Shapes of liquid drops obtained using symbolic computation

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Abstract

The first aim of this paper is to show how two free boundary problems arising from fluid mechanics can be solved with a domain perturbation method. The second aim is to analyse the range of validity of the series solutions. The analysis will aim at identifying the location and the nature of the singularities characterizing these series. After expansion, the equations obtained are linear, but at each stage the length of the expressions grows exponentially. Herein we implement techniques for the automatic generation of hierarchical expression sequences and we present several tools for reducing the combinatorial blow-up of the expressions arising in these two problems. © 2005 Elsevier Ltd. All rights reserved.

Keywords: Free boundary; Perturbation method; Combinatorial blow-up

1. Introduction

In fluid mechanics as well as in other areas of applied mathematics it is very rare to obtain analytical solutions of systems of partial differential equations for complex problems. Nevertheless analytical or exact solutions can be very useful, especially for the calibration of numerical calculations. Increasingly, numerical computation in physics is
preferred to any other analytical investigation. However, some results can only be obtained with analytical techniques. The contribution of the present work is twofold: the first aim is to develop a perturbation technique in a symbolic computational language for a type of free boundary problems; the second aim is to study the validity of the method.

The method used here, called domain perturbation, is essentially the one initiated by Hadamard (1968) and the first to have applied it were Garabedian and Shiffer (1952–1953); Joseph and Fosdick (1972). It consists in taking as principal unknown the displacement field from a known position to the unknown one. For some free boundary problems a parameter $\varepsilon$, usually dimensionless, is involved, while for a particular value of this parameter, an equilibrium position is known, say $\Omega_0$. All the quantities, i.e. the domain $\Omega$ occupied by the fluid and the mechanical quantities: velocity, pressure, etc., depend upon $\varepsilon$. Since $\Omega$ must be determined as a part of the solution to the problem, the region occupied by the fluid is mapped onto the domain $\Omega_0$, which remains fixed throughout this procedure. The equations are then expressed on $\Omega_0$, and one obtains a system of $n$ partial differential equations $F_i(u, \varepsilon) = 0$, $i = 1, \ldots, n$, depending on the parameter $\varepsilon$. Thanks to the implicit function theorem (cf. Appendix A) one seeks the solution as a perturbation series. This method has been successfully used in several free boundary problems (Brancher and Séro-Guillaume, 1983; Schwartz and Whitney, 1981; Séro-Guillaume and Er-Riani, 1999; Sattinger, 1976; Van Dyke, 1975).

By hand, perturbation expansions are terminated at the second or third term. Higher order perturbation equations and their solutions are more difficult to obtain. Consequently, on many occasions, this technique is abandoned in favour of numerical computations. Recently, however, new computer algebra systems have appeared offering the possibility to manipulate symbolic expressions on computer, as asymptotic expansions. Moreover the domain perturbation technique has two main limitations. The first one, called combinatorial blow-up, is related to a general feature of perturbation techniques: the number of terms to be calculated at each order grows at least as an exponential. If the calculations are made with a symbolic computation language on a computer, it can exhaust the memory. Let us briefly describe the reasons for this blow-up. Let us assume that the equation $G(u, \varepsilon) = 0$ has to be solved by the implicit function theorem around the point $(u = u_0, \varepsilon = 0)$. If the series solution is denoted by $u = \sum_{n \geq 0} \frac{\varepsilon^n u_n}{n!}$, then $u_n$ are the solutions to the linear problem:

$$L_0(u_n) = A_n(0, u_0, \ldots, u_{n-1}) \quad (1)$$

where $L_0$ is the linear operator $\frac{\partial G}{\partial u_0}(u_0, 0)$, and $A_n$ can be calculated recursively. The $n$th derivative $u^{(n)} = \frac{\partial^n u}{\partial \varepsilon^n}(\varepsilon)$ satisfies the relation

$$\frac{\partial G}{\partial \varepsilon}(u(\varepsilon), \varepsilon) u^{(n)} = A_n \left( \varepsilon, u^{(0)}, u^{(1)}, \ldots, u^{(n-1)} \right). \quad (2)$$

If (2) is differentiated with respect to $\varepsilon$ one obtains the recurrence relation

$$A_{n+1} = \frac{dA_n}{d\varepsilon} - \frac{\partial}{\partial u} \left( \frac{\partial G}{\partial u} \right) u^{(1)} u^{(n)} - \frac{\partial}{\partial \varepsilon} \left( \frac{\partial G}{\partial u} \right) u^{(n)} \quad (3)$$
where \( \frac{d}{d\varepsilon} \) is the total derivative, that is
\[
\frac{d}{d\varepsilon} = \frac{\partial}{\partial\varepsilon} + \sum_{i>0} u^{(i+1)} \frac{\partial}{\partial u^{(i)}}.
\]

Setting \( \varepsilon = 0 \) in (2) we get (1).

The successive orders can be calculated with a symbolic calculus code. We have stipulated the iterative structure to emphasize the fact that the left hand side of (1) is the same for all orders, which means that if one can solve the linear Eq. (1) at first order, the successive relations can be solved at any order, the inverse of the operator \( L_0 \) being calculated once. But the number of terms on the right hand side grows as a factorial. There is a combinatorial blow-up for the number of terms with respect to the order. Then it is usually possible to obtain only a limited number of terms. The second limitation is related to the domain of validity of the series obtained. Few studies, see Schwartz and Whitney (1981) for example, have been devoted to this point; in particular the implicit function theorem ensures the analyticity of the solutions but does not give any idea of the convergence radius of these solution series. The radius of convergence is the distance from the origin to the nearest singularity.

A useful tool for extracting this singularity is the Domb–Sykes method Domb and Sykes (1957). Indeed, suppose that a function \( f(\varepsilon) \) has the form
\[
f(\varepsilon) = \sum_{n=0}^{\infty} c_n \varepsilon^n = \begin{cases} 
(\varepsilon_0 - \varepsilon)^\gamma g(\varepsilon) & \text{for } \gamma \neq 0, 1, \ldots, n, \ldots \\
(\varepsilon_0 - \varepsilon)^\gamma \ln(\varepsilon_0 - \varepsilon) g(\varepsilon) & \text{for } \gamma = 0, 1, \ldots, n, \ldots 
\end{cases}
\]
where \( g(\varepsilon) \) is an analytic function. The form (4) is a general possible form of singularity. Then the radius of convergence \( \varepsilon_0 \) may be found with the d’Alembert ratio as
\[
\varepsilon_0 = \lim_{n \to \infty} \left| \frac{c_n}{c_{n-1}} \right|
\]
where \( \varepsilon_0 \) is a singularity of \( f(\varepsilon) \) as stipulated by Fabry’s theorem (Dienes, 1957). This singularity must be real and positive in order to have a physical sense. Therefore the signs of \( c_n \) are fixed (Van Dyke, 1974). Only a finite number of coefficients \( c_n \) are known, so it is difficult to obtain precisely the limit in (5). Domb and Sykes have suggested that the inverse ratio \( c_n/c_{n-1} \) has the following expansion:
\[
a_n = \left| \frac{c_n}{c_{n-1}} \right| = \frac{1}{\varepsilon_0} \left[ 1 - \frac{1}{n} + \gamma \right] + o \left( \frac{1}{n} \right) = h(1/n).
\]
Thus, the intersection of the straight line \( a_n = h(1/n) \) with the axis \( 1/n = 0 \) is exactly \( 1/\varepsilon_0 \) and the slope of this line gives the exponent \( \gamma \). Unfortunately \( a_n \) is often a slowly converging sequence. A good way to improve the convergence is to use the Richardson extrapolation (Bender and Orszag, 1978), which is appropriate for this kind of sequence and can be defined as follows. If \( a_n \) can be written in the following form:
\[
a_n = Q_0 + \frac{Q_1}{n} + \frac{Q_2}{n^2} + \cdots,
\]
then the Richardson extrapolation is
\[
q_{mn} = \sum_{k=0}^{m} \frac{(-1)^{k+m} a_{n+k} (n+k)^m}{k! (m-k)!}.
\]
This new sequence has a quicker convergence to the limit $Q_0$. Of course the greater the number of terms in the perturbation series, the better the determination of the radius of convergence. Consequently, the problem is to seek a maximum of coefficients of the perturbation expansions to get a more precise value of $\varepsilon_0$. In this work, results are expressed in a power series in a small parameter $\varepsilon$ and have the form

$$f(\varepsilon) = \sum_{n=0}^{N} c_n \varepsilon^n.$$  

The singularity is factored out multiplicatively, by setting

$$f(\varepsilon) = A(\varepsilon)(\varepsilon - \varepsilon_0)^\alpha$$

while the function $A(\varepsilon)$ is regular in the neighbourhood of $\varepsilon = \varepsilon_0$. The use of the Domb–Sykes method improved by the Richardson extrapolation allows us to find the nearest singularity and its type. Recalling the blow-up of the number of coefficients, it is desirable to derive a method which saves computer memory as much as possible. It is shown how a substitution method can contribute to saving at least half the computer memory.

This paper is organized as follows. Section 2 is devoted to the equilibrium shape of a rotating liquid drop with the method described above. This problem has a simple structure. The only equation to be solved is the equilibrium equation, which expresses that the pressure excess between the fluids, due to rotation, is balanced by surface tension. It will serve as an example, for which we will reveal the method of substitution used to obtain the expansion of the equations. In Section 3, the shape of a bubble in the potential flow of non-viscous fluid will be studied. The structure of this problem is much more complicated. The equilibrium equation expresses that jump pressure induced by the flow is balanced by the interfacial tension, but the pressure is dependent on the velocity so the flow equation must be solved. In both cases the convergence domain of the series will be determined using the Sykes–Domb method. We find an encouraging agreement of the results obtained with the analytical method and the results obtained with numerical calculations. In the conclusion we will discuss the limitations of the method and compare the number of terms obtained with and without the substitution method.

2. The equilibrium shape of a rotating liquid drop

In addition to applications in astronomy and mechanics, the theory of rotating drops found applications in nuclear physics with the drop model of the atomic nucleus (Swiatecki, 1974; Evans, 1955) and in chemistry for measuring the interfacial tension coefficients of surfactants (Ono and Kondo, 1960; Vonnegut, 1942). Plateau (Appell, 1932) was the first to make experiments on rotating drops. He observed in his investigations that if there is no rotation then the drop has a spherical form while under the influence of rotation the sphere flattens at the poles and the oblateness increases with the angular velocity $\omega$.

2.1. Formulation

Let us consider a non-viscous liquid drop, of volume $V_0$, held by the action of surface tension in a second non-viscous liquid and made to rotate about an axis. The effect of
gravity is neglected. It is known (Appell, 1932; Chandrasekhar, 1965) that no steady motion can exist unless the fluid rotates as a rigid body. We will therefore assume that the angular velocity \( \omega \) is constant. When \( \omega \) is zero, the drop is spherical. For \( \omega > 0 \), the drop occupies a domain \( \Omega \). A spherical coordinate system \((r, \theta, \psi)\) has to be assumed, with the origin at the centre of the drop and with the \(z\) axis as the rotation axis as shown in Fig. 1. The equilibrium profile of the drop is given by the equation \( r = Rg(\theta, \psi, \varepsilon) \), where \( R \) is the radius of the sphere defined in terms of the drop volume: \( V_0 = \frac{4}{3} \pi R^3 \).

We assume an equatorial plane of reflective symmetry perpendicular to the rotation axis. So function \( g \) must verify the condition

\[
g(\pi - \theta, \psi, \varepsilon) = g(\theta, \psi, \varepsilon)
\]

and consequently the drop shape is defined on the rectangular domain \( 0 \leq \psi \leq \pi \) and \( 0 \leq \theta \leq \pi/2 \).

Eq. (9) yields the relation

\[
\frac{\partial g}{\partial \theta} \left( \frac{\pi}{2}, \psi, \varepsilon \right) = 0.
\]

The drop satisfies the constraint that it encloses a fixed volume, i.e.

\[
\int_0^\pi \int_0^{\pi/2} g^3 \sin \theta \, d\theta \, d\psi = C\text{te}.
\]

The balance of surface tension and hydrodynamic pressure determines the shape of the drop. The pressure distributions in the two media are given by

\[
p_1 = p_{10} + 1/2 \rho_1 \omega^2 r^2 \sin^2 \theta
\]

and

\[
p_2 = p_{20} + 1/2 \rho_2 \omega^2 r^2 \sin^2 \theta.
\]

In (12) and (13) \( p_{10}, p_{20} \) are the pressures of the two media on the rotation axis. Indices 1 and 2 correspond to the inside of the drop and to the liquid outside the drop respectively.
The interface equilibrium condition is the Laplace relation:

\[ p_1 - p_2 = \sigma C \quad (14) \]

where \( C \) is the mean curvature of the interface and \( \sigma \) the surface tension. Eliminating \( p_1 \) and \( p_2 \) from (12)–(14) yields the Young–Laplace equation for meniscus shape:

\[ \sigma C = \Delta p_0 + \frac{1}{2} \Delta \rho \omega^2 r^2 \sin^2 \theta \quad (15) \]

where \( \Delta \rho = \rho_1 - \rho_2 \) is the difference of densities between the rotating liquid and the ambient liquid, and \( \Delta p_0 = p_{10} - p_{20} \) is the pressure difference between the two media, at the axis of the rotation. It is convenient to use the radius \( R \) as a length scale and to employ dimensionless variables:

\[ C^* = RC, \quad r = Rr^*. \]

Thus Eq. (15) becomes

\[ \varepsilon g^2 \sin^2 \theta - C^* + k = 0 \quad (16) \]

where \( \varepsilon = \frac{R^3 \omega^2 \Delta \rho}{2 \sigma} \) is the rotational bond number and \( k = \frac{R \Delta \rho_0}{\sigma} \) is called the reference pressure. It is calculated by constraining the drop volume to be a fixed amount \( V_0 \). Then it is an unknown constant depending only on \( \varepsilon \). Henceforth the asterisks will be dropped for the sake of simplicity. Let us note that the sphere \( \Omega_0 \), given by the equation \( g(\theta, \psi, 0) = 1 \), is a solution to the problem. We know from experiments that the solution is axisymmetrical for small values of \( \varepsilon \); thus for \( \varepsilon \neq 0 \) one can look for an axisymmetrical solution. Therefore \( g \) is independent of \( \psi \). In terms of the interface equation, the curvature \( C \) can be written as

\[ C = -\frac{g_0 g - 2g_0^2 - g^2}{(g^2 + g_0^2)^{3/2}} - \frac{g_0 \cos \theta - g \sin \theta}{g \sin \theta (g^2 + g_0^2)^{1/2}} \quad (17) \]

The subscript denotes partial differentiation, i.e. \( g_0 = \frac{\partial g}{\partial \theta} \). Using the implicit function theorem (see Appendix A), one can show that there is a unique analytic solution in \( \varepsilon \) to the problem (11)–(16). Therefore \( g(\theta, \varepsilon) \) and \( k \) must be written as

\[ g(\theta, \varepsilon) = \sum_{n \geq 0} \varepsilon^n f_n(\theta) \quad (18) \]

\[ k = \sum_{n \geq 0} \varepsilon^n h_n. \quad (19) \]

The straightforward method for obtaining the successive governing equations for determining \((f_n, h_n)\) is substituting (18) and (19) into (11)–(16) and collecting like powers of \( \varepsilon \). Doing this, the curvature can be written as

\[ C = 2 + \sum_{n \geq 1} \varepsilon^n C_n. \quad (20) \]
With for example
\[
\begin{align*}
C_1 (\theta) &= - \left( f_1'' + \frac{\cos \theta}{\sin \theta} f_1' + 2 f_1 \right) = -L f_1 \\
C_2 (\theta) &= -L f_2 + 2 f_1'' f_1 + 2 f_1' f_1 \frac{\cos \theta}{\sin \theta} + 2 f_1^2
\end{align*}
\]
expressing compactly the equations to be solved, it is convenient to introduce the differential operator
\[L = \frac{d^2}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{d}{d\theta} + 2i d.\]
As was stipulated in the introduction, \( f_n \) are solutions to the ordinary differential equation hierarchy
\[L f_n = A( f_0, f_1, \ldots, f_{n-1}; h_0, \ldots, h_{n-1}), \quad f_0 = 1.\]
These equations are made subject to condition (11) in order to calculate \( h_n \). Equations like (23) can be solved analytically using a basis of Legendre polynomials \( P_n(x) \), each \( f_n \) being written as (cf. Appendix A)
\[f_n(\theta) = \sum_{m \geq 0} \alpha_n^m P_{2m}(\xi), \quad \xi = \cos \theta.
\]
The calculations of coefficients \((f_n, h_n)\) were carried out entirely on a PC computer with a Pentium processor rated at 2.4 GHz; with 1 Go of RAM. With a naïve use of a computer algebra system, the explicit expressions for the coefficients \( \alpha_n^m \) suffer from combinatorial growth and the computer runs out of memory in the middle of a calculation at order 14. Obviously the main difficulty of this problem lies in the calculation of the power series of the curvature (20).

There are several approaches for avoiding the intermediate expression swell. The first one consists of noticing that expression (23) leads to a natural construction of computation sequences or straight-line programs as defined by Corless et al. (1997) and formalized in Maple by the procedure CompSeq. This procedure generates subexpressions that can be printed separately and are identified by the Maple command ‘optimize’. But the difficulty of this construction lies in the fact that at each stage, it is necessary to determine effectively an explicit expression for the second member of (23). This is due to the non-linearity of the mean curvature operator. So this approach appears to be inappropriate. The second approach is to break the calculation into pieces, and succeed in doing each piece on its own. This can take the calculation a little further, but cannot avoid the combinatorial blow-up. In view of this it might seem that an alternative approach for avoiding the intermediate expression swell would be the substitution of terms, as we shall see below. One notes that the curvature can be written as
\[C = - \frac{1}{g \sin \theta} \frac{d}{d\theta} \left[ \frac{g\theta}{(g^2 + g_\theta^2)^{1/2}} \right] + \frac{2}{(g^2 + g_\theta^2)^{1/2}}.
\]
Let us set \( E_\alpha = (x^2 + y^2)^\alpha; \alpha \) is a real number that will be required to take the values \( \alpha = -1, -1/2, \) and \( x = x(\epsilon) \) and \( y = y(\epsilon) \) are analytic functions. One has to perform the
substitutions \(x = g\) and \(y = g\theta\) to expand \(C\). If we write \(x_n = \frac{d^n x}{d\epsilon^n}\) and \(y_n = \frac{d^n y}{d\epsilon^n}\), then we can show that

\[
\frac{\partial^n E_\alpha}{\partial \epsilon^n} = F_n(x_0, \ldots, x_n, y_0, \ldots, y_n) E_{\alpha - n}.
\]

The \(F_n\) are polynomial functions depending on the \(2n + 2\) variables \((x_0, \ldots, x_n, y_0, \ldots, y_n)\) and satisfying the following recurrence relations:

\[
\begin{align*}
F_{n+1}(x_0, \ldots, x_{n+1}, y_0, \ldots, y_{n+1}) &= \frac{dF_n}{d\epsilon} E_1(x_0, y_0) + F_n F_1, \quad n \geq 1 \\
F_1(x_0, x_1, y_0, y_1) &= 2\alpha (x_0 x_1 + y_0 y_1).
\end{align*}
\]

The total derivative of \(F_n\) is given by

\[
\frac{dF_n}{d\epsilon} = \sum_{i=0}^{n} x_{i+1} \frac{\partial F_n}{\partial x_i} (x_0, \ldots, x_n, y_0, \ldots, y_n) + y_{i+1} \frac{\partial F_n}{\partial y_i} (x_0, \ldots, x_n, y_0, \ldots, y_n).
\]

It is now easy to calculate \(F_n\) recursively and hence the expressions for the successive derivatives of \(E_\alpha\). By substitution of \(\epsilon = 0, x_m = m! f_m(\theta)\) and \(y_m = m! f'_m(\theta)\) for \(m = 0, 1, \ldots\) into expression (25), one gets the expansion series for \(E_\alpha\) and then that for the curvature. With Maple, for example, one can calculate the power series expansion of the curvature directly by using the Taylor expansion procedure. This approach allowed us to get no more than 14 terms. Subdividing expressions into smaller ones permitted us to get 12 more terms. Using substitution techniques, as we see above, permitted us to get more than 200 terms. For the calculation of the expression (20) related to the power series expansion of the curvature, it will indeed be judicious to substitute intermediate expressions, instead of making direct calculations.

2.2. Solution to perturbation equations

The substitution of relations (18) and (19) into (11) and (16) yields a sequence of perturbation problems. Let us examine the first few orders:

\textbf{Order 0}

At order 0 Eq. (16) gives \(h_0 = 2\).

\textbf{Order 1}

At order 1, system (11)–(16) is

\[
Lf_1 + \sin^2 \theta + h_1 = 0 \quad (26)
\]

\[
\int_0^{\pi/2} f_1 \sin \theta \, d\theta = 0. \quad (27)
\]

From relation (24) \(f_1\) may be written as a series in Legendre polynomials:

\[
f_1(\theta) = \sum_{m \geq 0} \alpha_m^n P_{2m}(\xi).
\]
Substituting this relation into (26) and (27) will give (cf. Appendix A)

$$\alpha_0 = 0, \quad \alpha_1 = -\frac{1}{6}, \quad \alpha_m = 0 \text{ for } m \geq 2.$$ 

So the solution is

$$f_1(\theta) = -\frac{1}{6} P_2(\xi), \quad h_1 = -\frac{2}{3}.$$ 

**Order 2**

At order 2 the use of relations (11)–(16) yields

$$L f_2 - 2 f_2^2 - 2 f_2'' f_1 + 2 f_1 \sin^2 \theta - \frac{2 f_1 f'_1 \cos \theta}{\sin \theta} + h_2 = 0$$

$$\int_0^{\pi/2} f_2 \sin \theta \, d\theta = -\int_0^{\pi/2} f_1^2 \sin \theta \, d\theta.$$ 

The solution is

$$f_2(\theta) = \frac{1}{1260} [18 P_4(\xi) - 25 P_2(\xi) - 7], \quad h_2 = -\frac{4}{45}.$$ 

Similarly we can obtain higher order terms. Then, in spherical coordinates, the radial position of the interface is given at order 3 by

$$r = g(\theta, \varepsilon) = 1 - \frac{\varepsilon}{6} P_2(\xi) + \frac{\varepsilon^2}{1260} [18 P_4(\xi) - 25 P_2(\xi) - 7]$$

$$- \frac{\varepsilon^3}{124740} [54 P_6(\xi) - 351 P_4(\xi) + 561 P_2(\xi) + 154] + O(\varepsilon^4).$$

In Fig. 2, we plot the cross-section of the drop for various values of the rotational Bond number, using a 37-term series for $g$.

### 2.3. Study of the radius of convergence

The previous section dealt with the analytic structure of the solution series (18). Once this is accomplished, the next step is to identify the nature and location of the nearest singularity limiting the convergence of these series. In order to do that, we employ the Domb–Sykes method combined with the Richardson extrapolation procedure. This will be first applied to the aspect ratio $a(\varepsilon) = g(0, \varepsilon)/g(\pi/2, \varepsilon) = R_{\text{max}}/R_{\text{min}}$, which, as noted, is the ratio of polar and equatorial radii of the drop. The series of the aspect ratio is $a(\varepsilon) = \sum_{n \geq 0} c_n \varepsilon^n$, where the first 36 coefficients $c_n$ are tabulated in Table 1.

We suppose that the singularity $\varepsilon_0$ of $a(\varepsilon)$ is multiplicative, so $a(\varepsilon)$ takes the form (8). We observe that the pattern of signs appearing in the series $a(\varepsilon)$ is fixed, so the singularity $\varepsilon_0$ lies on the positive axis (Aziz and Benzies, 1976).

In Fig. 3, the $c_n/c_{n-1}$ for the aspect ratio are plotted.

One can see in Fig. 3 that the hypothesis needed for applying the Richardson extrapolation appears to be verified. The direct extrapolation to the value $1/n = 0$ gives $\varepsilon_0 = 2.2747$, whereas Richardson’s procedure (7) gives the estimation $\varepsilon_0 = 2.2742$. We note that five digits are valid because they do not change in the Richardson extrapolation.
Fig. 2. Axially symmetric equilibrium shapes for a rotating drop, for different values of the rotational bond number.

Table 1
Coefficients $c_n$ of the aspect ratio of the rotating drop

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_n$</th>
<th>$n$</th>
<th>$c_n$</th>
<th>$n$</th>
<th>$c_n$</th>
<th>$n$</th>
<th>$c_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.250000$</td>
<td>10</td>
<td>$-0.1494 \times 10^{-5}$</td>
<td>19</td>
<td>$-0.3484 \times 10^{-9}$</td>
<td>28</td>
<td>$-0.1194 \times 10^{-12}$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>11</td>
<td>$-0.5687 \times 10^{-6}$</td>
<td>20</td>
<td>$-0.1418 \times 10^{-9}$</td>
<td>29</td>
<td>$-0.4982 \times 10^{-13}$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.3125 \times 10^{-2}$</td>
<td>12</td>
<td>$-0.2192 \times 10^{-6}$</td>
<td>21</td>
<td>$-0.5794 \times 10^{-10}$</td>
<td>30</td>
<td>$-0.2081 \times 10^{-13}$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.8928 \times 10^{-3}$</td>
<td>13</td>
<td>$-0.8544 \times 10^{-7}$</td>
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<td>$-0.2375 \times 10^{-10}$</td>
<td>31</td>
<td>$-0.8713 \times 10^{-14}$</td>
</tr>
<tr>
<td>5</td>
<td>$-0.2548 \times 10^{-3}$</td>
<td>14</td>
<td>$-0.3359 \times 10^{-7}$</td>
<td>23</td>
<td>$-0.9769 \times 10^{-11}$</td>
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</tr>
<tr>
<td>6</td>
<td>$-0.8615 \times 10^{-4}$</td>
<td>15</td>
<td>$-0.1331 \times 10^{-7}$</td>
<td>24</td>
<td>$-0.4029 \times 10^{-11}$</td>
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<td>$-0.1533 \times 10^{-14}$</td>
</tr>
<tr>
<td>7</td>
<td>$-0.3024 \times 10^{-4}$</td>
<td>16</td>
<td>$-0.5310 \times 10^{-8}$</td>
<td>25</td>
<td>$-0.1666 \times 10^{-11}$</td>
<td>34</td>
<td>$-0.6448 \times 10^{-15}$</td>
</tr>
<tr>
<td>8</td>
<td>$-0.1083 \times 10^{-4}$</td>
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<td>$-0.2131 \times 10^{-8}$</td>
<td>26</td>
<td>$-0.6906 \times 10^{-12}$</td>
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<td>$-0.2714 \times 10^{-15}$</td>
</tr>
<tr>
<td>9</td>
<td>$-0.3983 \times 10^{-5}$</td>
<td>18</td>
<td>$-0.8596 \times 10^{-9}$</td>
<td>27</td>
<td>$-0.2869 \times 10^{-12}$</td>
<td>36</td>
<td>$-0.1144 \times 10^{-15}$</td>
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</table>

Table 2
Richardson extrapolation of the sequence $c_n$ of the aspect ratio $a(\varepsilon)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
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<td>0.439772</td>
<td>0.439749</td>
<td>0.439729</td>
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</tbody>
</table>

(cf. Table 2), so the calculations have been stopped at order 36. The results are given in Table 2.
With similar calculations, we proceeded to the extraction of singularities of the equatorial and polar radii and the reference pressure. We note that they have the same value as the aspect ratio singularity.

2.4. Discussion and comparison with previous results

The results show that there is a maximum value of the Bond number $\varepsilon_0 = 2.2742$ above which there is no axially symmetric drop. This value is somewhat smaller than the approximate value (2.4544) obtained by Chandrasekhar (1965) with elliptical integrals, but close to that found by Brown and Scriven (1980) (2.3222) where the deformation of an axisymmetrical drop is computed numerically. The results also show that as $\varepsilon$ increases, the drop becomes oblate, spreading out in the direction normal to the rotation axis and contracting in the direction of the rotation axis. At $\varepsilon_0$ the drop shape changes from axisymmetric to two lobed, and just beyond that value to four lobed.

3. The shape of a bubble in a uniform flow

3.1. Formulation

The deformation of a gas bubble or a liquid drop in a uniform flow of constant velocity $V_\infty$ in the absence of gravity is computed using a perturbation scheme. The flow is assumed to be inviscid and incompressible. We consider solutions which represent a steady flow caused by rectilinear motion of the bubble with constant velocity $V_\infty$. We therefore assume that the flow is irrotational. The shape of the bubble is governed by the balance between inviscid pressure forces and surface tension. The problem will be set in a reference frame attached to the bubble. Then the fluid velocity at infinity is $V_\infty = -V_\infty e_z$, where $e_z$ is a unit vector along axis $z$. Let $v = V_\infty + \nabla u$ be the velocity of the flow.
and \( u \) the potential due to the presence of the bubble, with \( u \to 0 \), and \( \nabla u \to 0 \) at infinity. The incompressibility condition \( \nabla \cdot \mathbf{v} = 0 \) and the slip condition at the boundary of the bubble give the governing equations for the fluid flow at the unknown position \( \Omega \):

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega^c \\
\frac{\partial u}{\partial n} &= -\mathbf{V}_\infty \cdot \mathbf{n} \quad \text{on } \partial \Omega
\end{aligned}
\]  

(28)

where \( \Omega^c \) is the exterior domain of \( \Omega \), \( \mathbf{n} \) is the unit outward normal of \( \partial \Omega \). As the shape of the calculation domain is unknown, (28) is a free boundary problem and one has to add one more equation for solving such a problem. This will be done using the momentum equation. It is assumed that there is no motion inside the bubble, so that \( p_0 \), the pressure inside the drop, is constant. The momentum equations reduce to the Bernoulli equation

\[
\rho \left( \frac{1}{2} |\nabla u|^2 + \mathbf{V}_\infty \cdot \mathbf{u}_\infty \right) - \sigma C = p_\infty - p_0 = \Delta p.
\]

(29)

For \( \mathbf{V}_\infty = 0 \) the equilibrium shape is a sphere of radius \( R \). We consider spherical polar coordinates \((r, \theta, \psi)\) with the origin at the centre of the bubble. In terms of dimensionless variables \( u = V_\infty R \tilde{u}, r = R \tilde{r} \), Eqs. (28) and (29) become

\[
\begin{aligned}
\Delta \tilde{u} &= 0 \quad \text{in } \Omega^c \\
\frac{\partial \tilde{u}}{\partial n} &= -e \cdot \mathbf{n} \quad \text{on } \Omega \\
\frac{R \rho V_\infty^2}{\sigma} \left[ \frac{1}{2} |\nabla \tilde{u}|^2 - \mathbf{V}_\infty \cdot \mathbf{e}_z \right] - R \tilde{C} &= \frac{R(p_\infty - p_0)}{\sigma} = \frac{R \Delta p}{\sigma} = k.
\end{aligned}
\]  

(30)

(31)

Let us set \( \varepsilon = R \rho V_\infty^2 / \sigma \), \( \varepsilon \) is the Weber number which is the ratio of inertial forces to interfacial ones. The reference pressure \( k \) is an unknown constant depending on \( \varepsilon \), and will be determined by the volume constraint: \( V = V_0 \). If \( \varepsilon = 0 \), the equilibrium shape is a spherical domain \( \Omega_0 \) taken as a fixed reference domain. If \( \varepsilon \neq 0 \), the equilibrium configuration is \( \Omega \). Therefore \( \varepsilon \) will be considered as an expansion parameter for this problem. Hereafter we will suppress tildes on \( u, C \) and \( r \). Let us describe briefly the domain perturbation method, cited above. Once Eqs. (30) and (31) are transported back to the sphere, we get a new system of partial differential equations, say \( F(T, X, \varepsilon) = 0 \), written for the known domain \( \Omega_0 \). The set of all the transformations \( X = T(X, \varepsilon) \) can be considered as a one-parameter group of transformations, and so we can say that \( X = (\mathbf{Id} + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots)(X) \) where \( \mathbf{Id} \) is the identity. Let \( T \) be the transformation field from the initial equilibrium position \( \Omega_0 \), i.e. for \( \varepsilon = 0 \) to the actual
equilibrium position $\Omega$, i.e. for $\varepsilon \neq 0$, see Fig. 4. The form of the transformation field is chosen:

$$\mathbf{T} = r g(\theta, \psi, \varepsilon) \mathbf{e}_r.$$  \hspace{1cm} (32)

Therefore the interface shape function is given by $r = g(\theta, \psi, \varepsilon)$. If $V_\infty$ is changed to $-V_\infty$ the configuration is unchanged; then by symmetry

$$g(\pi - \theta, \psi, \varepsilon) = g(\theta, \psi, \varepsilon).$$  \hspace{1cm} (33)

The equations transported back to the reference configuration $\Omega_0$ are

$$\begin{align*}
\nabla \cdot \left( \frac{\det \mathbf{T}'}{|\mathbf{T}'|} \mathbf{T}'^{-1} \mathbf{T}'^{-1} \nabla u \right) &= 0 \quad \text{for } r > 1 \\
\mathbf{T}'^{-1} \mathbf{\nabla} u \cdot \mathbf{T}'^{-1} \mathbf{e}_z &= \mathbf{T}'^{-1} \mathbf{e}_r \cdot \mathbf{T}'^{-1} \mathbf{n} \quad \text{for } r = 1 \\
\varepsilon \left[ \frac{1}{2} |\mathbf{T}'^{-1} \nabla u|^2 - \mathbf{T}'^{-1} \mathbf{\nabla} u \cdot \mathbf{T}'^{-1} \mathbf{e}_r \right] - C &= k \quad \text{for } r = 1
\end{align*}$$  \hspace{1cm} (34)

where $\mathbf{T}'$, $\mathbf{T}'^{-1}$, $\mathbf{T}'^T$ denote respectively the Jacobian, the inverse and the transpose of the Jacobian $\mathbf{T}'$. The reference pressure $k$ is obtained by constraining the drop volume to have a constant value $V_0$ as in (11):

$$\int_0^\pi \int_0^{\pi/2} g^3 \sin \theta \, d\theta \, d\psi = Cte.$$  

For axisymmetrical shapes the radial shape function $g(\theta, \psi, \varepsilon)$ depends only on the azimuthal angle $\theta$; the first equation of (34) can be written as

$$F_1 r + \frac{2}{r} F_1 \theta + \frac{1}{r} F_2 \psi + \frac{\cos \theta}{r \sin \theta} F_2 = 0 \quad \text{for } r > 1$$  \hspace{1cm} (35)

where

$$F_1 (r, \theta) = \left( g + \frac{g_\theta^2}{g} \right) u_r - \frac{g_\theta}{r} u_\theta, \quad F_2 (r, \theta) = \frac{g}{r} u_\theta - g_\theta u_r.$$  

The boundary condition in ((34).2) becomes

$$\left( g^2 + g_\theta^2 \right) u_r - gg_\theta u_\theta = \left( g^2 + g_\theta^2 \right) \cos \theta + gg_\theta \sin \theta \quad \text{for } r = 1.$$  \hspace{1cm} (36)
Relation ((34).3) takes the following form:

\[ \varepsilon \left\{ \frac{1}{2} \left[ u_r^2 + F_3^2 \right] - u_r \cos \theta + F_3 \left[ \sin \theta + \frac{g \theta u_r}{g} \right] \right\} - (C + k) g^2 = 0 \]

for \( r = 1 \) \hspace{1cm} (37)

with: \( F_3 (r, \theta) = u_\theta - \frac{g \theta u_r}{g} \).

As in the previous case for the rotating drop, one can note that for \( \varepsilon = 0 \) there is a solution: \( g(\theta, 0) = 1 \), and \( u(r, \theta, 0) \). Using the implicit function theorem we can assert the existence of a unique solution depending only on \( r \), \( \theta \) and analytic in \( \varepsilon \) (Séro-Guillaume and Er-Riani, 1999). Then the solution to the problem of \( u, k \) and \( g \) will be sought as series in \( \varepsilon \):

\[
\begin{cases}
  u(r, \theta, \varepsilon) = \sum_{n \geq 0} \varepsilon^n v_n (r, \theta) \\
  g(\theta, \varepsilon) = \sum_{n \geq 0} \varepsilon^n f_n (\theta) \\
  k = \sum_{n \geq 0} \varepsilon^n h_n.
\end{cases}
\]

(38)

3.2. Solution to the perturbation equations

One has to substitute relations (38) into the governing Eqs. (36)–(38), collect the power of \( \varepsilon \) and obtain a sequence of differential equations and boundary conditions. The substitution, especially for the curvature, will not be done directly but will be done with the method revealed in Section 2. We note that the system of equations is uncoupled at each order. In fact the equations of the potential at order \( m + 1 \) depend on solutions of the equilibrium equation at order \( m \), so we can solve alternatively equations for the flow and equations corresponding to equilibrium equation. At each order, we first integrate the equation derived from (34) with respect to \( v_n \) subject to the boundary condition (36). Secondly we insert this value into (37) to get the expression for \( f_n \), which contains an unknown constant of pressure \( h_n \). This constant will be determined by the condition of volume conservation (11). Furthermore these equations turn out to have solutions as linear combinations of Legendre polynomials (cf. Appendix A). Then we expand all quantities in terms of these polynomials as

\[ v_n (r, \theta) = \sum_{m \geq 0} a_n^m (r) P_m (\xi), \quad f_n (\theta) = \sum_{m \geq 0} \alpha_n^m P_{2m} (\xi). \]

We derive equations for \( a_n^m (r) \) and \( \alpha_n^m \) which are subject to condition (11), enabling us to calculate \( h_n \). We include for illustration the lowest orders.

Order 0:
Eqs. (34)–(36) would look at order 0 as follows:

\[
\begin{cases}
  \Delta v_0 = 0 \quad \text{for } r > 1 \\
  v_{0r} = \cos \theta \quad \text{for } r = 1 \\
  v_0 \rightarrow 0 \quad \text{at infinity}
\end{cases}
\]
so \( v_0 (r, \theta) = -\frac{\cos \theta}{2r^2} \).

At order 0, Eq. (37) gives \( h_0 = -2 \).

**Order 1:**

We find for order 1

\[
\begin{align*}
\Delta v_1 &= -f_1 v_{0r} + f_1'' v_0 r - \frac{\cos \theta f_1 v_0}{r} - \frac{2 f_1 v_{0r}}{r} - \frac{f_1 v_{0\theta \theta}}{r} \\
&= \frac{2 f_1 v_{0\theta r}}{r} - \frac{\cos \theta f_1 v_0}{\sin \theta} f_1 v_{0\theta} & \text{for } r > 1 \\
v_1 r &= (v_{0\theta} + \sin \theta) f_1' + 2 f_1 (\cos \theta - v_0) & \text{for } r = 1.
\end{align*}
\]

Eq. (37) reads

\[
\frac{1}{2} v_0^2 - v_0 r \cos \theta + \frac{v_0^2 \theta}{2} + v_{0\theta} \sin \theta - C_1 (\theta) = h_1 \text{ for } r = 1.
\]

By inserting the value of \( v_0 (r, \theta) \) back in (40), and taking into account relation (11), one obtains

\[
f_1 (\theta) = -\frac{3}{16} P_2 (\xi), \quad h_1 = \frac{1}{4}.
\]

By inserting the values of \( v_0 (r, \theta) \) and \( f_1 (\theta) \) in relations (39), one obtains the system

\[
\begin{align*}
\Delta v_1 &= \frac{9}{16 r^4} \left( 5 \cos^3 \theta - 3 \cos \theta \right) = \frac{9}{8 r^4} P_3 (\xi) \\
v_1 r &= \frac{27}{32} \left( \cos \theta - \cos^3 \theta \right) = \frac{27}{80} \left( P_1 (\xi) - P_3 (\xi) \right) & \text{for } r = 1 \\
v_1 &\to 0, & \text{at infinity}.
\end{align*}
\]

One can develop \( v_1 (r, \theta) \) on the basis of Legendre polynomials as

\[
v_1 (r, \theta) = \sum_{n \geq 0} a_n^1 (r) P_n (\xi).
\]

To solve the linear system (41) to obtain \( v_1 (r, \theta) \) it should be noted that

\[
\Delta v_1 = \sum_{n \geq 0} \left( \frac{d^2 a_n^1}{dr^2} + 2 \frac{d a_n^1}{r dr} - \frac{n (n + 1)}{r^2} a_n^1 \right) P_n (\xi) = \frac{9}{8 r^4} P_3 (\xi).
\]

Moreover, the boundary condition for \( r = 1 \) becomes

\[
v_1 r = \sum_{n \geq 0} \frac{d a_n^1}{dr} P_n (\xi) = \frac{27}{80} P_1 (\xi) - \frac{27}{80} P_3 (\xi).
\]

Finally we obtain

\[
a_1^1 = -\frac{27}{160 r^2}, \quad a_3^1 = \frac{9}{8} \left( \frac{1}{8 r^4} - \frac{1}{10 r^2} \right), \quad \text{and for } n \neq 3, \quad a_n^1 = 0.
\]
so

\[ v_1(r, \theta) = -\frac{27}{160r^2} P_1(\xi) + 9 \left( \frac{1}{8r^4} - \frac{1}{10r^2} \right) P_3(\xi). \]

**Order 2:**

The equilibrium equation leads to

\[
2(v_0r - \cos \theta)v_{1r} + 2(v_0\theta + \sin \theta)(v_{1\theta} - 2f_1 v_0r)
+ 2f_1'v_0\theta - 4C_1(\theta) - C_2(\theta) - f_1 = h_2.
\]

By inserting the expressions for \( f_1(\theta), v_0(r, \theta) \) and \( v_1(r, \theta) \) in (43), one obtains

\[ L f_2 + \frac{1}{256} \left( 531 \cos^3 \theta - 594 \cos^2 \theta + 147 \right) = k_2. \]

The solution to this equation is

\[ f_2(\theta) = \frac{1}{4480} \left( 118P_4(\xi) - 405P_2(\xi) - 63 \right) \]

while (11) gives \( k_2 = 3/16. \)

In Table 3 we given the first coefficients of the expansion of the aspect ratio of the drop

\[ b(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n d_n. \]

3.3. Discussion

Sykes–Domb’s graph combined with Richardson’s procedure, as shown in Fig. 5, gives the critical value \( \varepsilon_0 = 1.2572. \) Miksis et al. (1981) have studied numerically the shape of a deformed bubble in a uniform flow, a problem very similar to the one treated here. However, they do not keep the volume constant. They observed similar shapes but for greater values of the Weber numbers. They obtained a maximum Weber number above which the solution fails to exist. This value corresponds to \( \varepsilon_0 = 1.615, \) a value which is close to the one obtained here. Meiron (1989) studied numerically exactly the same question as in this paper. He used a collocation method for small values of the Weber number. He obtained the value 1.5246 for the singularity. This value is greater than the one we have obtained. He noted difficulties in the convergence of his calculations for values of \( \varepsilon \) close to 1.2. This is a frequent problem in this type of numerical calculation because the calculated critical values are generally singular values. The numerical algorithms are
often unstable for such calculations. It has to be noted that for such problems symbolic computation enables one to avoid this problem and reveals itself to be superior to numerical computation. As a last point, let us single out the fact that for the drop in a uniform flow the critical value $\varepsilon_0$ corresponds to a subcritical bifurcation, which means that above this value there is no stable equilibrium position; therefore the drop breaks up. This is one of the atomization processes; see Lefebvre (1989). The accurate determination of this critical value is useful for the calibration of nozzles producing calibrated drops.

4. Conclusion

With the calculation of the shape, velocity and pressure to a higher order, the domain perturbation method has been shown to be a useful tool for the calculation of drop or bubble shapes, and more generally for free boundary problems for which this method can be applied. The literature contains many perturbation solutions (Aziz and Benzies, 1976; Van Dyke, 1974; Mack and Bishop, 1988) and their calculations stopped because of the exponential growth of expressions; indeed these expressions increase in size so rapidly that they exhaust the memory of any machine. This phenomenon happens frequently in intermediate calculations in computer algebra calculations and it is known as intermediate expression swell or combinatorial explosion. It is commonly present in many algebra algorithms like Gaussian elimination, the Euclidean algorithm and computation of the polynomial Greatest Common Divisor. It is also characterized by the appearance of huge numbers. In our case, it appears in the number of terms required for the computation of the hierarchical sequences increasing exponentially as the number of inputs increases. It is difficult to give general rules for solving this problem. However, it is necessary to try to either simplify
or regroup terms. This raises the question, still open, of simplification and factorization processes. Some related works arising in fluid mechanics, as in lubrication and convection, such as in (Corless and Jeffrey, 1990; Corless et al., 1997), deal with the manipulation of expansion series, and show how to turn very large expressions, once generated, into more compact and useful computation sequences. But the problems considered are linear, whereas the two problems treated here are strongly non-linear, and one of them is non-local, i.e. the equations depend effectively on the geometric domain. With a straightforward use of Maple we could only obtain 14 terms of the solution in the case of the rotating drop and 10 terms in the case of the bubble in a uniform flow. But making the substitution of the curvature’s expansion as in (25) we obtained more coefficients in the two problems. In order to see the memory gain obtained by our substitution we have plotted in Fig. 6 the RAM occupied by the computation as a function of the number of terms when the calculations are made with and without substitution. We can note that the proposed method saves about half the memory. The first curve can be fitted with $y = A_1 \exp(B_1 N) = 0.0294 \exp(0.5180N)$, where $N$ is the number of terms in the solution to the shape of a drop in uniform flow, while the second curve is fitted with $y = A_2 \exp(B_2 N) = 0.3544 \exp(0.3613N)$. So the fact that $B_1$ is bigger than $B_2$ shows the saving of memory.

In conclusion, it is worth noting that simplification of computation sequences can be an efficient tool for obtaining compact expressions especially for perturbation methods but, usually, large expression simplification can exhaust the memory of the computer without giving a conclusive result. So minor issues, like substitutions made using relation (25), have a significant effect on the performance of any computer algebra system. One of the main advantages of our approach is the fact that it considerably reduces the problem of intermediate expression swell.
Acknowledgements

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Appendix A

Existence and uniqueness by the implicit function theorem

Let us recall the following theorem:

**Implicit function theorem:** Let $E$ and $F$ be complex Banach spaces and let $G : E \times \mathbb{C} \to F$, where $\mathbb{C}$ denotes the complex numbers. Suppose that $G$ is continuously Frechet differentiable in a neighbourhood of $(\phi_0, 0)$. Suppose in addition that $G(\phi_0, 0) = 0$ and $\frac{\partial G}{\partial \phi}(\phi_0, 0)$ is an isomorphism from $E$ to $F$. Then there exists an analytic $E$-valued function $\phi(\varepsilon)$ defined for sufficiently small $\varepsilon$ such that $G(\phi(\varepsilon), \varepsilon) = 0$. The proof of this theorem goes back to Graves and Hilderbrandt (1923) and was generalized by Nash (1956) and Moser (1961) to non-linear differential problems. We shall apply the implicit function theorem in these Banach spaces for problem one:

$$E = \{ \phi = (g, k), g \in C^2([0, \pi/2] \times [0, \pi]), g_0'(\pi/2, \varepsilon) = 0 \}$$

$$F = \{ \zeta \in C^0([0, \pi/2] \times [0, \pi]) \times \mathbb{R} \}.$$

Eqs. (11) and (16) can be written as

$$F(\phi, \varepsilon) = (F_1(\phi, \varepsilon), F_2(\phi, \varepsilon)) = (\varepsilon g^2 \sin^2 \theta - C + k, \int_0^{\pi/2} g^3 \sin \theta \, d\theta - 1) = 0.$$  

It is obvious that $F(\phi_0, 0) = 0$, where $\phi_0 = (1, 2)$. But we need to check that $\frac{\partial F}{\partial \phi}(\phi_0, 0)$ is an isomorphism from $E$ to $F$.

We obtain

$$\frac{\partial F}{\partial \phi}(\phi_0, 0) = \begin{pmatrix} \frac{\partial F_1}{\partial \phi}(\phi_0, 0) & 1 \\ \frac{\partial F_2}{\partial \phi}(\phi_0, 0) & 0 \end{pmatrix}$$

where

$$\frac{\partial F_1}{\partial g}(\phi_0, 0)(\eta, h) = \frac{\partial^2 g}{\partial \theta^2} + \frac{\cos \theta \, \partial \eta}{\sin \theta} + 2 \eta + h = \zeta \quad \text{(A.1)}$$

$$\frac{\partial F_2}{\partial g}(\phi_0, 0)(\eta, h) = 3 \int_0^{\pi/2} \eta \sin \theta \, d\theta = m. \quad \text{(A.2)}$$

Let us verify by a direct calculation that this system always has a unique solution in $E$. Indeed if we change the variables,

$$x = \cos \theta, \eta(\theta) = \Phi(x), \zeta(\theta) = \Psi(x).$$
then Eq. (A.1) yields
\[
(1 - x^2) \Phi'' + 2x \Phi' + 2 \Phi + h = \zeta. \tag{A.3}
\]

However it is known that Legendre polynomials \( P_n(x) \) defined by the recurrence relation
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( x^2 - 1 \right)^n
\]
are orthogonal in the interval \([-1, 1]\) and constitute a basis of \( L^2(-1, 1) \) (cf. Courant and Hilbert (1953)). In addition they are solutions of the eigenvalue problem:
\[
(1 - x^2) y'' + 2xy' - n(n + 1)y = 0. \tag{A.4}
\]

We represent the functions \( \eta \) and \( \zeta \) as a linear combination of Legendre polynomials:
\[
\Phi = \sum_{n \geq 0} a_n P_n(x), \quad \Psi = \sum_{n \geq 0} b_n P_n(x).
\]
The condition \( \eta(\pi/2) = 0 = \Phi'(0) \) implies that \( a_{2n+1} = 0, \ n \geq 0 \). So \( \Phi = \sum_{n \geq 0} a_{2n} P_{2n}(x) \).

By inserting the expression for \( \Phi \) in relation (A.3), we get the expression for \( a_n \):
\[
\begin{cases}
a_0 + h = b_0 \\
a_{2n} = \frac{b_{2n}}{2n(2n + 1) - 1} & \text{for } n \geq 1.
\end{cases} \tag{A.5}
\]

Furthermore (A.2) can be written as
\[
3 \sum_{n \geq 0} a_{2n} \int_0^{\pi/2} P_{2n}(\cos \theta) \sin \theta \, d\theta = -3 \sum_{n \geq 0} a_{2n} \int_0^1 P_{2n}(x) P_0(x) \, dx = -3/2 \sum_{n \geq 0} a_{2n} \int_{-1}^1 P_{2n}(x) P_0(x) \, dx = m.
\]

Then \( a_0 = -m/3 \). We get \( h \) from the first relation of (A.5). We have applied the implicit theorem to problem one for the sake of simplicity but the proof is exactly the same for problem two.

References


