JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **215,** 388–414 (1997)
ARTICLE NO. AY975636

Stability of Travelling Waves for a Cross-Diffusion Model

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Submitted by Colin Rogers

Received September 30, 1996

A cross-diffusion model of mono-species forest with two age classes is considered in this paper. By C_0-semigroup theory and a detailed spectral analysis, the asymptotic stability of travelling waves is proved.

1. INTRODUCTION

A simple mathematical model of mono-species forest with two age classes which takes account of seed production and dispersal was first presented in [1],

$$\begin{cases} u_t = \delta\beta w - \gamma(v)u - fu \\ v_t = fu - hv \\ w_t = \alpha v - \beta w + p w_{xx} \end{cases} \tag{1.1}$$

where $\gamma(v) = a(v-b)^2 + c$ with positive a, b, c.

By means of an asymptotic procedure, (1.1) is then reduced to the following lower-dimensional reaction-cross-diffusion model [1]

$$\begin{cases} u_t = \rho v - (v-1)^2 u - su + v_{xx} \\ v_t = u - hv. \end{cases} \tag{1.2}$$

* Research supported by National Science Foundation of China and Sichuan Youth Science and Technology Foundation.

0022-247X/97 $25.00

The existence and stability of standing waves of (1.2) has been obtained in [1] (however, the proof of stability in [1] is far from complete), in which a numerical analysis of the existence of the travelling waves was also given. By analytic methods, the existence of other travelling waves of (1.2) was proved in [2], and a detailed relation between speed and parameters ρ, h, s was obtained.

Here, we first briefly state the main results in [2, 1].

Define

$$\text{region } 0 = \{(\rho, h)/0 < \rho < sh\},$$
$$\text{region } 1 = \{(\rho, h)/sh < \rho < (s+1)h\},$$
$$\text{region } 2 = \{(\rho, h)/\rho > (s+1)h\}.$$

In region 0, (1.2) has only one equilibrium point $(0,0)$; in region 1, (1.2) has three equilibrium points $(0,0)$, (u_-, v_-), and (u_+, v_+); in region 2, (1.2) has two nonnegative equilibrium points $(0,0)$ and (u_+, v_+), with

$$v_{\pm} = 1 \pm \left(\frac{\rho - sh}{h}\right)^{1/2}, \qquad u_{\pm} = hv_{\pm}.$$

We further divide region 1 into two subregions D_{11} and D_{12} as

$$D_{11} = \left\{(\rho, h)/sh < \rho < sh + \frac{1}{9}h\right\},$$
$$D_{12} = \left\{(\rho, h)/sh + \frac{1}{9}h < \rho < (s+1)h\right\}.$$

It is shown in [1] that for $\rho = (s + 1/9)h$, there exists a standing wave $(U(x), V(x))$ of (1.2) with the explicit form

$$\begin{cases} U = hV \\ V = \frac{4}{3}\left[1 + \exp\left(-\frac{4}{3}\sqrt{h/2}\,x\right)\right]^{-1}, \end{cases} \qquad \rho = \left(s + \frac{1}{9}\right)h. \tag{1.3}$$

THEOREM 1 [2]. *For any* $(\rho, h) \in D_{11} \cup D_{12}$, *there exists a unique* $c_1^*(\rho, h)$ *such that* (1.2) *has a travelling wave solution* $(U(x + c_1^* t), V(x + c_1^* t))$ *connecting* $(0,0)$ *and* (u_+, v_+), *where* V *is a monotone increasing* *function and*

$$0 > c_1^*(\rho, h) > -\frac{\sqrt{h}}{\sqrt{(s+h)^2 + h}} \geq -\frac{1}{\sqrt{4s+1}}, \qquad (\rho, h) \in D_{11}, \tag{1.4}$$

$$0 < c_1^*(\rho, h) < \frac{2\sqrt{h}}{\sqrt{(s+h)^2 + 4h}} \leq \frac{1}{\sqrt{s+1}}, \qquad (\rho, h) \in D_{12}. \tag{1.5}$$

Furthermore, for fixed s and h, $c_1^*(\rho, h)$ *is monotone increasing in* ρ. *For fixed s and* ρ, $c_1^*(\rho, h)$ *is monotone decreasing in h for* (ρ, h) *in* D_{12}.

In [2], existence of travelling wave solutions of (1.2) connecting (u_-, v_-) and (u_+, v_+) with $(\rho, h) \in$ region 1 was also obtained for any $c \in [\bar{c}_1(\rho, h), 1)$. For $(\rho, h) \in$ region 2, the existence of travelling waves of (1.2) for any $c \in [\bar{c}_2(\rho, h), 1)$ was similarly obtained in [2].

In this paper, we shall consider the stability of travelling waves obtained in Theorem 1 of [2] and (1.3), i.e., the stability of travelling waves connecting $(0, 0)$ and (u_+, v_+), for (ρ, h) in region 1.

We note that due to the cross-diffusion term, (1.2) is no longer a parabolic system, in fact, (1.2) is related to a wave equation with dissipative term, so the theory of analytic semigroups is not valid for (1.2). Furthermore, the classical theory of stability of travelling waves for parabolic systems [3, 4] (e.g., [4, p. 215, Theorem 1.1]) cannot be applied directly to (1.2). Therefore, to deal with the problems with cross-diffusion terms we have to develop a suitable setting, subtle estimates, and spectral analysis.

In this paper, combining the theory of the C_0-semigroup [5, 7] with some basic ideas in [3], by a series of detailed spectral analysis, we show that the travelling waves obtained in Theorem 1 are exponentially stable with shift in a suitable space. In some sense, we offer a setting for dealing with more general systems in the context of the C_0-semigroup, although the needed spectral estimates and the splitting of the space is not easy to obtain for non-parabolic systems. The proof of stability is also valid for the standing wave of (1.2), which also overcomes the shortcoming of the proof in [1]. In [1], only the estimates of the eigenvalues for the linearized system are obtained.

The plan of this paper is as follows.

In Section 2, the local existence of solutions of (1.2) is proved and the main result (Stability Theorem) is stated. In Section 3, a series of detailed spectral results are proved. In Section 4, the proof of the main result is given. Some basic lemmas in Section 4 are proved in the Appendix.

2. THE LOCAL EXISTENCE AND MAIN RESULTS

Consider the initial value problem

$$\begin{cases} u_t = \rho v - (v-1)^2 u - su + v_{xx} \\ v_t = u - hv \\ u(0,x) = u_0(x) \\ v(0,x) = v_0(x). \end{cases} \tag{2.1}$$

For (ρ, h) in region 1, there exists a unique travelling wave solution $(U(x+ct), V(x+ct))$ of (2.1) connecting $(0,0)$ and (u_+, v_+) (see Theorem 1 and (1.3)).

By introducing a new variable $\xi = x + ct$, (2.1) can be rewritten as

$$\begin{cases} u_t = -cu_\xi + \rho v - (v-1)^2 u - su + v_{\xi\xi} \\ v_t = -cv_\xi + u - hv \\ u(0,\xi) = u_0(\xi) \\ v(0,\xi) = v_0(\xi). \end{cases} \tag{2.2}$$

Obviously, $(U(\xi), V(\xi))$ is a stationary solution of (2.2). We define the exponential stability of $(U(x+ct), V(x+ct))$ as follows.

DEFINITION 2.1. The travelling wave solution $(U(x+ct), V(x+ct))$ of (2.1) is said to be exponentially stable with shift according to the norm of $\| \ \|_X$, if there exists $\delta_0 > 0$ such that for any (u_0, v_0) with $\|(u_0 - U, v_0 - V)\|_X \le \delta_0$, there exists a unique solution $(u(t,\xi), v(t,\xi))$ of (2.2) on $(0, +\infty)$, with $(u(t,\xi) - U(\xi), v(t,\xi) - V(\xi)) \in X$ and satisfies

$$\left\| \left(u(t,\xi) - U(\xi+\xi_0), v(t,\xi) - V(\xi+\xi_0) \right) \right\|_X \le Me^{-\beta' t},$$

where ξ_0 is a number depending on (u_0, v_0); $M > 0$ and $\beta' > 0$ are independent of t, ξ_0, and (u_0, v_0).

To prove the stability of $(U(\xi), V(\xi))$, we first consider the local existence of solution of (2.2) in some suitable space. Define $X = L_2(R) \times H^1(R)$, $X_0 = H^1(R) \times H^2(R)$, and

$$Y = \{(u,v)/(u-U, v-V) \in X\},$$

$$Y_0 = \{(u,v)/(u-U, v-V) \in X_0\}.$$

It is easy to prove that $(U,V) \in C^1(R) \times C^2(R)$ and that (U,V) tends to $(0,0)$ at $-\infty$ and tends to (u_+, v_+) at $+\infty$ exponentially, respectively.

THEOREM 2.1. *For any* $(u_0(\xi), v_0(\xi)) \in Y$ *there exists a unique mild solution* $(u(t,\xi), v(t,\xi)) \in C([0,t_0), Y)$ *of* (2.2) *on some* $(0, t_0)$; *furthermore, if* $(u_0(\xi), v_0(\xi)) \in Y_0$, *then*

$$(u(t,\xi), v(t,\xi)) \in C\big([0,t_0), Y_0\big) \times C^1\big([0,t_0), Y\big).$$

Denote

$$w_1(t,\xi) = u(t,\xi) - U(\xi), \qquad w_2(t,\xi) = v(t,\xi) - V(\xi).$$

Then $(w_1(t,\xi), w_2(t,\xi))$ satisfies

$$\begin{cases} w_{1t} = -cw_{1\xi} + w_{2\xi\xi} + f_0(w_1, w_2) \\ w_{2t} = -cw_{2\xi} + w_1 + g_0(w_1, w_2) \\ w_1(0,\xi) = w_{10}(\xi) \\ w_2(0,\xi) = w_{20}(\xi), \end{cases} \tag{2.3}$$

where

$$\begin{cases} f_0(w_1, w_2) = \rho w_2 - s w_1 - (V-1)^2 w_1 - 2U(V-1)w_2 - Uw_2^2 \\ \qquad\qquad\qquad -2(V-1)w_1 w_2, \\ g_0(w_1, w_2) = -hw_2. \end{cases} \tag{2.4}$$

To prove Theorem 2.1, we only need to prove the local existence of solution of (2.3) and (2.4).

Let

$$A = \begin{bmatrix} -c\dfrac{\partial}{\partial\xi} & \dfrac{\partial^2}{\partial\xi^2} \\ 1 & -c\dfrac{\partial}{\partial\xi} \end{bmatrix}. \tag{2.5}$$

Then (2.3) can be written as

$$\begin{cases} w_t = Aw + F_0(w) \\ w(0) = w_0, \end{cases} \tag{2.6}$$

with

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad F_0(w) = \begin{bmatrix} f_0(w_1, w_2) \\ g_0(w_1, w_2) \end{bmatrix}.$$

LEMMA 2.1. *The linear operator* A: $X_0 \to X$ *is an infinitesimal generator of a* C_0-semigroup $T(t)$ *on* X *satisfying*

$$\|T(t)\|_{X \to X} \leq e^t. \tag{2.7}$$

Proof. Obviously $D(A) = X_0$, and $\overline{D(A)} = X$. It is only needed to prove that the resolvent set $\rho(A)$ of A contains the ray $(1, \infty)$ and

$$\left\|(\lambda I - A)^{-1}\right\|_{X \to X} \leq \frac{1}{\lambda - 1}, \qquad \text{for } \lambda > 1. \tag{2.8}$$

For any $(f, g) \in X$, $\lambda > 1$, let w_2 be the solution of

$$-(1 - c^2)w_2'' + 2c\lambda w_2' + \lambda^2 w_2 = g^* \triangleq f + \lambda g + cg' \in L^2(R). \tag{2.9}$$

By Fourier transformation, it is easy to prove that there exists a unique solution $w_2 \in H^2(R)$ of (2.9).

Let $w_1 = \lambda w_2 + cw_2' - g$. Then $(w_1, w_2) \in H^1 \times H^2$ is the unique solution of the equation

$$(\lambda I - A)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \triangleq F. \tag{2.10}$$

Furthermore, it follows from (2.10) that

$$\begin{aligned}
\|F\|_X^2 &= \|f\|_{L_2}^2 + \|g\|_{L_2}^2 + \|g'\|_{L_2}^2 \\
&= \int_R \Big(|\lambda w_1 + cw_1' - w_2''|^2 + |\lambda w_2 - w_1 + cw_2'|^2 \\
&\qquad\qquad + |\lambda w_2' - w_1' + cw_2''|^2 \Big)\, d\xi \\
&\geq \lambda^2 \Big(\|w_1\|_{L_2}^2 + \|w_2\|_{L_2}^2 + \|w_2'\|_{L_2}^2 \Big) - 2\lambda \operatorname{Re} \int_R \big(w_1 \overline{w_2} \big)\, d\xi \\
&> (\lambda - 1)^2 \Big(\|w_1\|_{L_2}^2 + \|w_2\|_{L_2}^2 + \|w_2'\|_{L_2}^2 \Big), \qquad \text{for } \lambda > 1,
\end{aligned}$$

thus

$$\left\|(\lambda I - A)^{-1}\right\|_{X \to X} \leq \frac{1}{\lambda - 1}, \qquad \text{for } \lambda > 1,$$

which completes the proof of Lemma 2.1.

Proof of Theorem 2.1. Note that $H^1(R) \hookrightarrow L_\infty(R)$, thus

$$F_0(w) = \begin{pmatrix} f_0(w_1, w_2) \\ g_0(w_1, w_2) \end{pmatrix} : X \to X$$

and $F_0(w)$ is lipschitz continuous for any $w \in X$.

By Lemma 2.1 and the theory of the C_0-semigroup [5], it follows that for any $w_0 \in X$, there exists a unique mild solution $w(t) \in C([0, t_0), X)$ of (2.6); furthermore, if $w_0 \in D(A) = X_0$, then

$$w(t) \in C([0, t_0), X_0) \cap C^1([0, t_0), X).$$

This completes the proof of Theorem 2.1.

Now we state the main result of this paper.

THEOREM 2.2 (Stability Theorem). *For any* $(\rho, h) \in$ *region* 1, *the travelling wave solution* $(U(x+ct), V(x+ct))$ *of* (2.1) *obtained in Theorem* 1 *and* (1.3) *is exponentially stable with shift in norm of* $L_2(R) \times H^1(R)$.

3. SOME PRELIMINARY SPECTRAL RESULTS

For any fixed $(\rho, h) \in$ region 1, let $(U(x+ct), V(x+ct))$ be the travelling wave solution obtained in Theorem 1 and (1.3). Denote $\xi = x + ct$. Then $(U(\xi), V(\xi))$ is a stationary solution of (2.2).

Linearizing (2.2) at $(U(\xi), V(\xi))$, we obtain

$$\begin{cases} u_t = \rho v - (V-1)^2 u - 2U(V-1)v - su + v_{\xi\xi} - cu_\xi \triangleq L_1(u, v) \\ v_t = u - hv - cv_\xi \triangleq L_2(u, v). \end{cases} \tag{3.1}$$

Define an operator L: $X_0 \to X$ as

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} L_1(u, v) \\ L_2(u, v) \end{pmatrix},$$

thus

$$L = \begin{pmatrix} -(V-1)^2 - s - c\dfrac{\partial}{\partial \xi} & \rho - 2U(V-1) + \dfrac{\partial^2}{\partial \xi^2} \\ 1 & -h - c\dfrac{\partial}{\partial \xi} \end{pmatrix}. \tag{3.2}$$

Let $w_1(t,\xi) = u(t,\xi) - U(\xi)$, $w_2(t,\xi) = v(t,\xi) - V(\xi)$. Then $w(t,\xi) = (w_1(t,\xi), w_2(t,\xi))$ satisfies

$$\begin{cases} w_t = Lw + O(|w|^2) \\ w(0,\xi) = w_0(\xi). \end{cases} \tag{3.3}$$

In this section, we shall obtain some detailed spectral estimates for the operator L.

THEOREM 3.1. *For any fixed* $(\rho, h) \in$ *region* 1, *there exists* $\beta_0 > 0$, *such that*

$$\sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(L) \setminus \{0\}\} \leq -\beta_0, \tag{3.4}$$

and 0 *is the simple eigenvalue of* L, *where* $\sigma(L)$ *denotes the spectral set of* L.

We note that for any $\lambda \in \rho(L) = \mathbf{C} \setminus \sigma(L)$, the following property holds: $\forall\ (f,g) \in X$, there exists a unique solution $(u,v) \in X_0$ of the problem

$$(\lambda I - L)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{3.5}$$

It follows from (3.5) that (u,v) satisfies

$$u = (\lambda + h)v + cv' - g, \tag{3.6}$$

and

$$v'' - b(\xi,\lambda)v' + d(\xi,\lambda)v = \frac{1}{1-c^2}\left[f + (V-1)^2 g + sg + cg'\right], \tag{3.7}$$

with

$$b(\xi,\lambda) = \frac{c}{1-c^2}\left[2\lambda + s + h + (V-1)^2\right]$$

$$d(\xi,\lambda) = \frac{1}{1-c^2}\Big[\rho - (\lambda+h)(\lambda+s) - 2U(V-1) - (\lambda+h)(V-1)^2\Big].$$

Define the operator $B(\lambda)$: $H^2(R) \to L_2(R)$ as

$$B(\lambda)v = v'' - b(\xi,\lambda)v' + d(\xi,\lambda)v. \tag{3.8}$$

Equations (3.6)–(3.8) imply that

$$\lambda \in \rho(L), \qquad \text{iff } 0 \in \rho(B(\lambda)), \tag{3.9}$$

and

$$\lambda \in \sigma_p(L), \qquad \text{iff } 0 \in \sigma_p(B(\lambda)). \tag{3.10}$$

LEMMA 3.1. *For any fixed* $(\rho, h) \in$ *region* 1, *there exists* $\alpha_0 > 0$, *such that*

$$\sigma_p(L) \setminus \{0\} \subset \{\lambda / \operatorname{Re} \lambda \le -\alpha_0\}, \tag{3.11}$$

and 0 *is a simple eigenvalue of* L.

To prove Lemma 3.1, we need the following spectral results about $B(\lambda)$.

PROPOSITION 3.1. *For any fixed* $(\rho, h) \in D_{11} \cup D_{12}$, *if for some* λ *with* $\operatorname{Re} \lambda \ge -\delta_0$ ($a_0 > \delta_0 > 0$ *small enough depending only on* ρ, h, s), *there exists a solution* $v(\xi, \lambda) \in H^2(R)$ *of*

$$B(\lambda) v(\xi, \lambda) = 0, \tag{3.12}$$

then $v(\xi, \lambda)$ *must decay to zero exponentially as* $\xi \to \pm\infty$ *with exponential rate* $\sigma_{\pm}(\lambda)$, *respectively,*

$$\sigma_+(\lambda) = \operatorname{Re}\left(\frac{b_+(\lambda) - \sqrt{b_+^2(\lambda) - 4d_+(\lambda)}}{2} \right),$$

$$\sigma_-(\lambda) = \operatorname{Re}\left(\frac{b_-(\lambda) + \sqrt{b_-^2(\lambda) - 4d_-(\lambda)}}{2} \right),$$

$$b_+(\lambda) = b(+\infty, \lambda), \qquad d_+(\lambda) = d(+\infty, \lambda),$$

$$b_-(\lambda) = b(-\infty, \lambda), \qquad d_-(\lambda) = d(-\infty, \lambda).$$

Proof. Note that for any fixed $(\rho, h) \in D_{11} \cup D_{12}$, there exists a $\delta_0 > 0$ small enough depending only on ρ, h, s such that for $\operatorname{Re} \lambda \ge -\delta_0$,

$$\operatorname{Re} b_+(\lambda) < 0, \qquad \operatorname{Re} b_-(\lambda) < 0, \text{ if } c < 0,$$

$$\operatorname{Re} b_+(\lambda) > 0, \qquad \operatorname{Re} b_-(\lambda) > 0, \text{ if } c > 0.$$

First we prove that

$$\operatorname{Re}\left(\sqrt{b_+^2 - 4d_+}\right) > |\operatorname{Re}(b_+)| > 0. \tag{3.13}$$

Denote

$$b_+ = b_1 + ib_2, \qquad d_+ = d_1 + id_2, \qquad \sqrt{b_+^2 - 4d_+} = c_1 + ic_2, c_1 \ge 0,$$

and

$$\lambda = \lambda_1 + i\lambda_2,$$

where b_1, b_2, d_1, d_2, λ_1, and λ_2 are real numbers satisfying

$$b_1 = \frac{c}{1-c^2}\left[2\lambda_1 + s + h + (v_+ - 1)^2\right], \qquad b_2 = \frac{2\lambda_2 c}{1-c^2},$$

$$d_1 = \frac{1}{1-c^2}\Big[\rho - hs - 2u_+(v_+ - 1) - h(V-1)^2 - \lambda_1(h+s) - \lambda_1(v_+ - 1)^2 - \lambda_1^2 + \lambda_2^2\Big],$$

$$d_2 = -\frac{\lambda_2}{1-c^2}\left[2\lambda_1 + s + h + (v_+ - 1)^2\right] = \frac{c^2-1}{2c^2} b_1 b_2.$$

Note that

$$b_1^2 - b_2^2 - 4d_1 = c_1^2 - c_2^2, \qquad b_1 b_2 - 2d_2 = c_1 c_2, \tag{3.14}$$

and

$$b_1 b_2 - 2d_2 = \frac{1}{c^2} b_1 b_2, \qquad d_1 < \frac{1}{1-c^2}\lambda_2^2.$$

By (3.14), we have

$$c_1^2 = \frac{\left(b_1^2 - b_2^2 - 4d_1\right) + \sqrt{\left(b_1^2 - b_2^2 - 4d_1\right)^2 + 4(b_1 b_2 - 2d_2)^2}}{2}$$

and

$$c_1^2 - b_1^2 = \frac{-b_1^2 - b_2^2 - 4d_1 + \sqrt{\left(b_1^2 - b_2^2 - 4d_1\right)^2 + (4/c^4) b_1^2 b_2^2}}{2}.$$

(i) If $b_1^2 + b_2^2 + 4d_1 < 0$, obviously we have $c_1^2 - b_1^2 > 0$, thus (3.13) holds.

(ii) If $b_1^2 + b_2^2 + 4d_1 = 0$, obviously $c_1^2 - b_1^2 \geq 0$. It is easy to see that if $c_1^2 - b_1^2 = 0$, then $b_1 = 0$, which is impossible. Thus (3.13) holds

(iii) If $b_1^2 + b_2^2 + 4d_1 > 0$, then

$$c_1^2 - b_1^2 = \frac{-\left(b_1^2 + b_2^2 + 4d_1\right)^2 + \left(b_1^2 - b_2^2 - 4d_1\right)^2 + (4/c^4)b_1^2 b_2^2}{2\left(\left(b_1^2 + b_2^2 + 4d_1\right) + \sqrt{\left(b_1^2 - b_2^2 - 4d_1\right)^2 + (4/c^4)b_1^2 b_2^2}\right)}.$$

Note that

$$\begin{aligned}
-\left(b_1^2 + b_2^2 + 4d_1\right)^2 &+ \left(b_1^2 - b_2^2 - 4d_1\right)^2 + \frac{4}{c^4} b_1^2 b_2^2 \\
&= 4b_1^2\left(-b_2^2 - 4d_1 + \frac{1}{c^4} b_2^2\right) \\
&> 16 b_1^2\left[\left(\frac{1}{c^4} - 1\right)\frac{\lambda_2^2 c^2}{(1 - c^2)^2} - \frac{\lambda_2^2}{1 - c^2}\right] \geq 0,
\end{aligned}$$

thus $c_1^2 - b_1^2 > 0$, which completes the proof of (3.13).

Similarly, we can prove that

$$\operatorname{Re}\left(\sqrt{b_-^2 - 4d_-}\right) > |\operatorname{Re} b_-| > 0. \tag{3.15}$$

Therefore $v(\xi, \lambda)$ must decay to zero exponentially as $\xi \to \pm\infty$ with exponential rate $\sigma_\pm$, respectively. This completes the proof of Proposition 3.1.

Let $v(\xi)$ be a solution of (3.12) for some λ. Define

$$\hat{v}(\xi) = v(\xi)\exp\left(-\frac{1}{2}\int_0^\xi b(s, \lambda)\, ds\right). \tag{3.16}$$

Then $\hat{v}(\xi)$ satisfies

$$\hat{B}(\lambda)\hat{v} \triangleq \hat{v}_{\xi\xi} - D(\xi, \lambda)\hat{v} = 0, \tag{3.17}$$

with

$$\begin{aligned}
D(\xi, \lambda) &= -d(\xi, \lambda) - \frac{1}{2} b'(\xi, \lambda) + \frac{1}{4} b^2(\xi, \lambda) \\
&= \frac{1}{(1 - c^2)^2}\left[\lambda^2 + \lambda\left(h + s + (V - 1)^2\right)\right] + E(\xi).
\end{aligned}$$

It follows from Proposition 3.1 and (3.13)–(3.16) that $\hat{v}(\xi)$ decays to zero exponentially at infinity with the exponential rate $\gamma_\pm =$

$\operatorname{Re}(\mp \sqrt{b_\pm^2 - 4d_\pm}/2)$, respectively, which implies that if for some λ with $\operatorname{Re}\lambda \geq -\delta_0$, $0 \in \sigma_p(B(\lambda))$, then for the same λ with $\operatorname{Re}\lambda \geq -\delta_0$, $0 \in \sigma_p(\hat{B}(\lambda))$.

Note that for $\rho = (s + 1/9)h$, we have $c = 0$ and $B(\lambda) \equiv \hat{B}(\lambda)$.

PROPOSITION 3.2. *If for some λ with* $\operatorname{Re}\lambda \geq -\delta_0$, $0 \in \sigma_p(\hat{B}(\lambda))$, *then λ must be a real number.*

Proof. If $0 \in \sigma_p(\hat{B}(\lambda))$, for some λ with $\operatorname{Re}\lambda \geq -\delta_0$, let $v(\xi) \in H^2(R)$ satisfy

$$v_{\xi\xi} - D(\xi, \lambda)v = 0.$$

Then

$$\int_R |v_\xi|^2 \, d\xi + \int_R D(\xi, \lambda)|v|^2 \, d\xi = 0,$$

thus

$$\operatorname{Im} \int_R D(\xi, \lambda)|v|^2 \, d\xi$$
$$= \frac{(\operatorname{Im}\lambda)}{(1 - c^2)^2} \int_R \left(2\operatorname{Re}\lambda + h + s + (V - 1)^2\right)|v|^2 \, d\xi = 0,$$

which implies $\operatorname{Im}\lambda = 0$; this completes the proof of Proposition 3.2.

In the following, we only need to prove the non-existence of positive constant λ for $0 \in \sigma_p(\hat{B}(\lambda))$.

By contradiction, assume there exists a positive constant λ for $0 \in \sigma_p(\hat{B}(\lambda))$.

Note that

$$D_\lambda(\xi, \lambda) > 0, \qquad \textit{for } \lambda \geq 0. \tag{3.18}$$

and

$$-\left(\widehat{V_\xi}\right)_{\xi\xi} + D(\xi, 0)\widehat{V_\xi} = 0,$$

with $\widehat{V_\xi}(\xi) = V_\xi(\xi)\exp(-\frac{1}{2}\int_0^\xi b(s,0)\,ds) > 0$.

The Theorem of Sturm–Liouville assures us that $\lambda = 0$ is the first simple eigenvalue for the linear eigenvalue problem

$$-v_{\xi\xi} + D(\xi, 0)v = \lambda v, \tag{3.19}$$

with eigenfunction $\widehat{V_\xi}(\xi)$.

Thus for any $\psi(\xi) \in H^1(R)$, we have

$$\int_R \left(|\psi_\xi|^2 + D(\xi, 0)|\psi|^2 \right) d\xi \geq 0. \tag{3.20}$$

On the other hand, let $v(\xi, \lambda)$ satisfy

$$v_{\xi\xi}(\xi, \lambda) - D(\xi, \lambda) v(\xi, \lambda) = 0.$$

Then it follows from (3.18) that

$$\int_R \left(\left| v_\xi(\xi, \lambda) \right|^2 + D(\xi, 0) \left| v(\xi, \lambda) \right|^2 \right) d\xi$$

$$= \int_R (-D(\xi, \lambda) + D(\xi, 0)) \left| v(\xi, \lambda) \right|^2 d\xi < 0,$$

which contradicts (3.20).

Therefore, there exists no positive λ for (3.17), furthermore, 0 is a simple eigenvalue for (3.19); the same results hold for (3.12). This completes the proof of Lemma 3.1.

LEMMA 3.2. *For any fixed* $(\rho, h) \in$ *region* 1, *there exists* $\beta_1 > 0$ *such that if* $0 \in \sigma_{\text{ess}}(B(\lambda))$ $\forall\, \lambda \in S_0$, *then* $S_0 \subset \{\lambda / \operatorname{Re} \lambda \leq -\beta_1\}$.

To prove Lemma 3.2, we first consider the following operator $B_0(\lambda)$: $H^2(R) \to L_2(R)$,

$$B_0(\lambda) = \frac{d^2}{d\xi^2} - b_0(\xi, \lambda) \frac{d}{d\xi} + d_0(\xi, \lambda), \tag{3.21}$$

with

$$b_0(\xi, \lambda) = \begin{cases} b(-\infty, \lambda), & \xi < 0 \\ b(+\infty, \lambda), & \xi > 0, \end{cases}$$

$$d_0(\xi, \lambda) = \begin{cases} d(-\infty, \lambda), & \xi < 0 \\ d(+\infty, \lambda), & \xi > 0. \end{cases}$$

Note that

$$0 \in \sigma_{\text{ess}}(B(\lambda)) \qquad \text{if and only if } 0 \in \sigma_{\text{ess}}(B_0(\lambda)). \tag{3.22}$$

Let

$$S_- = \left\{ \lambda / -\tau^2 - i\tau b(-\infty, \lambda) + d(-\infty, \lambda) = 0, \text{ for some real } \tau \right\}, \tag{3.23}$$

and

$$S_+ = \{\lambda / -\tau^2 - i\tau b(+\infty, \lambda) + d(+\infty, \lambda) = 0, \text{ for some real } \tau\}. \quad (3.24)$$

By computation, we have

$$S_- = \left\{\lambda \Big/ \left(\operatorname{Re}\lambda + \frac{h+s+1}{2}\right)^2 + \frac{\operatorname{Im}^2\lambda}{c^2} = a_-\right\}$$
$$\cup \left\{\lambda / \operatorname{Re}\lambda = -\frac{s+h+1}{2}\right\},$$

with

$$a_- = \frac{(h+s+1)^2}{4} + \rho - (s+1)h > 0$$

and

(i) If $a_+ = (h + \rho/h)^2/4 - 2(\rho - sh) - 2\sqrt{h(\rho - sh)} > 0$, then

$$S_+ = \left\{\lambda \Big/ \left(\operatorname{Re}\lambda + \frac{(h+\rho/h)^2}{2}\right)^2 + \frac{\operatorname{Im}^2\lambda}{c^2} = a_+\right\}$$
$$\cup \left\{\lambda / \operatorname{Re}\lambda = -\frac{\rho/h + h}{2}\right\};$$

(ii) If $a_+ \leq 0$, then

$$S_+ = \left\{\lambda / \operatorname{Re}\lambda = -\frac{\rho/h + h}{2}\right\}.$$

Obviously,

$$\sup\{\operatorname{Re}\lambda;\ \lambda \in S_- \cup S_+\} \leq -\beta_1, \quad (3.25)$$

with

$$\beta_1 = \min\left\{\sqrt{(s+1)h - \rho},\ \sqrt{2(\rho - sh) + 2\sqrt{h(\rho - sh)}},\ \frac{h + \rho/h}{2}\right\}.$$

Define

$$P = \{\lambda / \operatorname{Re}\lambda > -\beta_1\}. \quad (3.26)$$

P is an open connected set in $\mathbf{C} \setminus (S_+ \cup S_-)$. It follows from [3, p. 138, Lemma 2] that either

(i) $0 \in \sigma(B_0(\lambda))$ for all λ in P, or (3.27)

(ii) $0 \in \rho(B_0(\lambda))$ for all λ in P, except at isolated points, at which 0 is an eigenvalue. (3.28)

LEMMA 3.3. *For any fixed* $(\rho, h) \in$ *region* 1, *if* $\operatorname{Re} \lambda > 0$, *then*

$$0 \in \rho\big(B_0(\lambda)\big), \qquad \text{and} \qquad \big\|B_0^{-1}(\lambda)\big\|_{L_2(R)\to H^2(R)} \le C(\lambda).$$

Proof. For $(v_1, v_2) \in H^1(R) \times H^1(R)$, define an operator $\overline{B}_0(\lambda)$: $H^1(R) \times H^1(R) \to L_2(R) \times L_2(R)$ as

$$\overline{B}_0(\lambda)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' + A_0(\xi, \lambda)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

with

$$A_0(\xi, \lambda) = \begin{cases} A_+(\lambda), & \xi > 0 \\ A_-(\lambda), & \xi < 0, \end{cases}$$

$$A_\pm(\lambda) = \begin{bmatrix} 0 & -1 \\ d_\pm(\lambda) & -b_\pm(\lambda) \end{bmatrix},$$

and

$$b_\pm(\lambda) = b(\pm\infty, \lambda), \qquad d_\pm(\lambda) = d(\pm\infty, \lambda).$$

Let $\sigma_+^\pm(\lambda)$ and $\sigma_-^\pm(\lambda)$ be eigenvalues of $A_+(\lambda)$ and $A_-(\lambda)$, respectively. Then

$$\sigma_+^\pm = \frac{-b_+(\lambda) \pm \sqrt{b_+^2(\lambda) + 4d_+(\lambda)}}{2},$$

and

$$\sigma_-^\pm = \frac{-b_-(\lambda) \pm \sqrt{b_-^2(\lambda) = 4d_-(\lambda)}}{2}.$$

For $\operatorname{Re} \lambda > 0$, Proposition 3.1 assures us that

$$\operatorname{Re} \sigma_\pm^+ > 0 \qquad \text{and} \qquad \operatorname{Re} \sigma_\pm^- < 0. \tag{3.29}$$

Let E_+, E_- be the projections corresponding to the eigenvalues of $A_\pm(\lambda)$ in the right half-plane. Then (3.29) assures us that

$$R(E_+) = \{k(1, -\sigma_+^+), k \in \mathbf{C}\},$$

$$R(I - E_-) = \{k(\sigma_-^+, 1), k \in \mathbf{C}\},$$

and

$$\dim R(E_+) = \dim R(I - E_-) = 1. \tag{3.30}$$

Furthermore, (3.29) implies

$$R(E_+) \cap R(I - E_-) = \{0\}. \tag{3.31}$$

It follows from [3, p. 137, Lemma 1] that for Re $\lambda > 0$, $\overline{B}_0(\lambda)$ is invertible, and $\|(\overline{B}_0)^{-1}(\lambda)\|$ is bounded. Thus for any $f \in L_2(R)$, there exists a unique $(v_1, v_2) \in H^1(R) \times H^1(R)$, such that

$$\overline{B}_0(\lambda)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix},$$

and

$$\|(v_1, v_2)\|_{H^1(R)\times H^1(R)} \leq C\|f\|_{L_2(R)}. \tag{3.32}$$

Note that $v_1' = v_2$. Then v_1 satisfies

$$B_0(\lambda)v_1 = f,$$

and this with (3.32) completes the proof of Lemma 3.3.

Lemma 3.3 assures us that (3.28) holds; thus

$$\text{if } 0 \in \sigma_{ess}(B_0(\lambda)), \text{ then } \lambda \in \mathbf{C} \setminus P. \tag{3.33}$$

Lemma 3.2 follows from (3.22), (3.26), and (3.33).

Finally Lemmas 3.1–3.2 and (3.9) imply Theorem 3.1.

THEOREM 3.2. *The linear operator L: $X_0 \to X$ is an infinitesimal generator of a C_0-semigroup $T_L(t)$ on X satisfying*

$$\|T_L(t)\|_{X\to X} \leq e^{\omega_0 t}, \tag{3.34}$$

with $\omega_0 = \|-\rho + 2U(V-1)\|_\infty + 1$.

Proof. Theorem 3.1 assures us that the resolvent set $\rho(L)$ of L contains the ray $(0, +\infty)$.

It remains to prove that there exists $\omega_0 > 0$, such that

$$\|(\lambda I - L)^{-1}\|_{X\to X} \leq \frac{1}{\lambda - \omega_0}, \qquad \text{for } \lambda > \omega_0. \tag{3.35}$$

For any given $(f, g) \in X$, $\lambda > 0$, let $(u, v) \in X_0$ satisfy

$$(\lambda I - L)\binom{u}{v} = \binom{f}{g}.$$

Define

$$\binom{f_1}{g_1} \triangleq -L\binom{u}{v} = \begin{pmatrix} \left(s + (V-1)^2 + c\dfrac{\partial}{\partial \xi}\right)u \\ + \left(-\rho + 2U(V-1) - \dfrac{\partial^2}{\partial \xi^2}\right)v \\ -u + hv + cv' \end{pmatrix}.$$

Note that

$$\begin{aligned}
&\int_R \left(|f|^2 + |g|^2 + |g'|^2\right) d\xi \\
&\quad = \int_R \left(|\lambda u + f_1|^2 + |\lambda v + g_1|^2 + |\lambda v' + g_1'|^2\right) d\xi \\
&\quad \geq \lambda^2 \int_R \left(|u|^2 + |v|^2 + |v'|^2\right) d\xi + 2\lambda \operatorname{Re} \int_R \left(u\overline{f_1} + v\overline{g_1} + v'\overline{g_1'}\right) d\xi \\
&\quad \geq \lambda^2 \left(\|u\|^2 + \|v\|^2 + \|v'\|^2\right) - 2C_1 \lambda \|u\| \|v\| \\
&\quad \geq (\lambda - C_1)^2 \left(\|u\|^2 + \|v\|^2 + \|v'\|^2\right) \qquad \text{for } \lambda \geq C_1, \qquad (3.36)
\end{aligned}$$

with

$$C_1 = \|-\rho + 2U(V-1)\|_\infty + 1.$$

Thus (3.36) implies that (3.35) holds with $\omega_0 = C_1$; this completes the proof of Theorem 3.2.

Let X_2 be a subspace in X, and $X_2^0 = X_2 \cap X_0$. Define an operator L_2: $X_2^0 \to X_2$ as

$$L_2 w = Lw, \qquad \text{for } w \in X_2^0,$$

and define

$$\|w\|_{X_2} = \|w\|_X.$$

In the following, we assume

$$\rho(L) \subset \rho(L_2), \qquad \text{and} \qquad 0 \in \rho(L_2). \qquad (3.37)$$

Then it follows from Theorem 3.1 that

$$\operatorname{Re}\{\sigma(L_2)\} \le -\beta_0. \tag{3.38}$$

THEOREM 3.3. *Under the assumption of* (3.37), *the* L_2 *generate a* C_0-*semigroup* $T_2(t)$ *on* X_2 *satisfying*

$$\|T_2(t)\|_{X_2 \to X_2} \le M_0 e^{-\beta t}, \qquad \text{for } t > 0, \tag{3.39}$$

for some $M_0 > 0$, $\beta > 0$.

Note that (3.37) implies that Theorem 3.2 is valid for L_2. It remains to prove (3.39).

It follows from [7] (see also [6]) that (3.39) holds if and only if the following two conditions holds

(i) $\sup\{\operatorname{Re}\lambda, \lambda \in \sigma(L_2)\} < 0,$ (3.40)

(ii) $\sup_{\operatorname{Re}\lambda \ge 0} \|(\lambda - L_2)^{-1}\|_{X_2 \to X_2} < +\infty.$ (3.41)

By (3.37) and Theorem 3.1, obviously (3.40) holds.

In the following, we only need to prove (3.41) holds.

By Theorem 3.2 and the theory of the C_0-semigroup [5], we have

$$\left\|(\lambda - L_2)^{-1}\right\| \le \frac{1}{(\operatorname{Re}\lambda) - \omega_0}, \qquad \operatorname{Re}\lambda > \omega_0.$$

Thus

$$\left\|(\lambda - L_2)^{-1}\right\| \le 1, \qquad \text{for } \lambda \in Q_1 = \{\lambda / \operatorname{Re}\lambda \ge \omega_0 + 1\}. \tag{3.42}$$

Note that $\{\lambda/\operatorname{Re}\lambda \ge 0\} \subset \rho(L_2)$. Thus for any fixed $n > 0$, there exists C_n such that

$$\left\|(\lambda - L_2)^{-1}\right\| \le C_n, \qquad \lambda \in P_n = \{\lambda/0 \le \operatorname{Re}\lambda \le \omega_0 + 1, |\operatorname{Im}\lambda| \le n\}. \tag{3.43}$$

To complete the proof of Theorem 3.3, we need to prove the following results.

LEMMA 3.4. *There exists a constant* $0 < M_0 < +\infty$, *such that for any* $\lambda \in Q_2 = \{\lambda/0 \le \operatorname{Re}\lambda \le \omega_0 + 1, |\operatorname{Im}\lambda| \ge 1\}$,

$$\left\|(\lambda I - L_2)^{-1}\right\|_{X_2 \to X_2} \le M_0. \tag{3.44}$$

Furthermore, there exists a constant $0 < M_0^* < +\infty$, *such that* $\forall\ \lambda \in Q_2$, *and* $\forall\ (f, g) \in X_2$, *if* (u, v) *is a solution of*

$$(\lambda I - L_2)(u, v) = (f, g), \tag{3.45}$$

then

$$\|u\|_{L_2(R)} + \|\lambda v\|_{L_2(R)} + \|v'\|_{L_2(R)} \leq M_0^*\big(\|(f,g)\|_{X_2}\big), \qquad \lambda \in Q_2. \quad (3.46)$$

Proof. Obviously for $\lambda \in Q_2$, (3.46) implies (3.45), thus we only need to prove (3.46).

By contradiction, assume (3.46) doesn't hold. Then there exists $\{\lambda_n\} \in Q_2$, and $(u_n, v_n) \in X_2^0$ such that

$$\|u_n\|_{L_2} + \|\lambda_n v_n\|_{L_2} + \|v_n'\|_{L_2} = 1, \qquad n = 1, 2, \cdots; \quad (3.47)$$

and

$$(\lambda_n I - L_2)(u_n, v_n) \to 0, \qquad \text{in } X_2, \text{ as } n \to \infty. \quad (3.48)$$

Relation (3.43) imply that $\{\text{Im } \lambda_n\}$ must be unbounded. Also note that $0 \leq \text{Re } \lambda_n \leq \omega_0 + 1$, thus we can choose a subsequence of $\{\lambda_n\}$, also denoted by $\{\lambda_n\}$, such that

$$\text{Re } \lambda_n \to \lambda_1 \geq 0, \qquad \text{Im } \lambda_n \to +\infty, \text{ as } n \to \infty; \quad (3.49)$$

or

$$\text{Re } \lambda_n \to \lambda_1 \geq 0, \qquad \text{Im } \lambda_n \to -\infty, \text{ as } n \to \infty. \quad (3.50)$$

Without losing generality, let (3.49) hold.

By (3.48), we have

$$\left(\lambda_n + s + (V-1)^2 + c\frac{\partial}{\partial \xi}\right)u_n + \left(-\rho + 2U(V-1) - \frac{\partial^2}{\partial \xi^2}\right)v_n \to 0, \qquad \text{in } L_2(R), \quad (3.51)$$

and

$$-u_n + \left(\lambda_n + h + c\frac{\partial}{\partial \xi}\right)v_n \to 0 \qquad \text{in } H^1(R). \quad (3.52)$$

Multiply (3.51) by $\overline{u_n}$, then integrate it on R and we have

$$\text{Re} \int_R \left[\left(\lambda_n + s + (V-1)^2\right)|u_n|^2 + cu_n'\overline{u_n}\right] d\xi$$

$$+ \text{Re} \int_R (-\rho + 2U(V-1))\, v_n\overline{u_n}\, d\xi$$

$$+ \text{Re} \int_R v_n'\overline{u_n'}\, d\xi \to 0, \qquad \text{as } n \to \infty.$$

Note that (3.47) implies

$$\left| \operatorname{Re} \int_R (-\rho + 2U(V-1)) v_n \overline{u_n} \, d\xi \right| \le C \|v_n\|_{L_2} \|u_n\|_{L_2}$$
$$\le \frac{C}{|\lambda_n|} \|\lambda_n v_n\|_{L_2} \|u_n\|_{L_2} \le \frac{C_1}{|\lambda_n|}.$$

Thus

$$\int_R \left(\lambda_1 + s + (V-1)^2 \right) |u_n|^2 \, d\xi + \operatorname{Re} \int_B v_n' \overline{u_n'} \, d\xi \to 0, \qquad \text{as } n \to \infty. \tag{3.53}$$

By (3.52), we also have

$$-u_n' + (\lambda_n + h) v_n' + c v_n'' \to 0, \qquad \text{in } L_2(R), \text{ as } n \to \infty. \tag{3.54}$$

Multiply (3.54) by $\overline{v_n'}$, then integrate it over R and we have

$$-\operatorname{Re} \int_R u_n' \overline{v_n'} \, d\xi + (\lambda_1 + h) \int_R |v_n'|^2 \, d\xi \to 0, \qquad \text{as } n \to \infty. \tag{3.55}$$

By (3.53) and (3.55), we further have

$$\int_R \left(\lambda_1 + s + (V-1)^2 \right) |u_n|^2 \, d\xi + (\lambda_1 + h) \int_R |v_n'|^2 \, d\xi \to 0, \qquad \text{as } n \to \infty.$$

Thus,

$$\|u_n\|_{L_2} \to 0, \qquad \|v_n'\|_{L_2} \to 0, \text{ as } n \to \infty. \tag{3.56}$$

Relations (3.52) and (3.56) further imply

$$(\lambda_n + h) v_n \to 0, \qquad \text{in } L_2(R), \text{ as } n \to \infty;$$

thus

$$\|\lambda_n v_n\|_{L_2(R)} \to 0, \qquad \text{as } n \to \infty. \tag{3.57}$$

Relation (3.56) and (3.57) contradict (3.47), which completes the proof of Lemma 3.4 and Theorem 3.3.

4. THE PROOF OF THE STABILITY THEOREM

In this section, we shall give the proof of Theorem 2.2. Before proving Theorem 2.2, we need to obtain some further results about the operator L except those in Section 3.

LEMMA 4.1. *For any fixed* $(\rho, h) \in$ *region* 1, $N(L) = \operatorname{span}\{(U'(\xi), V'(\xi))\}$ *and*

$$N(L) \cap \overline{R(L)} = \{0\}. \tag{4.1}$$

The proof is given in the Appendix.

Define

$$L^* = \begin{bmatrix} -s - (V-1)^2 + c\dfrac{\partial}{\partial \xi} & 1 \\ \rho - 2U(V-1) + \dfrac{\partial^2}{\partial \xi^2} & -h + c\dfrac{\partial}{\partial \xi} \end{bmatrix}.$$

Obviously, L^* is the adjoint operator of L.

Note that

$$X = N(L^*) \oplus \overline{R(L)}.$$

Then Lemma 4.1 implies

$$N(L^*) \neq \{0\}.$$

Furthermore, we have

LEMMA 4.2. *For any* $(\rho, h) \in$ *region* 1,

$$\dim N(L^*) = 1.$$

The proof of Lemma 4.2 is given in the Appendix.

By Lemmas 4.1–4.2, it is easy to prove the following results.

LEMMA 4.3. *For any* $(\rho, h) \in$ *region* 1,

$$X = N(L) \oplus \overline{R(L)}, \tag{4.2}$$

that is, for any fixed $w \in X$, *there exist a unique* $w_0 \in N(L)$ *and a unique* $w_1 \in \overline{R(L)}$ *such that*

$$w = w_0 + w_1.$$

Note that $X = N(L^*) \oplus \overline{R(L)}$ and

$$\langle \nu, w \rangle = 0, \qquad \forall\, \nu \in N(L^*)\ \forall\, w \in \overline{R(L)}. \tag{4.3}$$

Lemmas 4.1–4.3 and (4.3) further imply that for $\nu_0 \in N(L^*)$, $\|\nu_0\|_X = 1$,

$$\langle \nu_0, W_0 \rangle \neq 0, \tag{4.4}$$

where $W_0(\xi) = (U'(\xi), V'(\xi))$.

Denote

$$X_2 = \overline{R(L)}, \qquad X_2^0 = \overline{R(L)} \cap H^2(R), \tag{4.5}$$

and define an operator L_2: $X_2^0 \to X_2$ as

$$L_2 w = Lw, \qquad w \in X_2. \tag{4.6}$$

LEMMA 4.4. *For any* $(\rho, h) \in$ *region* 1, *if* L_2 *is defined by* (4.5)–(4.6), *then*

$$0 \in \rho(L_2), \qquad \text{and} \qquad \rho(L) \subset \rho(L_2).$$

Proof. Lemma 4.3 implies that L_2 is an onto operator and $N(L_2) = \{0\}$, thus $0 \in \rho(L_2)$.

For any $\lambda \in \rho(L)$, and any $f \in X_2$, there exists a unique $w \in X_0$ such that

$$(\lambda I - L)w = f.$$

By the fact that

$$\lambda \neq 0, \qquad \text{and} \qquad \lambda w = Lw + f \in X_2,$$

we have $w \in X_2 \cap X_0 = X_2^0$, which completes the proof of Lemma 4.4.

Lemma 4.4 assures that Theorem 3.3 holds, thus we have

THEOREM 4.1. *For any* $(\rho, h) \in$ *region* 1, *operator* L_2 *is defined by* (4.5)–(4.6). *Then* L_2 *generate a* C_0-semigroup $e^{L_2 t}$ *on* X_2 *satisfying*

$$\|e^{L_2 t}\|_{X_2 \to X_2} \leq Ce^{-\beta t} \qquad \text{for } t > 0, \tag{4.7}$$

for some $\beta > 0$.

Now we turn to the proof of the Stability Theorem.

Proof of Theorem 2.2. For any fixed $(\rho, h) \in$ region 1, let $W_0(\xi) = (U(\xi), V(\xi))$ $(\xi = x + ct)$ be the travelling wave solution obtained in Theorem 1 and (1.3). For any $(u_0, v_0) \in Y$, if $\|u_0 - U\|_{L_2} + \|v_0 - V\|_{H^1} < \delta$, it follows from Theorem 2.1 that there exists a unique local solution $(u(t, \xi), v(t, \xi)) \in Y$ of (2.2).

Denote $w(t, \xi) = (u(t, \xi) - U(\xi), v(t, \xi) - V(\xi))$. Then (2.2) becomes

$$\frac{dw}{dt} = Lw + F(w), \tag{4.8}$$

with $F(0)=0$, $F'(0)=0$. Note that for any fixed $\sigma \in R$, $W_0(\xi+\sigma) - W_0(\xi)$ satisfies

$$L\big(W_0(\xi+\sigma) - W_0(\xi)\big) + F\big(W_0(\xi+\sigma) - W_0(\xi)\big) = 0.$$

As in [3], introducing two new variables $(\sigma(t), y(t,\xi)) \in R \times X_2$ such that

$$w(t,\xi) = W_0(\xi + \sigma(t)) - W_0(\xi) + y(t,\xi), \tag{4.9}$$

then (4.8) becomes

$$W_0'(\xi+\sigma)\frac{d\sigma}{dt} + \frac{dy}{dt} = Ly + F\big(W_0^*(\xi,\sigma) + y(t,\xi)\big) - F\big(W_0^*(\xi,\sigma)\big), \tag{4.10}$$

with $W_0^*(\xi,\sigma) = W_0(\xi+\sigma) - W_0(\xi)$.

Let $\sigma(t)$ satisfy

$$\frac{d\sigma}{dt} = \phi(\sigma, y), \tag{4.11}$$

with

$$\phi(\sigma,y) = \langle \nu_0, F\big(W_0^*(\cdot,\sigma) + y\big) - F(W_0^*(\cdot,\sigma))\rangle / \langle \nu_0, W_0'(\cdot + \sigma)\rangle. \tag{4.12}$$

If $\sigma(t)$ is small, then (4.4) and (4.10)–(4.12) assure us that

$$\langle y, \nu_0\rangle = 0, \qquad \text{i.e., } y \in X_2,$$

and y satisfies

$$\frac{dy}{dt} = L_2 y + G(\sigma, y), \tag{4.13}$$

with

$$G(\sigma,y) = E_2\big\{F\big(W_0^*(\cdot,\sigma) + y\big) - F\big(W_0^*(\cdot,\sigma)\big) - W_0'(\cdot+\sigma)\phi(\sigma,y)\big\}.$$

Thus ϕ and G are C^1 functions with

$$|\phi(\sigma,y)| + \|G(\sigma,y)\|_{X_2} \le \gamma(\rho)\|y\|_{X_2}, \qquad \text{when } |\sigma| + \|y\|_{X_2} \le \rho, \tag{4.14}$$

and $\gamma(\rho) \to 0$ as $\rho \to 0$.

Suppose $|\sigma(0)| + \|y(0)\|_{X_2}$ is small; as long as $|\sigma(t)|$ remains less than $\delta > 0$, by virtue of Theorem 4.1 and by the standard argument, we can prove that

$$\| y(t) \|_{X_2} \le K e^{-\beta' t} \| y(0) \|_{X_2}, \qquad 0 < \beta' < \beta, \tag{4.15}$$

and so

$$\left| \frac{d\sigma}{dt}(t) \right| = O(e^{-\beta' t}).$$

Thus $|\sigma(t)| < \delta$ for all $t > 0$, and there exists σ_∞ such that

$$|\sigma(t) - \sigma_\infty| + \| y(t) \|_{X_2} = O(e^{-\beta' t}).$$

Thus

$$\| u(t, \xi) - U(\xi + \sigma_\infty) \|_{L_2} + \| v(t, \xi) - V(\xi + \sigma_\infty) \|_{H^1} = O(e^{-\beta' t}), \tag{4.16}$$

which completes the proof of Theorem 2.2.

APPENDIX

Proof of Lemma 4.1. Lemma 3.1 assures us that $N(L) = \operatorname{span}\{(U'(\xi), V'(\xi))\}$. To prove (4.1), by contradiction, assume $(U', V') \in \overline{R(L)}$. Then there exists a sequence $\{(u_n, v_n)\}_1^\infty \subset X_0$ satisfying

$$L\begin{pmatrix} u_n \\ v_n \end{pmatrix} - \begin{pmatrix} U' \\ V' \end{pmatrix} \to 0, \qquad \text{in } X, \text{ as } n \to \infty.$$

Thus

$$\left(-s - (V-1)^2 - c\frac{\partial}{\partial \xi} \right) u_n + \left(\rho - 2U(V-1) + \frac{\partial^2}{\partial \xi^2} \right) v_n \to hV' + cV'', \quad \text{in } L_2(R); \tag{A.1}$$

$$v_n - h v_n - c v_n' \to V' \qquad \text{in } H^1(R). \tag{A.2}$$

Substituting (A.2) into (A.1), we have

$$B_0 v_n - \frac{\left(h + s + (V-1)^2\right)V' + 2cV''}{1 - c^2} \to 0 \qquad \text{in } L_2(R).$$

with $B_0 = B(0)$.

Let $\phi_1 = V'$. We note that

$$B_0\phi_1 = 0, \qquad \text{and} \qquad \phi_1(\xi) \neq 0,\ \xi \in R.$$

Let $v_n(\xi) = C_{1n}(\xi)\phi_1(\xi) \in H^2(R)$. Then $C_{1n}(\xi)$ satisfies

$$C_n''\phi_1 + C_n'(2\phi_1' - b\phi_1) - \frac{\left(h + s + (V-1)^2\right)\phi_1 + 2c\phi_1'}{1-c^2} \to 0, \quad \text{in } L_2(R). \tag{A.3}$$

Define

$$\phi_1^* = \phi_1 \exp\left(-\int_0^\xi b(s)\,ds\right),$$
$$(\phi_1')^* = \phi_1' \exp\left(-\int_0^\xi b(s)\,ds\right),$$

and

$$\widehat{\phi_1} = \phi_1 \exp\left(-\frac{1}{2}\int_0^\xi b(s)\,ds\right),$$

Lemma 3.1 implies that $\phi_1^*, (\phi_1')^*, \widehat{\phi_1} \in L_2(R)$.

Multiplying (A.3) by ϕ_1^*, we have

$$\left[C_n'\left(\widehat{\phi_1}\right)^2\right]' - \frac{(h + s + (V-1)^2 + b(\xi)c}{1-c^2}\left(\widehat{\phi_1}\right)^2 - \frac{c}{1-c^2}\left(\widehat{\phi_1}\right)' \to 0, \quad \text{in } L_1(R). \tag{A.4}$$

Note that

$$v_n'(\xi) = C_n'(\xi)\phi_1(\xi) + C_n(\xi)\phi_1'(\xi) \in H^1(R).$$

Then

$$C_n'(\xi)\left(\widehat{\phi_1}\right)^2(\xi) = \phi_1^*(\xi)v_n' - v_n(\xi)(\phi_1')^* \in L_1(R).$$

Integrating (A.4) over R, we have

$$\frac{1}{1-c^2}\int_{-\infty}^{+\infty}\left[\left(h + s + (V-1)^2 + b(\xi)c\right]\left(\widehat{\phi_1}\right)^2(\xi)\,d\xi = 0,$$

i.e.,

$$\frac{1}{1-c^2}\int_{-\infty}^{+\infty}\left(1+\frac{c^2}{1-c^2}\right)\left[(h+s+(V-1)^2\right]\left(\widehat{\phi_1}\right)^2(\xi)\,d\xi=0.$$

Thus $\widehat{\phi_1}\equiv 0$, which is impossible. This completes the proof of Lemma 4.1.

Proof of Lemma 4.2. Since $N(L^*)\neq\{0\}$, let $w_0=(u_0,v_0)\in N(L^*)$. Then

$$v_0=\left(s+(V-1)^2\right)u_0-cu_0',$$

and $u_0\in N(B_0^*)$, with B_0^* the adjoint operator of B_0.

We only need to prove that 0 is the simple eigenvalue of B_0^*.

Define $B^*(\lambda)$ as

$$B^*(\lambda)=\frac{d^2}{d\xi^2}+b(\xi,\lambda)\frac{d}{d\xi}+d(\xi,\lambda).$$

Obviously, $B^*(\lambda)$ is the adjoint operator of $B(\lambda)$, and $B_0^*=B^*(0)$.

Along the lines of the proof of Lemmas 3.1–3.2, we can similarly prove the following lemma.

LEMMA A.1. *For any* $(\rho,h)\in$ *region* 1, *if there exists* $v\in N(B^*(\lambda))$ *for some* λ *with* $\operatorname{Re}\lambda\geq 0$, *then for the same* λ *there exists* $v^*\in N(\widehat{B^*}(\lambda))$ *with*

$$\widehat{B^*}(\lambda)=\frac{d^2}{d\xi^2}+\left(d(\xi,\lambda)-\frac{1}{2}b'(\xi,\lambda)-\frac{1}{4}b^2(\xi,\lambda)\right)\triangleq\frac{d^2}{d\xi^2}+D^*(\xi,\lambda),$$

and vice versa. Furthermore, λ *must be real.*

It follows from Lemma A.1 that 0 is the simple eigenvalue of B_0^* if and only if 0 is the simple eigenvalue of $\widehat{B^*}(0)$.

In the following, we prove that 0 is the simple eigenvalue of $\widehat{B^*}(0)$.

By contradiction, assume 0 is not a simple eigenvalue of $\widehat{B^*}(0)$. Then the first eigenvalue of $-\widehat{B^*}(0)$ must be negative, and denoted by $\sigma_1^0<0$.

The first eigenvalue of $-\widehat{B^*}(\lambda)$ is denoted by $\sigma_1(\lambda)$, and

$$\sigma_1(\lambda)=\min_{\|v\|_{L_2}=1}\int_R\left(|v'|^2+D^*(\xi,\lambda)|v|^2\right)d\xi.$$

Note that $D_\lambda^*(\xi,\lambda)>0$ for $\lambda\geq 0$, thus $\sigma_1(\lambda)$ is increasing in λ.

Also note that

$$D^*(\xi,\lambda)\geq M(\lambda),\qquad \forall\,\xi\in R,\text{ and } M(\lambda)\to+\infty,\text{ as }\lambda\to+\infty.$$

Then there exists $\lambda_0>0$, such that

$$\sigma_1(\lambda_0)>0.$$

Thus the continuity of $\sigma_1(\lambda)$ implies that there exists λ^*, with $0 < \lambda^* < \lambda_0$, such that

$$\sigma_1(\lambda^*) = 0,$$

i.e., $0 \in \sigma_p(\widehat{B^*}(\lambda^*))$, thus $0 \in \sigma_p(B^*(\lambda))$, which further implies

$$\lambda^* \in \sigma_p(L^*), \qquad \text{with } \lambda^* > 0.$$

By the fact that if $\lambda^* \in \sigma(L^*)$, then $\overline{\lambda^*} \in \sigma(L)$, which contradicts Theorem 3.1, this completes the proof of Lemma 4.2.

ACKNOWLEDGMENTS

The author is grateful to Professor P. Fife for introducing her to the problem studied here and for the useful discussion with him. The author is also grateful to Professor Q. X. Ye and the anonymous referee for their kind suggestions.

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