenerate, so it is not possible to determine the existence of boundary value solutions. The complete symbol is

$$\det \sigma(B) = \left(i\bar{u}X + i\bar{v}Y + \frac{Y^2}{\mathrm{Re}} \right)(X^2+Y^2). \tag{28}$$

Thus as before we have a hypoelliptic equation which has roots unbounded in the complex plane, and is therefore not well-posed as an initial-value problem.

To attempt to determine exactly how the equations interact to preclude the existence of initial-value solutions we considered the simplified equation set,

$$G(u) = -p_x, \tag{29}$$

$$F(v) = -p_y, \tag{30}$$

$$u_x + v_y = 0, \tag{31}$$

where G and F are general differential operators. We then investigated, numerically when necessary, the roots of all possible forms of these operators.

Obtaining the determinant of the complete symbol of the above equations set we get,

$$\det(\rho) = \hat{G}X^2 + \hat{F}Y^2. \tag{32}$$

Clearly whenever G and F are equal the above simplified equation set will retain the ill-posed character of the full equations. After an extensive investigation of the roots of various forms of F and G it has been found that all but a few degenerate combinations lead to negative imaginary roots. Examples of such degenerate forms are,

$$G = {}_{\cdot x}, \quad F = {}_{\cdot y},$$

$$G = {}_{\cdot x}, \quad F = {}_{\cdot yy}.$$

It is apparent that an ill-posed initial-value problem will result whenever the V momentum equation acts to couple the x derivative terms in the U momentum and continuity equations in a physically realistic manner. This is equivalent to saying that as long as the pressure field is able to act globally to adjust the flow so that it satisfies continuity the equations are ill-posed as an initial-value problem.

If viscous effects are confined to a thin layer for the flow past a stationary solid surface the governing equations can be simplified further to produce equation set C,

$$\bar{u}u_x + \bar{v}u_y = -p_x + \frac{1}{\mathrm{Re}}u_{yy}, \tag{33}$$

$$p_y = 0, \tag{34}$$

$$u_x + v_y = 0. \tag{35}$$

In this case

$$\det \sigma(C) = \left(i\bar{u}X + i\bar{v}Y + \frac{Y^2}{\mathrm{Re}}\right)(Y^2). \tag{36}$$

It can be seen that $\det \sigma(C)$ has the roots $X = iY^2/u - vY/u$ and $(X, 0)$. The first of these roots has a positive imaginary part, provided $\bar{u}$ is positive. It is evident, from the second of the above roots, that the symbol possesses a singularity at the point $Y = 0$ which prevents the equation set being well-posed as an initial-value problem. If this singularity can be removed then it is apparent that the equation set will be well-posed as an initial-value problem in space, with the sign of $\bar{u}$ determining if x or $-x$ is the positive time-like direction. A method of removing the singularity by restricting the solution space to only those functions with compact support in x is suggested.

Consider that u is the solution to a single PDE in (λ, η) space, and that we wish to prove that the equation is well-posed as an initial-value problem in the positive time-like direction. We express u in terms of its initial data $p(\lambda)$ as,

$$u = (2\pi)^{-1}\int e^{i\eta\Gamma}\hat{p}(\Gamma)Q(\lambda, \Gamma)\,d\Gamma. \tag{37}$$

In equation (37) Q is represented as

$$Q(\lambda, \Gamma) = (2\pi)^{-1}\int_C \frac{e^{i\lambda z}}{\sigma(z, \Gamma)}\,dz, \tag{38}$$

where C goes once around each root of $\sigma = 0$.

It is evident that it is not possible to obtain a unique u for a given initial data, due to the multiplicity of zeros that occur at $\Gamma = 0$. If it is possible to remove the single point $\Gamma = 0$ then the resulting solution will be unique, providing that any remaining zeros are bounded below in the complex plane. A method for removing the $\Gamma = 0$ singularity by requiring that any solution must have compact support in η is given.

The variable $u(\lambda, \eta)$ is represented as $U_0 + U_r$, where

$$U_0 = (2\pi)^{-1}\int_{\delta < |\Gamma|} e^{i\eta\Gamma}\hat{p}(\Gamma)Q(\lambda, \Gamma)\,d\Gamma,\ U_r = R(\lambda, \eta), \tag{39}$$

with Q is defined as above. Provided the initial data is of class C_0^{m+n+1}, the integrand of equation (39) is of class C^m and we may take the limit of equation (39) as δ goes to zero to obtain U_0 of class C_0^{m+n+1}. Therefore u will have compact support if U_r does. Therefore we set $R(\lambda, \eta)$ to zero, and

$$u = \lim_{\delta\to 0} U_0.$$

Thus provided the roots of $\sigma(z, \Gamma), \Gamma \neq 0$ are bounded below in the complex plane, and the initial data $p(\lambda)$ is sufficiently smooth, an initial-value solution will exist.

The additional requirement that only solutions with compact support are admissable indicates that equations of this form are not well-posed as pure initial-value problems. This is in accord with the fact that the wave equation $U_{xx} - U_{yy} = 0$, is not well-posed for initial data specified on a characteristic, requiring data also specified on one more line, as the $(X, 0)$ root is typical of hyperbolic systems when the coordinate directions are coincident with the characteristic directions. The requirement that the solution have compact support is equivalent to setting the solution to zero for large $|\eta|$, i.e. imposing the boundary-value $u = 0$ at large $|\eta|$.

We have therefore shown initial-value solutions do exist for such a system provided we limit our solution space to solutions with compact support. In practice this is accomplished by specifying appropriate boundary values for v and p on one line of the form $y = \text{constant}$.

That the full steady-state Navier–Stokes equations, set A, equations (13)–(15), are not well-posed as an initial-value problem is well-known. System B, equations (25)–(27), is often referred to as the "parabolized" Navier–Stokes equations, due to the fact that equations (25) and (26) taken individually [i.e. if $p(x, y)$ were given], are parabolic. However as the symbolic analysis indicates, when the system is considered as a whole it is still hypoelliptic, but does not have initial-value solutions. Equation set C, equations (33)–(35), is also frequently referred to as "parabolic", the justification being, as we have shown, that it will be well-posed as an initial-value problem. Nonetheless it is still not a true parabolic system, being in some sense a mixture of parabolic and hyperbolic types.

4.2.1. Reduced Navier–Stokes equations in cylindrical coordinates. A reduced Navier–Stokes equation set is commonly used to enable internal swirling flow to be solved using single sweep schemes. The equation set that is used is as follows [9]. The Navier–Stokes equations are expressed in nondimensional axisymmetric form in cylindrical coordinates, (x, r, θ) with corresponding velocity components (u, v, w), as follows,

$$uu_x + vu_r = -p_x + \frac{1}{\mathrm{Re}}\left(u_{rr} + \frac{u_r}{r}\right), \tag{40}$$

$$\frac{ww}{r} = p_r, \tag{41}$$

$$uw_x + vw_r + \frac{vw}{r} = \frac{1}{\mathrm{Re}}\left(w_{rr} + \frac{w_r}{r} - \frac{w}{r^2}\right), \tag{42}$$

$$u_x + v_r + \frac{v}{r} = 0. \tag{43}$$

Obtaining the complete symbol of the above equation set as before we get,

$$\det \sigma(A) = \left(i\bar{u}X + i\bar{v}R + \frac{R^2 - iR/r}{\mathrm{Re}}\right)\left(i\bar{u}X + i\bar{v}R + \frac{i\bar{v}}{r} + \frac{R^2 - iR/r + 1/r^2}{\mathrm{Re}}\right)\left(R^2 - \frac{i}{r}\right).$$

Clearly there are no negative imaginary roots, if we can once again preclude the singular point $R = 0$ and require $\bar{u}$ to be positive. Therefore the equation set is well-posed as an initial-value problem in space.

As can be seen the equation set is similar to that obtained in cartesian coordinates, except that the cross-stream pressure gradient is no longer set equal to zero, but retains its component that balances the centrifugal force, which will not in general be insignificant. This term actually arises from the Christoffel symbol associated with the vv_r term. The fact that this term may be retained without affecting the existence of initial value solutions is because the condition of axisymmetry is enforced, and therefore there can be no θ dependent pressure continuity interaction.

4.2.2. Method of Briley. Another interesting reduced form of the Navier–Stokes equations to consider is that presented by Briley and McDonald [10]. In this paper the authors split the cross-stream velocity into a potential component and a rotational component in the following manner.

$$v = v^{\phi} + v^{\psi}, \quad w = w^{\phi} + w^{\psi},$$

where $\cdot^{\phi}$ is the potential component, and $\cdot^{\psi}$ is the rotational component. They relate these components as,

$$v^{\phi} = \phi_y, \quad w^{\phi} = \phi_z, \quad v^{\psi} = \psi_z, \quad w^{\psi} = -\psi_y,$$

with v the velocity in the y direction, and w the velocity in the z direction.

Briley and McDonald further correct the velocities by using a known potential flow, which introduces down stream influence. However since the potential flow is already known, and not solved concurrently, it does not affect the present analysis.

The authors demonstrate that if the streamwise diffusion terms, and the potential component of the cross-stream velocity, are dropped, on an order of magnitude basis, the resulting equation set is well-posed as an initial-value problem in the dominant velocity direction, the x direction. The equation set is as follows,

$$uu_x + vu_y + wu_z = -p_x + \frac{1}{\mathrm{Re}}(u_{yy} + u_{zz}),$$

$$u(\beta v^{\phi} + v^{\psi})_x + v(\beta v^{\phi} + v^{\psi})_y + w(\beta v^{\phi} + v^{\psi})_z = -py - \frac{1}{\mathrm{Re}}(\Omega_z),$$

$$u(\beta w^{\phi} + w^{\psi})_x + v(\beta w^{\phi} + w^{\psi})_y + w(\beta w^{\phi} + w^{\psi})_z = -pz + \frac{1}{\mathrm{Re}}(\Omega_y),$$

$$u_x + (v^{\phi})_y + (w^{\phi})_z = 0.0,$$

$$(v^{\phi})_z - (w^{\phi})_y = 0.0,$$

$$(v^{\psi})_y + (w^{\psi})_z = 0.0,$$

$$-(v^{\psi})_z + (w^{\psi})_y = \Omega.$$

Symbolic analysis is applied to the systems of equations in the following way. After linearizing, and taking Fourier transforms, as before, the above equations set may be written in matrix form as,

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 & iX & 0 \\ 0 & \beta L_2 & L_2 & 0 & 0 & iY & \dfrac{iZ}{\mathrm{Re}} \\ 0 & 0 & 0 & \beta L_2 & L_2 & iZ & \dfrac{-iY}{\mathrm{Re}} \\ iX & iY & 0 & iZ & 0 & 0 & 0 \\ 0 & iZ & 0 & -iY & 0 & 0 & 0 \\ 0 & 0 & iY & 0 & iZ & 0 & 0 \\ 0 & 0 & -iZ & 0 & iY & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v}^{\phi} \\ \hat{v}^{\psi} \\ \hat{w}^{\phi} \\ \hat{w}^{\psi} \\ \hat{p} \\ \hat{\Omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Where in the above expression we have,

$$L_1 = \bar{u}iX + \bar{v}iY + \bar{w}iZ + \frac{1}{\mathrm{Re}}(Y^2 + Z^2),$$

$$L_2 = \bar{u}iX + \bar{v}iY + \bar{w}iZ.$$

Once again obtaining the symbol of the above equation set we get,

$$\det(\rho) = L_1^2(Y^2 + Z^2)^2 + L_1\beta L_2 X^2(Y^2 + Z^2).$$

It is clear that if we set β to be zero then the above equation will have roots of the form,

$$X = i\frac{Y^2 + Z^2}{\bar{u}\,\mathrm{Re}} - \frac{\bar{v}}{\bar{u}}y - \frac{\bar{w}}{\bar{u}}z.$$

Hence there are no negative imaginary roots and so the problem is well-posed as an initial-value problem in the positive x direction, provided $\bar{u}$ is positive.

Briley and McDonald found that if β is set to be 1 then the inviscid version of the equation set will be ill-posed as an initial-value problem in the positive x direction, but found, using a first order characteristics analysis, that the viscous version is well-posed. When β is set equal to 1 in the above expression it has not been possible to obtain an analytic representation for the roots. However numerical experimentation has demonstrated the existence of a root unbounded below in the complex plane, and therefore we suggest the above equation set, with $\beta = 1$, is ill-posed as an initial-value problem in the positive x direction.

It is of interest to consider the method of Briley and McDonald, with $\beta = 0$, in axisymmetric cylindrical coordinates. If we write the equations in cylindrical coordinates, and enforce the condition of axisymmetry, we obtain,

$$w^{\phi} = 0,$$

$$v^{\psi} = 0,$$

$$\Omega = (w^{\psi})_r + \frac{w^{\psi}}{r},$$

and we thus obtain the set of equations,

$$uu_x + vu_r = -p_x + \frac{1}{\mathrm{Re}}\left(u_{rr} + \frac{u_r}{r}\right),$$

$$u(v^{\psi})_x + v(v^{\psi})_r - \frac{w(w^{\psi})}{r} = -p_r + \frac{1}{\mathrm{Re}}\left((v^{\psi})_{rr} + \frac{(v^{\psi})_r}{r} - \frac{(v^{\psi})}{r^2}\right),$$

$$u(w^{\psi})_x + v(w^{\psi})_r + \frac{v(w^{\psi})}{r} = \frac{1}{\mathrm{Re}}\left((w^{\psi})_{rr} + \frac{(w^{\psi})_r}{r} - \frac{(w^{\psi})}{r^2}\right),$$

$$u_x + (v^{\phi})_r + \frac{(v^{\phi})}{r} = 0.0.$$

Since v^{ψ} and w^{ϕ} are zero we see that this is the same equation set as equations (40)–(43) above with $v = v^{\phi}$ and $w = w^{\psi}$. Evidently the approximation made for internal swirling flow to enable the Navier–Stokes equations to be well-posed as an initial-value problem is a specialization of the general method of Briley and McDonald.

5. CONCLUDING REMARKS

Our analysis started with the classical theory of scalar second order PDEs, which are divided into elliptic, parabolic and hyperbolic categories, based on the sign of the discrimant. That such a classification is complete, and divides all second order PDEs into those that must be solved as boundary-value problems, and those that may be solved as initial-value problems, makes it a useful tool in determining numerical methods for solving such equations. However, as we have pointed out, problems arise in the extension of such a classification to higher order systems. If we are to satisfactorily deal with higher order systems we must be able to determine whether such systems are well-posed as initial-value problems. This has been the motivation behind our development of the symbolic analysis presented in this paper. In particular it is used to analyse some of the various forms of the Navier–Stokes equations currently employed in the field of computational fluid dynamics.

In Section 2 the constraint that must be placed on the roots of the symbol in order to guarantee the existence of initial-value solutions is given without proof; an extremely good exposition of this proof is given in Schecter [7]. Section 3 establishes the usefulness of this analytic tool, by showing that it is readily extended to higher order systems, such as the Navier–Stokes equations (Section 4).

As has been shown symbolic analysis is a valuable tool for determining when higher order systems of PDEs provide well-posed initial-value problems. In relation to steady fluid flow problems, reduced forms of the Navier–Stokes equations are analysed to see if they permit stable computational solutions to be obtained in a single march in the time-like direction.

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