Finite element analysis for parametrized nonlinear equations around turning points

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Abstract

Nonlinear equations with parameters are called parametrized nonlinear equations. In this paper, a priori error estimates of finite element solutions of parametrized nonlinear elliptic equations on branches around turning points are considered. Existence of a finite element solution branch is shown under suitable conditions on an exact solution branch around a turning point. Also, some error estimates of distance between exact and finite element solution branches are given. It is shown that error of a parameter is much smaller than that of functions. Approximation of nondegenerate turning points is also considered. We show that if a turning point is nondegenerate, there exists a locally unique finite element nondegenerate turning point. At a nondegenerate turning point an elaborate error estimate of the parameter is proved.

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1. Introduction

Let $A$, $B$ be Banach spaces and $A \subset \mathbb{R}^n$ a bounded interval. Let $F : A \times A \to B$ be a smooth operator. The nonlinear equations

$$F(\lambda, u) = 0$$

with parameter $\lambda \in A$ is called parametrized nonlinear equations.

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In [17,18] a thorough theory of a priori error estimates of finite element solutions of the following parametrized strongly nonlinear problems has been developed:

\[ F(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times H_0^1(\Omega), \]

\[ \langle F(\lambda, u), v \rangle := \int_{\Omega} [a(\lambda, x, u(x)), \nabla u(x)) \cdot \nabla v(x) + f(\lambda, x, u(x)), \nabla u(x))v(x)] \, dx, \quad \forall v \in H_0^1(\Omega), \]

where \( \Omega \subset \mathbb{R}^d \) \( (d = 1, 2, 3) \) is a bounded domain with the piecewise \( C^2 \) boundary \( \partial \Omega \), and \( a: \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) and \( f: \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) are sufficiently smooth functions. Here, Eq. (1.1) is called strongly nonlinear if \( a(\lambda, x, y, z) (\lambda \in \Lambda, x \in \Omega, y \in \mathbb{R}, z \in \mathbb{R}^d) \) is nonlinear with respect to \( z \). Otherwise, it is called mildly nonlinear.

Since Eq. (1.1) is defined in divergence form, finite element solutions to (1.1) are defined in a natural way.

In [8,9,13] Fink and Rheinboldt have shown that some subset of the solutions to (1.1) form an one-dimensional smooth manifold without boundaries, if the nonlinear operator defined by (1.1) is Fréchet differentiable and Fredholm of index 1. They have also shown that corresponding finite element solutions form an one-dimensional smooth manifold. In this paper we denote by \( \mathcal{M}_0 \) and \( \mathcal{M}_h \) the exact solution manifold of (1.1) and the corresponding finite element solution manifold, respectively.

Here, a linear operator \( P \in \mathcal{L}(A, B) \) is called Fredholm if (1) the dimension of \( \text{Ker} \) \( P \) is finite, (2) \( \text{Im} \) \( P \) is closed, (3) the dimension of \( \text{Coker} \) \( P := B/\text{Im} \) \( P \) is finite. If \( P \in \mathcal{L}(A, B) \) is Fredholm, its index \( \text{ind} \) \( P \) is defined by \( \text{ind} \) \( P := \dim \text{Ker} \) \( P \) \( - \) \( \dim \text{Coker} \) \( P \). Let \( U \subset A \) be open and \( F: U \rightarrow B \) Fréchet differentiable. \( F \) is called Fredholm in \( U \) if its Fréchet derivative \( \text{DF}(u) \) is Fredholm at any \( u \in U \). It is shown that \( \text{ind} \) \( \text{DF}(u) \) is constant in each connected component of \( U \). Hence, we define the index of \( F \) by \( \text{ind} F := \text{ind} \text{DF}(u) \).

In [17,18], it is shown that, under reasonable conditions, for each compact subset \( \mathcal{M}_0 \subset \mathcal{M}_0 \), there exists a locally unique compact subset \( \mathcal{M}_h \subset \mathcal{M}_h \) such that \( \mathcal{M}_0 \) is approximated uniformly by \( \mathcal{M}_h \), if triangulation of \( \Omega \) is sufficiently fine. Moreover, several a priori error estimates are obtained. For other prior works on the error analysis of finite element solutions of parametrized nonlinear equations, see [3–5,7–9,11–13] and references therein.

The aim of this paper is to refine the error analysis on branches around turning points (Fig. 1).

![Fig. 1. Nondegenerate and degenerate turning points.](image-url)
A point \((\lambda, u) \in \mathcal{M}_0\) is called a turning point if the partial Fréchet derivative \(D_u F(\lambda, u) \in \mathcal{L}(A, B)\) at \((\lambda, u)\) is not an isomorphism.

To develop a refined error analysis around a turning point, we introduce a slightly different formulation of the problem from that in [17], and show a theorem which is similar to [18, Theorem 8.6; 17, Corollary 7.8]. Next, we obtain an elaborate error estimate of parameter. In the following we explain the basic ideas of this paper.

In the error analysis of parametrized nonlinear equations, we have the following difficulty. Suppose that we are approaching a turning point during continuation process of a solution branch. Since we cannot fix the parameter \(\lambda\) around a turning point in (1.1), \(\lambda\) should be treated as an unknown parameter. Hence, correspondence of an approximated solution to an exact solution becomes ambiguous in such a situation.

Recently, many authors have overcome this difficulty in the following manner. We introduce a \((nonlinear, in general)\) functional \(\mathcal{S}(\lambda; u) : A \times A \rightarrow \mathbb{R}\), and consider the following problem:

\[
H(\gamma; \lambda, u) := (\mathcal{S}(\lambda; u) - \gamma, F(\lambda, u)) = (0, 0) \in \mathbb{R} \times A, \tag{1.2}
\]

where \(H : \mathbb{R} \times A \times A \rightarrow \mathbb{R} \times B\). We expect that the partial Fréchet derivative \(D_{(\gamma, \lambda; u)} H(\gamma; \lambda, u) \in \mathcal{L}(\mathbb{R} \times A, \mathbb{R} \times B)\) is an isomorphism at a turning point \((\lambda, u)\) and in its neighborhood. In Section 2, it will be shown that, if \(D_{(\gamma, \lambda; u)} H(\gamma; \lambda, u) = (0, 0)\) at \((\lambda, u) \in \mathcal{M}_0\), then the above partial Fréchet derivative is an isomorphism. If we could find a good definition of such \(\mathcal{S}\), then the solution branch would now be parametrized by \(\gamma\).

Finite element solutions \((\lambda_h, u_h)\) would be defined by

\[
H_h(\gamma; \lambda, u) := (\mathcal{S}(\lambda; u) - \gamma, F_h(\lambda, u)) = (0, 0), \tag{1.3}
\]

where \(F_h\) is an approximation of \(F\). In this setting the correspondence of an exact solution \((\lambda, u)\) and a finite element solution \((\lambda_h, u_h)\) is represented by \(\mathcal{S}(\lambda_h; u_h) = \gamma = \mathcal{S}(\lambda, u)\).

In the above setting we will show that, even around a turning point, there exists a locally unique finite element solution branch near an exact solution branch under suitable conditions. Also, some error estimates of distance between the exact and finite element solution branches are given.

Next, we will consider an elaborate error estimate of parameter \(\lambda\). In error analysis of the finite element method (1.3) for (1.2) around a turning point, we would have error estimates such as

\[
|\lambda - \lambda_h| + \|u - u_h\|_A \leq Ch'.
\]

In many practical computation, it is usually observed that the error \(|\lambda - \lambda_h|\) is much smaller than \(\|u - u_h\|_A\), or \(Ch'\).

A typical and well-known example of this phenomenon is finite element approximation of the eigenvalue problems

\[
-\Delta u = \lambda u, \quad u \in H_0^1(\Omega). \tag{1.4}
\]

Let \((\lambda, u)\) be an eigen-pair of (1.4) and \((\lambda_h, u_h)\) its finite element approximation. Suppose that the eigenvalue \(\lambda\) is simple. Then we have an error estimate such as

\[
|\lambda - \lambda_h| \leq C\|u - u_h\|_{H_0^1},
\]

where \(C\) is a positive constant independent of \(h\) (see, for example, [1] and [14, Chapter 6]).

We will show that a similar estimate hold for the finite element solutions \((\lambda_h, u_h)\) of (1.3) under the condition that \(D_u F(\lambda, u)\) is self-adjoint. To obtain a similar estimate we introduce an auxiliary
equation. Let \( z \) and \( z_h \) be the exact and finite element solutions to the auxiliary equation. We will show that the error \( |\lambda - \lambda_h| \) is estimated as

\[
|\lambda - \lambda_h| \leq C\|u - u_h\|_A(\|u - u_h\|_A + \|z - z_h\|_A)
\]

around a turning point, where \( C \) is a positive constant independent of \( h \).

Occasionally, a turning point on the exact solution manifold \( \mathcal{M}_0 \) has a certain physical meaning, and, in such a case, computing its precise value will become important. If a turning point \( (\lambda_0, u_0) \in \mathcal{M}_0 \) is nondegenerate (see Section 3 for its definition), we can show that the associated finite element solution manifold also has a locally unique nondegenerate turning point \( (\lambda_h^0, u_h^0) \in \mathcal{M}_h \). The error \( |\lambda_0 - \lambda_h^0| \) is estimated accurately by a similar manner as above.

In Sections 2 and 3 we develop our theory in an abstract setting. In Section 4, we apply the abstract theorems obtained in Sections 2 and 3 to the practical equation (1.1). In Section 5, we apply the abstract theorems to a simple eigenvalue problem and show that well-known results of finite element analysis for eigenvalue problems are also proven by our approach.

2. Abstract formulation

In this section, we formulate our problem in an abstract setting, and show a theorem which claims existence of a locally unique solution branch of a discretized problem. The setting in this section is slightly different from that of [17].

For the stage of our analysis we first introduce functional spaces.

(A1) There are Banach spaces, \( V, W, \) and \( X_p \) (\( 1 \leq p \leq \infty \)), where \( X_2 \) is a Hilbert space, such that \( V \subset X_\infty \subset X_p \) (\( 1 \leq p \leq \infty \)) and \( W \subset X'_p \subset X'_q \) (\( 1 \leq q \leq \infty \)). Here, \( X'_q \) is the dual space of \( X_q \) and we suppose that all inclusions are continuous. We also suppose that \( X_r \) is dense in \( X_p \) if \( 1 \leq p < r < \infty \).

Let \( F: A \times X_p \rightarrow X'_q \) \((1/p + 1/q = 1)\) be a nonlinear map, where \( A \subset \mathbb{R} \) is an interval. We consider the parametrized nonlinear equation \( F(\lambda, u) = 0 \). Since we will suppose that \( F \) is strongly nonlinear, the domain and the range should be taken carefully. In many cases, \( F \) is not Fréchet differentiable on \( A \times X_p \), \( p < \infty \), and should be restricted to a certain subspace to make it differentiable.

We also need extensions and restrictions of the Fréchet derivatives \( D_F(\lambda, v), D_F(\lambda, v) \) etc., at \( (\lambda, v) \). When we need to specify the domain of, say, \( D_F(\lambda, v) \) clearly, we will write such as \( D_F(\lambda, v) \in \mathcal{L}(P, Q) \). This means that \( D_F(\lambda, v) \) now denotes its extension (or restriction) whose domain is \( P \) and range is in \( Q \).

Now, we take certain \( p \geq 2 \) and \( q \) with \( 1/p + 1/q = 1 \), and fix them. We then assume the following:

(A2) The restriction of \( F \) to \( A \times X_\infty \), denoted by \( F \) again, is a Fréchet differentiable map from \( A \times X_\infty \) to \( X'_p \). For any \( \lambda \in A \) and \( v \in X_\infty \) the derivative \( DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_\infty, X'_p) \) can be extended to \( DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q) \) and it is locally Lipschitz continuous on \( A \times X_\infty \): i.e., for any bounded convex set \( C \subset A \times X_\infty \) there exists a positive constant \( C_1(C) \) such that

\[
\|DF(\lambda_1, v) - DF(\lambda_2, w)\|_{\mathcal{L}(\mathbb{R} \times X_p, X'_q)} \leq C_1(C)(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})
\]

for arbitrary \( (\lambda_1, v), (\lambda_2, w) \in C \).
Lemma 2.1. (1) For any \((\lambda,u) \in \mathcal{R}(F,\mathcal{S})\), \(\dim \ker D_uF(\lambda,u)\) is at most 1.

(2) For \((\lambda,u) \in \mathcal{R}(F,\mathcal{S})\), we have either

Case 1: \(\ker D_uF(\lambda,u) = \{0\}\) and \(D_uF(\lambda,u) \in \operatorname{im} D_uF(\lambda,u)\), or

Case 2: \(\dim \ker D_uF(\lambda,u) = 1\), and \(D_uF(\lambda,u) \notin \operatorname{im} D_uF(\lambda,u)\).

For the proof, see [18, Section 4].

We introduce a nonlinear functional \(\rho: A \times X_p \to \mathbb{R}\) and assume that

(A4) The restriction of \(\rho\) to \(A \times X_\infty\), denoted by \(\rho\) again, is Fréchet differentiable.

(A5) For \((\lambda,u) \in A \times X_\infty\), the Fréchet derivative \(D\rho(\lambda,u) \in \mathcal{L}(\mathbb{R} \times X_\infty, \mathbb{R})\) can be extended to \(D\rho(\lambda,u) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R}) = \mathbb{R} \times X'_p\), and it is locally Lipschitz continuous on \(A \times X_\infty\), i.e., for any bounded convex set \(C \subset A \times X_\infty\), there exists a positive constant \(C_2(C)\) such that

\[\|D\rho(\lambda_1,v) - D\rho(\lambda_2,w)\|_{\mathbb{R} \times X'_p} \leq C_2(C)(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})\]

for any \((\lambda_1,v), (\lambda_2,w) \in C\).

(A6) Let \((\lambda,u) \in \mathcal{S}\) and \(D_uF(\lambda,u) \in \mathcal{L}(X_p,X'_q)\). We suppose that if \(D_uF(\lambda,u)\psi = f\) for \(\psi \in X_p\) and \(f \in W\), then \(\psi \in V\).

Lemma 2.2. Assume that (A1)–(A6) are valid. Suppose that there is \((\lambda_0,u_0) \in \mathcal{R}(F,\mathcal{S})\) such that \(D_2F(\lambda_0,u_0) \neq 0\) in \(W\). From (A3), there exists \((\mu_0,\psi_0) \in \mathbb{R} \times V\) such that \(\ker D_2F(\lambda_0,u_0) = \text{span}\{\mu_0,\psi_0\}\). We assume that \(D\rho(\lambda_0,u_0)(\mu_0,\psi_0) \neq 0\) in \(\mathbb{R}\). Define \(G: A \times W \to \mathbb{R} \times V\) by \(G(\lambda,u) := (\rho(\lambda,u) - \gamma,F(\lambda,u))\), where \(\gamma \in \mathbb{R}\). Then, \(DG(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times W, \mathbb{R} \times V)\) is an isomorphism. Moreover, \(DG(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)\) is an isomorphism as well.

Proof. From the assumptions we find that \(\ker D_2F(\lambda_0,u_0) \cap \ker D\rho(\lambda_0,u_0) = \{(0,0)\}\). This implies that \(\ker DG(\lambda_0,u_0)\) is trivial and \(DG(\lambda_0,u_0)\) is one-to-one.

Since \(DF(\lambda_0,u_0)\) is onto, for any \(g \in W\), there is \((v,\phi) \in \mathbb{R} \times V\) such that \(DF(\lambda_0,u_0)(v,\phi) = g\). Since \(D\rho(\lambda_0,u_0)(\mu_0,\psi_0) \neq 0\), for any \(t \in \mathbb{R}\), there is \(\alpha \in \mathbb{R}\) such that \(D\rho(\lambda_0,u_0)((v,\phi) + \alpha(\mu_0,\psi_0)) = t\). This yields that \(DG(\lambda_0,u_0)\) is onto. Therefore, \(DG(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)\) is an isomorphism.

To show that \(DG(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)\) is an isomorphism, we first show that \(DF(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)\) is onto. Since \(DF(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)\) is Fredholm with index 1 by (A3), we only have to show that the dimension of \(\ker DF(\lambda_0,u_0) \subset \mathbb{R} \times X_p\) is 1.

Let \((\mu,\psi) \in \mathbb{R} \times X_p\) be such that \(DF(\lambda_0,u_0)(\mu,\psi) = 0 \in X'_q\). This is also written as \(D_uF(\lambda_0,u_0)\psi = -\mu D_2F(\lambda_0,u_0)\). Since \(D_uF(\lambda_0,u_0) \in W\) and (A6), we conclude that \(\psi \in W\) and \(\dim \ker (DF(\lambda_0,u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)) = 1\).
Using this fact, we show that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)$ is an isomorphism by the exactly same manner as above. □

**Corollary 2.3.** Assume that (A1)–(A6) are valid. Suppose that there exists $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{I})$ such that $F(\lambda_0, u_0) = 0$, $\rho(\lambda_0, u_0) = \gamma_0$, and $D_z F(\lambda_0, u_0) \neq 0$. Suppose also that $\operatorname{Ker} DF(\lambda_0, u_0) \cap \operatorname{Ker} D_p(\lambda, u_0) = \{(0,0)\}$. Define $H : \mathbb{R} \times A \times V \to \mathbb{R} \times W$ by $H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u))$.

Then, we have $H(\gamma_0, \lambda_0, u_0) = (0,0)$ and $D_{(\lambda, u)} H(\gamma_0, \lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)$ is an isomorphism. Therefore, by the implicit function theorem, there exist a positive constant $\varepsilon$ and a $C^1$ map $(\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \ni \gamma \mapsto (\hat{\lambda}(\gamma), u(\gamma)) \in A \times V$ such that $(\hat{\lambda}(\gamma_0), u(\gamma_0)) = (\lambda_0, u_0)$ and $H(\gamma, \hat{\lambda}(\gamma), u(\gamma)) = (0,0)$ for any $\gamma$. That is, the solution manifold of the equation $F(\lambda, u) = 0$ is parametrized by $\gamma = \rho(\lambda, u)$ around $(\lambda_0, u_0)$.

To define discretized solutions of $F(\lambda, u) = 0$, we introduce the finite-dimensional subspaces $S_k \subset X_\infty$ which are parametrized by $h$, $0 < h < 1$ with the following properties:

(A7) There exists a real $r \geq 0$ and a positive constant $C_3$ independent of $h$ such that

$$\|v_h\|_{X_\infty} \leq C_3 h \|v_h\|_{X_p}, \quad \forall v_h \in S_h.$$  

The relations of Banach spaces are depicted in the following:

$$A \times S_h \ni \lambda \mapsto (\lambda, u_h) \mapsto F(\lambda, u_h) \subset \mathbb{R} \times X_p \subset \mathbb{R} \times X_2$$

The finite element solution $(\lambda_h, u_h) \in A \times S_h$ is defined naturally by

$$\langle F(\lambda_h, u_h), v_h \rangle = 0, \quad \forall v_h \in S_h,$$

where $\langle \cdot, \cdot \rangle$ is the duality pair of $X'_q$ and $X_2$. We derive an equivalent definition of the finite element solutions which is more convenient in the error analysis.

Let $Q \in \mathcal{L}(X_2, X'_2)$ be a self-adjoint operator, that is, $\langle Qu, v \rangle = \langle Qv, u \rangle$ for all $u, v \in X_2$. Suppose that there exists a positive constant $\alpha$ such that

$$\langle Qv, v \rangle \geq \alpha \|v\|_{X_2}^2, \quad \forall v \in X_2. \quad (2.1)$$

We define $\langle u, v \rangle_Q := \langle Qu, v \rangle$. It is easy to show that $\langle \cdot, \cdot \rangle_Q$ is an inner product and the norm $\|v\|_Q := (v, v)_Q^{1/2}$ is equivalent to the original norm $\|v\|_{X_2}$. It is also easy to show that $Q \in \mathcal{L}(X_2, X'_2)$ is an isomorphism.

We define the canonical projection $\hat{P}_h : X_2 \to S_h$ by $\langle \psi - \hat{P}_h \psi, v_h \rangle_Q = 0$ for all $v_h \in S_h$. Obviously, we have that $(u, \hat{P}_h v) = (\hat{P}_h u, v) \subset \mathcal{L}(h, v) = 0$ for all $u, v \in X_2$. As in [18, Section 6] it follows from the definitions that $(\lambda_h, u_h)$ is a finite element solution if and only if $\langle Q\hat{P}_h Q^{-1} F(\lambda_h, u_h), v \rangle = 0$ for all $v \in X_2$.

Following Fink and Rheinboldt [8,9,13] we define the approximation of $F(\lambda, u)$ by

$$F_h(\lambda, u) := (I - P_h)Qu + P_h F(\lambda, u), \quad P_h := Q \hat{P}_h Q^{-1}, \quad (2.2)$$
Theorem 2.4. Assume that (A1)–(A7) are valid. Suppose that there exists \((\lambda_0, u_0) \in \mathcal{A}(F, I)\) such that \(F(\lambda_0, u_0) = 0\), \(\rho(\lambda_0, u_0) = 0\), and \(D_2F(\lambda_0, u_0) \neq 0\). Suppose also that \(\text{Ker } D\phi(\lambda_0, u_0) \cap \text{Ker } D\psi(\lambda_0, u_0) = \{(0, 0)\}\). Then, by Corollary 2.3, there exist a positive constant \(\varepsilon_0\) and a \(C^1\) map \([\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0] \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in A \times V\) such that \((\lambda(\gamma), u(\gamma)) = (\lambda_0, u_0), \gamma = \rho(\lambda(\gamma), u(\gamma)), \text{ and } F(\lambda(\gamma), u(\gamma)) = 0\). We also assume that \((\lambda(\gamma), u(\gamma)) \in \mathcal{A}(F, I)\) for all \(\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]\). We also assume that there exists the projection \(\Pi_h : X_p \rightarrow S_h\) for each \(h > 0\) such that, for all \(\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]\),

\[
\lim_{h \to 0} h^{-\varepsilon} \|u(\gamma) - \Pi_h u(\gamma)\|_{X_p} = 0,
\]

\[
\lim_{h \to 0} \|u(\gamma) - \Pi_h u(\gamma)\|_{X_{\infty}} = 0
\]

and the above convergences are uniform.

We, on the other hand, suppose that \(D^\prime_2 F(\lambda_0, u_0)\) is decomposed into \(D_2 F(\lambda_0, u_0) = Q + R\), where \(Q \in \mathcal{L}(X_p, X_p')\) is the principal part which is self-adjoint and satisfies (2.1), and \(R \in \mathcal{L}(X_p, X_p')\) is compact. The discretized nonlinear map \(F_h : X_p \rightarrow X_q\) and the projection \(P_h : X_q \rightarrow X_q\) is defined by (2.2). We suppose that

\[
\lim_{h \to 0} \|\psi - P_h \psi\|_{X_q'} = 0, \quad \forall \psi \in X_q'.
\]

Then, for sufficiently small \(h > 0\), there exist a positive constant \(\varepsilon_1 \leq \varepsilon_0\) and a unique map \([\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1] \ni \gamma \mapsto (\lambda_h(\gamma), u_h(\gamma)) \in A \times S_h\) such that \(F_h(\lambda_h(\gamma), u_h(\gamma)) = 0\) for all \(\gamma \in [\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1]\). Moreover, we have the estimate

\[
|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - \Pi_h u(\gamma)\|_{X_p} \leq K_1 \|u(\gamma) - \Pi_h u(\gamma)\|_{X_p}
\]

for all \(\gamma \in [\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1]\), where \(K_1\) is a positive constant independent of \(h\) and \(\gamma\).

Proof. The proof of Theorem 2.4 is quite similar to those of [17, Theorem 7.7; 18, Theorem 8.4]. Hence, we give here a sketch of the proof.

Step 1: We define \(H, H_h : \mathbb{R} \times A \times V \to \mathbb{R} \times W\) by

\[
H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u)), \quad H_h(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F_h(\lambda, u)),
\]

where \(F_h(\lambda, u)\) is defined by (2.2).

We claim that there exist positive constants \(\varepsilon_1, C_4\), independent of \(h > 0\) and \(\gamma \in [\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1]\), such that, for sufficiently small \(h > 0\),

\[
\|D(\lambda, u)H(\gamma, \lambda, u)\|_{\mathbb{R} \times X_p} \geq C_4(\|\mu\|_{X_p} + \|\nu_h\|_{X_p}), \quad \forall (\mu, \nu_h) \in \mathbb{R} \times S_h.
\]

From Corollary 2.3, for any \(\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]\), \(D(\lambda, u)H(\gamma, \lambda(\gamma), u(\gamma)) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)\) is an isomorphism, and is extended to an isomorphism \(D(\lambda, u)H(\gamma, \lambda(\gamma), u(\gamma)) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X_q)\). Set

\[
\omega := \max_{\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]} \|D(\lambda, u)H(\gamma, \lambda(\gamma), u(\gamma))\|_{\mathbb{R} \times X_q}\]
We write

\[ D(\lambda, u)H(\gamma, \lambda, \Pi u)(\mu, v) = (D \rho(\lambda, u(\gamma))(\mu, v), DF(\lambda, u(\gamma))(\mu, v), 0) \]

\[ + (D \rho(\lambda, \Pi u(\gamma)) - D \rho(\lambda, u(\gamma))(\mu, v), 0) \]

\[ + (0, DF(\lambda, \Pi u(\gamma)) - DF(\lambda, u(\gamma))(\mu, v)). \]

On the first and second term of the right-hand side of the above formula, we have

\[ \| (D \rho(\lambda, u(\gamma)), DF(\lambda, u(\gamma)))(\mu, v) \|_{\mathbb{R} \times X_p} \geq \omega^{-1}(\| \mu \| + \| v \|_{X_p}). \]

\[ \| (D \rho(\lambda, \Pi u(\gamma)) - D \rho(\lambda, u(\gamma))(\mu, v) \| \leq \epsilon(h)(\| \mu \| + \| v \|_{X_p}), \]

respectively, where \( \epsilon(h) \to 0 \) as \( h \to 0 \).

On the third term, we write

\[ (DF(\lambda, \Pi u(\gamma)) - DF(\lambda, u(\gamma)))(\mu, v) = -\mu(I - P_h)D \rho(\lambda, u(\gamma)) \]

\[ + \mu P_h(D \rho(\lambda, \Pi u(\gamma)) - D \rho(\lambda, u(\gamma))) \]

\[ + P_h(D \rho(\lambda, \Pi u(\gamma)) - D \rho(\lambda, u(\gamma)))v_h \]

\[ - (I - P_h)(-Q + D \rho(\lambda, u_0))v_h \]

\[ + (I - P_h)(D \rho(\lambda, u_0) - D \rho(\lambda, u(\gamma)))v_h. \]

Estimating the each term of the right-hand side of the above equation, we can show that

\[ \| (DF(\lambda, \Pi u(\gamma)) - DF(\lambda, u(\gamma)))(\mu, v) \|_{X_p} \leq (\omega^{-1}/2 + \delta(h))(\| \mu \| + \| v \|_{X_p}) \]

for any \( \gamma \in [\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1] \), where \( \epsilon_1 \) is sufficiently small and \( \delta(h) \to 0 \) as \( h \to 0 \). Note that in the above estimates the assumption that \( D \rho F(\lambda_0, u_0) = Q + R \) and \( R \) is compact is used to show

\[ \lim_{h \to 0} \| (I - P_h)(-Q + D \rho F(\lambda_0, u_0)) \|_{X_p} = \lim_{h \to 0} \| (I - P_h)R \|_{X_p} = 0. \]

Gathering the above estimates the claim is proved with \( C_4 := \omega^{-1}/3 \) for sufficiently small \( h \).

**Step 2:** We check the following: (1) \( \lim_{h \to 0} h^{-r}\| H(\gamma, \lambda, \Pi u(\gamma)) \|_{\mathbb{R} \times X_p} = 0 \) and the above convergence is uniform with respect to \( \gamma \in [\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1] \).

(2) \( \| D \rho H(\gamma, \lambda, \Pi u(\gamma)) \|_{\mathbb{R} \times X_p} = 1 \).

(3) For any bounded convex set \( \mathcal{C} \subset \mathbb{R} \times \Lambda \times S_h \) we have

\[ \| D H(\gamma, \lambda_1, u_1) - D H(\gamma, \lambda_2, u_2) \|_{\mathbb{R} \times X_p} \leq C(\mathcal{C})h^{-r}(|\gamma_1 - \gamma_2| + |\lambda_1 - \lambda_2| + \| u_1 - u_2 \|_{X_p}) \]

for any \( (\gamma_i, \lambda_i, u_i) \in \mathcal{C}, i = 1, 2 \), where \( C(\mathcal{C}) := \max\{ C_1(\mathcal{C}), C_2(\mathcal{C}) \} \).

**Step 3:** Now we apply [3, Theorem 1.1] to \( f := H_h \) in the following situation:

\[ A := \mathbb{R} \text{ with norm } h^{-r}|\gamma|, \]

\[ B := \mathbb{R} \times S_h \subset \mathbb{R} \times X_p \text{ with norm } h^{-r}(|\lambda| + \| w_h \|_{X_p}), \]

\[ E := \mathbb{R} \times S_h \subset \mathbb{R} \times X_1 \text{ with norm } h^{-r}(|\lambda| + \| Q w_h \|_{X_1}), \]

\[ S := [\gamma_0 - \epsilon_1, \gamma_0 + \epsilon_1], \]

\[ y(\gamma) := (\lambda(\gamma), \Pi u(\gamma)). \]
From Steps 1 and 2, all assumptions of [3, Theorem 1.1] are satisfied, and all claims of Theorem 2.4 follows immediately. □

Note that, as mentioned before, the solution branch \((\lambda_{h}(\gamma), u_{h}(\gamma))\) \(\in \mathcal{S}\) whose existence is proved in Theorem 2.4 is the finite element solution branch under the assumption that \(Q \in \mathcal{L}(X_{p}, X'_{q}) \subset \mathcal{L}'(X_{2}, X'_{2})\) appeared in Theorem 2.4 is self-adjoint and satisfies (2.1).

3. Elaborate error estimates of the parameter \(\lambda\)

In this section we give elaborate error estimates of the parameter \(\lambda\). To do this we need more assumptions.

(A8) The nonlinear maps \(F : A \times X_{\infty} \to X'_{\infty}\) and \(\rho : A \times X_{\infty} \to \mathbb{R}\) are of \(C^{2}\) class.

(A9) For any \((\lambda, u) \in \mathcal{S} \subset A \times W, D_{u}F(\lambda, u) \in \mathcal{L}'(X_{2}, X'_{2})\) is self-adjoint.

Now, let \((\lambda, u) \in \mathcal{R}(F, \mathcal{S})\) be a solution of \(F(\lambda, u) = 0\) at which all assumptions of Theorem 2.4 and (A8), (A9) hold. Let \((\lambda_{h}, u_{h}) \in A \times S_{h}\) be the corresponding finite element solution with \(\rho(\lambda_{h}, u_{h}) = \rho(\lambda, u)\).

We consider the following auxiliary problem: find \((\eta, z) \in \mathbb{R} \times X_{p}\) such that

\[
\langle (D_{u}F^{0})z, v \rangle = \eta(D_{u}\rho^{0}, v), \quad \forall v \in X_{p},
\]

\[
\langle D_{z}F^{0}, z \rangle - \eta D_{z}\rho^{0} = 1, \tag{3.1}
\]

where \(D_{u}F^{0} := D_{u}F(\lambda, u), D_{u}\rho^{0} := D_{u}\rho(\lambda, u), \) etc.

**Lemma 3.1.** Suppose that all assumptions of Theorem 2.4 and (A8), (A9) hold. Then, Eq. (3.1) has an unique solution \((\eta, z) \in \mathbb{R} \times X_{p}\).

**Proof.** Recall that we have either

Case 1: \(\text{Ker } D_{u}F(\lambda, u) = \{0\}\) and \(D_{z}F(\lambda, u) \in \text{Im } D_{u}F(\lambda, u)\), or

Case 2: \(\text{dim } \text{Ker } D_{u}F(\lambda, u) = 1\), and \(D_{z}F(\lambda, u) \not\in \text{Im } D_{u}F(\lambda, u)\).

Suppose that we are in Case 1. Then, \(\text{Ker } DF(\lambda, u) = \text{span}\{(1, -(D_{u}F^{0})^{-1}(D_{z}F^{0}))\}\). By the assumption we have \(D_{\rho}^{0}(1, -(D_{u}F^{0})^{-1}(D_{z}F^{0})) \neq 0\), that is \(D_{z}\rho^{0} - \langle D_{u}\rho^{0}, (D_{u}F^{0})^{-1}(D_{z}F^{0}) \rangle \neq 0\).

Let \(\eta := \langle D_{u}\rho^{0}, (D_{u}F^{0})^{-1}(D_{z}F^{0}) \rangle - D_{z}\rho^{0} \rangle^{-1}\) and \(z := \eta(D_{u}F^{0})^{-1}(D_{u}\rho^{0})\). Since \(D_{u}F^{0}\) is self-adjoint by (A9), we have

\[\eta\langle D_{z}F^{0}, (D_{z}F^{0})^{-1}(D_{u}\rho^{0}) \rangle = \eta\langle D_{u}\rho^{0}, (D_{u}F^{0})^{-1}(D_{z}F^{0}) \rangle\]

and \(\langle D_{z}F^{0}, z \rangle - \eta D_{z}\rho^{0} = \eta(D_{u}\rho^{0}, (D_{u}F^{0})^{-1}(D_{z}F^{0}) \rangle - D_{z}\rho^{0} \rangle = 1\). Hence \((\eta, z)\) is a solution of (3.1). Uniqueness is proved by the same manner.

Now, suppose that we have Case 2. Then, there exists \(\psi_{0} \in V\) such that \(\text{Ker } DF(\lambda, u) = \text{span}\{(0, \psi_{0})\}\) and \(\langle D_{u}\rho^{0}, \psi_{0} \rangle \neq 0\).

Since \(DF(\lambda, u)\) is onto, there exists \((\theta, \phi) \in \mathbb{R} \times X_{p}\) such that

\[
\theta\langle D_{z}F^{0}, v \rangle + \langle (D_{u}F^{0})\phi, v \rangle = \langle D_{u}\rho^{0}, v \rangle, \quad \forall v \in X_{2} \tag{3.2}
\]

and \(\theta\) is determined uniquely.
We claim that $D_{a}\rho^{0} \not\in \text{Im}(D_{a}F^{0})$. If $D_{a}\rho^{0} \in \text{Im}(D_{a}F^{0})$, then there would exist $w \in X_{p}$ such that $(D_{a}F^{0})w = D_{a}\rho^{0}$. Hence, we have

$$0 \neq \langle D_{a}\rho^{0}, \psi_{0} \rangle = \langle (D_{a}F^{0})w, \psi_{0} \rangle = \langle (D_{a}F^{0})\psi_{0}, w \rangle = 0$$

and obtain a contradiction. Therefore, we conclude that $D_{a}\rho^{0} \not\in \text{Im}(D_{a}F^{0})$ and $\theta \neq 0$.

Letting $v := \psi_{0}$ in (3.2), we have $\theta \langle D_{a}F^{0}, \psi_{0} \rangle = \langle D_{a}\rho^{0}, \psi_{0} \rangle \neq 0$. Hence, we conclude $\langle D_{a}F^{0}, \psi_{0} \rangle = \langle D_{a}\rho^{0}, \psi_{0} \rangle/\theta \neq 0$. We thus immediately notice that $(0, \omega_{\psi_{0}})$ with $\omega := \langle D_{a}F^{0}, \psi_{0} \rangle^{-1}$ is a solution of (3.1). Again, the uniqueness is shown by the same manner. 

It is obvious that we may apply Theorem 2.4 to Eq. (3.1) with the following setting:

$$F(\eta, z) := (D_{a}F^{0})z - \eta(D_{a}\rho^{0}), \quad \rho(\eta, z) := \langle D_{a}F^{0}, z \rangle - \eta(D_{a}\rho^{0})$$

and obtain

**Lemma 3.2.** For sufficiently small $h > 0$, there exists the unique finite element solution $(\eta_{h}, z_{h}) \in \mathbb{R} \times S_{h}$ of (3.1) such that

$$\langle (D_{a}F^{0})z_{h}, v_{h} \rangle = \eta_{h} \langle D_{a}\rho^{0}, v_{h} \rangle, \quad \forall v \in S_{h},$$

$$\langle D_{a}F^{0}, z_{h} \rangle - \eta_{h}D_{a}\rho^{0} = 1.$$ 

Moreover, we have the estimate

$$|\eta - \eta_{h}| + \|z - z_{h}\|_{X_{p}} \leq C\|z - \Pi_{h}z\|_{X_{p}},$$

where $C$ is a positive constant independent of $h$.

Let $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$ is a solution of $F(\lambda, u) = 0$ which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and $(\lambda_{h}, u_{h}) \in A \times S_{h}$ the corresponding finite element solution. By Taylor’s theorem and $\langle F(\lambda_{h}, u_{h}), v_{h} \rangle = \langle F(\lambda, u), v_{h} \rangle = 0$ for any $v_{h} \in S_{h}$, we have

$$0 = (\lambda_{h} - \lambda)\langle D_{a}F^{0}, v_{h} \rangle + \langle (D_{a}F^{0})(u_{h} - u), v_{h} \rangle + \frac{1}{2}(\lambda_{h} - \lambda)^{2}\langle D_{aa}F^{0}, v_{h} \rangle$$

$$+ (\lambda_{h} - \lambda)(\langle D_{aa}F^{0})(u_{h} - u), v_{h} \rangle + \frac{1}{2}\langle (D_{aa}F^{0})(u_{h} - u)^{2}, v_{h} \rangle,$$

(3.3)

where

$$D_{aa}F^{0} := \int_{0}^{1}(1 - s)D_{aa}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))ds,$$

$$(D_{aa}F^{0})(u_{h} - u) := \int_{0}^{1}(1 - s)D_{aa}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))(u_{h} - u)ds,$$

$$(D_{aa}F^{0})(u_{h} - u)^{2} := \int_{0}^{1}(1 - s)D_{aa}F(\lambda + s(\lambda_{h} - \lambda), u + s(u_{h} - u))(u_{h} - u)^{2}ds.$$ 

Letting $v := u - u_{h}$ in (3.1), we obtain

$$\langle (D_{a}F^{0})z, u - u_{h} \rangle = \langle (D_{a}F^{0})(u - u_{h}), z \rangle = \eta\langle D_{a}\rho^{0}, u - u_{h} \rangle.$$
Since
\[ 0 = \rho(\lambda, u_h) - \rho(\lambda, u) \]
\[ = (\lambda - \lambda)(D_2 \rho^0) + \langle D_u \rho^0, u_h - u \rangle + \frac{1}{2}(\lambda - \lambda)^2 (D_{\lambda \lambda} \rho^0) + (\lambda - \lambda)(D_{\lambda u} \rho^0)(u_h - u) + \frac{1}{2}(D_{uu} \rho^0)(u_h - u)^2, \]
where
\[ D_{\lambda \lambda} \rho^0 := \int_0^1 (1 - s)D_{\lambda \lambda} \rho(\lambda + s(\lambda - \lambda), u + s(u_h - u)) \, ds, \]
\[ (D_{\lambda u} \rho^0)(u_h - u) := \int_0^1 (1 - s)(D_{\lambda u} \rho(\lambda + s(\lambda - \lambda), u + s(u_h - u)), u_h - u) \, ds, \]
\[ (D_{uu} \rho^0)(u_h - u)^2 := \int_0^1 (1 - s)(D_{uu} \rho(\lambda + s(\lambda - \lambda), u + s(u_h - u))(u_h - u), u_h - u) \, ds, \]
we have
\[ \langle (D_u F^0)(u - u_h), z \rangle = -\eta(\lambda - \lambda_h)(D_2 \rho^0) + \frac{\eta}{2}(\lambda - \lambda_h)^2(D_{\lambda \lambda} \rho^0) \]
\[ + \eta(\lambda - \lambda_h)(D_{\lambda u} \rho^0)(u_h - u) + \frac{\eta}{2}(D_{uu} \rho^0)(u_h - u)^2. \]

It follows from (3.3) with \( v_h := z_h \) (recall that \( (\eta, z) \in \mathbb{R} \times S_h \) is the finite element solution of (3.1)) and (3.4) that
\[ (\lambda - \lambda_h)(\langle D_1 F^0, z \rangle - \eta(D_2 \rho^0) + B_h) = \langle (D_u F^0)(u - u_h), z - z_h \rangle \]
\[ + \frac{1}{2} \langle (D_{uu} F^0)(u - u_h)^2, z_h \rangle - \frac{\eta}{2}(D_{uu} \rho^0)(u - u_h)^2, \]
where \( \lim_{h \to 0} B_h = 0 \). Therefore, we have proved the following theorem:

**Theorem 3.3.** Let \((\lambda, u) \in \mathcal{F}(F, \mathcal{F})\) be a solution of \( F(\lambda, u) = 0 \) which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let \((\lambda_h, u_h) \in \Lambda \times S_h\) be the corresponding finite element solution. Let \((\eta, z) \in \mathbb{R} \times X_p\) and \((\eta_h, z_h) \in \mathbb{R} \times S_h\) be the exact and the finite element solutions of (3.1).

Then, for sufficiently small \( h > 0 \), we have the following elaborate error estimate of \( |\lambda - \lambda_h|\):
\[ |\lambda - \lambda_h| \leq C_h \left| \langle (D_u F^0)(u - u_h), z - z_h \rangle + \frac{1}{2} \langle (D_{uu} F^0)(u - u_h)^2, z_h \rangle - \frac{\eta}{2}(D_{uu} \rho^0)(u - u_h)^2 \right|, \]
where \( D_u F^0 := D_u F(\lambda, u), \)
\[ (D_{uu} F^0)(u - u_h)^2 := \int_0^1 (1 - s)D_{uu} F(\lambda + s(\lambda - \lambda), u + s(u_h - u))(u - u_h)^2 \, ds, \]
\[ (D_{uu} \rho^0)(u - u_h)^2 := \int_0^1 (1 - s)(D_{uu} \rho(\lambda + s(\lambda - \lambda), u + s(u_h - u))(u - u_h), u - u_h) \, ds, \]
and \( C_h \) is a positive constant such that \( \lim_{h \to 0} C_h = 1 \).
Sometimes, one may want to compute a turning point itself. For such a purpose we are able to develop a similar analysis as above. Let \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) be a turning point of the equation \(F(\lambda, u) = 0\) at which the assumptions of Theorem 2.4 and (A8), (A9) hold. That is, \(F(\lambda_0, u_0) = 0\), \(DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, W)\) is onto, and \(D_u F(\lambda_0, u_0) \not\in \text{Im} D_u F(\lambda_0, u_0)\). It then follows from the proof of Lemma 3.1 that (3.1) has an unique solution \((0, z_0) \in \mathbb{R} \times X_p\) at \((\lambda_0, u_0)\):

\[
\langle D_u F(\lambda_0, u_0) z_0, v \rangle = 0, \quad \forall v \in X_p,
\]

\[
\langle D_z F(\lambda_0, u_0), z_0 \rangle = 1.
\]

We consider the nonlinear map \(K: A \times V \times X_p \to \mathbb{R} \times W \times X_q'\) defined by

\[
K(\lambda, u, z) := \begin{pmatrix}
\langle D_z F(\lambda, u), z \rangle - 1 \\
F(\lambda, u) \\
D_u F(\lambda, u) z
\end{pmatrix}.
\]

At a turning point \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) the equation \(K(\lambda, u, z) = (0, 0, 0)\) has the solution \((\lambda_0, u_0, z_0) \in A \times V \times X_p\). A turning point \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) is called nondegenerate, if

\[D_{uu} F(\lambda_0, u_0) \psi_0 \psi_0 \notin \text{Im} D_u F(\lambda_0, u_0),\]

where \(\{\psi_0\} \subset X_p\) is the basis of \(\text{Ker} D_u F(\lambda_0, u_0)\) (see [4, Section 4]). For a nondegenerate turning point, we have the following lemma. For the proof of the lemma, see [4,15].

**Lemma 3.4.** Let \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) be a turning point at which the assumptions of Theorem 2.4 and (A8), (A9) hold. Then, \((\lambda_0, u_0)\) is a nondegenerate turning point if and only if the Fréchet derivative \(DK(\lambda_0, u_0, z_0) \in \mathcal{L}(\mathbb{R} \times V \times X_p, \mathbb{R} \times W \times X_q')\) is an isomorphism, where \(z_0 \in X_p\) is the solution of (3.5) and the nonlinear map \(K\) is defined by (3.6).

From Lemma 3.4, the results in [16] can be applied to the equation \(K(\lambda, u, z) = (0, 0, 0)\) at a nondegenerate turning point \((\lambda_0, u_0)\) and obtain the following lemma:

**Lemma 3.5.** Let \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) is a nondegenerate tuning point. Then, for sufficiently small \(h > 0\), there exist a locally unique finite element solution \((\lambda_0^h, u_0^h, z_0^h) \in \mathbb{R} \times (S_h)^2\) such that

\[
\langle D_z F_h(\lambda_0^h, u_0^h), z_0^h \rangle = 1,
\]

\[
F_h(\lambda_0^h, u_0^h) = 0,
\]

\[
D_u F_h(\lambda_0^h, u_0^h) z_0^h = 0,
\]

where \(F_h\) is the nonlinear map defined by (2.2). The finite element solution \((\lambda_0^h, u_0^h)\) is a nondegenerate turning point on the finite element solution manifold \(\mathcal{M}_h\).

Moreover, we have the following error estimate:

\[
|\lambda_0 - \lambda_0^h| + \|u_0 - u_0^h\|_{X_p} + \|z_0 - z_0^h\|_{X_p} \leq C(\|u_0 - \Pi_h u_0\|_{X_p} + \|z_0 - \Pi_h z_0\|_{X_p}),
\]

where \(C\) is a positive constant independent of \(h\), and \(\Pi_h : X_p \to S_h\) is the projection which appears in Theorem 2.4.
Now, we develop a similar elaborate error estimate for \(|\lambda_0 - \lambda_0^h|\). Again, let \((\lambda_0, u_0) \in \mathscr{R}(F, \mathcal{H})\) be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and \((\lambda_0^h, u_0^h) \in \Lambda \times S_h\) the corresponding finite element solution. By Taylor’s theorem and \(\langle F(\lambda_0^h, u_0^h), v_h \rangle = \langle F(\lambda_0, u_0), v_h \rangle = 0\) for any \(v_h \in S_h\), we have

\[
0 = (\lambda_0^h - \lambda_0) \langle D_1 F(\lambda_0, u_0), v_h \rangle + \langle D_0 F(\lambda_0, u_0)(u_0^h - u_0), v_h \rangle + \frac{1}{2} (\lambda_0^h - \lambda_0)^2 \langle D_{zz} F^0, v_h \rangle + (\lambda_0^h - \lambda_0)(\langle D_{za} F^0, (u_0^h - u_0), v_h \rangle + \frac{1}{2} \langle D_{uu} F^0, (u_0^h - u_0)^2, v_h \rangle, \tag{3.7}
\]

where

\[
D_{zz} F^0 := \int_0^1 (1 - s)D_{zz} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0)) \, ds,
\]

\[
(D_{za} F^0)(u_0^h - u_0) := \int_0^1 (1 - s)D_{za} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0^h - u_0) \, ds,
\]

\[
(D_{uu} F^0)(u_0^h - u_0)^2 := \int_0^1 (1 - s)D_{uu} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0^h - u_0)^2 \, ds.
\]

Letting \(v := u_0 - u_0^h\) in (3.5), we obtain

\[
\langle D_0 F(\lambda_0, u_0)z_0, u_0 - u_0^h \rangle = \langle D_0 F(\lambda_0, u_0)(u_0 - u_0^h), z_0 \rangle = 0.
\]

Plugging this equation into (3.7) with \(v_h := z_0^h\), we obtain

\[
(\lambda_0 - \lambda_0^h)(\langle D_2 F(\lambda_0, u_0), z_0 \rangle + B_h) = \langle D_0 F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h \rangle + \frac{1}{2} \langle (D_{uu} F^0)(u_0 - u_0^h)^2, z_0 \rangle,
\]

where \(\lim_{h \to 0} B_h = 0\). Therefore, we have proved the following theorem:

**Theorem 3.6.** Let \((\lambda_0, u_0) \in \mathscr{R}(F, \mathcal{H})\) be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let \((\lambda_0^h, u_0^h) \in \Lambda \times S_h\) be the corresponding nondegenerate turning point on the finite element solution branch \(\mathcal{M}_h\). Let \(z_0 \in X_p\) and \(z_0^h \in S_h\) be the exact and the finite element solutions which appear in Lemmas 3.4 and 3.5.

Then, for sufficiently small \(h > 0\), we have the following elaborate error estimate of \(|\lambda_0 - \lambda_0^h|\):

\[
|\lambda_0 - \lambda_0^h| \leq C_h \left[ \langle D_0 F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h \rangle + \frac{1}{2} \langle (D_{uu} F^0)(u_0 - u_0^h)^2, z_0 \rangle \right],
\]

where

\[
(D_{uu} F^0)(u_0 - u_0^h)^2 := \int_0^1 (1 - s)D_{uu} F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0 - u_0^h)^2 \, ds
\]

and \(C_h\) is a positive constant such that \(\lim_{h \to 0} C_h = 1\).

**Remark 3.7.** Apparently, Lemma 3.5 and Theorem 3.6 are very similar to Brezzi et al. [4, Theorem 7]. The main difference is the tools used in [4] and in this paper. In [4] the Liapunov–Schmidt reduction is used to parametrize solution branches around turning points. On the other hand, so-called “bordering technique” is used throughout this paper. In [15], it is pointed out that bordering technique is closely related with the Liapunov–Schmidt reduction.
Therefore, as in [16–18], we define set \( W \) of \( C/VT \) spaces are denoted by where Assumption 4.1. (1) The mappings \( a : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( f : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) are sufficiently smooth functions.

In this section, we use the following notation. We denote by \( J_a(\lambda, x, y, z) \) and \( \nabla_z f(\lambda, x, y, z) \) the Jacobian matrix of \( a \) and the gradient of \( f \) with respect to \( z \in \mathbb{R}^d \). Partial derivatives with respect to \( \lambda \) and \( y \in \mathbb{R} \) are denoted such as \( a_\lambda(\lambda, x, y, z) \) and \( f_y(\lambda, x, y, z) \), for example.

Also, the usual Sobolev spaces are denoted by \( H^1_0(\Omega) \) and \( W^{1,p}(\Omega) \), \( 1 \leq p \leq \infty \). The space \( W^{1,p}_0(\Omega) \), \( 1 \leq p \leq \infty \), is defined by 
\[
W^{1,p}_0(\Omega) := \{ v \in W^{1,p}(\Omega) \mid u = 0 \text{ on } \partial \Omega \}.
\]
The space \( W^{-1,p}(\Omega) \), \( 1 < p \leq \infty \), is the dual space of \( W^{1,q}_0(\Omega) \), \( 1/p + 1/q = 0 \). The usual Hölder spaces are denoted by \( C^x(\Omega) \) and \( C^{1,z}(\Omega) \), \( 0 < x < 1 \).

In (4.1) the nonlinear operator \( F \) is defined as \( F : A \times H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \). To make \( F \) being well-defined and differentiable, we have to impose a strong growth condition on the function \( a \).

Therefore, as in [16–18], we define \( F \) as \( F : A \times W^{1,\infty}_0(\Omega) \rightarrow W^{-1,\infty}(\Omega) \). To ensure differentiability of \( F \) we impose the following conditions to \( a \) and \( f \):

**Assumption 4.1.** (1) The mappings \( a : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( f : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) are of \( C^2 \) class.

(2) The Jacobian matrix \( J_a(\lambda, x, y, z) \) is symmetric for all \( \lambda \in A \), \( x \in \Omega \), \( y \in \mathbb{R} \), and \( z \in \mathbb{R}^d \).

By a simple computation we obtain the following lemma:

**Lemma 4.2.** Under Assumption 4.1 the operator \( F : A \times W^{1,\infty}_0(\Omega) \rightarrow W^{-1,\infty}(\Omega) \) defined by (4.1) is of \( C^2 \) class.

Moreover, its Fréchet derivative \( DF(\lambda, u) \) can be extended \( DF(\lambda, u) \in \mathcal{L}(W^{1,p}_0, W^{-1,p}) \) (\( \forall p \), \( 1 < p \leq \infty \)), which is locally Lipschitz continuous on \( A \times W^{1,\infty}_0(\Omega) \): for an arbitrary bounded convex set \( \mathcal{C} \subset A \times W^{1,\infty}_0(\Omega) \), there exists a positive constant \( C_1(\mathcal{C}) \) such that for any \( (\mu, u), (\tau, w) \in \mathcal{C} \),
\[
\|DF(\mu, u) - DF(\tau, w)\|_{\mathcal{L}(W^{1,p}_0, W^{-1,p})} \leq C_1(\mathcal{C}) (|\mu - \tau| + \|u - w\|_{W^{1,p}_0, \infty}).
\]

Unfortunately, we may not expect that the Fréchet derivative \( D_a F(\lambda, u) \in \mathcal{L}(W^{1,\infty}_0, W^{-1,\infty}) \) could be an isomorphism if \( d \geq 2 \). Therefore, we introduce the following Banach spaces:

4. Strongly nonlinear boundary value problem

In this section we consider the following problem. Let \( \Omega \subset \mathbb{R}^d \) (\( d = 1, 2, 3 \)) be a bounded domain. Our problem is to find \( u \in H^1_0(\Omega) \) such that 
\[
\langle F(\lambda, u), v \rangle := \int_{\Omega} (a(\lambda, x, u, \nabla u) \cdot \nabla v + f(\lambda, x, u, \nabla u)v) \, dx = 0, \quad \forall v \in H^1_0(\Omega),
\]
where \( a : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( f : A \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) are sufficiently smooth functions.

By a simple computation we obtain the following lemma:
Define \( C^{1,2}_0(\bar{\Omega}) \subset C^{1,2}(\bar{\Omega}) \) by
\[
C^{1,2}_0(\bar{\Omega}) := \{ v \in C^{1,2}(\bar{\Omega}) \mid v = 0 \text{ on } \partial \Omega \}.
\]
Define \( W_s(\Omega) \subset H^{-1}(\Omega) \) by
\[
W_s(\Omega) := \left\{ \sum_{i=1}^{d} f_{is_i} \mid f_i \in C^s(\bar{\Omega}) \right\},
\]
where \( f_{is_i} \) is the partial derivative of \( f_i \) in the sense of distribution with respect to \( x_i \), \((x_1, \ldots, x_d) \in \Omega \). In other words, \( F \in W_s(\Omega) \) is an element of \( H^{-1}(\Omega) \) which is represented by
\[
\langle F, v \rangle = -\int_{\Omega} f \cdot \nabla v \, dx, \quad \forall v \in H^1_0(\Omega),
\]
where \( f \in C^s(\bar{\Omega})^d \). The norm of \( W_s(\Omega) \) is defined by
\[
\| F \|_{W_s} := \inf_{F = \text{div } f} \{ \| f \|_{C^s(\Omega)^d} \}
\]
for \( F \in W_s(\Omega) \). Then, we have

**Lemma 4.3.** With the norm \( \| \cdot \|_{W_s} \), \( W_s(\Omega) \) is a Banach space.

With Assumption 4.1 the operator \( F \) can be regarded as \( F : C^{1,2}_0(\bar{\Omega}) \to W_s(\Omega) \).

**Lemma 4.4.** Under Assumption 4.1, the operator \( F : C^{1,2}_0(\bar{\Omega}) \to W_s(\Omega) \) is continuously differentiable.

To make the Fréchet derivative \( D_uF(\cdot, u) \in \mathcal{L}(C^{1,2}_0, W_s) \) isomorphic, we require the following conditions to the domain \( \Omega \).

Let \( p^* \) be taken such as
\[
p^* \geq 2 \quad \text{if } d = 1,
\]
\[
p^* > d \quad \text{if } d > 1
\]
and fixed. Let \( \alpha := 1 - d/p^* \). Suppose that \( d \times d \)-matrix \( \beta(x) = (\beta_{ij}(x)) \in (C^s(\bar{\Omega}))^{d \times d} \) satisfies the strong ellipticity condition: there exists a positive constant \( \delta \) such that
\[
\sum_{i,j=1}^{d} \beta_{ij}(x)\xi_i\xi_j \geq \delta |\xi|^2, \quad \forall x \in \bar{\Omega}, \, \forall \xi \in \mathbb{R}^d.
\]

The linear differential operator \( Q \) and \( L \) are defined by
\[
Q u := -\sum_{i,j=1}^{d} (\beta_{ij}(x) u_{x_j})_{x_i}, \quad Lu := \sum_{i,j=1}^{d} \beta_{ij}(x) u_{x_i x_j}.
\]

**Assumption 4.5.** For the fixed \( p^* \) which is taken as (4.2), \( \alpha := 1 - d/p^* \), and a given \( \beta(x) = (\beta_{ij}(x)) \in (C^s(\bar{\Omega}))^{d \times d} \) with the strong ellipticity condition (4.3), \( Q \in \mathcal{L}(W^{1,p^*}_0, W^{-1,p^*}) \), \( Q \in \mathcal{L}(C^{1,2}_0, W_s) \), and \( L \in \mathcal{L}(H^1_0 \cap W^{2,p^*}, L^{p^*}) \) are isomorphisms. We also assume that the Laplacian \( \Delta \in \mathcal{L}(H^1_0 \cap W^{2,p^*}, L^{p^*}) \) is an isomorphism.
Of course, in the case of \( d = 1 \), we do not need to impose Assumption 4.5. In the case of \( d = 2, 3 \), Assumption 4.5 is satisfied for any \( p^* \), \( 1 < p^* < \infty \) and any \( \beta(x) \in (C^\alpha(\bar{\Omega}))^{d \times d} \), if the boundary \( \partial \Omega \) is of \( C^{1,1} \) class (see \cite[Theorems 8.34, 9.15]{10}).

Let \( (\lambda, u) \in A \times C^{1,2}_0(\bar{\Omega}) \) be a solution of \( F(\lambda, u) = 0 \). Suppose, moreover, that \( \beta(x) := J_\lambda a(\lambda, x, u(x), \nabla u(x)) \in (C^\alpha(\bar{\Omega}))^{d \times d} \) satisfies the strong ellipticity condition (4.3). Then, by Assumption 4.5, the operator \( Q \in \mathcal{L}(W_0^{1,p^*}, W^{-1,p^*}) \) defined by, for \( \psi \in W_0^{1,p^*}(\Omega) \),

\[
\langle Q\psi, v \rangle := \int_\Omega (\beta(x) \nabla \psi(x)) \cdot \nabla v(x) \, dx, \quad \forall v \in W_0^{1,p^*}(\Omega)
\]

is an isomorphism.

**Lemma 4.6.** Suppose that Assumptions 4.1 and 4.5 hold. Define the subset \( \mathcal{S} \subset A \times C^{1,2}_0(\bar{\Omega}) \) by

\[
\mathcal{S} := \{ (\lambda, u) \in A \times C^{1,2}_0(\bar{\Omega}) \mid \beta(x) := J_\lambda a(\lambda, x, u, \nabla u) \text{ satisfies } (4.3) \}.
\]

Then, (1) \( \mathcal{S} \) is open in \( A \times C^{1,2}_0(\bar{\Omega}) \), and \( F : \mathcal{S} \to W_2(\Omega) \) is a nonlinear Fredholm operator of index 1.

(2) For \( (\lambda, u) \in \mathcal{S} \), \( DF(\lambda, u) \in \mathcal{L}(\mathbb{R} \times W_0^{1,p^*}, W^{-1,p^*}) \) is Fredholm of index 1.

(3) Let \( (\lambda, u) \in \mathcal{S} \), \( \phi \in W_0^{1,p^*}(\Omega) \), and \( g \in W_2(\Omega) \). If \( D_\lambda F(\lambda, u) \phi = g \), then we have \( \phi \in C^{1,2}_0(\bar{\Omega}) \).

**Proof.** All statements of Lemma 4.6 follow from Assumption 4.5. \( \square \)

For the regularity of the solutions of \( F(\lambda, u) = 0 \), we have the following lemma:

**Lemma 4.7.** Suppose that Assumptions 4.1 and 4.5 hold. Let \( (\lambda, u) \in \mathcal{S} \) satisfy \( F(\lambda, u) = 0 \). Then, we have \( u \in W_2^{2,p^*}(\Omega) \).

**Proof.** See \cite[Section 5]{16}. \( \square \)

Let \( x_0 \in \Omega \) be an inner point which is taken certainly. We define the functional \( \rho : A \times W_0^{1,p^*}(\Omega) \to \mathbb{R} \) by

\[
\rho(\lambda, u) := u(x_0).
\]

Then, from Lemma 2.2, we have the following lemma:

**Lemma 4.8.** Suppose that there exists \( (\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S}) \) such that \( F(\lambda_0, u_0) = 0 \), \( \gamma_0 := u_0(x_0) \), \( D_\lambda F(\lambda_0, u_0) \neq 0 \), and \( \psi_0(x_0) \neq 0 \), where \( \{ (\mu_0, \psi_0) \} \) is the basis of \( \text{Ker} DF(\lambda_0, u_0) \). Then, the map \( H(\gamma, \lambda, u) := (u(x_0) - \gamma, F(\lambda, u)) \) satisfies that \( H(\gamma_0, \lambda_0, u_0) = (0, 0) \) and its partial Fréchet derivative \( D_{(\lambda, u)} H(\gamma_0, \lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times C^{1,2}_0(\bar{\Omega}), \mathbb{R} \times W_2) \) is an isomorphism. Moreover, its extension \( D_{(\lambda, u)} H(\gamma_0, \lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times W_0^{1,p^*}, \mathbb{R} \times W^{-1,p^*}) \) is an isomorphism as well.

**Proof.** We check the assumptions of Lemma 2.2. From the setting, (A1) obviously holds. By Lemmas 4.2 and 4.6 that (A2), (A3), and (A6) hold. Since \( \rho : A \times W_0^{1,p^*}(\Omega) \to \mathbb{R} \) is linear, (A4) and (A5) are satisfied clearly. The condition \( \psi_0(x_0) \neq 0 \) means that \( \text{Ker} D\rho(\lambda_0, x_0) \cap \text{Ker} DF(\lambda_0, u_0) \neq \{(0, 0)\} \). Thus, by Lemma 2.2, all statements of the lemma follow immediately. \( \square \)
Let \( \{ A_h \} \) be a family of triangulation of \( \Omega \), which is parametrized by \( h > 0 \), and \( h \to 0 \). On \( \{ A_h \} \) we define, in a certain way, a family of finite-dimensional spaces \( S_h \subset W_0^{1,\infty}(\Omega) \subset W_0^{1,p^*}(\Omega) \).

Since (4.1) is defined in divergent form, the finite element solution \( (\tilde{\lambda}_h, u_h) \in \mathcal{A} \times S_h \) is naturally defined by
\[
\langle F(\tilde{\lambda}_h, u_h), v_h \rangle = 0, \quad \forall v_h \in S_h.
\]

For \( u \in \mathcal{S} \), the bilinear form \( A(\psi, \phi) \) defined by
\[
A(\psi, \phi) := \int_{\Omega} (\beta(x) \nabla \psi) \cdot \nabla \phi \, dx
\]
is an inner product of \( H_0^1(\Omega) \), since \( \beta(x) := \int \alpha(\lambda, x, u, \nabla u) \) satisfies (4.3) and is symmetric. We define the canonical projection \( \tilde{P}_h : H_0^1(\Omega) \to S_h \) by
\[
A(\psi - \tilde{P}_h \psi, v_h) = 0, \quad \forall v_h \in S_h
\]
for \( \psi \in H_0^1(\Omega) \). Then, the projection \( P_h : W^{-1,p^*}(\Omega) \to W^{-1,p^*}(\Omega) \) is defined by
\[
P_h := Q\tilde{P}_h Q^{-1}.
\]  

For the family of triangulation \( \{ A_h \} \) and finite element spaces \( \{ S_h \} \) we require the following:

**Assumption 4.9.** Under Assumptions 4.1 and 4.5, we assume that

1. For each \( h > 0 \), \( A_h \) satisfies \( \bigcup_{T \in A_h} \bar{T} = \bar{\Omega} \), and the finite element space satisfies \( S_h \subset W_0^{1,\infty}(\Omega) \).
2. For any \( \beta(x) = (\beta_{ij}(x))_{i,j=1,\ldots,d} \) which satisfies \( \beta_{ij} \in W^{1,p^*}(\Omega) \), \( \beta_{ij} = \beta_{ji} \), and (4.3), the canonical projection \( P_h \in \mathcal{L}(W^{-1,p^*}, W^{-1,p^*}) \) defined by (4.6) satisfies
\[
\lim_{h \to 0} \| \psi - P_h \psi \|_{W^{-1,p^*}} = 0, \quad \forall \psi \in W^{-1,p^*}(\Omega).
\]

3. The family of triangulation \( \{ A_h \} \) is regular [6, p. 131], and satisfies the inverse assumption [6, p. 135].

On the matters related to Assumption 4.9(2) see [2, Chapter 7].

Recall that \( \mathcal{M}_0 \subset \mathcal{R}(F, \mathcal{S}) \) is the solution manifold defined by
\[
\mathcal{M}_0 := \{ (\lambda, u) \in \mathcal{R}(F, \mathcal{S}) | F(\lambda, u) = 0 \}.
\]

**Theorem 4.10.** Suppose that Assumptions 4.1, 4.5, and 4.9 hold with \( p^* \) which is taken as (4.2) and \( \alpha := 1 - d/p^* \). Let \( \hat{\mathcal{M}}_0 \subset \mathcal{M}_0 \) be a connected compact subset with the following properties:

1. \( D_2 F(\lambda, u) \neq 0 \) for any \( (\lambda, u) \in \hat{\mathcal{M}}_0 \).
2. There exists \( x_0 \in \Omega \) such that \( \psi(x_0) \neq 0 \) for all \( (\lambda, u) \in \hat{\mathcal{M}}_0 \), where \( \{ (\mu, \psi) \} \) is the basis of \( \mathrm{Ker} \text{DF}(\lambda, u) \).

Then, \( \hat{\mathcal{M}}_0 \) is parametrized by \( \gamma = u(x_0) \). We assume without loss of generality that the above \( x_0 \in \Omega \) is a nodal point of \( S_h \) for all sufficiently small \( h > 0 \).

Then, for sufficiently small \( h > 0 \), there exists the corresponding locally unique finite element solution branch \( \hat{\mathcal{M}}_h \) which is parametrized by the same \( \gamma \), that is, \( u_h(\gamma)(x_0) = \gamma \) and
\[
\langle F(\tilde{\lambda}_h(\gamma), u_h(\gamma)), v_h \rangle = 0, \quad \forall v_h \in S_h.
\]
for any \((\lambda_0(\gamma), u_0(\gamma))\) ∈ \(\tilde{\mathcal{M}}_h\). Moreover, the following estimates hold:

\[
\begin{align*}
|\lambda_h(\gamma) - \lambda_0(\gamma)| + ||u_h(\gamma) - \Pi_h u(\gamma)||_{W_0^{1,p'}} & \leq K_1 ||u(\gamma) - \Pi_h u(\gamma)||_{W_0^{1,p'}}, \\
|\lambda_h(\gamma) - \lambda_0(\gamma)| + ||u_h(\gamma) - u(\gamma)||_{W_0^{1,p'}} & \leq K_2 ||u(\gamma) - \Pi_h u(\gamma)||_{W_0^{1,p'}}, \\
|\lambda_h(\gamma) - \lambda_0(\gamma)| + ||u_h(\gamma) - u(\gamma)||_{W_0^{1,p'}} & \leq K_3 h^{1/d/p}
\end{align*}
\]

for all \((\lambda(\gamma), u(\gamma))\) ∈ \(\tilde{\mathcal{M}}_0\), \((\lambda_h(\gamma), u_h(\gamma))\) ∈ \(\tilde{\mathcal{M}}_h\), where \(\Pi_h : W_0^{1,p'}(\Omega) \to S_h\) is the usual interpolant projection as in [6, Theorem 16.1]. Here, \(K_1, K_2,\) and \(K_3\) are independent of \(h > 0\) and \(\gamma\).

**Proof.** We put the present situation into the setting of Theorem 2.4. Let \(X_p := W_0^{1,p'}(\Omega),\) \(X_p' := W^{-1,p'}(\Omega),\) \(X_2 := H_0^1(\Omega),\) and \(X_2' := H^{-1}(\Omega).\) Let \(V := C_0^{1,2}(\Omega)\) and \(W := W_2(\Omega).\) The nonlinear operator \(F\) is defined by (4.1), \(\mathcal{S}\) is defined by (4.4), and the functional \(\rho\) is define by (4.5).

We check the conditions of Theorem 2.4. Take any \((\lambda_0, u_0)\) ∈ \(\tilde{\mathcal{M}}_0\). Assumptions (A1)–(A6) have been checked in the proof of Lemma 4.8. Also, it is checked that \(\text{Ker} \, DF(\lambda_0, u_0) \cap \text{Ker} \, D\rho(\lambda_0, u_0) = \{(0,0)\} \). By Assumption 4.9(3), (A7) is valid with \(r := d/p\).

By Lemma 4.7 we have that \(u \in W^{2,p'}(\Omega)\) for any \((\lambda, u) \in \tilde{\mathcal{M}}_0.\) Hence, we have \(||u - \Pi_h u||_{W_0^{1,p'}} \leq Ch||u||_{W^{2,p'}}\) and \(||u - \Pi_h u||_{W_0^{1,\infty}} \leq Ch^{1/d/p}||u||_{W^{2,p'}}\) (see [6, Theorem 16.2]). Therefore, (2.3) and (2.4) are satisfied. By Assumption 4.9, the projection \(P_h\) defined by (4.6) satisfies (2.5).

Therefore, by Theorem 2.4, there exist a constant \(\epsilon_1\) and a locally unique \(C^2\) map \(u_0(x_0) - \epsilon_1, u_0(x_0) + \epsilon_1\) \(\gamma \mapsto (\lambda_0(\gamma), u_0(\gamma)) \in \mathcal{A} \times S_h\) such that \(u(\gamma)(x_0) = \gamma\) and

\[\langle F(\lambda(\gamma), u(\gamma)), v_h \rangle = 0, \quad \forall v_h \in S_h.\]

Moreover, the error estimates hold for any \(\gamma \in [u_0(x_0) - \epsilon_1, u_0(x_0) + \epsilon_1]\). Finally, by connectedness and compactness of \(\mathcal{M}_0\), we can pick up finite such points \((\lambda_0, u_0) \in \tilde{\mathcal{M}}_0\) and conclude that there exists the locally unique finite element solution branch \(\tilde{\mathcal{M}}_h\) on which the error estimates hold. \(\square\)

For obtaining the elaborate error estimates of \(\lambda\), we assume

**Assumption 4.11.** For all \((\lambda, x, y, z) \in \mathcal{A} \times \Omega \times \mathbb{R} \times \mathbb{R}^d, a_j(\lambda, x, y, z) = \nabla_z f(\lambda, x, y, z).\)

It is easy to check that \(D_u F(\lambda, u) \in \mathcal{L}(H_0^1, H_0^1)\) is self-adjoint under Assumption 4.11.

Let \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) be a solution of \(F(\lambda, u) = 0\) at which all assumptions of Theorem 4.10 and Assumption 4.11 hold. We consider the following auxiliary equation: find \((\eta, z) \in \mathbb{R} \times W_0^{1,p'}(\Omega)\) such that

\[
\int_\Omega ((\alpha(x)\nabla z) \cdot \nabla v + \beta(x) \cdot (v\nabla z + z\nabla v) + \delta(x)z v) \, dx = \eta v(x_0), \quad \forall v \in W_0^{1,p'}(\Omega),
\]

\[
\int_\Omega (a_j(\lambda_0, x, u_0, \nabla u_0) \cdot \nabla z + f_j(\lambda_0, x, u_0, \nabla u_0) z) \, dx = 1,
\]

where

\[
\alpha(x) := J_a(\lambda_0, x, u_0(x), \nabla u_0(x)), \quad \delta(x) := f_j(\lambda_0, x, u_0(x), \nabla u_0(x)),
\]

\[
\beta(x) := a_j(\lambda_0, x, u_0(x), \nabla u_0(x)) = \nabla_z f(\lambda_0, x, u_0(x), \nabla u_0(x)).
\]
By Lemma 3.1 there exists a unique solution \((\eta, z) \in \mathbb{R} \times W_0^{1, p^*_\infty}(\Omega)\) to (4.7). Also from Lemma 3.2 we know that there exists the corresponding unique finite element solution \((\eta_h, z_h) \in \mathbb{R} \times S_h\). By Theorem 3.3 we obtain

**Theorem 4.12.** Let \((\lambda, u) \in \mathcal{R}(F, \mathcal{S})\) be a solution of \(F(\lambda, u) = 0\) which satisfies the assumptions of Theorem 4.10. Also, suppose that Assumption 4.11 holds. Let \((\lambda_h, u_h) \in \Lambda \times S_h\) be the corresponding finite element solution with \(u(x_0) = u_h(x_0)\). Let \((\eta, z) \in \mathbb{R} \times W_0^{1, p^*_\infty}(\Omega)\) and \((\eta_h, z_h) \in \mathbb{R} \times S_h\) be the exact and finite element solutions to (4.7).

Then, we have the following elaborate error estimates for \(\lambda\): for sufficiently small \(h > 0\) there exists a positive constant \(C\) independent of \(h\) such that

\[
|\lambda - \lambda_h| \leq C(\|u - u_h\|_{H^1_0(\Omega)} + \frac{1}{2} \max \{|\eta|\} \|u - u_h\|_{H^1_0(\Omega)}).
\]

Next, we consider elaborate error estimate of a nondegenerate turning point. Let \((\lambda_0, u_0) \in \mathcal{M}_0\) be a turning point. We suppose that the assumptions of Theorem 4.12 are satisfied. Then, by Lemma 3.4, \((\lambda_0, u_0)\) is nondegenerate if and only if the following equation has the isolate solution \((\lambda_0, u_0, z_0) \in \Lambda \times C_0^{0, z}(\bar{\Omega}) \times W_0^{1, p^*_\infty}(\Omega)\): A solution of a nonlinear equation is called isolated if the Fréchet derivative of the associated nonlinear operator is an isomorphism between certain Banach spaces at the solution.)

\[
\int_{\Omega} (a(\lambda, x, u, \nabla u) \cdot \nabla v + f(\lambda, x, u, \nabla u)v) \, dx = 0, \quad \forall v \in H^1_0(\Omega),
\]

\[
\int_{\Omega} ((a(x) \nabla z \cdot \nabla v + \beta(x) \cdot (v \nabla z + z \nabla v)) \, dx = 0, \quad \forall v \in H^1_0(\Omega),
\]

\[
\int_{\Omega} (a(\lambda, x, u, \nabla u) \cdot \nabla z + f(\lambda, x, u, \nabla u)z) \, dx = 1, \quad \text{(4.8)}
\]

where

\[
a(x) := J_z a(\lambda, x, u(x), \nabla u(x)), \quad \delta(x) := f_z(\lambda, x, u(x), \nabla u(x)),
\]

\[
\beta(x) := a_z(\lambda, x, u(x), \nabla u(x)) = \nabla_z f(\lambda, x, u(x), \nabla u(x)).
\]

Finally, by Lemma 3.5 and Theorem 3.6 we obtain

**Theorem 4.13.** Let \((\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})\) be a turning point which satisfies the assumptions of Theorem 4.12. We suppose that \((\lambda_0, u_0)\) is nondegenerate. Then, for sufficiently small \(h > 0\), there exists the corresponding locally unique nondegenerate turning point \((\lambda_h^0, u_h^0) \in \mathcal{M}_h\) on the finite element solution manifold. Moreover, we have the following elaborate error estimates for \(\lambda^0_0\): there exists a positive constant \(C\) independent of \(h\) such that

\[
|\lambda_0 - \lambda_h^0| \leq C(\|u_0 - u_h^0\|_{H^1_0} + \|z_h^0\|_{H^1_0} + \frac{1}{2} \|z_h^0\|_{W_0^{1, p^*_\infty}} \|u_0 - u_h^0\|_{H^1_0}),
\]

where \((\lambda_0, u_0, z_0) \in \Lambda \times C_0^{0, z}(\bar{\Omega}) \times W_0^{1, p^*_\infty}(\Omega)\) is the isolated solution of (4.8) and \((\lambda_h^0, u_h^0, z_h^0) \in \Lambda \times (S_h)^2\) is the locally unique finite element solution.
Remark 4.14. In this section we only deal with homogeneous Dirichlet boundary condition. We, however, are able to deal with more general boundary conditions. For example, consider the following equation: find \((\lambda, u) \in \mathbb{R} \times H^1(\Omega)\) such that

\[
\langle \tilde{F}(\lambda, u), v \rangle := \int_{\Omega} \left( a(\lambda, x, u, \nabla u) \cdot \nabla v + f(\lambda, x, u, \nabla u) v \right) \, dx = 0, \quad \forall v \in H^1_0(\Omega),
\]

\[
u = g \quad \text{on} \quad \partial \Omega.
\]

Then, we define the nonlinear map \(F : \mathbb{R} \times W^{1,p}(\Omega) \to W^{-1,1,p}(\partial \Omega) \times W^{-1,1,p}(\partial \Omega)\) by \(F(\lambda, u) := (\tilde{F}(\lambda, u), \gamma u - g)\), where \(\gamma : W^{1,p}(\Omega) \to W^{-1,1,p}(\partial \Omega)\) is the trace operator. We can develop a similar theory as above if \(D_u F(\lambda, u) \in \mathcal{L}(W^{1,p}(\Omega), W^{-1,1,p}(\partial \Omega) \times W^{-1,1,p}(\partial \Omega))\) can be an isomorphism.

By the same manner we can deal with a system of parametrized equations with general boundary conditions.

5. Remarks on eigenvalue problems

In this section we apply our results to an eigenvalue problem, and give alternate proofs of well-known results on finite element approximation of eigenvalue problems.

Let \(\Omega \subset \mathbb{R}^d (d = 1, 2, 3)\) be a bounded domain. We consider the following problem: find \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)\) such that

\[
- \Delta u = \lambda u \quad \text{in} \quad \Omega.
\] (5.1)

Let \(\lambda \in \mathbb{R}\) be a simple eigenvalue of (5.1). To determine the corresponding eigenfunction \(u \in H^1_0(\Omega)\), we introduce the functional \(\rho : \mathbb{R} \times H^1_0(\Omega) \to \mathbb{R}\) by

\[
\rho(\lambda, u) := \int_{\Omega} u^2 \, dx.
\]

To apply the results obtained in Sections 2 and 3 to the eigenvalue problem (5.1), we define the operator \(F : \mathbb{R} \times H^1_0(\Omega) \to H^{-1}(\Omega)\) by

\[
\langle F(\lambda, u), v \rangle := \int_{\Omega} \left( \nabla u \cdot \nabla v - \lambda uv \right) \, dx, \quad \forall v \in H^1_0(\Omega),
\]

that is,

\[
a(\lambda, x, u, \nabla u) := \nabla u, \quad f(\lambda, x, u, \nabla u) := - \lambda u.
\]

Clearly, assumptions (A1)–(A6), (A8), and (A9) are satisfied with the setting

\[
V = X_\infty = X_p := H^1_0(\Omega), \quad W = X'_1 = X'_q := H^{-1}(\Omega), \quad \mathcal{S} := \mathbb{R} \times H^1_0(\Omega).
\]

With \(r = 0\), (A7) is also valid.

Let \((\lambda, u)\) be an eigenpair of (5.1), that is, \(u \neq 0\) and \(F(\lambda, u) = 0\). It is easy to see that, in this case, the condition \((\lambda, u) \in \mathcal{S}(F, \mathcal{S})\) is equivalent to that the eigenvalue \(\lambda\) is simple.

On the regularity of the boundary \(\partial \Omega\), the triangulation \(\{ \Delta_h \}\) of \(\Omega\), and the finite element spaces \(\{ S_h \}\) on it, we impose the following conditions:

**Assumption 5.1.** (1) For each \(h > 0\), \(\Delta_h\) satisfies \(\bigcup_{r \in \Delta_h} \bar{T} = \bar{\Omega}\), and the finite element space satisfies \(S_h \subset H^1_0(\Omega)\).
(2) The family of triangulation \( \{ T_k \} \) is regular [6, p. 131].

(3) There exists a positive constant \( \delta < 1 \) such that \(-\Delta \in \mathcal{L}(H^1_0 \cap H^{1+\delta}, H^{-1+\delta}) \) is an isomorphism.

(4) For the canonical projection \( \Pi_h : H^1_0(\Omega) \to S_h \) defined by
\[
\int_{\Omega} \nabla (u - \Pi_h u) \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in S_h, \tag{5.2}
\]
we have, with same constant \( \delta \) in (3),
\[
\lim_{h \to 0} \| u - \Pi_h u \|_{H^1_0} = 0, \quad \text{for } u \in H^1_0(\Omega),
\]
\[
\| u - \Pi_h u \|_{H^1_0} \leq Ch^\delta \| u \|_{H^{1+\delta}}, \quad \text{for } u \in H^1_0(\Omega) \cap H^{1+\delta}(\Omega).
\]

On the matters related to Assumption 5.1, see [2, Chapter 12].

Under Assumption 5.1 we conclude that, for \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega), F(\lambda, u) = 0 \) implies \( u \in H^{1+\delta}(\Omega) \). Hence, we have checked all assumptions of Theorem 2.4 and have proved

**Theorem 5.2.** Suppose that Assumption 5.1 holds. Let \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega) \) be an eigen-pair such that \( \lambda \) is simple and \( \rho(\lambda, u) \neq 0 \).

Then, there exists a unique finite element solution \((\lambda_h, u_h) \in \mathbb{R} \times S_h \) such that \( \rho(\lambda_h, u_h) = \rho(\lambda, u) \) and
\[
\int_{\Omega} (\nabla u_h \cdot \nabla v_h - \lambda_h u_h v_h) \, dx = 0, \quad \forall v_h \in S_h.
\]

Moreover, the following estimate hold:
\[
|\lambda_h - \lambda| + \| u_h - \Pi_h u \|_{H^1_0} \leq K_1 \| u - \Pi_h u \|_{H^1_0},
\]
\[
|\lambda_h - \lambda| + \| u_h - u \|_{H^1_0} \leq K_2 \| u - \Pi_h u \|_{H^1_0},
\]
where \( \Pi_h : H^1_0(\Omega) \to S_h \) is the canonical projection defined by (5.2), and \( K_1, K_2 \) are positive constants independent of \( h > 0 \).

Now, let us consider an elaborate error estimate of the eigenvalue \( \lambda \). Let \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega) \) is an eigen-pair of (5.1) such that \( \lambda \) is simple and \( \rho(\lambda, u) = 1 \). The auxiliary equation (3.1) becomes
\[
\int_{\Omega} (\nabla z \cdot \nabla v - \lambda z v) \, dx = 2\eta \int_{\Omega} uv \, dx, \quad \forall v \in H^1_0(\Omega), \quad \int_{\Omega} uz \, dx = 1. \tag{5.3}
\]

Since \( \lambda \) is a simple eigenvalue of (5.1), we have \( \dim \ker D_\lambda F(\lambda, u) = 1 \). Hence, from the proof of Lemma 3.1, we conclude \( \eta = 0 \) in (5.3). Thus, we realize that there exists \( \mu \in \mathbb{R} \) such that \( z = \mu u \).

By the fact that \( \int_{\Omega} u^2 \, dx = 1 \) and \( \int_{\Omega} uz \, dx = 1 \), we find that (5.3) coincide with (5.1) and \( z = u \).

Therefore, from Theorem 3.3 and its proof, we obtain
\[
(\lambda - \lambda_h) (\langle D_\lambda F(\lambda, u), u_h \rangle - \langle D_\lambda u F(\lambda, u)(u_h - u), u_h \rangle) = \langle D_\lambda F(\lambda, u)(u_h - u), u_h - u \rangle.
\]

Since
\[
\langle D_\lambda F(\lambda, u), u_h \rangle = - \int_{\Omega} uu_h \, dx \quad \text{and} \quad \int_{\Omega} u^2 \, dx = 1,
\]
\[
\langle D_\lambda u F(\lambda, u)(u_h - u), u_h \rangle = - \int_{\Omega} (u_h - u) u_h \, dx.
\]

we obtain
\[ \lambda_h - \lambda = \int_\Omega (|\nabla (u - u_h)|^2 - \lambda (u - u_h)^2) \, dx, \]  
which is essentially equivalent to Eq. (46) in [14, Section 6.3]. By (5.4) we have proved the following well-known result by a different approach:

**Theorem 5.3.** Let \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)\) be an eigen-pair such that \(\rho(\lambda, u) = 1\) and \(\lambda\) is simple. Let \((\lambda_h, u_h) \in \mathbb{R} \times S_h\) be the corresponding finite element eigen-pair with \(\rho(\lambda_h, u_h) = 1\).

Then, there exists a positive constant \(C\) independent of \(h > 0\) such that
\[ \lambda_h - \lambda \leq C \|u - u_h\|_{H^1_0}. \]

**References**