Strong Maximum Principles for Parabolic Nonlinear Problems with Nonlocal Inequalities Together with Arbitrary Functionals

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1. INTRODUCTION

In this paper we give a theorem on strong maximum principles for problems with a diagonal system of nonlinear parabolic functional-differential inequalities and with nonlocal inequalities together with arbitrary functionals. The diagonal system of the inequalities considered here is of the form

\[ u_i(t, x) \leq f_i(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u) \quad (i = 1, \ldots, m), \]  

where \((t, x) \in D \subseteq (t_0, t_0 + T) \times \mathbb{R}^n\) and \(D\) is one of six relatively arbitrary sets more general than the cylindrical domain \((t_0, t_0 + T) \times D_0 \subseteq \mathbb{R}^{n+1}\). The symbol \(u\) denotes the mapping

\[ u: \bar{D} \ni (t, x) \rightarrow u(t, x) = (u^1(t, x), \ldots, u^n(t, x)) \in \mathbb{R}^n, \]

\(\bar{D}\) is an arbitrary set contained in \((-\infty, t_0 + T) \times \mathbb{R}^n\) such that \(D \subseteq \bar{D}\). The right-hand sides \(f^i\) \((i = 1, \ldots, m)\) of system (1.1) are functionals of \(u\); \(u^i_x(t, x) = \text{grad}_x u^i(t, x)\) \((i = 1, \ldots, m)\) and \(u^i_{xx}(t, x)\) \((i = 1, \ldots, m)\) denote the

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matrices of second order derivatives with respect to $x$ of $u'(t, x)$ $(i = 1, \ldots, m)$.

The nonlocal inequalities, considered here, are of the form

$$[u'(t_0, x) - K'] + \sum_{i \in I_*} h_i(x)[F'(x, u') - K']$$

$$\leq 0 \quad \text{for} \quad x \in S_{t_0} \ (j = 1, \ldots, m),$$

(1.2)

where $K' (j = 1, \ldots, m)$ are some constants, $I_*$ is a subset of a countable set $I$ of natural indices, $t_0 < T_{2i-1} < T_{2i} \leq t_0 + T \ (i \in I)$, $h_i: S_{t_0} \rightarrow (-\infty, 0]$ $(i \in I_*)$ are some functions, $F': S_{t_0} \times C([T_{2i-1}, T_{2i}] \times S_{t_0}) \rightarrow R \ (i \in I_*)$ are given mappings satisfying some assumptions, and

$$S_{t_0} := \text{int}\{x \in R^n: (t_0, x) \in \bar{D}\}.$$

The results obtained in this paper are a continuation and direct generalizations of those given by the author [24]. Moreover, some results obtained here are direct generalizations of results given by Chabrowski [5, 6]. Finally, some results obtained in the paper are indirect generalizations of those given by the author [1], Szarski [10], Walter [11], Redheffer and Walter [9], Lakshmikantham and Leela [5], and Ladde et al. [7]. The method of the proof of the main theorem in this paper is similar to the method used in [3, 4], and for ease in comparison of these methods we use in this article similar notation as in [3, 4].

2. PRELIMINARIES

The notation and definitions given in this section are valid throughout this paper.

We use the following notation: $R = (-\infty, \infty)$, $R_+ = (-\infty, 0]$, $N = \{1, 2, \ldots\}$, $x = (x_1, \ldots, x_n) \ (n \in N)$.

For any vectors $z = (z^1, \ldots, z^m) \in R^m$, $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^m) \in R^m$ we write

$$z \leq \bar{z} \quad \text{if} \quad z^i \leq \bar{z}^i \ (i = 1, \ldots, m).$$

Let $t_0$ be a real finite number and let $0 < T < \infty$. A set $D \subset \{(t, x): t > t_0, x \in R^n\}$ (bounded or unbounded) is called a set of type (P) if:

1. The projection of the interior of $D$ on the $t$-axis is the interval $(t_0, t_0 + T)$.

2. For every $(\bar{t}, \bar{x}) \in D$ there is a positive $r$ such that

$$\left\{(t, x): (t - \bar{t})^2 + \sum_{i=1}^n (x_i - \bar{x}_i)^2 < r, t < \bar{t}\right\} \subset D.$$
For any \( t \in [t_0, t_0 + T] \) we define the following sets:

\[
S_t = \begin{cases} \text{int}\{x \in R^n: (t_0, x) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in R^n: (t, x) \in D\} & \text{for } t \neq t_0 \end{cases}
\]

and

\[
\sigma_t = \begin{cases} \text{int}[\bar{D} \cap (\{t_0\} \times R^n)] & \text{for } t = t_0, \\ D \cap (\{t\} \times R^n) & \text{for } t \neq t_0. \end{cases}
\]

It is easy to see, by condition 2 of the definition of a set of type \((P)\), that \( S_t \) and \( \sigma_t \) are open sets in \( R^n \) and \( R^{n+1} \), respectively.

Let \( \bar{D} \) be a set contained in \(( -\infty, t_0 + T] \times R^n \) such that \( \bar{D} \subset \bar{D} \). We introduce the following sets:

\[
\partial_t D := \bar{D} \setminus D \quad \text{and} \quad \Gamma := \partial_t D \setminus \sigma_t.
\]

For an arbitrary fixed point \((\bar{t}, \bar{x}) \in D\) we denote by \( S^-(\bar{t}, \bar{x}) \) the set of points \((t, x) \in D\) that can be joined with \((\bar{t}, \bar{x})\) by a polygonal line contained in \( D \) along which the \( t \)-coordinate is weakly increasing from \((t, x)\) to \((\bar{t}, \bar{x})\).

By \( Z_m(\bar{D}) \) we denote the space of continuous in \( \bar{D} \) mappings

\[
w: \bar{D} \ni (t, x) \mapsto w(t, x) = (w^1(t, x), \ldots, w^m(t, x)) \in R^m.
\]

In the set of mappings bounded from above in \( \bar{D} \) and belonging to \( Z_m(\bar{D}) \) we define the functional

\[
[w]_t = \max_{i=1,\ldots,m} \sup_{i \leq t} \{0, w^i(t, x): (t, x) \in \bar{D}, i \leq t\}, \quad \text{where} \quad t \leq t_0 + T.
\]

By \( X \) we denote a fixed subset (not necessarily a linear subspace) of \( Z_m(\bar{D}) \) and by \( M_{n \times n}(R) \) we denote the space of real square symmetric matrices \( r = [r_{jk}]_{n \times n} \).

A mapping \( u \in X \) is called regular in \( D \) if \( u'_t, u'_x = \text{grad}_x u^i, u'_{xx} = [u'_{x_i x_j}]_{n \times n} (i = 1, \ldots, m) \) are continuous in \( D \).

Let the mappings

\[
f^i: D \times R^m \times R^n \times M_{n \times n}(R) \times Z_m(\bar{D}) \ni (t, x, z, q, r, w) \mapsto f^i(t, x, z, q, r, w) \in R \quad (i = 1, \ldots, m)
\]

be given and let the operators \( P_i (i = 1, \ldots, m) \) be defined by the formulae

\[
P_i u(t, x) = u'_i(t, x) - f^i(t, x, u(t, x), u'_x(t, x), u''_x(t, x), u), \quad u \in X, (t, x) \in D \quad (i = 1, \ldots, m).
\]
A regular mapping $u$ in $D$ is called a solution of the system of the functional-differential inequalities

$$P_i u(t, x) \leq 0, \quad (t, x) \in D \ (i = 1, \ldots, m) \quad (2.1)$$

in $D$ if (2.1) is satisfied.

For any set $Z \subset \bar{D}$ and for a mapping $u \in X$ we use the symbol $\max_{(t, x) \in Z} u(t, x)$ in the sense:

$$\left( \max_{(t, x) \in Z} u^1(t, x), \ldots, \max_{(t, x) \in Z} u^n(t, x) \right).$$

Let us define the set (compare [3, 4])

$$\mathcal{S} = \bigcup_{i \in I} (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}),$$

where $I$ is a countable set of all such mutually different natural numbers that:

(i) $t_0 < T_{2i-1} < T_{2i} \leq t_0 + T$ for $i \in I$ and $T_{2i-1} \neq T_{2j-1}, T_{2i} \neq T_{2j}$ for $i, j \in I$, $i \neq j$;

(ii) $T_0 := \inf \{ T_{2i-1} : i \in I \} > t_0$ if $\text{card } I = \aleph_0$;

(iii) $S_t \supseteq S_{t_0}$ for every $t \in \bigcup_{i \in I} [T_{2i-1}, T_{2i}]$;

(iv) $S_t \supseteq S_{t_0}$ for every $t \in [T_0, t_0 + T]$ if $\text{card } I = \aleph_0$.

An unbounded set $D$ of type (P) is called a set of type (PS$_F$) if:

(a) $\mathcal{S} \neq \emptyset$,

(b) $\Gamma \cap \sigma_{t_0} \neq \emptyset$.

Let $\mathcal{S}_*$ denote a nonempty subset of $\mathcal{S}$. We define the following set:

$$I_* = \{ i \in I : (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}) \subset \mathcal{S}_* \}.$$
ASSUMPTION (F). We say that the functions $F^i: S_{t_0} \times C([T_{2^i-1}, T_{2^i}]) \times S_{t_0} \to R \ (i \in I_*)$ satisfy Assumption (F) if for every fixed point $\tilde{x} \in S_{t_0}$ and for every fixed $j \in \{1, \ldots, m\}$ the inequalities

$$F^i(\tilde{x}, u^j) \leq \max_{t \in [T_{2^i-1}, T_{2^i}]} u^i(t, \tilde{x}) \quad (i \in I_*) \quad (2.2)$$

are satisfied.

3. STRONG MAXIMUM PRINCIPLES WITH NONLOCAL INEQUALITIES TOGETHER WITH ARBITRARY FUNCTIONALS IN SETS OF TYPES ($P_{SF}$) OR ($P_{SB}$)

Our main result is the following theorem on strong maximum principles with nonlocal inequalities together with integrals in sets of types ($P_{SF}$) or ($P_{SB}$):

**THEOREM 3.1.** Assume that:

1. $D$ is a set of type ($P_{SF}$) or ($P_{SB}$).
2. The mappings $f^i \ (i = 1, \ldots, m)$ are weakly increasing with respect to $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m \ (i = 1, \ldots, m)$. Moreover, there is a positive constant $L$ such that the inequalities

$$f^i(t, x, z, q, r, w) - f^i(t, x, \tilde{z}, \tilde{q}, \tilde{r}, \tilde{w})$$

$$\leq L \left( \max_{k=1, \ldots, m} |z^k - \tilde{z}^k| + |x| \sum_{j=1}^n |q^i - \tilde{q}^j| + |x|^2 \sum_{j,k=1}^n |r_{jk} - \tilde{r}_{jk}| + [w - \tilde{w}] \right)$$

are satisfied for all $(t, x) \subset D, z, \tilde{z} \subset R^m, q, \tilde{q} \subset R^n, r, \tilde{r} \subset M_{n \times n}(R), w, \tilde{w} \subset X, \sup_{(t, x) \in D} [w(t, x) - \tilde{w}(t, x)] < \infty \ (i = 1, \ldots, m)$.
3. The mapping $u \in X$ is a solution of system (2.1) in $D$.
4. The maximum of $u$ on $\Gamma$ is attained. Moreover,

$$K = (K^1, \ldots, K^m) := \max_{(t, x) \in \Gamma} u(t, x) \quad (3.1)$$

and $K \in X$.
5. The inequalities

$$[u^i(t_0, x) - K^i] + \sum_{i \in I_*} h_i(x)[F^i(x, u^i) - K^i]$$

$$\leq 0 \quad \text{for} \ x \in S_{t_0} \ (j = 1, \ldots, m) \quad (3.2)$$
are satisfied, where \( F^i : S_{\text{in}} \times C([T_{2i-1}, T_{2i}], S_{t_0}) \to R \ (i \in I_*) \) are given mappings satisfying Assumption (F) and \( h_i : S_{t_0} \to R \ (i \in I_*) \) are given functions such that 
\[-1 \leq \sum_{i \in I_*} h_i(x) \leq 0 \text{ for } x \in S_{t_0} \text{ and, additionally, if card } I_* = \mathbb{R}_0, \text{ then the series } \sum_{i \in I_*} h_i(x) F^i(x, u^i) \ (j = 1, \ldots, m) \text{ are convergent for } x \in S_{t_0}.
\]

(6) The maximum of \( u \) in \( \tilde{D} \) is attained. Moreover,
\[ M = (M^1, \ldots, M^m) := \max_{(t, x) \in \tilde{D}} u(t, x) \quad (3.3) \]
and \( M \in X. \)

(7) The inequalities
\[ f^i(t, x, M, 0, 0, M) \leq 0 \quad \text{for } (t, x) \in D \ (i = 1, \ldots, m) \]
are satisfied.

(8) The mappings \( f^i \ (i = 1, \ldots, m) \) are parabolic with respect to \( u \) in \( D \) and uniformly parabolic with respect to \( M \) in any compact subset of \( D \) (cf. [2] or [3]).

Then
\[ \max_{(t, x) \in D} u(t, x) = \max_{(t, x) \in \Gamma} u(t, x). \quad (3.4) \]

Moreover, if there is a point \((\tilde{t}, \tilde{x}) \in D\) such that \( u(\tilde{t}, \tilde{x}) = \max_{(t, x) \in \tilde{D}} u(t, x) \), then
\[ u(t, x) = \max_{(t, x) \in \Gamma} u(t, x) \quad \text{for } (t, x) \in S^-(\tilde{t}, \tilde{x}). \]

Proof. We shall prove Theorem 3.1 for a set of type \( (P_{SF}) \) only since the proof of this theorem for a set of type \( (P_{SB}) \) is analogous.

If \( \sum_{i \in I_*} h_i(x) = 0 \text{ for } x \in S_{t_0} \), then Theorem 3.1 from this paper is a consequence of Theorem 3.1 of [4]. Therefore, we shall prove Theorem 3.1 only in the case where the following condition holds:
\[ -1 \leq \sum_{i \in I_*} h_i(x) < 0 \quad \text{for } x \in S_{t_0}. \quad (3.5) \]

Assume, therefore, that \( (3.5) \) holds and, since we shall argue by contradiction, suppose
\[ M \neq K. \quad (3.6) \]

But, from \( (3.1) \) and \( (3.3) \), we have
\[ K \leq M. \quad (3.7) \]
Consequently, by (3.6) and (3.7), we obtain

\[ K < M. \]  \hspace{1cm} (3.8)

Observe, from Assumption (6), that the following condition holds:

There is \((t^*, x^*) \in \bar{D}\) such that \(u(t^*, x^*) = M := \max_{(t, x) \in \bar{D}} u(t, x). \) \hspace{1cm} (3.9)

By (3.9), by Assumption (4), and by (3.8), we have

\[(t^*, x^*) \in \bar{D} \setminus \Gamma = D \cup \sigma_{t_0}. \]  \hspace{1cm} (3.10)

An argument analogous to the one in the proof of Theorem 4.1 from [2] or Theorem 1 from [3] yields

\[(t^*, x^*) \notin D. \]  \hspace{1cm} (3.11)

Conditions (3.10) and (3.11) give

\[(t^*, x^*) \in \sigma_{t_0}. \]  \hspace{1cm} (3.12)

Simultaneously, by the definition of sets \(I\) and \(I_\ast\), we must consider the following cases:

(A) \( I_\ast \) is a finite set, i.e., without loss of generality there is a number \(p \in \mathbb{N}\) such that \(I_\ast = \{1, \ldots, p\}\).

(B) \( \text{card} I_\ast = \aleph_0. \)

First we shall consider case (A). And so, since \(u \in C(\bar{D})\), it follows that for every \(j \in \{1, \ldots, m\}\) and \(i \in I_\ast\), there is \(T'_i \in [T_{2j-1}, T_{2j}]\) such that

\[ u'(T'_i, x^*) = \max_{t \in [T_{2j-1}, T_{2j}]} u'(t, x^*). \]  \hspace{1cm} (3.13)

Consequently, by (3.2), by Assumption (F), by (3.13), and by the inequality

\[ u(t, x^*) < u(t_0, x^*) \quad \text{for} \quad t \in \bigcup_{i=1}^p [T_{2j-1}, T_{2j}] \]

being a consequence of (3.9), (3.12), and of (a)(i), (a)(iii) of the definition of a set of type \((P_{sr})\), we have
\[ 0 \geq [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[F'(x^*, u') - K'] \]
\[ \geq [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[u'(T_i, x^*) - K'] \]
\[ \geq [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[u'(t_0, x^*) - K'] \]
\[ = [u'(t_0, x^*) - K'] \left[ 1 + \sum_{i=1}^{p} h_i(x^*) \right]. \]

Hence
\[ u(t_0, x^*) \leq K \quad \text{if} \quad 1 + \sum_{i=1}^{p} h_i(x^*) > 0. \] (3.14)

Then, from (3.8) and (3.12), we obtain a contradiction of (3.14) with (3.9).

Assume now
\[ \sum_{i=1}^{p} h_i(x^*) = -1. \] (3.15)

Observe that for every \( j \in \{1, \ldots, m\} \) there is a number \( l_j \in \{1, \ldots, p\} \) such that
\[ u'(T_{l_j}, x^*) = \max_{i=1, \ldots, p} u'(T_i, x^*). \] (3.16)

Consequently, by (3.15), (3.16), (3.13), by Assumption (F), and by (3.2), we obtain
\[ u'(t_0, x^*) - u'(T_{l_j}, x^*) \]
\[ = [u'(t_0, x^*) - K'] - [u'(T_{l_j}, x^*) - K'] \]
\[ = [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[u'(T_{l_j}, x^*) - K'] \]
\[ \leq [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[u'(T_{l_j}, x^*) - K'] \]
\[ \leq [u'(t_0, x^*) - K'] + \sum_{i=1}^{p} h_i(x^*)[F'(x^*, u') - K'] \]
\[ \leq 0 \quad (j = 1, \ldots, m). \]
Hence

\[ u'(t_0, x^*) \leq u'(T_{j_0}^i, x^*) \quad (j = 1, \ldots, m) \text{ if } \sum_{i=1}^{p} h_i(x^*) = -1. \quad (3.17) \]

Since, by (a)(i) of the definition of a set of type \((P_{SR})\), \(T_{j_0}^i > t_0 \quad (j = 1, \ldots, m)\), we get from (3.12) that condition (3.17) contradicts (3.9). This completes the proof of (3.4) if \(I_*\) is a finite set.

It remains to investigate case (B). Analogously as in the proof of (3.4) in case (A), by (3.2), by Assumption (F), by (3.13), and by the inequality

\[ u(t, x^*) < u(t_0, x^*) \quad \text{for } t \in \bigcup_{i \in I_*} [T_{2i-1}, T_{2i}] \]

being a consequence of (3.9), (3.12), and of (a)(i), (a)(iii) of the definition of a set of type \((P_{SR})\), we have

\[
0 \geq [u'(t_0, x^*) - K'] + \sum_{i \in I_*} h_i(x^*) [F^j(x^*, u^j) - K^j] \\
\geq [u'(t_0, x^*) - K'] + \sum_{i \in I_*} h_i(x^*) [u'(T_{j_0}^i, x^*) - K^j] \\
\geq [u'(t_0, x^*) - K'] + \sum_{i \in I_*} h_i(x^*) [u'(t_0, x^*) - K^j] \\
= [u'(t_0, x^*) - K'] \left[ 1 + \sum_{i \in I_*} h_i(x^*) \right] \quad (j = 1, \ldots, m).
\]

Hence

\[ u(t_0, x^*) \leq K \quad \text{if } 1 + \sum_{i \in I_*} h_i(x^*) > 0. \quad (3.18) \]

Then, from (3.8) and (3.12), we obtain a contradiction of (3.18) with (3.9).

Assume now

\[ \sum_{i \in I_*} h_i(x^*) = -1. \quad (3.19) \]

Let

\[ \tilde{T}_{j_0}^i = \inf_{i \in I_*} \tilde{T}_{j_0}^i \quad (j = 1, \ldots, m). \quad (3.20) \]

Since \(u \in C(\bar{D})\) and since, by (3.12) and by (a)(iv), (a)(ii) of the definition of a set of type \((P_{SR})\), \(x^* \in S_i\) for every \(t \in [T_0, t_0 + T]\) if \(\text{card } I = \aleph_0\), it
follows from (3.20) that for every \( j \in \{1, \ldots, m\} \) there is a number \( \hat{i}_j \in [\hat{T}^j, t_0 + T] \) such that

\[
u^j(\hat{i}_j, x^*) = \max_{t \in [\hat{T}^j, t_0 + T]} \nu^j(t, x^*). \tag{3.21}\]

Consequently, by (3.19), (3.21), (3.13), by Assumption (F) and by Assumption (5), we have

\[
u^j(t_0, x^*) - \nu^j(\hat{i}_j, x^*)
= \left[ \nu^j(t_0, x^*) - K^j \right] - \left[ \nu^j(\hat{i}_j, x^*) - K^j \right]
= \left[ \nu^j(t_0, x^*) - K^j \right] + \sum_{i \in I_*} h_i(x^*) \left[ \nu^j(i, x^*) - K^j \right]
\leq \left[ \nu^j(t_0, x^*) - K^j \right] + \sum_{i \in I_*} h_i(x^*) \left[ \nu^j(\hat{T}^j, x^*) - K^j \right]
\leq \left[ \nu^j(t_0, x^*) - K^j \right] + \sum_{i \in I_*} h_i(x^*) \left[ F^j(x^*, u^j) - K^j \right]
\leq 0 \quad (j = 1, \ldots, m).
\]

Hence

\[\nu^j(t_0, x^*) \leq \nu^j(\hat{i}_j, x^*) \quad (j = 1, \ldots, m) \quad \text{if} \quad \sum_{i \in I_*} h_i(x^*) = -1. \tag{3.22}\]

Since, by (a)(ii) of the definition of a set of type \((P_{SF})\), \( \hat{i}_j > t_0 \) \((j = 1, \ldots, m)\), we get from (3.12) that condition (3.22) contradicts condition (3.9). This completes the proof of equality (3.4).

The second part of Theorem 3.1 is a consequence of equality (3.4) and of Lemma 3.1 from [2] or Theorem 3.1 from [1]. Therefore, the proof of Theorem 3.1 is complete.

### 4. Remarks

**Remark 4.1.** It is easy to see that functionals \( F^i : S_{t_0} \times C([T_{2i-1}, T_{2i}] \times S_{t_0}) \to R \ (i \in I_*) \) given by the formulae

\[F^i(x, u^j) = u^j(\cdot, x)|_{[T_{2i-1}, T_{2i}]} \quad (i \in I_*, j = 1, \ldots, m) \tag{4.1}\]

or

\[F^i(x, u^j) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x) \, d\tau \quad (i \in I_*, j = 1, \ldots, m) \tag{4.2}\]

satisfy Assumption (F).
Nonlocal inequalities (1.2) together with functionals (4.2) were considered in [4] and nonlocal inequalities (1.2) together with particular cases of functionals (4.1) were considered in [3].

**Remark 4.2.** It is easy to see, by the proof of Theorem 3.1 from this paper and by the proofs of Theorems 1 and 4.1 from papers [3] and [2], respectively, that if the functions $h_i \ (i \in I_*)$ from Assumption (5) of Theorem 3.1 satisfy the condition

$$\left[ \sum_{i \in I_*} h_i(x) = 0 \right] - 1 < \sum_{i \in I_*} h_i(x) \leq 0 \quad \text{for} \quad x \in S_{t_0},$$

then it is sufficient to assume in this theorem that $D$ is only an unbounded set of type (P) satisfying condition (b) of the definition of a set of type $(P_{SR})$ or $D$ is only a bounded set of type (P), i.e., according to the terminology introduced in [2], $D$ is a set of type $(P_F)$ or $(P_B)$, respectively. $D$ is only an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii), and (b) of the definition of a set of type $(P_{SR})$ or $D$ is only a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii) of the definition of a set of type $(P_{SR})$. Moreover, if $I_*$ is a finite set and

$$-1 \leq \sum_{i \in I_*} h_i(x) \leq 0 \quad \text{for} \quad x \in S_{t_0},$$

then it is sufficient to assume in Theorem 3.1 that $D$ is only an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii), and (b) or $D$ is only a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii).

**Remark 4.3.** If $D$ is a set of type $(P_{SB})$ and if $\bar{D} = \bar{D}$, then the first parts of Assumptions (4) and (6) of Theorem 3.1 are trivially satisfied since $u, v \in C(\bar{D})$ and $\Gamma$ is the bounded and closed set in this case.

**Remark 4.4.** Analogously as in [3] (cf. [3, Theorem 2]) we can obtain a theorem on strong minimum principles with the following nonlocal inequalities together with arbitrary functionals,

$$[v^j(t_0, x) - k^j] + \sum_{i \in I_*} h_i(x)[F^i(x, u^i) - k^i]$$

$$\geq 0 \quad \text{for} \quad x \in S_{t_0} \ (j = 1, \ldots, m), \quad (4.3)$$

in sets of types $(P_{SR})$ and $(P_{SB})$. 
5. PHYSICAL INTERPRETATIONS OF PROBLEMS CONSIDERED

Theorem 3.1 can be applied to descriptions of physical phenomena in which we can measure sums of temperatures of substances or sums of amounts of substances according to the formulae

\[ u'(t_0, x) + \sum_{i \in I_*} h_i(x) F'(x, u') \quad \text{for} \quad x \in S_{t_0} \quad (j = 1, \ldots, m) \]

\((h_i (i \in I_*)) \) are known functions). For example, Theorem 3.1 can be applied to the description of a diffusion phenomenon of a small amount of a gas in a transparent tube, under the assumption that the diffusion is observed by the surface of this tube. The measurement \( u(t_0, x) \) \((m = 1)\) of a small amount of the gas at the initial instant \( t_0 \) is usually less precise than the measurement

\[ u(t_0, x) + \sum_{i \in I_*} h_i(x) F'(x, u) \quad \text{for} \quad x \in S_{t_0} \quad (m = 1), \]

where

\[ F'(x, u) \quad \text{for} \quad x \in S_{t_0} \quad (i \in I_*, \ m = 1) \]

are amounts of this gas on the intervals \([T_{2i-1}, T_{2i}]\) \((i \in I_*)\), respectively. Therefore, Theorem 3.1 seems to be more useful in some physical applications than Theorem 4.1 from [2] on strong maximum principles with initial inequalities of the form:

\[ u(t_0, x) \leq K \quad \text{for} \quad x \in S_{t_0}. \]

Particularly, for functionals \( F^i \) defined by the formulae

\[ F^i(x, u) = u(T_{2i}, x) \quad \text{for} \quad x \in S_{t_0} \quad (m = 1) \]

or

\[ F^i(x, u) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u(\tau, x) \, d\tau \quad \text{for} \quad x \in S_{t_0} \quad (m = 1) \]

physical interpretations were given in [3, 4].

If \( I_* = \{1\}, \ T_1 = t_0 + T - \Delta t, \ 0 < \Delta t < T, \ T_2 = t_0 + T, \ F^1 = F, \ -1 \leq h_1(x) = -h(x) \leq 0 \ \text{for} \ x \in S_{t_0}, \ \text{and} \ m = 1, \ \text{then the nonlocal conditions} \]

\[ u'(t_0, x) + \sum_{i \in I_*} h_i(x) F'(x, u') = 0 \quad \text{for} \quad x \in S_{t_0} \quad (j = 1, \ldots, m) \]
are reduced to the condition

\[ u(t_0, x) = h(x) F(x, u) \quad \text{for} \quad x \in S_{t_0} \quad (m = 1), \tag{5.1} \]

where \( F: S_{t_0} \times C([-T_1, T_2] \times S_{t_0}) \rightarrow \mathbb{R} \), and this condition can be used in the description of heat effects in atomic reactors. It is easy to see, by (5.1), that if \( u(t_0, x) \) is interpreted as the given temperature in an atomic reactor at the initial instant \( t_0 \), then the atomic reaction is the safest for \( 1 \approx h(x) \leq 1 \) and this reaction is the most dangerous for \( 0 < h(x) \approx 0 \). In the case if \( h(x) = 1 \) for \( x \in S_{t_0} \), formula (5.1) is reduced to the condition:

\[ u(t_0, x) = F(x, u) \quad \text{for} \quad x \in S_{t_0} \quad (m = 1). \tag{5.2} \]

Moreover, if

\[ F(x, u) = u(t_0 + T, x) \quad \text{for} \quad x \in S_{t_0} \quad (m = 1), \]

then (5.2) is reduced to the periodic condition (see [3]). Finally, if

\[ F(x, u) = \frac{1}{\Delta t} \int_{t_0 + T - \Delta t}^{t_0 + T} u(\tau, x) \, d\tau \quad \text{for} \quad x \in S_{t_0} \quad (m = 1), \]

then (5.2) is reduced to the modified periodic condition (see [4]).

**Remark 5.1.** The considerations from Section 5 concerning Theorem 3.1 are also true for the strong minimum principles with nonlocal inequalities (4.3) (cf. Remark 4.4).

**References**

2. L. Byszewski, Strong maximum and minimum principles for parabolic functional-differential problems with initial inequalities \( u(t_0, x)(z) \leq K \), *Ann. Polon. Math.* 52 (1990), 79–85.

