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Functional–Differential Equations with Compressed Arguments and Polynomial Coefficients: Asymptotics of the Solutions

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Functional–differential equations with linearly compressed arguments and polynomial coefficients are considered. We prove, under some mild restrictions on the coefficients, that each solution $y(t)$ of such an equation, satisfying estimate $|y(t)| \le C \exp\{\gamma \ln^2 |t|\}$ $(t \to \infty)$, where $0 < \gamma < \tilde{\gamma}$, is polynomial.

1. INTRODUCTION

Recently there has been considerable interest in problems concerning functional–differential equations (FDE) with linearly transformed arguments of the form

$$y^{(m)}(t) = \sum_{j=0}^{l} \sum_{k=0}^{m-1} a_{jk} y^{(k)}(\alpha_j t + \beta_j) \tag{1.1}$$

$$a_{jk} \in \mathbb{C}, \qquad \alpha_j, \beta_j \in \mathbb{R}, \; -\infty < t < \infty$$

[1, 4–11, 13–17, 19–24]. Such equations form a wide and natural class of general FDE [2, 12] and have diverse applications in areas ranging from number theory to astrophysics [10].

Provided $A = \max |\alpha_j| < 1$, each solution of (1.1) is an entire function of order zero and is hence unbounded both on $\mathbb{R}_+$ and $\mathbb{R}_-$ [16, 6]. This result cannot be strengthened in general due to the existence of polynomial

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solutions of (1.1). It was proved in [5, 6] that a necessary and sufficient condition for the existence of polynomial solutions is that

$$\sum_{j=0}^{l} a_{j0}\alpha_j^n = 0 \tag{1.2}$$

for some $n \in \mathbb{N}$.

Under the assumption that Eq. (1.1) has no polynomial solutions and $\beta_j = 0$ for all $j = 0, \ldots, l$, one can prove a stronger result. Roughly speaking, every nontrivial solution of (1.1) grows as $t \to \infty$ faster then $\exp\{\gamma \ln^2 t$ for some $\gamma > 0$. More precisely, every solution of Eq. (1.1) which satisfies estimate

$$|y(t)| \le C \exp\{\gamma \ln^2(1 + |t|)\} \tag{1.3}$$

for some $C > 0$ and

$$\gamma < \tilde{\gamma} = \frac{1}{2\,|\ln \alpha|}, \tag{1.4}$$

where $\alpha = \min |\alpha_j|$, vanishes identically [5, 6].

The goal of this paper is to prove a similar result for FDE with polynomial coefficients. The paper continues the study of FDE with linearly compressed arguments and polynomial coefficients initiated in [4, 17, 22, 23].

2. Asymptotic Behavior of the Solutions

Let us consider the equation

$$y^{(m)}(t) = \sum_{j=0}^{l} \sum_{k=0}^{m-1} \sum_{\nu=0}^{r} a_{jk\nu} t^{\nu} y^{(k)}(\alpha_j t). \tag{2.1}$$

Denote

$$\alpha = \min_{0 \le j \le l} |\alpha_j|, \qquad A = \max_{0 \le j \le l} |\alpha_j|$$

and assume that $A < 1$.

We begin with the following general

Theorem 1 [4, 23]. *Every (classical) solution of* (2.1) *is an entire function of order zero.*

It is a well-known fact from complex analysis that any entire function of order $\rho < \frac{1}{2}$ is unbounded on any ray in the complex plane [18]. Therefore an immediate consequence of Theorem 1 is

COROLLARY 1. *Every nonconstant solution of* (2.1) *is unbounded both on* R_+ *and* R_-.

The main result of the paper is

THEOREM 2. *Suppose that* $A < 1$ *and, beginning with some* n:

$$\sum_{j=0}^{l} a_{jor}\alpha_j^n \neq 0 \qquad (n \geq N_0). \tag{2.2}$$

Then each nontrivial solution of Eq. (2.1), *which satisfies the estimate*

$$|y(t)| \leq C \exp\{\gamma \ln^2(1 + |t|)\} \tag{2.3}$$

for all $t \in R_+$ *(or all* $t \in R_-$*), where* $C > 0$ *and*

$$\gamma < \bar{\gamma} = \frac{1}{2\,|\ln \alpha|}, \tag{2.4}$$

is a polynomial.

Proof of Theorem 2. The proof is based on the Wiman–Valiron theorem [18] and Lemma 1, following below. We divide the proof into 4 steps.

(a) Let us differentiate Eq. (2.1) n times, where $n > r$. We obtain

$$\begin{aligned} y^{(m+n)}(t) &= \sum_{j=0}^{l}\sum_{k=0}^{m-1}\sum_{\nu=0}^{r} a_{jk\nu}[t^\nu y^{(k)}(\alpha_j t)]^{(n)} \\ &= \sum_{j=0}^{l}\sum_{k=0}^{m-1}\sum_{\nu=0}^{r} a_{jk\nu}\frac{n!\,\nu!}{(n-\nu)!\,\nu!}\alpha_j^{n-\nu}y^{(k+n-\nu)}(\alpha_j t) \\ &\quad + \left[\sum_{j=0}^{l}\sum_{k=0}^{m-1}\sum_{\nu=0}^{r} a_{jk\nu}\frac{n!}{(n-\nu+1)!}\alpha_j^{n-\nu+1}t y^{(k+n-\nu+1)}(\alpha_j t)\right. \\ &\quad \left. + \cdots + \sum_{j=0}^{l}\sum_{k=0}^{m-1}\sum_{\nu=0}^{r} a_{jk\nu}\alpha_j^n t^\nu y^{(k+n)}(\alpha_j t)\right]. \end{aligned} \tag{2.5}$$

Denote $b_n = y^{(n)}(0)$ and insert $t = 0$ in (2.5). Taking into account the fact that the expression in the brackets on the right-hand side of (2.5) vanishes for $t = 0$, we obtain

$$b_{m+n} = \sum_{j=0}^{l} \sum_{k=0}^{m-1} \sum_{\nu=0}^{r} a_{jk\nu} \frac{n!}{(n-\nu)!} \alpha_j^{n-\nu} b_{k+n-\nu} \tag{2.6}$$

or

$$\begin{aligned} b_{n+m+r} = &\sum_{\nu=0}^{r-1} \frac{(n+r)!}{(n+r-\nu)!} \sum_{k=1}^{m-1} \sum_{j=0}^{l} a_{jk\nu} \alpha_j^{n+r-\nu} b_{n+k+(r-\nu)} \\ &+ \frac{(n+r)!}{n!} \left(\sum_{j=0}^{l} a_{jor} \alpha_j^n \right) b_n . \end{aligned} \tag{2.7}$$

LEMMA 1. *Suppose* $\{b_n\}_0^\infty$ *is a nonfinite solution of the difference equation* (2.7) (*i.e., there exist infinitely many* $b_n \neq 0$ $(n = 0, 1, \ldots)$).

Suppose also that beginning with some $N \geq N_0$ *inequality* (2.2) *holds. Then for each sufficiently large* $N (N \geq N_1 \geq N_0)$ *there exists an* n: $N(m+r) + 1 \leq n \leq (N+1)(m+r)$ *such that*

$$|b_n| \geq D^n \alpha^{n^2/2} \tag{2.8}$$

In order not to interrupt the presentation, the proof of Lemma 1 will be given at the end of the paper.

Now we deduce Theorem 2 from Lemma 1 by means of the Wiman–Valiron theorem.

(b) Suppose that $y(t)$ is not polynomial. We shall demonstrate that in this case (2.3) is satisfied only if $y(t) \equiv 0$.

According to Theorem 1 [4, 23], every solution $y(t)$ of Eq. (2.1) can be extended to the complex plane as an entire function $y(z)$ of zero order:

$$y(z) = b_0 + \frac{b_1}{1!} z + \cdots + \frac{b_n}{n!} z^n + \cdots .$$

Denote, as usual,

$$m(r) = \min_{|z|=r} |y(z)|, \qquad M(r) = \max_{|z|=r} |y(z)|.$$

It is obvious that

$$|y(t)| \geq m(r), \qquad |t| = r. \tag{2.9}$$

According to the Wiman–Valiron theorem, for every entire function of order $\rho < 1$ and every $\varepsilon > 0$ there exists a sequence $r_i \to \infty$ such that

$$m(r_i) \geq M(r_i)^{[\cos \pi\rho - \varepsilon]}. \tag{2.10}$$

In particular, for an entire function of order $\rho = 0$

$$m(r_i) \geq M(r_i)^{1-\varepsilon} \tag{2.11}$$

It follows from (2.9), (2.11), and the Cauchy inequality that

$$|y(t)| \geq M(r)^{1-\varepsilon} \geq \sup_{0 \leq n \leq \infty} \left(\frac{|b_n|}{n!} r^n\right)^{1-\varepsilon} \tag{2.12}$$

for $t = \pm r_i$.

By our assumption, $y(x)$ is not a polynomial, that is, $\{b_n\}$ is a nonfinite sequence and Lemma 1 can be applied to estimate b_n from below. Then one obtains from (2.12) and (2.8) that

$$|y(t)| \geq \sup_{n \in \mathfrak{M}} \left[\frac{(Dr)^n}{n!\,(1/\alpha)^{n^2/2}}\right]^{1-\varepsilon}, \tag{2.13}$$

where $\mathfrak{M} = \{n_1, n_2, \ldots\}$ is a sequence of integers such that

$$0 < n_{i+1} - n_i \leq 2s; \qquad N_1 s + 1 \leq n_i; \qquad n_i \xrightarrow[i \to \infty]{} \infty. \tag{2.14}$$

Here and further $s = m + r$.

By Stirling's formula

$$n!\,(1/\alpha)^{n^2/2} \leq c_1 (1/\alpha)^{((1+\varepsilon)/2)n^2}.$$

for each $\varepsilon > 0$ and some $c_1 = c_1(\varepsilon) > 0$. Thus

$$|y(t)| \geq c_1 \left[\sup_{n \in \mathfrak{M}} \frac{(Dr)^n}{E^{n^2/2}}\right]^{1-\varepsilon}, \tag{2.15}$$

with $E = (1/\alpha)^{1+\varepsilon}$.

(c) In order to find $\sup_{n \in \mathfrak{M}} [(Dr)^n / E^{n^2/2}]$ we introduce the function

$$f(x) = \frac{(Dr)^x}{E^{x^2/2}} = e^{-(\ln E/2)x^2 + (\ln Dr)x} \tag{2.16}$$

of the continuous argument x. The function $f(x)$ attains its maximal value together with the quadratic polynomial

$$\varphi(x) = -\left(\frac{\ln E}{2}\right) x^2 + (\ln Dr)x$$

at the point $x_{\max} = \ln(Dr)/\ln E$. For r sufficiently large $x_{\max} > 0$ and according to (2.14) one can find n_{i_0} such that

$$0 < n_{i_0} - x_{\max} \leq s. \tag{2.17}$$

Therefore

$$\sup_{n \in m} \frac{(Dr)^n}{E^{n^2/2}} \geq f(n_{i_0}) \geq f(x_{\max} + 2s), \tag{2.18}$$

with the second inequality fulfilled since $y(x)$ is monotonically decreasing for $x > x_{\max}$. But

$$\begin{aligned} f(x_{\max} + 2s) &= e^{-2s^2 \ln E} e^{(1/2 \ln E)\ln^2 Dr} \\ &= e^{-2s^2 \ln E} \cdot e^{\ln^2 D/2 \ln E} \cdot D^{\ln r/\ln E} \cdot e^{\ln 2 r/2 \ln E} \geq C e^{(1-\varepsilon)\ln^2 r/2 \ln E}. \end{aligned} \tag{2.19}$$

From (2.15), (2.18), (2.19), and by the definition of $E = ((1/\alpha)^{+\varepsilon})$ it follows that for each nonpolynomial solution $y(t)$ we have

$$|y(t)| \geq C e^{(1-\varepsilon)^2/(1+\varepsilon)\cdot(1/2|\ln \alpha|)\ln^2 r} \geq C e^{((1-\delta)/2|\ln \alpha|)\ln^2 r} \tag{2.20}$$

for any $\delta > 0$ and big enough $t = \pm r_n$.

This completes the proof modulo Lemma 1.

(d) *Proof of Lemma* 1. Rewrite (2.7) in the form

$$\begin{aligned} b_n = \frac{n!}{(n+r)! \sum_{j=0}^{l} a_{jor} \alpha_j^n} \Bigg[b_{n+m+r} - \sum_{\nu=0}^{r-1} \frac{(n+r)!}{(n+r-\nu)!} \\ \times \sum_{k=1}^{m-1} \sum_{j=0}^{l} a_{jk\nu} \alpha_j^{n+r-\nu} b_{n+k+(r-\nu)} \Bigg] \end{aligned} \tag{2.21}$$

and observe that (2.2) implies the inequality

$$\left| \sum_{j=0}^{l} a_{jor} \alpha_j^n \right| \geq D_1 \alpha^n, \qquad n \geq N_1 \tag{2.22}$$

for N_1 sufficiently large. It follows from (2.21) and (2.22) that

$$|b_n| \leq D_2 \alpha^{-n} \max_{1 \leq i \leq s} |b_{n+i}|, \qquad n \geq N_1, \tag{2.23}$$

with $s = m + r$ (and we can assume without loss of generality that $D_2 \geq 1$).

Denoting

$$M_{N+1} = \max\{|b_{sN+1}|, \ldots, |b_{s(N+1)}|\}$$

we obtain

$$M_{N_1} \le D_2^{sN'} \alpha^{-\{s(N_1+N')+[s(N_1+N')-1]+\cdots+(sN_1+1)\}} \cdot M_{N+N'} \tag{2.24}$$

for any positive integer N'. In order to prove (2.24) successively put n equal $s(N_1 + N')$, $s(N_1 + N') - 1, \ldots, sN_1 + 1$, in (2.23), whence

$$|b_{s(N_1+N')}| \le D_2 \alpha^{-s(N_1+N')} M_{N_1+N'},$$

$$|b_{s(N_1+N')-1}| \le D_2 \alpha^{-s(N_1+N')+1} \max\{D_2 \alpha^{-s(N_1+N')}, M_{N_1+N'}, M_{N_1+N'}\}$$

$$= D_2^2 \alpha^{-s(N_1+N')} \alpha^{-s(N_1+N')+1} M_{N_1+N'},$$

$$\ldots$$

$$|b_{sN_1+1}| \le D_2^{sN'} \alpha^{-s(N_1+N')} \cdots \alpha^{-(sN_1+1)} M_{N_1+N'},$$

which proves (2.24).

It follows from (2.24) that

$$\begin{aligned} M_{N_1+N'} &\ge D_2^{-s(N'+N_1)} \alpha^{[1+\cdots+s(N'+N_1)]} M_{N_1} \\ &\ge D_3^{s(N'+N_1)} \alpha^{(1/2)[s(N_1+N')]^2} M_{N_1} \end{aligned} \tag{2.25}$$

for some $D_3 > 0$ (without loss of generality we may assume that $D_3 < 1$).

Note that $M_{N_1} \neq 0$, because otherwise (according to (2.7)) $b_n = 0$ for every $n \ge sN_1$, which is in contradiction with our assumption about the nonfinitness of $\{b_n\}_{n=0}^{\infty}$. Now (2.8) follows from (2.25). This proves the lemma.

Remark 1. Strict inequality $A < 1$ is essential for the validity of Theorem 2. In fact, consider the equation

$$y'(t) = ay(\alpha t) + by(t); \qquad 0 < \alpha < 1; \qquad |a| < |b| \tag{2.26}$$

with $A = \max |\alpha_j| = 1$. It was proved by T. Kato and J. B. McLeod [14,Th.3(i)] that $\lim_{t\to\infty} y(t) = 0$ for any solution of (2.26), in contrast with the statement of Theorem 2.

Remark 2. A special case of Theorem 2, relating to the equation

$$y'(t) = ay(At), \qquad 0 < A < 1$$

has been proved by N. G. de Bruijn [3] (see also [14]).

Remark 3. It was proved in [5] that, provided $A < 1$, every solution of (1.1) satisfies (2.3) with $\gamma = m/(2|\ln A|)$. In a similar manner, for FDE (2.1) with polynomial coefficients analogous upper bound of the solutions can be obtained. Provided $A < 1$, there exists $\tilde{\gamma}_1 > (\tilde{\gamma})$ such that any solution of (2.1) satisfies (2.3), where $\gamma > \tilde{\gamma}_1$.

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