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Existence and Relaxation Results for Nonlinear Second-Order Multivalued Boundary Value Problems in $\mathbb{R}^N$

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In this paper we study second order differential inclusions with nonlinear boundary conditions. Our formulation is general and incorporates as special cases well-known problems such as the Dirichlet (Picard), Neumann, and periodic problems.

$W^{1,2}(T, \mathbb{R}^N)$-norm (strong relaxation theorem). Finally we examine the Dirichlet problem when the multivalued right-hand side does not depend on the derivative of x and satisfies a general growth hypothesis and a sign-type condition. For this problem we prove existence results and a relaxation theorem.

Key Words: maximal monotone map; compact operator; compact embedding; Leary–Schauder alternative theorem; continuous selector; extremal solution; relaxation theorem; Aumann's selection theorem.

INTRODUCTION

In [17], Kandilakis–Papageorgiou obtained some existence theorems for second-order differential inclusions in $\mathbb{R}^N$ with boundary conditions involving a maximal monotone operator. In that paper the authors developed the L^p-theory ($1 < p < \infty$) for the problem and their approach was based on the theory of maximal monotone operators. In this paper we consider the same multivalued boundary value problem and present the L^1-existence theory for it. At the same time we weaken some of the hypotheses employed by Kandilakis–Papageorgiou. Moreover, since the use of maximal monotone operators is no longer possible, we need to develop a different approach, which nevertheless culminates to the eventual use of the Leray–Schauder alternative theorem (in both its single-valued and multivalued formulations). Our work also extends those of Pruszko [26], Frigon–Granas [11], Erbe–Krawcewicz [9], Kravvaritis–Papageorgiou [22] and Frigon [29]. Of these works, only Erbe–Krawcewicz use nonlinear boundary conditions. However they work within L^2 and their solutions

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belong in the Hilbert space $W^{2,2}(T, \mathbb{R}^N)$. In addition they limit their study to the "convex" problem (i.e., the multivalued right-hand side has convex values). In contrast, here we examine both the convex and the nonconvex problems. However, we go beyond these problems and establish the existence of "extremal" solutions (i.e., solutions moving through the extreme points of the multivalued right-hand side). We also show that the extremal solutions are dense in the solution set of the "convex" problem (relaxation theorem). Only Kravvaritis–Papageorgiou [22] considered the problem of existence of extremal solutions and proved a relaxation theorem. Our results here extend the corresponding results of Kravvaritis–Papageorgiou. Our formulation is general enough to incorporate as special case problems which have been studied previously, like the Dirichlet (Picard) problem, the Neumann problem, the periodic problem, and the Sturm–Liouville problem. In a series of corollaries we indicate how the above mentioned problems can be obtained as particular cases of our formulation. Finally in the context of a single-valued right-hand side, our results here extend those by Erbe–Palamides [10], Granas–Guenther–Lee [12], Knobloch [20], and Knobloch–Schmitt [21].

2. PRELIMINARIES

In this section we fix our terminology and notation and briefly recall some basic definitions and facts from multivalued analysis that we shall need in the sequel.

Let X be a Banach space. By $P_{f(c)}(X)$ (resp. $P_{k(c)}(X)$), we denote the collection of all nonempty, closed (and convex) (resp. nonempty, compact (and convex)) subsets of X. If X is separable, then a multifunction $F: T = [0, b] \to P_f(X)$ is said to be "measurable," if for all $x \in X$, the $\mathbb{R}_+$-valued function $t \to d(x, F(t)) = \inf[\|x - v\| : v \in F(t)]$ is measurable. In fact due to the completeness of the Lebesgue σ-field $\mathscr{L}(T)$ with respect to the Lebesgue measure on T, this definition of measurability is equivalent to saying that $GrF = \{(t, v) \in T \times X : v \in F(t)\} \in \mathscr{L}(T) \times B(X)$, where $B(X)$ is the Borel σ-field of X (graph measurability). In general, however, we can only say that measurability implies graph measurability. For details we refer to the survey paper of Wagner [28].

Given $F: T \to P_f(X)$ and $1 \leqslant p \leqslant \infty$, by S_F^p we denote the selectors of $F(\cdot)$ which belong in the Lebesgue–Bochner space $L^p(T, X)$; i.e., $S_F^p = \{f \in L^p(T, X) : f(t) \in F(t) \text{ a.e. on } T\}$. In general this set may be empty. However a straightforward application of Aumann's selection theorem (see Wagner [28, Theorem 5.10]) proves that S_F^p is nonempty if and only if $\inf\{\|v\| : v \in F(t)\} \in L^p(T)$. The set S_F^p is closed, and is also convex if and only if for almost all $t \in T$, $F(t)$ is convex and bounded if and only if $|F(t)| = \sup[\, \|v\| : v \in F(t)]$ belongs in $L^p(T)$. Finally the set S_F^p is decomposable in the sense that if $(f_1, f_2, A) \in S_F^p \times S_F^p \times \mathscr{L}(T)$, then $\chi_A f_1 + \chi_{A^c} f_2 \in S_F^p$.

On $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric," by setting

$$h(A, B) = \max[\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\|], \qquad A, B \in P_f(X).$$

It is well-known (see Klein–Thompson [19]) that $(P_f(X), h)$ is a complete metric space and that $(P_k(X), h)$ is a closed and separable subspace of it. In addition $(P_{f_c}(X), h)$ and $(P_{kc}(X), h)$ are closed subspaces of $(P_f(X), h)$ and $(P_k(X), h)$, respectively. A multifunction $F: X \to P_f(X)$ is said to be "Hausdorff continuous" ("h-continuous") if it is continuous from X into the metric space $(P_f(X), h)$. Let Y, Z be Hausdorff topological spaces and let $G: Y \to 2^Z \setminus \{\emptyset\}$. We say that $G(\cdot)$ is upper semicontinuous (usc) (resp. lower semicontinuous (lsc)), if for all $C \subseteq Z$ nonempty closed, $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ resp. $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in Y. When Z is regular, for a closed valued $G(\cdot)$, upper semicontinuity implies that the graph of $G(GrG = \{(y, z) \in Y \times Z : z \in G(y)\})$ is closed in $Y \times Z$ with the product topology. The converse is not true in general. It is true if $G(\cdot)$ is locally compact. If Y, Z are both metric spaces, then the above definition of lower semicontinuity is equivalent to saying that for all $z \in Z$, $y \to d_Z(z, G(y)) = \inf[d_Z(z, v) : v \in G(y)]$ is upper semicontinuous as an $\mathbb{R}_+$-valued function. Also, lower semicontinuity is equivalent to saying that if $y_n \to y$ in Y as $n \to \infty$, then $G(y) \subseteq \underline{\lim}\, G(y_n) = \{z \in Z : \lim d_Z(z, G(y_n)) = 0\} = \{z \in Z : z = \lim z_n,\ z_n \in G(y_n),\ n \geqslant 1\}$. For further details on these and related concepts, we refer to DeBlasi–Myjak [5] and Klein–Thompson [19].

3. EXISTENCE THEOREMS

Let $T = [0, b]$. We consider the following nonlinear, multivalued, second-order boundary value problem

$$\left\{ \begin{array}{l} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ (x'(0), -x'(b)) \in \xi(x(0), x(b)) \end{array} \right\}. \tag{1}$$

Here $F: T \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ and $\xi: \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map. We are looking for solutions of (1) in the Sobolev space $W^{2,1}(T, \mathbb{R}^N)$. We prove existence theorems for both the convex and non-convex problems.

We start with a nonconvex existence theorem. Previously, only Pruszko [26], Frigon–Granas [11], and Kravvaritis–Papageorgiou [22] studied the nonconvex case, but only for the Dirichlet problem and with overall more restrictive hypotheses on the data. There is also the interesting work of Frigon [29] who examined problems with linear boundary conditions and established existence results under slightly weaker conditions on the multifunction $F(t, x, y)$. Our hypotheses on the multifunction $F(t, x, y)$ are the following.

$H(F)_1$. *$F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that*

(i) *$(t, x, y) \to F(t, x, y)$ is graph measurable;*

(ii) *for almost all $t \in T$, $(x, y) \to F(t, x, y)$ is lsc;*

(iii) *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$, and all $v \in F(t, x, y)$*

$$(v, x)_{\mathbb{R}^N} \geqslant -a\,\|x\|^2 - \beta\,\|x\|\,\|y\| - c(t)\,\|x\|$$

with $a, \beta \geqslant 0$ and $c \in L^1(T)_+$;

(iv) *there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$, we have*

$$\inf[(v, x)_{\mathbb{R}^N} + \|y\|^2 : v \in F(t, x, y),\ \|x - x_0\| + \|y - y_0\| < \delta] \geqslant c;$$

(v) *$|F(t, x, y)| = \sup[\|v\| : v \in F(t, x, y)] \leqslant \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\,\|y\|$ a.e. on T, with $\sup_{0 \leqslant r \leqslant k} \gamma_1(t, r) \leqslant \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^1(T)$ and $\sup_{0 \leqslant r \leqslant k} \gamma_2(t, r) \leqslant \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^\infty(T)$.*

Remark. Hypothesis $H(F)_1$ (iii) weakens hypothesis $H(F)_3$ (iv) of Kandilakis–Papageorgiou [17]. Also note that when $F(t, x, y)$ is single-valued and continuous in all three variables, hypothesis $H(F)_1$ (iv) reduces to the well-known “Nagumo–Hartman condition.”

The hypotheses on the multifunction $\xi(\cdot, \cdot)$ are the following.

$H(\xi)_1$. *$\xi: \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0, 0) \in \xi(0, 0)$ such that one of the following holds:* (i) *for every $(a', \beta') \in \xi(a, \beta)$, $(a', a)_{\mathbb{R}^N} \geqslant 0$, $(\beta', \beta)_{\mathbb{R}^N} \geqslant 0$; or* (ii) *$dom(\xi) = \{(a, \beta) \in \mathbb{R}^N \times \mathbb{R}^N : a = \beta\}$.*

THEOREM 1. If *hypotheses $H(F)_1$ and $H(\xi)_1$ hold,* then *problem* (1) *has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$.*

Proof. Let $D = \{x \in W^{2,1}(T, \mathbb{R}^N) : (x'(0), -x'(b)) \in \xi(x(0), x(b))\}$ and let $\hat{L} : D \subseteq L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ be defined by $\hat{L}(x) = -x''$. Then let $L = I + \hat{L}$.

Claim 1. $L: D \subseteq L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ is one to-one.

Suppose $L(x_1) = L(x_2)$. Then $-x_1''(t) + x_1(t) = -x_2''(t) + x_2(t)$ a.e. on T and so $(x_2 - x_1)''(t) = (x_2 - x_1)(t)$ a.e on T. We take the inner product with $(x_2 - x_1)(t)$ and then integrate over T. We have

$$\int_0^b (x_2''(t) - x_1''(t), x_2(t) - x_1(t))_{\mathbb{R}^N} \, dt = \|x_2 - x_1\|_2^2 \geqslant 0. \tag{2}$$

On the other hand for Green's identity we have with $z = x_2 - x_1$,

$$\int_0^b (z''(t), z(t))_{\mathbb{R}^N} \, dt = (z'(b), z(b))_{\mathbb{R}^N} - (z'(0), z(0))_{\mathbb{R}^N} - \|z'\|_2^2. \tag{3}$$

But $(x_1'(0), -x_1'(b)) \in \xi(x_1(0), x_1(b))$ and $(x_2'(0), -x_2'(b)) \in \xi(x_2(0), x_2(b))$. So exploiting the monotonicity of ξ, we have $-(x_2'(b) - x_1'(b), x_2(b) - x_1(b))_{\mathbb{R}^N} + (x_2'(0) - x_1'(0), x_2(0) - x_1(0))_{\mathbb{R}^N} = -(z'(b), z(b))_{\mathbb{R}^N} + (z'(0), z(0))_{\mathbb{R}^N} \geqslant 0$. Using that in (3), we obtain

$$\int_0^b (x_2''(t) - x_1''(t), x_2(t) - x_1(t))_{\mathbb{R}^N} \, dt \leqslant 0. \tag{4}$$

Combining (2) and (4), we conclude that $\|x_1 - x_2\|_2 = 0$, hence $x_1 = x_2$, which proves the claim.

Claim 2. $R(L) = R(I + \hat{L}) = L^1(T, \mathbb{R}^N)$ (i.e., L is onto).

To prove this claim, we need to show for every $h \in L^1(T, \mathbb{R}^N)$, the boundary value problem

$$\left\{ \begin{array}{l} -x''(t) + x(t) = h(t) \text{ a.e. on } T \\ (x'(0), -x'(b)) \in \xi(x(0), x(b)) \end{array} \right\} \tag{5}$$

has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$. If $h \in C(T, \mathbb{R}^N)$, then the existence of a solution for problem (5), follows from the work of Kandilakis–Papageorgiou [17]. In the general case let $h \in L^1(T, \mathbb{R}^N)$ and let $\{h_n\}_{n \geqslant 1} \subseteq C(T, \mathbb{R}^N)$ be such that $h_n \to h$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. Let $x_n \in W^{2,1}(T, \mathbb{R}^N)$ be the unique solution (5) corresponding to h_n. Taking the inner product with $x_n(t)$ and integrating over T, we have

$$\int_0^t (-x_n''(t), x_n(t))_{\mathbb{R}^N} \, dt + \|x_n\|_2^2 \leqslant \|h_n\|_1 \, \|x_n\|_\infty. \tag{6}$$

Apply Green's formula on the first term of (6). We obtain

$$\int_0^b (-x_n''(t), x_n(t))_{\mathbb{R}^N}\, dt$$

$$= -(x_n'(b), x_n(b))_{\mathbb{R}^N} + (x_n'(0), x_n(0))_{\mathbb{R}^N} + \|x_n'\|_2^2 \geqslant \|x_n'\|_2^2 \qquad (7)$$

since $(x_n'(0), -x_n'(b)) \in \xi(x_n(0), x_n(b))$, $n \geqslant 1$, and $(0, 0) \in \xi(0, 0)$. Using (7) in (6), we have

$$\|x_n'\|_2^2 + \|x_n\|_2^2 = \|x_n\|_{1,2}^2 \leqslant \|h_n\|_1 \, \|x_n\|_\infty, \qquad n \geqslant 1, \qquad (8)$$

where by $\|\cdot\|_{1,2}$ we denote the norm of the Sobolev space $W^{1,2}(T, \mathbb{R}^N)$. But recall that $W^{1,2}(T, \mathbb{R}^N)$ is embedded continuously in $C(T, \mathbb{R}^N)$. So we can find $\rho > 0$ such that $\|x_n\|_\infty \leqslant \rho \|x_n\|_{1,2}$ for all $n \geqslant 1$. So from (8) it follows that $\|x_n\|_{1,2} \leqslant \rho \sup_{n \geqslant 1} \|h_n\|_1 < \infty$ for all $n \geqslant 1$. Hence $\{x_n\}_{n \geqslant 1}$ is bounded in $W^{1,2}(T, \mathbb{R}^N)$. From this and Eq. (5), we see at once that $\{x_n''\}_{n \geqslant 1}$ is uniformly integrable. Therefore $\{x_n\}_{n \geqslant 1}$ is bounded in $W^{2,1}(T, \mathbb{R}^N)$. Recall that $W^{2,1}(T, \mathbb{R}^N)$ is embedded compactly in $W^{1,1}(T, \mathbb{R}^N)$. From this, the Dunford–Pettis theorem, and by passing to a subsequence if necessary, we may assume that $x_n'' \xrightarrow{w} g$ in $L^1(T, \mathbb{R}^N)$, $x_n \to x$ in $W^{1,1}(T, \mathbb{R}^N)$, $x_n'(t) \to x'(t)$ a.e. on T, and $x_n(t) \to x(t)$ for all $t \in T$ as $n \to \infty$. From these we infer at once that $x'' = g$ and so $x \in W^{2,1}(T, \mathbb{R}^N)$. Moreover we have that $x_n'(t) \to x'(t)$ for all $t \in T$ as $n \to \infty$. Indeed, we know that $x_n'(t) \to x'(t)$ for all $t \in T \backslash Z$, with Z being a Lebesgue-null subset of T. For $t \in Z$, we can find $\{t_m\}_{m \geqslant 1} \subseteq T \backslash Z$ such that $t_m \to t$ as $m \to \infty$. For every $m \geqslant 1$, we have $x_n'(t_m) \to x'(t_m)$ as $n \to \infty$. Since $x' \in C(T, \mathbb{R}^N)$, we have $x'(t_m) \to x'(t)$ as $m \to \infty$. Invoking Corollary 1.18 [1, p. 37 of Attouch], we can find $n \to m(n)$ increasing (not necessarily strictly) to infinity such that $x_n'(t_{m(n)}) \to x'(t)$ as $n \to \infty$. So we have $\|x_n'(t) - x'(t)\| \leqslant \|x_n'(t) - x_n'(t_{m(n)})\| + \|x_n'(t_{m(n)}) - x'(t)\| \leqslant \int_{t \wedge t_{m(n)}}^{t \vee t_{m(n)}} \|x_n''(s)\|\, ds + \|x_n'(t_{m(n)}) - x'(t)\| \to 0$ as $n \to \infty$ since $\{x_n''\}_{n \geqslant 1}$ is uniformly integrable. Note that $\xi(\cdot, \cdot)$ being maximal monotone, has closed graph. So in the limit as $n \to \infty$, we obtain

$$\left\{ \begin{array}{l} -x''(t) + x(t) = h(t) \text{ a.e. on } T \\ (x'(0), -x'(b)) \in \xi(x(0), x(b)) \end{array} \right\}$$

which proves the claim.

From Claims 1 and 2, it follows that $L^{-1} = (I + \hat{L})^{-1} : L^1(T, \mathbb{R}^N) \to D \subseteq L^1(T, \mathbb{R}^N)$ exists.

Claim 3. $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subseteq W^{1,1}(T, \mathbb{R}^N)$ is compact (i.e., continuous and maps bounded sets in $L^1(T, \mathbb{R}^N)$ into relatively compact sets in $W^{1,1}(T, \mathbb{R}^N)$).

Let $y_n \to y$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$ and let $x_n = L^{-1}(y_n)$, $n \geqslant 1$. Then $x_n \in D$ and $-x_n'' + x_n = y_n$, $n \geqslant 1$. From the proof of Claim 2, we know that $x_n \to x$ in $W^{1,1}(T, \mathbb{R}^N)$ as $n \to \infty$ and $-x''(t) + x(t) = y(t)$ a.e on T, $(x'(0), -x'(b)) \in \xi(x(0), x(b))$. Moreover, it is clear from that argument that L^{-1} maps bounded sets into relatively compact set, hence is a compact operator.

Next let $N : W^{1,1}(T, \mathbb{R}^N) \to 2^{L^1(T, \mathbb{R}^N)}$ be the multivalued Nemitsky operator corresponding to $-F$ and defined by

$$N(x) = \{v \in L^1(T, \mathbb{R}^N) : -v(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T\}$$

The next claim establishes the properties of $N(\cdot)$. Analogous results can be found in Frigon [29], Kandilakis–Papageorgiou [17], and Papageorgiou [23].

Claim 4. $N(\cdot)$ has nonempty, closed, decomposable values and is lsc.

The closedness and decomposability of the values of $N(\cdot)$ are easy to check. For the nonemptiness, note that if $x \in W^{1,1}(T, \mathbb{R}^N)$, then $\phi(t, v) = (t, x(t), x'(t), v)$ is a measurable map from $T \times \mathbb{R}^N$ into $T \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$. Because $F(\cdot, \cdot, \cdot)$ is graph measurable, we have $\phi^{-1}(GrF) \in \mathscr{L}(T) \times B(\mathbb{R}^N)$. But note that $\phi^{-1}(GrF) = GrF(\cdot, x(\cdot), x'(\cdot))$. So we can apply Aumann's selection theorem (see Wagner [28, Theorem 5.10]) and obtain $v : T \to \mathbb{R}^N$ a measurable map such that $-v(t) \in F(t, x(t), x'(t))$ a.e. on T. By virtue of hypothesis $H(F)_1$ (v), $v \in L^1(T, \mathbb{R}^N)$. Therefore for every $x \in W^{1,1}(T, \mathbb{R}^N)$, $N(x) \neq \emptyset$. To check the lower semicontinuity of $N(\cdot)$, we need to show that for every $u \in L^1(T, \mathbb{R}^N)$, $x \to d(u, N(x))$ is an upper semicontinuous $\mathbb{R}_+$-valued function defined on $W^{1,1}(T, \mathbb{R}^N)$. To this end, we have

$$\begin{aligned} d(u, N(x)) &= \inf[\, \|u - v\|_1 : v \in N(x)] \\ &= \inf\left[\int_0^b \|u(t) - v(t)\| \, dt : v \in N(x) \right] \\ &= \int_0^b \inf[\, \|u(t) - v\| : -v \in F(t, x(t), x'(t))] \, dt \\ &\qquad \text{(see Hiai–Umegaki [15, Theorem 2.2])} \\ &= \int_0^b d(u(t), -F(t, x(t), x'(t))) \, dt. \end{aligned}$$

We shall show that for every $\lambda \geqslant 0$, the superlevel set $\{x \in W^{1,1}(T, \mathbb{R}^N) : d(u, N(x)) \geqslant \lambda\} = U_\lambda$ is closed. For this purpose let $\{x_n\}_{n \geqslant 1} \subseteq U_\lambda$ and assume that $x_n \to x$ in $W^{1,1}(T, \mathbb{R}^N)$. By passing to a subsequence if necessary, we may assume that $x_n'(t) \to x'(t)$ and $x_n(t) \to x(t)$ a.e. on T as

$n \to \infty$. By virtue of hypothesis $H(F)_1$ (ii), $(x, y) \to d(u(t), -F(t, x, y))$ is an upper semicontinuous R_+-valued function. So via Fatou's Lemma, we have

$$\lambda \leqslant \overline{\lim}\, d(u, N(x_n)) = \overline{\lim} \int_0^b d(u(t), -F(t, x_n(t), x_n'(t)))\, dt$$
$$\leqslant \int_0^b \overline{\lim}\, d(u(t), -F(t, x_n(t), x_n'(t)))\, dt$$
$$\leqslant \int_0^b d(u(t), -F(t, x(t), x'(t)))\, dt = d(u, N(x)).$$

Therefore $x \in U_\lambda$ and this proves the lower semicontinuity of $N(\cdot)$.

Claim 4 allows us to apply Theorem 3 of Bressan–Colombo [3] and obtain $g : W^{1,1}(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ a continuous map such that $g(x) \in N(x)$ for every $x \in W^{1,1}(T, \mathbb{R}^N)$. Let $g_1(x) = g(x) + x$. Then it is clear that to finish our proof, we need to solve the following fixed point problem:

$$x = L^{-1} g_1(x). \tag{9}$$

Clearly a solution of (9) also solves problem (1). To produce $x \in D$ which solves (9) we shall use the Leray–Schauder alternative theorem.

By virtue of Claim 3 and the continuity of $g_1(\cdot)$, $L^{-1} g_1 : W^{1,1}(T, \mathbb{R}^N) \to D \subseteq W^{1,1}(T, \mathbb{R}^N)$ is a compact map. Next we shall show that the set $\Gamma = \{x \in D : x = \lambda L^{-1} g_1(x),\ 0 < \lambda < 1\}$ is bounded in $W^{1,1}(T, \mathbb{R}^N)$. To this end let $x \in \Gamma$. For some $0 < \lambda < 1$, we have

$$L\left(\frac{1}{\lambda} x\right) = g_1(x)$$
$$\Rightarrow \quad -x''(t) = \lambda g(x)(t) + (\lambda - 1)\, x(t) \text{ a.e. on } T$$
$$\left(\frac{1}{\lambda} x'(0), -\frac{1}{\lambda} x'(b)\right) \in \xi\left(\frac{1}{\lambda} x(0), \frac{1}{\lambda} x(b)\right).$$

We take the inner product with $x(t)$ and then integrate over T. So we have:

$$\int_0^b (-x''(t), x(t))_{\mathbb{R}^N}\, dt = \lambda \int_0^b (g(x)(t), x(t))_{\mathbb{R}^N}\, dt + (\lambda - 1)\, \|x\|_2^2. \tag{10}$$

Once again from Green's identity and the boundary conditions, we have

$$\|x'\|_2^2 \leqslant \int_0^b (-x''(t), x(t))_{\mathbb{R}^N}\, dt. \tag{11}$$

Also from Hypothesis $H(F)_1$(iii), we have

$$\lambda \int_0^b (g(t), x(t))_{\mathbb{R}^N}\, dt \leqslant \lambda a\, \|x\|_2^2 + \lambda\beta\, \|x\|_2\, \|x'\|_2 + \lambda\, \|c\|_1\, \|x\|_\infty. \quad (12)$$

Claim 5. For all $x \in D$ such that $x = \lambda L^{-1} g(x)$ for some $0 < \lambda < 1$, we have $\|x\|_\infty \leqslant M$, where $M > 0$ is as in hypothesis $H(F)_1$ (iv).

Let $r(t) = \|x(t)\|^2$. Let $t_0 \in T$ be the point where $r(\cdot)$ attains its maximum and suppose that $r(t_0) > M^2$. First assume that $0 < t_0 < b$. Then $r'(t_0) = 2(x'(t_0), x(t_0))_{\mathbb{R}^N} = 0$ and so by virtue of Hypothesis $H(F)_1$ (iv), there exist $c, \delta > 0$ such that for almost all $t \in T$,

$$\inf[(v, x)_{\mathbb{R}^N} + \|y\|^2 : v \in F(t, x, y),\ \|x(t_0) - x\| + \|x'(t_0) - y\| < \delta] \geqslant c.$$

Since $x \in C^1(T, \mathbb{R}^N)$, we can find $\delta_1 > 0$ such that if $t \in [t_0, t_0 + \delta_1]$, we have

$$\|x(t_0) - x(t)\| + \|x'(t_0) - x'(t)\| < \delta.$$

Also $-g(x)(t) \in F(t, x(t), x'(t))$ a.e. on T. So for almost all $t \in (t_0, t_0 + \delta_1]$, we have

$$\lambda(-g(x)(t), x(t))_{\mathbb{R}^N} + \lambda\, \|x'(t)\|^2 \geqslant \lambda c$$

$$\Rightarrow \lambda(x''(t), x(t))_{\mathbb{R}^N} + (\lambda - 1)\, \|x(t)\|^2 + \lambda\, \|x'(t)\|^2 \geqslant \lambda c$$

$$\Rightarrow \lambda \int_{t_0}^t (x''(s), x(s))_{\mathbb{R}^N}\, ds + \lambda \int_{t_0}^t \|x'(s)\|^2\, ds$$

$$\geqslant \lambda c(t - t_0),\ t \in (t_0, t_0 + \delta_1] \qquad (\text{since } 0 < \lambda < 1).$$

Using Green's identity on the first integral, we obtain

$$\lambda \int_{t_0}^t (x''(s), x(s))_{\mathbb{R}^N}\, ds$$

$$= \lambda(x'(t), x(t))_{\mathbb{R}^N} - \lambda(x'(t_0), x(t_0))_{\mathbb{R}^N} - \lambda \int_{t_0}^t \|x'(s)\|^2\, ds$$

$$= \lambda(x'(t), x(t))_{\mathbb{R}^N} - \lambda \int_{t_0}^t \|x'(s)\|^2\, ds.$$

Therefore for all $t \in (t_0, t_0+\delta_1]$, we have

$$\lambda(x'(t), x(t))_{\mathbb{R}^N} \geqslant \lambda c(t-t_0) > 0$$
$$\Rightarrow r'(t) > 0$$
$$\Rightarrow r(t) > r(t_0),$$

which contradicts the choice of t_0. So $\|x(t_0)\| \leqslant M$.

Now assume that $t_0 = 0$. First assume that $H(\xi)_1$ (i) holds. Then $(x'(0), x(0))_{\mathbb{R}^N} \geqslant 0$. But on the other hand, since $t_0 = 0$ is a maximizer of $r(t)$ on T, we have $2(x'(0), x(0))_{\mathbb{R}^N} = r'(0) \leqslant 0$. Therefore $(x'(0), x(0))_{\mathbb{R}^N} = 0$ and so the previous argument applies. Next, if $H(\xi)_1$ (ii) holds, we have $r(0) = r(b)$ and so $r'(0) \leqslant 0$, $r'(b) \geqslant 0$. But since $(x'(0), -x'(b)) \in \xi(x(0), x(b))$ and $(0, 0) \in \xi(0, 0)$, we also have $(x'(b), x(b))_{\mathbb{R}^N} \leqslant (x'(0), x(0))_{\mathbb{R}^N} \Rightarrow 0 \leqslant r'(b) \leqslant r'(0) \leqslant 0 \Rightarrow r'(0) = 0$ and again the previous argument applies. Similarly we treat the case $t_0 = b$. So $\|x(t_0)\| \leqslant M$ and this proves the claim.

Next using (11), (12), and Claim 5, in (10), we obtain (recall $0 < \lambda < 1$)

$$\|x'\|_2^2 \leqslant \lambda a \|x\|_2^2 + \lambda\beta \|x\|_2 \|x'\|_2 + \lambda \|c\|_1 \|x\|_\infty$$
$$\leqslant aM^2 b + \beta M^{1/2} b^{1/2} \|x'\|_2 + M \|c\|_1,$$

from which we infer that there exists $M_3 > 0$ such that for all $x \in \Gamma$, we have $\|x'\|_2 \leqslant M_3$. This proves the boundedness of Γ in $W^{1,1}(T, \mathbb{R}^N)$. Invoking the Leray–Schauder alternative theorem, we obtain $x = L^{-1}g(x)$. Evidently $x \in W^{2,1}(T, \mathbb{R}^N)$ is a solution of problem (1). ∎

Next we present the "convex" result. For this, we shall need the following hypotheses on F.

$H(F)_2$. *$F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that*

(i) *for every $x, y \in \mathbb{R}^N$, $t \to F(t, x, y)$ admits a measurable selector;*

(ii) *for almost all $t \in T$, $(x, y) \to F(t, x, y)$ has a closed graph;*

(iii) *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$ we have*

$$(v, x) \geqslant -a \|x\|^2 - \beta \|x\| \|y\| - c(t) \|x\|$$

with $a, \beta \geqslant 0$ and $c \in L^1(T)_+$;

(iv) *there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$, we have*

$$\inf[(v, x) + \|y\|^2 : v \in f(t, x, y),\ \|x - x_0\| + \|y - y_0\| < \delta] \geqslant c;$$

(v) $|F(t, x, y)| = \sup[\, \|v\| : v \in F(t, x, y)] \leqslant \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\, \|y\|$ *a.e. on* T, *with* $\sup_{0 \leqslant r \leqslant k} \gamma_1(t, r) \leqslant \eta_{1,k}(t)$, $\eta_{1,k} \in L^1(T)$ *and* $\sup_{0 \leqslant r \leqslant k} \gamma_2(t, r) \leqslant \eta_{2,k}(t)$ *a.e. on* T, $\eta_{2,k} \in L^\infty(T)$.

Remark. Hypothesis $H(F)_2$(i) is satisfied, if for example for all $(x, y) \in \mathbb{R}^N$ $GrF(\cdot, x, y) \in \mathscr{L}(T) \times B(\mathbb{R}^N)$. Then the measurable selector is provided by Aumann's selection theorem.

In this case, because of Hypothesis $H(F)_2$ (ii), in general we cannot pass to a single-valued problem via a continuous selector. Therefore we need the multivalued version of the Leray–Schauder alternative theorem. For the convenience of the reader we recall it here:

THEOREM 2. If *X is a Banach space*, $C \subseteq X$ *is nonempty*, *closed*, *convex with* $0 \in C$ *and* $G : C \to P_{kc}(C)$ *is an usc multifunction which maps bounded sets into relatively compact sets*, then *one of the following two statements is true*:

(a) *The set* $\Gamma = \{x \in C : x \in \lambda G(x)$ *for some* $0 < \lambda < 1\}$ *is unbounded*; *or*

(b) $G(\cdot)$ *has a fixed point* (*i.e.*, *there exists* $x \in C$ *such that* $x \in G(x)$).

Since in the above theorem $G(\cdot)$ is required to have convex values, we need the following more restrictive hypotheses on $\xi(\cdot, \cdot)$.

$H(\xi)_2$. $\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ *is a maximal monotone map with convex graph*, $(0, 0) \in \xi(0, 0)$ *and such that one of the following conditions holds*: (i) *for all* $(a', \beta') \in \xi(a, \beta)$, $(a', a)_{\mathbb{R}^N} \geqslant 0$, $(\beta', \beta)_{\mathbb{R}^N} \geqslant 0$; *or* (ii) $dom(\xi) = \{(a, \beta) \in \mathbb{R}^N \times \mathbb{R}^N : a = \beta\}$.

THEOREM 3. If *Hypotheses* $H(F)_2$ *and* $H(\xi)_2$ *hold*, then *problem* (1) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Proof. The idea of the proof of this theorem is the same as that in Theorem 1, so we present only those particular points where the two proofs differ.

In this case the multivalued Nemitsky operator $N : W^{1,1}(T, \mathbb{R}^N) \to 2^{L^1(T, \mathbb{R}^N)}$ has nonempty closed, and convex values in $L^1(T, \mathbb{R}^N)$ and is usc from $W^{1,1}(T, \mathbb{R}^N)$ into $L^1(T, \mathbb{R}^N)$ furnished with the weak topology (denoted by $L^1(T, \mathbb{R}^N)_w$). The closedness and convexity of the values of $N(\cdot)$ are clear. To see the nonemptiness, we proceed as follows. Let $x \in W^{1,1}(T, \mathbb{R}^N)$ and let $\{s_n\}_{n \geqslant 1}$, $\{r_n\}_{n \geqslant 1}$ be two sequences of step functions such that $s_n(t) \to x(t)$, $r_n(t) \to x'(t)$, $\|s_n(t)\| \leqslant \|x(t)\|$ and $\|r_n(t)\| \leqslant \|x'(t)\|$ a.e. on T. Then by virtue of Hypothesis $H(F)_2$, for every $n \geqslant 1$ $t \to F(t, s_n(t), r_n(t))$ admits a measurable selector $f_n(t)$. From Hypothesis $H(F)_2$ (v), it follows that $\{f_n\}_{n \geqslant 1}$ is uniformly integrable. So by the

Dunford–Pettis theorem and by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. Then from Theorem 3.1 of Papageorgiou [23], we have

$$f(t) \in \overline{\text{conv}}\, \overline{\lim}\{f_n(t)\}_{n \geqslant 1} \subseteq \overline{\text{conv}}\, \overline{\lim}\, F(t, s_n(t), r_n(t)) \\ \subseteq F(t, x(t), x'(t)) \qquad \text{a.e. on } T,$$

the last inclusion being a consequence of Hypothesis $H(F)_2$ (ii). So $f \in N(x)$ and this proves the nonemptiness of the values of $N(\cdot)$.

Next we shall show that $N(\cdot)$ is usc from $W^{1,1}(T, \mathbb{R}^N)$ into $L^1(T, \mathbb{R}^N)_w$ (Again analogous results can be found in Frigon [29], Granas–Guenther–Lee [30], Kandilakis–Papageorgiou [17], and Papageorgiou [23].) To this end let C be a nonempty and weakly closed subset of $L^1(T, \mathbb{R}^N)$. We need to show that $N^-(C) = \{x \in W^{1,1}(T, \mathbb{R}^N) : N(x) \cap C \neq \varnothing\}$ is closed. To this let $\{x_n\}_{n \geqslant 1} \subseteq N^{-1}(C)$ and assume that $x_n \to x$ in $W^{1,1}(T, \mathbb{R}^N)$ as $n \to \infty$. At least for a subsequence we can have that $x_n'(t) \to x'(t)$ a.e. on T and $x_n(t) \to x(t)$ for all $t \in T$ as $n \to \infty$. Let $f_n \in N(x_n) \cap C$, $n \geqslant 1$. Then by virtue of hypothesis $H(F)_2$ (v) and the Dunford–Pettis theorem, we may assume that $f_n \xrightarrow{w} f \in C$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. As before we have

$$f(t) \in \overline{\text{conv}}\, \overline{\lim}\{f_n(t)\}_{n \geqslant 1} \subseteq \overline{\text{conv}}\, \overline{\lim}\, F(t, x_n(t), x_n'(t)) \\ \subseteq F(t, x(t), x'(t)) \qquad \text{a.e. on } T$$

$$\Rightarrow f \in N(x) \cap C;$$

(i.e., $N^-(C)$ is closed in $W^{1,1}(T, \mathbb{R}^N)$.

This proves the upper semicontinuity of $N(\cdot)$ from $W^{1,1}(T, \mathbb{R}^N)$ into $L^1(T, \mathbb{R}^N)_w$.

We consider the following fixed point problem, which is equivalent to problem (1):

$$x \in L^{-1} N_1(x), \tag{13}$$

where $N_1(x) = N(x) + x$. Recalling that $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subseteq W^{1,1}(T, \mathbb{R}^N)$ is compact (see the proof of Theorem 1), we see to $L^{-1}N_1 : W^{1,1}(T, \mathbb{R}^N) \to P_{kc}(W^{1,1}(T, \mathbb{R}^N))$ is usc and maps bounded sets into relatively compact sets. Moreover, because of the convexity of the graph of ξ (see Hypothesis $H(\xi)_2$), we easily check that $L^{-1}N_1$ has convex values. Set $\Gamma_1 = \{x \in D \subseteq W^{1,1}(T, \mathbb{R}^N) : x \in \lambda L^{-1} N_1(x)$ for some $0 < \lambda < 1\}$. Then arguing as in the proof of Theorem 1, we can show that Γ_1 is bounded. Invoking Theorem 2, we infer that the fixed problem (13) has a solution $x \in D \subseteq W^{2,1}(T, \mathbb{R}^N)$. Evidently this is a solution of (1).

Let S denote the solution set of (1) From the a priori bounds obtained in the proof of Theorem 1, we know that S is bounded in $W^{2,1}(T, \mathbb{R}^N)$.

Then if $\{x_n\}_{n\geqslant 1} \subseteq S$, we, have that $\{x''\}_{n\geqslant 1}$ is uniformly integrable and $\{x_n\}_{n\geqslant 1}$ is relatively compact in $W^{1,1}(T, \mathbb{R}^N)$ (recall that $W^{2,1}(T, \mathbb{R}^N)$ is embedded compactly in $W^{1,1}(T, \mathbb{R}^N)$). So as in the proof of Theorem 1, we can show that $x_n \xrightarrow{w} x$ in $W^{2,1}(T, \mathbb{R}^N)$ and via Theorem 3.1 of [23] we can check that $x \in S$. Therefore we conclude that S is weakly compact in $W^{2,1}(T, \mathbb{R}^N)$. ▮

If we strengthen the unilateral growth condition $H(F)_1$ (iii) (resp. $H(F)_2$ (iii), we can drop Hypothesis $H(F)_1$ (iv) (resp. $H(F)_2$ (iv) (the Nagumo–Hartman condition) and also conditions (i) and (ii) from hypothesis $H(\xi)_1$ (resp. $H(\xi)_2$). So for the nonconvex problem our hypotheses on F are the following:

$H(F)_3$. *$F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_f(\mathbb{R}^N)$ is a multifunction which satisfies hypotheses* $H(F)_1$ (i), (ii), (v), *and*

(iii)′ *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$, we have*

$$(v, x)_{\mathbb{R}^N} \geqslant a \, \|x\|^2 - \beta \, \|x\| \, \|y\| - c(t) \, \|x\|$$

with $a, \beta \geqslant 0$, $c \in L^1(T)$ and $\beta < \min\{a, 1\}$.

The hypotheses on $\xi(\cdot, \cdot)$ are now the following:

$H(\xi)_3$. *$\xi: \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0, 0) \in \xi(0, 0)$.*

THEOREM 4. If *hypotheses $H(F)_3$ and $H(\xi)_3$ hold*, then *problem* (1) *has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$.*

Proof. Again we are led to the fixed point problem $x = L^{-1} g_1(x)$ (see the proof of Theorem 1). To apply the Leray–Schauder alternative theorem, we need to show that Γ is bounded in $W^{1,1}(T, \mathbb{R}^N)$. To this end, for $x \in \Gamma$ we have

$$\begin{cases} -x''(t) = \lambda g(x)(t) + (\lambda - 1)\, x(t) \text{ a.e. on } T \\ \left(\frac{1}{\lambda} x'(0), -\frac{1}{\lambda} x'(b)\right) \in \xi\left(\frac{1}{\lambda} x(0), \frac{1}{\lambda} x(b)\right), \qquad 0 < \lambda < 1 \end{cases}$$

Take the inner product with $x(t)$ and then integrate over T. Using Green's identity and Hypothesis $H(\xi)_3$, we obtain

$$\|x'\|_2^2 \leqslant \int_0^b (-x''(t), x(t))_{\mathbb{R}^N}\, dt \leqslant \lambda \int_0^b (g(x)(t), x(t))_{\mathbb{R}^N}\, dt$$

$$\Rightarrow \|x'\|_2^2 \leqslant \lambda\beta \|x\|_2 \|x'\|_2 + \lambda \|c\|_1 \|x\|_\infty - \lambda a \|x\|_2^2 \quad \text{(hypothesis } H(F)_3 \text{ (iii)}')$$

$$\Rightarrow \lambda a \|x\|_2^2 + \|x'\|_2^2 \leqslant \lambda\beta \|x\|_2 \|x'\|_2 + \lambda \|c\|_1 \|x\|_\infty.$$

Let $\gamma = \min\{a, 1\}$. Then we have

$$\lambda\gamma \|x\|_{1,2}^2 \leqslant \lambda\beta \|x\|_{1,2}^2 + \lambda\theta \|c\|_1 \|x\|_{1,2}$$

where $\theta > 0$ is such that $\|\cdot\|_\infty \leqslant \theta \|\cdot\|_{1,2}$ (it exists since $W^{2,1}(T, \mathbb{R}^N)$ is embedded continuously in $C(T, \mathbb{R}^N)$). So we have

$$(\gamma - \beta) \|x\|_{1,2} \leqslant \theta \|c\|_1$$

$$\Rightarrow \|x\|_{1,2} \leqslant M_4 \qquad \text{for all} \quad x \in \Gamma$$

$$\text{(recall that by Hypothesis } H(F_3) \text{ (iii)}', \beta < \gamma).$$

Therefore Γ is bounded in $W^{1,1}(T, \mathbb{R}^N)$ (in fact in $W^{2,1}(T, \mathbb{R}^N)$) and so by the Leray–Schauder alternative theorem there exists $x \in D \subseteq W^{2,1}(T, \mathbb{R}^N)$ such that $x = L^{-1} g_1(x)$. Clearly $x \in W^{2,1}(T, \mathbb{R}^N)$ solves problem (1). ∎

Similarly we can have an alternative version of the convex existence result (Theorem 3). In this case the hypotheses on F are the following:

$H(F)_4$. *$F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction which satisfies Hypotheses* $H(F)_2$ (i), (ii), (v), *and*

(iii)′ *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$, we have*

$$(v, x)_{\mathbb{R}^N} \geqslant a \|x\|^2 - \beta \|x\| \|y\| - c(t) \|x\|$$

with $a, \beta \geqslant 0$, $c \in L^1(T)_+$ and $\beta < \min\{a, 1\}$.

Also the hypotheses on $\xi(\cdot, \cdot)$ are the following:

$H(\xi)_4$. *$\xi: \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with convex graph and* $(0, 0) \in \xi(0, 0)$.

THEOREM 5. If *Hypotheses $H(F)_4$ and $H(\xi)_4$ hold*, then *problem* (1) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Thus far we have always assumed that $(0, 0) \in \xi(0, 0)$. Now we shall remove this condition so that we can incorporate in our formulation problems with nonhomogeneous boundary conditions. Now we assume only that $(0, 0) \in dom(\xi)$. Then, in order to analyze this case, we proceed as follows: Let $(v, w) \in \xi(0,0)$ and let $k \in C^2(T, \mathbb{R}^N)$ be such that $k(0) = k(b) = 0$ and $k'(0) = v$, $k'(b) = -w$. To see how we can generate a function, let $\phi_1 \in C_0^2(\mathbb{R})$ with $\sup \phi_1 \subseteq (-b/3, b/3)$, $\phi_1(0) = 0$, $\phi_1'(0) = 1$. Also let $\phi_2 \in C_0^2(\mathbb{R})$ with $\sup \phi_2 \subseteq (2b/3, 4b/3)$, $\phi_2(b) = 0$, $\phi_2'(b) = -1$. Define $k(t) = \phi_1(t)\, v$ for $t \in [0, b/3)$, $k(t) = 0$ for $t \in [b/3, 2b/3]$ and $k(t) = \phi_2(t)\, w$ for $t \in (2b/3, b]$. Evidently $k \in C^2(T, \mathbb{R}^N)$ and has the desired properties. We fix such a $k \in C^2(T, \mathbb{R}^N)$. Using such a k, we modify appropriately the hypotheses on the data. First for the nonconvex problem:

$H(F)_5$. *$F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction which satisfies* $H(F)_1$ (i), (ii), (v), *and*

(iii)′ *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y) - k''(t)$, we have*

$$(v, x - k(t))_{\mathbb{R}^N} \geqslant -a\, \|x - k(t)\|^2 - \beta\, \|x - k(t)\|\, \|y - k'(t)\| - c(t)\, \|x - k(t)\|$$

with $a, \beta \geqslant 0$, and $c \in L^1(T)$;

(iv)′ *there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$, we have*

$$\inf[(v, x - k(t))_{\mathbb{R}^N} + \|y - k'(t)\|^2 : v \in F(t, x, y) - k''(t),$$
$$\|x - (x_0 + k(t))\| + \|y - (y_0 + k'(t))\| \leqslant \delta] \geqslant c.$$

The hypotheses on $\xi(\cdot, \cdot)$ take the following form:

$H(\xi)_5$. *$\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map such that one of the following holds:*

(i) *$(0, 0) \in dom(\xi)$ and for all $(a', \beta') \in \xi(a, \beta)$, we have $(a' - v, a)_{\mathbb{R}^N} \geqslant 0$, $(\beta' - w, \beta)_{\mathbb{R}^N} \geqslant 0$; or*

(ii) *$dom(\xi) = \{(a, \beta) \in \mathbb{R}^N \times \mathbb{R}^N : a = \beta\}$.*

THEOREM 6. If *Hypotheses $H(F)_5$ and $H(\xi)_5$ hold*, then *problem* (1) *has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$.*

Proof. Let $\xi_1 : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ be defined by $\xi_1(a, b) = \xi(a, \beta) - (v, w)$. Evidently $\xi_1(\cdot, \cdot)$ is maximal monotone, $dom(\xi_1) = dom(\xi)$ and $(0, 0) \in \xi_1(0, 0)$. Also let $F_1 : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ be defined by $F_1(t, x, y) = F(t, x + k(t), y + k'(t)) - k''(t)$. We consider the following nonlinear multivalued boundary value problem:

$$\left\{ \begin{array}{l} x''(t) \in F_1(t, x(t), x'(t)) \text{ a.e. on } T \\ (x'(0), -x'(b)) \in \xi_1(x(0), x(b)) \end{array} \right\}. \tag{14}$$

Note that from the definition of F_1 and Hypotheses $H(F)_5$ (i), (ii), and (v), we have that $(t, x, y) \to F_1(t, x, y)$ is graph measurable, for almost all $t \in T$, $(x, y) \to F_1(t, x, y)$ is lsc and for almost all $t \in T$ and all $x, y \in \mathbb{R}^N$, we have $|F_1(t, x, y)| = \sup[\|v\| : v \in F_1(t, x, y)] \leqslant \hat{\gamma}_1(t, \|x\|) + \hat{\gamma}_2(t, \|x\|)\, \|y\|$ with $\sup_{0 \leqslant r \leqslant k} \hat{\gamma}_1(t, r) \leqslant \hat{\eta}_{1,k}(t)$ a.e. on T, $\hat{\eta}_{1,k} \in L^1(T)$ and $\sup_{0 \leqslant r \leqslant k} \hat{\gamma}_2(t, r) \leqslant \hat{\eta}_{2,k}(t)$ a.e. on T, $\hat{\eta}_{2,k} \in L^\infty(T)$.

For almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F_1(t, x, y)$, by virtue of hypothesis $H(F)_5$ (iii)′, we have

$$(v, x)_{\mathbb{R}^N} \geqslant -a\, \|x\|^2 - \beta\, \|x\|\, \|y\| - c(t)\, \|x\|.$$

Finally, if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, by virtue of Hypothesis $H(F)_5$ (iv)′, we have

$$\begin{aligned}
&\inf[(v, x)_{\mathbb{R}^N} + \|y\|^2 : v \in F_1(t, x, y),\ \|x - x_0\| + \|y - y_0\| < \delta] \\
&\quad = \inf[(v, x)_{\mathbb{R}^N} + \|y\|^2 : v \in F(t, x + k(t), y + k'(t)) - k''(t), \\
&\qquad \|x - x_0\| + \|y - y_0\| < \delta] \geqslant c.
\end{aligned}$$

So we have checked that $F_1(t, x, y)$ satisfies Hypotheses $H(F)_1$. Similarly by its definition and because of Hypotheses $H(\xi)_5$, $\xi_1(a, \beta)$ satisfies Hypotheses $H(\xi)_1$. So we can apply Theorem 1 and obtain $x \in W^{2,1}(T, \mathbb{R}^N)$ which solves problem (14). It is easy then to check that $(x + k) \in W^{2,1}(T, \mathbb{R}^N)$ solves (1). ∎

Similarly we can have the other nonconvex existence result, in which the unilateral growth condition is strengthened, but we drop the Nagumo–Hartman condition and the alternative extra conditions on ξ are also removed. More precisely the hypotheses on F and ξ, are the following:

$H(F)_6$. *$F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction which satisfies Hypotheses $H(F)_1$ (i), (ii), (v), and*

(iii)′ *for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y) - k''(t)$*

$$(v, x - k(t))_{\mathbb{R}^N} \geqslant a\, \|x - k(t)\|^2 - \beta\, \|x - k(t)\|\, \|y - k'(t)\| - c(t)\, \|x - k(t)\|$$

with $a, \beta \geqslant 0$, and $c \in L^1(T)$ and $\beta < \min\{a, 1\}$.

$H(\xi)_6$. *$\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0, 0) \in dom(\xi)$.*

THEOREM 7. If *Hypotheses $H(F)_6$ and $H(\xi)_6$ hold*, then *problem* (1) *has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$.*

In an analogous manner, we can have the corresponding "convex" results. For the first, the hypotheses on F are the following:

$H(F)_7$. $F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ *is a multifunction which satisfies Hypotheses* $H(F)_2$ (i), (ii), (v) *and hypotheses* $H(F)_5$ (iii)$'$ *and* (iv)$'$.

THEOREM 8. If *Hypotheses* $H(F)_7$ *and* $H(\xi)_5$ *hold*, then *problem* (1) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

For the second result, the hypotheses on F are the following:

$H(F)_8$. $F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ *is a multifunction which satisfies Hypotheses* $H(F)_2$ (i), (ii), (v) *and Hypothesis* $H(F)_6$ (iii)$'$.

THEOREM 9. If *Hypotheses* $H(F)_8$ *and* $H(\xi)_6$ *hold*, then *problem* (1) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Now we will present some special cases of interest, which are incorporated in our general formulation.

Case a. Suppose $K_1, K_2 \in P_{fc}(\mathbb{R}^N)$ with $0 \in K_1 \cap K_2$ and consider the following multivalued boundary value problem:

$$\left.\begin{cases} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) \in K_1, \qquad x(b) \in K_2 \\ (x'(0), x(0))_{\mathbb{R}^N} = \sigma(x'(0), K_1), \qquad (-x'(b), x(b)) = \sigma(-x'(b), K_2) \end{cases}\right\}. \tag{15}$$

Here for every $v \in \mathbb{R}^N$ and $K \in P_{fc}(\mathbb{R}^N)$, $\sigma(v, K) = \sup[(v, k)_{\mathbb{R}^N} : k \in K]$. Also in what follows for every $k \in K$, by $N_K(k)$ we denote the normal cone to K at the point k; i.e., $N_K(k) = \{v \in \mathbb{R}^N (v, k)_{\mathbb{R}^N} = \sigma(v, K)\}$. Recall that $N_K(k) = T_K(k)^- =$ the negative polar cone of the tangent cone $T_K(k)$. Finally by $\delta_K(\cdot)$ we denote the indicator function of K; i.e.,

$$\delta_K = \begin{cases} 0 & \text{if } \ x \in K; \\ +\infty & \text{if } \ x \notin K. \end{cases}$$

For problem (15) let $\xi(a, \beta) = \delta_{K_1 \times K_2}(a, \beta) = N_{K_1 \times K_2}(a, \beta) = N_{K_1}(a) \times N_{K_2}(\beta)$. Note that if $(a', \beta') \in \xi(a, \beta)$, then we have $(a', a)_{\mathbb{R}^N} = \sigma(a', K_1) \geqslant 0$ and $(\beta', \beta)_{\mathbb{R}^N} = \sigma(\beta', K_2) \geqslant 0$. Moreover if both K_1 and K_2 are closed convex cones, then $\xi(a, \beta) = K_1^- \times K_2^-$ and thus $Gr\xi$ is convex. In addition in this case we have $(x'(0), x(0))_{\mathbb{R}^N} = 0$ and $(-x'(\beta), x(\beta))_{\mathbb{R}^N} = 0$. So when

K_1, K_2 are closed convex cones in $\mathbb{R}^N$, problem (15) takes the following form:

$$\left\{\begin{array}{ll} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T & \\ x(0) \in K_1, & x(b) \in K_2 \\ (x'(0), x(0))_{\mathbb{R}^N} = 0, & (x'(b), x(b))_{\mathbb{R}^N} = 0 \end{array}\right\}. \tag{16}$$

Then the previous theorems give us the following corollaries:

Corollary 10. If *Hypotheses* $H(F)_1$ *or* $H(F)_3$ *hold and* $K_1, K_2 \in P_{fc}(\mathbb{R}^N)$ *with* $0 \in K_1 \cap K_2$, then *problem* (15) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

Corollary 11. If *Hypotheses* $H(F)_2$ *or* $H(F)_4$ *hold and* K_1, K_2 *are proper closed, convex cones in* $\mathbb{R}^N$, then *problem* (16) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Remark. With the exception of the work of Kandilakis–Papageorgiou [17], none of the other papers mentioned in the introduction can accommodate problems with set theoretic boundary conditions.

Case b. Consider the classical Dirichlet problem:

$$\left\{\begin{array}{l} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) = x(b) = 0 \end{array}\right\}. \tag{17}$$

For this case $K_1 = K_2 = \{0\}$ and $\xi = \partial\delta_{K_1 \times K_2} = N_{K_1} \times N_{K_2} = \mathbb{R}^N \times \mathbb{R}^N$. So there are no constraints on $x'(0)$ and $x'(b)$. Therefore we can state the following corollaries:

Corollary 12. If *Hypotheses* $H(F)_1$ *or* $H(F)_3$ *hold*, then *problem* (17) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

Corollary 13. If *Hypotheses* $H(F)_2$ *or* $H(F)_4$ *hold*, then *problem* (17) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Case c. Consider the classical Neumann problem:

$$\left\{\begin{array}{l} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x'(0) = x'(b) = 0 \end{array}\right\}. \tag{18}$$

For this problem, we take $K_1 = K_2 = \mathbb{R}^N$ and $\xi = \partial\delta_{K_1 \times K_2} = N_{K_1} \times N_{K_2} = \{(0, 0)\}$. So we have the following corollaries:

Corollary 14. If *Hypotheses* $H(F)_1$ *or* $H(F)_3$ *hold*, then *problem* (18) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

Corollary 15. If *Hypotheses $H(F)_2$ or $H(F)_4$ hold*, then *problem* (18) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Remark We can also treat the nonhomogeneous Dirichlet problem with $x'(0) = x'(b) = v$. In this case $K_1 = K_2 = \mathbb{R}^N$ and $\xi = \partial\delta_{K_1 \times K_2} + (v, -v) = (v, -v)$. Then we use Hypotheses $H(F)_5$ or $H(F)_6$ (nonconvex problem) and $H(F)_7$ or $H(F)_8$ (convex problem), with $w = -v$ and $k \in C^2(T, \mathbb{R}^N)$ as before.

Case d. Consider the periodic problem:

$$\left\{\begin{array}{ll} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) = x(b), & x'(0) = x'(b) \end{array}\right\}. \tag{19}$$

In this case let $K = \{(x, y) \in \mathbb{R}^N : x = y\}$. Set $\xi = \partial\delta_K = K^\perp = \{(v, w) \in \mathbb{R}^N \times \mathbb{R}^N : v = -w\}$. Then the following corollaries hold:

Corollary 16. If *Hypotheses $H(F)_1$ or $H(F)_3$ hold*, then *problem* (19) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

Corollary 17. If *Hypotheses $H(F)_2$ or $H(F)_4$ hold*, then *problem* (19) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

We should point out that for Cases (b), (c), and (d) above, when Hypotheses $H(F)_1$ or $H(F)_2$ are in effect, existence results with slightly more general conditions can be found in Frigon [29].

Case e. Let $g_1, g_2 : \mathbb{R}^N \to \mathbb{R}^N$ be two nonexpansive maps such that $g_1(0) = g_2(0) = 0$. Consider the following boundary value problem:

$$\left\{\begin{array}{ll} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x'(0) = x(0) + g_1(x(0)), & -x'(b) = x(b) + g_1(x(b)) \end{array}\right\}. \tag{20}$$

Recall that $I + g_1$ and $I + g_2$ are maximal monotone maps on $\mathbb{R}^N$ and if $\xi = (I + g_1, I + g_2)$, then $\xi(\cdot, \cdot)$ is a maximal monotone map on $\mathbb{R}^N \times \mathbb{R}^N$, with $(0, 0) = \xi(0, 0)$. Note that if $(a', \beta') = \xi(a, \beta)$, then $(a', a)_{\mathbb{R}^N} = (a + g_1(a), a)_{\mathbb{R}^N} \geqslant \|a\|^2 - \|g_1(a)\| \, \|a\| \geqslant \|a\|^2 = 0$. Similarly $(\beta', \beta)_{\mathbb{R}^N} \geqslant 0$. So we can state the following corollary:

Corollary 18. If *Hypotheses $H(F)_3$ or $H(F)_5$ hold*, then *problem* (20) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

Case f. Our formulation also incorporates vector-valued Strum–Liouville boundary value problems:

$$\left\{\begin{array}{ll} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ Ax(0) - Bx'(0) = 0, & Cx(b) + Dx'(b) = 0. \end{array}\right\}. \tag{21}$$

Here A, B, C, $-D$ are nonnegative definite $N \times N$-matrices. We assume that B, D are invertible and furthermore that $B^{-1}A = AB^{-1}$ and $D^{-1}C = CD^{-1}$. We set $\xi(a, \beta) = (B^{-1}Aa, -D^{-1}C\beta)$. Since $A, B^{-1}, C, -D^{-1} \geqslant 0$, from the commutativity hypothesis and Halmos [14, p. 141], we have that $B^{-1}A \geqslant 0$ and $-D^{-1}C \geqslant 0$. Therefore $\xi(\cdot, \cdot)$ is maximal monotone. Hence we can have the following two corollaries:

COROLLARY 19. If *Hypotheses* $H(F)_1$ *or* $H(F)_3$ *hold*, then *problem* (21) *has a solution* $x \in W^{2,1}(T, \mathbb{R}^N)$.

COROLLARY 20. If *Hypotheses* $H(F)_2$ *or* $H(F)_4$ *hold*, then *problem* (21) *has a solution set which is nonempty and weakly compact in* $W^{2,1}(T, \mathbb{R}^N)$.

Remark. Theorems 6–9 allow to deal with the nonhomogeneous Strum–Liouville problem.

4. EXTREMAL SOLUTIONS

In this section we examine the following problem:

$$\left\{\begin{array}{ll} x''(t) \in ext\, F(t, x(t), x'(t)) \text{ a.e. on } T & \\ x(0) = x'(0) = 0, & -x'(b) \in \xi(x(b)) \end{array}\right\}. \tag{22}$$

Here *ext* $F(t, x, y)$ denotes the extreme points of the set $F(t, x, y)$. Recall that the set *ext* $F(t, x, y)$ need not be closed and the multifunction $(x, y) \to ext\, F(t, x, y)$ need not have any continuity properties, even if $(x, y) \to F(t, x, y)$ is regular enough. So the existence of solutions for problem (22) can not be deduced from one of the existence theorems in Section 3 and a new approach is necessary.

For simplicity in our calculations, throughout this section we assume $b = 1$, i.e., $T = [0, 1]$. Our hypotheses on $F(t, x, y)$ and $\xi(x)$ are the following:

$H(F)_9$. $F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ *is a multifunction such that*

(i) *for every* $x, y \in \mathbb{R}^N$, $t \to F(t, x, y)$ *is measurable*;

(ii) *for almost all* $t \in T$, $(x, y) \to F(t, x, y)$ *is h-continuous*;

(iii) *for almost all* $t \in T$ *and all* $x, y \in \mathbb{R}^N$, $|F(t, x, y)| = \sup\{\|v\| : v \in F(t, x, y)\} \leqslant a(t) + \beta(\|x\| + \|y\|)$, *with* $a \in L^1(T)$, $\beta < \frac{1}{2}$.

$H(\xi)_7$. $\xi : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *is a maximal monotone map with* $(0, 0) \in Gr\xi$.

THEOREM 21. If *Hypotheses* $H(F)_9$ *or* $H(\xi)_7$ *hold*, then *problem* (22) *has a solution* $x(\cdot) \in W^{2,1}(T, \mathbb{R}^N)$.

Proof. First we derive some *a* priori bounds for the solutions of (22), when *ext F* is replaced by *F*. So let $x \in W^{2,1}(T, \mathbb{R}^N)$ be such a solution. By definition we have

$$\left\{\begin{array}{ll} x''(t) = f(t) \text{ a.e. on } T = [0, 1] & \\ x(0) = x'(0) = 0, & -x'(1) \in \xi(x(1)) \end{array}\right\}$$

with $f \in L^1(T, \mathbb{R}^N)$, $f(t) \in F(t, x(t), x'(t))$ a.e. on T. We take the inner product with $x(t)$ and then integrate over $T = [0, 1]$. Using Green's formula, we have

$$\|x'\|_2^2 \leqslant \|f\|_1 \, \|x\|_\infty.$$

Note that for every $t \in T$, $x(t) = \int_0^t x'(s)\, ds \Rightarrow \|x\|_\infty \leqslant \|x'\|_1 \leqslant \|x'\|_2$ (recall $b = 1$). So

$$\|x'\|_2 \leqslant \|f\|_1.$$

Using Hypothesis $H(F)_9$ (iii), we have $\|f\|_1 \leqslant \|a\|_1 + \beta(\|x\|_1 + \|x'\|_1) \leqslant \|a\|_1 + \beta(\|x\|_2 + \|x'\|_2)$ and $\|x\|_2 \leqslant \|x\|_\infty \leqslant \|x'\|_2$. Hence $\|f\|_1 \leqslant \|a\|_1 + 2\beta \, \|x'\|_2$ and so $\|x'\|_2 \leqslant 1/(1 - 2\beta) \, \|a\|_1 = M_1$ (recall $\beta < \frac{1}{2}$). Also $\|x\|_2 \leqslant \|x\|_\infty \leqslant \|x'\|_2 \leqslant M_1$. Since for all $t \in T$, $x'(t) = \int_0^t f(s)\, ds \Rightarrow \|x'\|_\infty \leqslant \|f\|_1 \leqslant \|a\|_1 + M_1 = M_2$. Therefore, without any loss of generality, we may replace $F(t, x, y)$ by $\hat{F}(t, x, y) = F(t, r_{M_2}(x), r_{M_2}(y))$, where $r_{M_2}(\cdot)$ is the M_2-radial retraction in $\mathbb{R}^N$; i.e.,

$$r_{M_2}(z) = \begin{cases} z & \text{if } \|z\| \leqslant M_2 \\ \dfrac{M_2 z}{\|z\|} & \text{if } \|z\| > M_2 \end{cases} \qquad \text{(note } M_1 \leqslant M_2\text{)}.$$

Observe that $|\hat{F}(t, x, y)| = \sup\{\|v\| : v \in F(t, x, y)\} \leqslant a(t) + M_2 = \phi(t)$ a.e. on T, with $\phi \in L^1(T)$.

Let $V = \{u \in L^1(T, \mathbb{R}^N) : \|u(t)\| \leqslant \phi(t) \text{ a.e. on } T\}$. From the arguments in the proof of Theorem 1, we know that for every $u \in V$, the boundary value problem

$$\left\{\begin{array}{ll} x''(t) = u(t) \text{ a.e. on } T = [0, 1] & \\ x(0) = x'(0) = 0, & -x'(1) \in \xi(x(1)) \end{array}\right\} \tag{23}$$

has a unique solution. Moreover, $\|x'\|_2 \leqslant \|a\|_1 + M_2$ and $\|x\|_2 \leqslant \|x'\|_2$. Therefore $\|x\|_{1,2} \leqslant 2\|a\|_1 + 2M_2 = M$.

Now let $K = \overline{B}(O, M)$ (=the closed M-ball in $W^{1,2}(T, \mathbb{R}^N)$). Since $W^{1,2}(T, \mathbb{R}^N)$ is embedded compactly in $C(T, \mathbb{R}^N)$, K is also compact and convex as a subset of $C(T, \mathbb{R}^N)$ (in fact it is easy to see that the weak-$W^{1,2}(T, \mathbb{R}^N)$ topology is induced by the metric of $C(T, \mathbb{R}^N)$). Let $G : K \to P_{fc}(L^1(T, \mathbb{R}^N))$ be the multifunction defined by $G(x) = -S^1_{\hat{F}(\cdot, x(\cdot), x'(\cdot))}$. Invoking Theorem 1.1 of Tolstonogov [27], we can find $g : K \to L^1_w(T, \mathbb{R}^N)$ a continuous map such that $g(x) \in ext\, G(x)$ for all $x \in K$. Here $L^1_w(T, \mathbb{R}^N)$ denotes the space $L^1(T, \mathbb{R}^N)$ furnished with the "weak" norm $\|f\|_w = \sup[\|\int_{t_1}^{t_2} f(s)\, ds\| : 0 \leqslant t_1 \leqslant t_2 \leqslant 1]$. Let $p_K : W^{1,2}(T, \mathbb{R}^N) \to K$ be the metric projection on K in $W^{1,2}(T, \mathbb{R}^N)$. It is well-known that $p_K(\cdot)$ is nonexpansive. Let $\hat{g} = g \circ p_K$ and $\hat{g}_1 = \hat{g} + p_K : W^{1,2}(T, \mathbb{R}^N) \to L^1_w(T, \mathbb{R}^N)$. Evidently $\hat{g}_1$ is continuous.

Let $D = \{x \in W^{2,1}(T, \mathbb{R}^N) : x(0) = x'(0) = 0, \ -x'(1) \in \xi(x(1))\}$ and let $\hat{L} : D \subseteq L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ be defined by $\hat{L}(x) = -x''$. Set $L = I + \hat{L}$. From the proof of Theorem 1, we know that $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subseteq W^{1,2}(T, \mathbb{R}^N)$ is compact (the argument remains valid although here we consider $W^{1,2}(T, \mathbb{R}^N)$ instead of $W^{1,1}(T, \mathbb{R}^N)$ as the range of L^{-1}; see the proof of Theorem 1). We consider the map $\hat{g}_1 \circ L^{-1} : L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$. Our claim is that $\hat{g}_1 \circ L^{-1}(\cdot)$ is weakly continuous. To this end let $\{v_a\}_{a \in J} \subseteq L^1(T, \mathbb{R}^N)$ be a net such that $v_a \xrightarrow{w} v$ in $L^1(T, \mathbb{R}^N)$. Then $L^{-1}(v_a) \to L^{-1}(v)$ in $W^{1,2}(T, \mathbb{R}^N)$. So $p_K(L^{-1}(v_a)) \to p_K(L^{-1}(v))$ in $W^{1,2}(T, \mathbb{R}^N)$ and $\hat{g}(L^{-1}(v_a)) \to \hat{g}(L^{-1}(v))$ in $L^1_w(T, \mathbb{R}^N)$. Note that $\{L^{-1}(v_a)\}_{a \in J}$ is bounded in $W^{2,1}(T, \mathbb{R}^N)$, hence is bounded in $C^1(T, \mathbb{R}^N)$ (recall that $W^{2,1}(T, \mathbb{R}^N)$ is embedded continuously in $C^1(T, \mathbb{R}^N)$). So there exists $M_3 > 0$ such that $\|L^{-1}(v_a)\|_{C^1(T, \mathbb{R}^N)} \leqslant M_3$ for all $a \in J$. Also from the multivalued Scorza–Dragoni theorem (see, for example, Kisielewicz [18, Theorem 3.7, p. 45]), given $\varepsilon > 0$ we can find $T_\varepsilon \subseteq T$ compact such that $\lambda(T \backslash T_\varepsilon) \leqslant \varepsilon$ and $\hat{F}|_{T_\varepsilon \times \mathbb{R}^N \times \mathbb{R}^N}$ is h-continuous. So $\hat{F}(T_\varepsilon \times \overline{B}_{M_3} \times \overline{B}_{M_3}) = E$ is compact in $\mathbb{R}^N$ (here $\overline{B}_{M_3} = \{z \in \mathbb{R}^N : \|z\| \leqslant M_3\}$). Since $\hat{g}(L^{-1}(v_a))(t) \in E$ for almost all $t \in T_\varepsilon$ and all $a \in J$, we can apply the theorem of Gutman [13] and have that $\hat{g}(L^{-1}(v_a)) \xrightarrow{w} \hat{g}(L^{-1}(v))$ in $L^1(T, \mathbb{R}^N)$. So $\hat{g}_1 \circ L^{-1} : L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ is weakly continuous as claimed. Recalling that K viewed as a subspace of $C(T, \mathbb{R}^N)$ is compact, we can find $M_4 > 0$ such that $\|p_K(x)(t)\| \leqslant M_4$ for all $x \in W^{1,2}(T, \mathbb{R}^N)$ and all $t \in T$. So for all $v \in L^1(T, \mathbb{R})$, we have $\|\hat{g}_1(L^{-1}(v))(t)\| \leqslant \phi(t) + M_4$ a.e. on T, hence the range of $\hat{g}_1 \circ L^{-1}$ in $L^1(T, \mathbb{R}^N)_w$ is relatively compact, being uniformly integrable (Dunford–Pettis theorem). So we can apply Tichonov's theorem and obtain $v_1 \in L^1(T, \mathbb{R}^N)$ such that $v_1 = \hat{g}_1 \circ L^{-1}(v_1)$. Set $x = L^{-1}(v_1)$. Then we have

$$L(x) = \hat{g}_1(x)$$

$$\Rightarrow -x''(t) + x(t) = v(t) + p_K(x)(t) \text{ a.e. on } T$$

$$x(0) = x'(0) = 0, \qquad -x'(1) \in \xi(x(1))$$

with $v \in L^1(T, \mathbb{R}^N)$, $v = g(p_K(x)) \in ext\, G(p_K(x)) = -S^1_{ext\, \hat{F}(\cdot, p_K(x), p_K(x)'(\cdot))}$ (see Benamara [2]). So $v(t) \in -ext\, \hat{F}(t, p_K(x)(t), p_K(x)'(t))$ a.e. on T. Take the inner product with $x(t)$ and then integrate over $T = [0, 1]$. Using Green's formula, we obtain

$$\|x'\|_2^2 + \|x\|_2^2 \leqslant \|v\|_1 \|x\|_\infty + \|p_K(x)\|_2 \|x\|_2.$$

Recall that $\|x\|_\infty \leqslant \|x'\|_2$ and it is immediate from the definition of $p_K(\cdot)$, that $\|p_K(x)\|_2 \leqslant \|x\|_2$. So we have

$$\|x'\|_2 \leqslant \|v\|_1 \leqslant \|\phi\|_1 \leqslant \|a\|_1 + M_2.$$

Moreover, $\|x\|_2 \leqslant \|x'\|_2$. Therefore $\|x\|_{1,2} \leqslant 2\|a\|_1 + 2M_2 = M$. So $p_K(x) = x$ and

$$\left\{\begin{array}{ll} x''(t) = -v(t) \in ext\, F(t, x(t), x'(t)) \text{ a.e. on } T = [0, 1] \\ x(0) = x'(0) = 0, \qquad -x'(1) \in \xi(x(1)) \end{array}\right\},$$

i.e, $x \in W^{2,1}(T, \mathbb{R}^N)$ is a solution of (22). ∎

Remark. The study of extremal solutions for first order Cauchy problems was initiated by DeBlasi–Pianigiani in a series of remarkable papers [6–8], in which they developed the so-called "Baire category method." Their method inspired Tolstonogov to prove his selection theorem, which was crucial in the above proof.

5. RELAXATION THEOREMS

In the previous sections we proved existence theorems for the convex, nonconvex, and extremal problems. In this section we examine to what extent the convexification of the right-hand side of the inclusion introduces new solutions. More precisely, we want to find out if the solutions of the nonconvex (resp. of the extremal) problem are dense in those of the convex one. Such a result is known in the literature as "relaxation theorem" (resp. "strong relaxation theorem") and has important implications in optimal control theory. It is well-known that in order to have optimal state-control pairs, the system has to satisfy certain convexity requirements. If these conditions are not present, then in order to guarantee existence of optimal solutions we need to pass to an augmented system with convex structure by introducing the so-called relaxed (generalized, chattering) controls. The resulting relaxed problem has a solution. The relaxation theorems tell us that the relaxed optimal state can be approximated by original states, which are generated by a more economical set of controls that are much simpler to build. In particular "strong relaxation" theorems imply that this approximation can be achieved using states generated by bang-bang controls.

As in Section 4, in order to simplify our calculations, we assume that $b=1$, i.e., $T=[0,1]$. In conjunction with (22), we consider its convexified counterpart:

$$\left\{\begin{array}{ll} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T=[0,1] \\ x(0)=x'(0)=0, \qquad -x'(1) \in \xi(x(1)) \end{array}\right\}. \tag{24}$$

In what follows, we denote the solution sets of (22) and (24), by S_e and S_c respectively. Our goal is to investigate under what conditions S_e is dense in S_c for the $W^{1,2}(T, \mathbb{R}^N)$-topology. It is well-known that simple h-continuity of $F(t, \cdot, \cdot)$ is not enough. So we introduce the following hypotheses on $F(t, x, y)$.

$H(F)_{10}$. *$F: T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is multifunction such that*

(i) *for every $x, y \in \mathbb{R}^N$, $t \to F(t, x, y)$ is measurable;*

(ii) *for almost all $t \in T$ and all $x, y, x_1, y_1 \in \mathbb{R}^N$, $h(F(t,x,y), F(t, x_1, y_1)) \leqslant k(t)[\|x-x_1\| + \|y-y_1\|]$ with $k \in L^\infty(T)$, $\|k\|_\infty < \frac{1}{2}$;*

(iii) *for almost all $t \in T$ and all $x, y \in \mathbb{R}^N$, $|F(t,x,y)| = \sup\{\|v\| : v \in F(t,x,y)\} \leqslant a(t) + \beta(\|x\| + \|y\|)$ with $a \in L^1(T)$, $\beta < \frac{1}{2}$;*

THEOREM 22. If *Hypotheses $H(F)_{10}$ and $H(\xi)_7$ hold*, then $\bar{S}_e = S_c$ *the closure taken in $W^{1,2}(T, \mathbb{R}^N)$.*

Proof. From Theorem 21 we know that $S_e \neq \varnothing$. Let $x \in S_c$. By definition we can find $f \in L^1(T, \mathbb{R}^N)$, $f(t) \in F(t, x(t), x'(t))$ a.e on T, such that

$$\left\{\begin{array}{ll} x''(t) = f(t) \text{ a.e. on } T=[0,1] \\ x(0)=x'(0), \qquad -x'(1) \in \xi(x(1)) \end{array}\right\}.$$

From the a priori estimation conducted in the proof of Theorem 1, we know that we may assume that $|F(t,x,y)| \leqslant \phi(t)$ a.e. on T, $\phi \in L^1(T)$. Also let K be as in that proof. Given $y \in K$, define $\Delta_\varepsilon : T \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ by

$$\Delta_\varepsilon(t) = \{v \in \mathbb{R}^N : \|f(t) - v\| < \varepsilon + d(f(t), F(t, y(t), y'(t))), \\ v \in F(t, y(t), y'(t))\}.$$

Let $\gamma: T \times \mathbb{R}^N \to \mathbb{R}_+$ be defined by $\gamma(t,v) = \|f(t) - v\| - d(f(t), F(t, y(t), y'(t)))$. From Hypotheses $H(F)_{10}$ (i) and (ii) and from Theorem 3.3 of Papageorgiou [24], we know that $t \to F(t, y(t), y'(t))$ is measurable. Hence $\gamma(\cdot, \cdot)$ is a Caratheodory function (i.e., measurable in t, continuous in v), thus jointly measurable. So $Gr\Delta_\varepsilon = \{(t, v) \in T \times \mathbb{R}^N : \gamma(t, v) < \varepsilon\} \in \mathscr{L}(T) \times B(\mathbb{R}^N)$ where $\mathscr{L}(T)$ is the Lebesgue σ-field of T and $B(\mathbb{R}^N)$ is the Borel σ-field of $\mathbb{R}^N$. Then Aumann's selection theorem (see Wagner [28, Theorem 5.10]), implies the existence of a measurable function $v : T \to \mathbb{R}^N$ such that $v(t) \in \Delta_\varepsilon(t)$ a.e. on T. So if we define $R_\varepsilon : K \to 2^{L^1(T, \mathbb{R}^N)}$ by

$$R_\varepsilon(y) = \{v \in S^1_{F(\cdot, y(\cdot), y'(\cdot))} : \|f(t) - v(t)\| < \varepsilon + d(f(t), F(t, y(t), y'(t))), \text{ a.e. on } T\},$$

we see that $R_\varepsilon(\cdot)$ has nonempty and decomposable values. Moreover, from Proposition 4 of Bressan–Colombo [3], we know that $R(\cdot)$ is lsc. Therefore $y \to \overline{R_\varepsilon(y)}$ is lsc and has closed and decomposable values. So we can apply Theorem 3 of Bressan–Colombo [3] and produce $u_\varepsilon : K \to L^1(T, \mathbb{R}^N)$ a continuous map such that $u_\varepsilon(y) \in \overline{R_\varepsilon(y)}$ for all $y \in K$. Also from Theorem 1.1 of Tolstonogov [27], we know that we can find $v_\varepsilon : K \to L^1_w(T, \mathbb{R}^N)$ a continuous map such that $v_\varepsilon(y)(t) \in ext\, F(t, y(t), y'(t))$ a.e on T and $\|u_\varepsilon(y) - v_\varepsilon(y)\|_w \leqslant \varepsilon$ for all $y \in K$.

Now let $\varepsilon_n \downarrow 0$ and set $u_n = u_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$. We consider the following boundary value problem:

$$\left\{ \begin{array}{ll} x_n''(t) = v_n(x_n)(t) \text{ a.e. on } T = [0, 1] & \\ x_n(0) = x_n'(0) = 0, & -x_n'(1) \in \xi(x_n(1)) \end{array} \right\}. \tag{25}$$

Working exactly as in the proof of Theorem 21, via Tichonov's fixed point theorem, we obtain a solution $x_n \in W^{2,1}(T, \mathbb{R}^N), n \geqslant 1$, of (25). We see that $\{x_n''\}_{n \geqslant 1}$ is uniformly integrable and also $\{x_n\}_{n \geqslant 1} \subseteq K$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} \hat{x}$ in $W^{2,1}(T, \mathbb{R}^N)$ and $x_n \to \hat{x}$ in $W^{1,2}(T, \mathbb{R}^N)$ as $n \to \infty$. Moreover, arguing as in the proof of Theorem 1 (see the proof of Claim 2), we can check that $\hat{x}(0) = \hat{x}'(0) = 0$ and $-\hat{x}'(1) \in \xi(\hat{x}(1))$.

We have

$$x''(t) - x_n''(t) = f(t) - v_n(x_n)(t) \text{ a.e. on } T$$

$$\Rightarrow (x''(t) - x_n''(t), x_n(t) - x(t))_{\mathbb{R}^N} = (f(t) - v_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N} \text{ a.e. on } T$$

$$\Rightarrow \int_0^b (x''(t) - x_n''(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt = \int_0^b (f(t) - v_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt. \tag{26}$$

By virtue of Green's formula, we have

$$\|x' - x_n'\|_2^2 \leqslant \int_0^b (x''(t) - x_n''(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt. \tag{27}$$

Also we have

$$\begin{aligned}
&\int_0^b (f(t) - v_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \\
&\quad = \int_0^b (f(t) - u_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \\
&\quad\quad + \int_0^b (u_n(x_n)(t) - v_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt.
\end{aligned}$$

By construction $\|u_n(x_n) - v_n(x_n)\|_w \leqslant \varepsilon_n$, hence $u_n(x_n) - v_n(x_n) \to 0$ in $L_w^1(T, \mathbb{R}^N)$ as $n \to \infty$. As before, via Gutman's theorem, we also have that $u_n(x_n) - v_n(x_n) \xrightarrow{w} 0$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. So

$$\int_0^b (u_n(x_n)(t) - v_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \to 0 \qquad \text{as} \quad n \to \infty. \tag{28}$$

Also we have

$$\begin{aligned}
&\int_0^b (f(t) - u_n(x_n)(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \\
&\quad \leqslant \int_0^b \|f(t) - u_n(x_n)(t)\| \cdot \|x(t) - x_n(t)\|\, dt \\
&\quad \leqslant \int_0^b (\varepsilon_n + h(F(t, x(t), x'(t)), F(t, x_n(t), x_n'(t))))\, \|x(t) - x_n(t)\|\, dt \\
&\quad \leqslant \int_0^b (\varepsilon_n + k(t))(\|x(t) - x_n(t)\| + \|x'(t) - x_n'(t)\|)\, \|x(t) - x_n(t)\|\, dt \\
&\quad \to \int_0^b k(t)(\|x(t) - \hat{x}(t)\|^2 \\
&\quad\quad + \|x'(t) - \hat{x}'(t)\|)\, \|x(t) - \hat{x}(t)\|\, dt \qquad \text{as} \quad n \to \infty.
\end{aligned} \tag{29}$$

Using (27)–(29) in (26), in the limit as $n \to \infty$, we obtain

$$\|x' - \hat{x}'\|_2^2 \leqslant \|k\|_\infty\, \|x - \hat{x}\|_2^2 + \|k\|_\infty\, \|x' - \hat{x}'\|_2\, \|x - \hat{x}\|_2. \tag{30}$$

Note that $x(t)-\hat{x}(t)=\int_0^t (x'(s)-\hat{x}(s))\,ds$. Hence $\|x-\hat{x}\|_2\leqslant \|x-\hat{x}\|_\infty\leqslant \|x'-\hat{x}'\|_1\leqslant \|x'-\hat{x}'\|_2$. Using this in (30), we have

$$\|x'-\hat{x}'\|_2^2\leqslant 2\,\|k\|_\infty\,\|x'-\hat{x}'\|_2^2.$$

Since $\|k\|_\infty<\frac{1}{2}$, we infer that $x'=\hat{x}'$, but $x(0)=\hat{x}(0)=0$, so $x=\hat{x}$. Finally, note that $x_n\in S_e$ for all $n\geqslant 1$. Therefore $S_c\subseteq \bar{S}_e$ (closure in $W^{1,2}(T,\mathbb{R}^N)$ and since S_c is already closed in $W^{1,2}(T,\mathbb{R}^N)$, we conclude that $S_c=\bar{S}_e$. ∎

Remark. It is clear from the above proof that we also have $\bar{S}_e=S_c$, the closure in $C(T,\mathbb{R}^N)$.

6. THE DIRICHLET PROBLEM

In this last section of the paper, we prove existence and relaxation results for the Dirichlet problem, when F is independent of y, but it satisfies a general growth hypothesis and a generalized sign-type condition (see Hu–Papageorgiou [16]). With the exception of Kravvaritis–Papageorgiou [22] no other work deals with the relaxation problem. In Kravvaritis–Papageorgiou [22] although F depends on x', the Lipschitz and growth conditions are more restrictive.

We consider the following two problems

$$\left.\begin{cases} x''(t)\in F(t,x(t)) \text{ a.e. on } T=[0,1] \\ x(0)=x(1)=0 \end{cases}\right\} \tag{31}$$

and its convexified counterpart

$$\left.\begin{cases} x''(t)\in \overline{conv}\,F(t,x(t)) \text{ a.e. on } T=[0,1] \\ x(0)=x(1)=0 \end{cases}\right\}. \tag{32}$$

We shall denote by S the solution set of (31) and by S_c the solution set of (32). We have $S\subseteq S_c\subseteq W^{2,1}(T,\mathbb{R}^N)$. We start with a nonemptiness result for S_c. Our hypotheses on $F(t,x)$ are the following:

$H(F)_{11}$. *$F\colon T\times\mathbb{R}^N\to P_{kc}(\mathbb{R}^N)$ is a multifunction such that*

(i) *for all $x\in\mathbb{R}^N$, $t\to F(t,x)$ is measurable;*

(ii) *for almost all $t\in T$, $GrF(t,\cdot)=\{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N\colon v\in F(t,x)\}$ is closed;*

(iii) *for every $M>0$, we can find $\gamma_M\in L^1(T)$ such that for almost all $t\in T$ and all $\|x\|\leqslant M$ we have $|F(t,x)|=\sup\{\|v\|\colon v\in F(t,x)\}\leqslant\gamma_M(t)$;*

(iv) *there exists $\theta \in L^{\infty}(T)$ such that for almost all $t \in T$, all $\|x\| \geqslant \theta(t)$ and all $v \in F(t, x)$ we have $(v, x)_{\mathbb{R}^N} \geqslant 0$.*

Remark. Note that in the region $\|x\| < \theta(t)$, we do not require $F(t, x)$ to satisfy a unilateral condition as is the case in $H(F)_1$(iii). Existence results with F depending also on the derivative of x can be found in Frigon [29].

PROPOSITION 23. *If Hypotheses $H(F)_{11}$ hold, then S_c is nonempty and weakly compact in $W^{2,1}(T, \mathbb{R}^N)$.*

Proof. Set $D = W^{2,1}(T, \mathbb{R}^N) \cap W_0^{1,1}(T, \mathbb{R}^N)$ and let $\hat{L}$: $D \subseteq L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ be defined by $Lx = -x''$. If $L = I + \hat{L}$, we know (see the proof of Theorem 1) that L^{-1}: $L^1(T, \mathbb{R}^N) \to D \subseteq W_0^{1,1}(T, \mathbb{R}^N)$ is linear compact.

Let N_1: $W_0^{1,1}(T, \mathbb{R}^N) \to 2^{L^1(T, \mathbb{R}^N)}$ be defined by $N_1(x) = -S^1_{F(\cdot, x(\cdot))} + x$. As in the proof of Theorem 3, we can verify that $N_1(\cdot)$ has nonempty, closed, convex values and is bounded and usc into $L^1(T, \mathbb{R}^N)_w$. Then $L^{-1}N_1$: $W_0^{1,p}(T, \mathbb{R}^N) \to P_{kc}(W_0^{1,1}(T, \mathbb{R}^N))$ is usc and maps bounded sets into relatively compact sets.

Now let $x \in D$ such that $x \in \lambda L^{-1} N_1(x)$ for some $0 < \lambda < 1$. We have

$$L(x) = -\lambda f + \lambda x, \qquad f \in S^1_{F(\cdot, x(\cdot))}$$
$$\Rightarrow -x''(t) = -\lambda f(t) + (\lambda - 1)\, x(t) \text{ a.e. on } T$$
$$x(0) = x(1) = 0.$$

Take the inner product with $x(t)$ and then integrate over $T = [0, b]$. We have

$$\begin{aligned} \|x'\|_2^2 &\leqslant \lambda \int_0^1 (-f(t), x(t))_{\mathbb{R}^N}\, dt \\ &= \lambda \int_{\{|x| < \theta\}} (-f(t), x(t))_{\mathbb{R}^N}\, dt + \lambda \int_{\{|x| \geqslant \theta\}} (-f(t), x(t))_{\mathbb{R}^N}\, dt \\ &\leqslant \|\theta\|_\infty \|\gamma_{M_1}\| = M \qquad (M_1 = \|\theta\|_\infty). \end{aligned} \tag{33}$$

Since $\|x'\|_1$ is an equivalent norm of $W_0^{1,1}(T, \mathbb{R}^N)$, from (33) we infer that the solutions of $x \in \lambda L^{-1} N_1(x)$, $0 < \lambda < 1$, are bounded in $W_0^{1,1}(T, \mathbb{R}^N)$ by M, which is independent of λ. Applying Theorem 2, we obtain $x \in D$ such that $x \in L^{-1} N_1(x)$. Hence $x''(t) \in F(t, x(t))$ a.e. on T, $x(0) = x(b) = 0$. Finally, as in the last part of the proof of Theorem 3, we check that S_c is weakly compact in $W^{2,1}(T, \mathbb{R}^N)$. ∎

In a similar way, using the selection theorem of Bressan–Colombo [3] and the single-valued Leray–Schauder theorem (as in the proof of

Theorem 1), we can have a nonemptiness result for the set S. The hypotheses on $F(t, x)$ are the following:

$H(F)_{12}$. *$F: T \times \mathbb{R}^N \times \to P_k(\mathbb{R}^N)$ is a multifunction such that*

(i) *$(t, x) \to F(t, x)$ is graph measurable;*

(ii) *for almost all $t \in T$, $x \to F(t, x)$ is lsc;*

(iii) *for every $M > 0$, we can find $\gamma_M \in L^1(T)$ such that for almost all $t \in T$ and all $\|x\| \leqslant M$, we have $|F(t, x)| = \sup\{\|v\|: v \in F(t, x)\} \leqslant \gamma_M(t)$;*

(iv) *there exists $\theta \in L^\infty(T)$ such that for almost all $t \in T$, all $\|x\| \geqslant \theta(t)$ and all $v \in F(t, x)$, we have $(v, x)_{\mathbb{R}^N} \geqslant 0$.*

PROPOSITION 24. If *Hypotheses $H(F)_{12}$ hold*, then S *is a nonempty subset of $W^{2,1}(T, \mathbb{R}^N)$.*

Now we shall prove a relaxation theorem. For this we need stronger continuity hypotheses on $F(t, x)$.

$H(F)_{13}$. *$F: T \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that*

(i) *for all $x \in \mathbb{R}^N$, $t \to F(t, x)$ is measurable;*

(ii) *there exists $k \in L^1(T)$, with $\|k\|_1 < 1$ such that for almost all $t \in T$ and all $x, x_1 \in \mathbb{R}^N$, we have*

$$h(F(t, x), F(t, x_1)) \leqslant k(t) \|x - x_1\|;$$

(iii) *for every $M > 0$, we can find $\gamma_M \in L^1(T)$ such that for almost all $t \in T$ and all $\|x\| \leqslant M$, we have $|F(t, x)| = \sup\{\|v\|: v \in F(t, x)\} \leqslant \gamma_M(t)$;*

(iv) *there exists $\theta \in L^\infty(T)$ such that for all $t \in T$, all $\|x\| \geqslant \theta(t)$ and all $v \in F(t, x)$, we have $(v, x)_{\mathbb{R}^N} \geqslant 0$.*

THEOREM 25. If *Hypotheses $H(F)_{13}$ hold*, then $S_c = \overline{S}$ *the closure taken in $W_0^{1,1}(T, \mathbb{R}^N)$.*

Proof. Let $G(t, s)$ be the Green's function corresponding to the operator $L(x) = -x''$, $x \in D = W^{2,1}(T, \mathbb{R}^N) \cap W_0^{1,1}(T, \mathbb{R}^N)$ (i.e., the one-dimensional Laplacian with Dirichlet boundary conditions). We know that

$$G(t, x) = \begin{cases} t(1-s)\, I & \text{if} \quad 0 \leqslant t \leqslant s \leqslant 1 \\ s(1-t)\, I & \text{if} \quad 0 \leqslant s < t \leqslant 1 \end{cases}.$$

It is well-known that for every $v \in L^1(T, \mathbb{R}^N)$ the unique solution $x \in D$ of the Dirichlet problem $x''(t) = v(t)$ a.e. on T, $x(0) = x(b) = 0$, is given by $x(t) = \int_0^b G(t, s)\, v(s)\, ds$ for all $t \in T$. Let $K: L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ be the operator defined by $K(v)(t)=\int_0^b G(t,s)\,v(s)\,ds$. Via the Arzela–Ascoli theorem, we easily check that this map is compact.

Now let $x \in S_c$. Then by definition $x=K(v)$ with $v \in S^1_{\overline{conv}\,F(\cdot,x(\cdot))}$. Let $\varepsilon>0$ be given. Since $K(\cdot)$ is compact, we can find U a symmetric weak neighborhood of the origin in $L^1(T,\mathbb{R}^N)$ such that if $v-v_1\in U$, then $\|x-x_1\|_\infty \leqslant \varepsilon$ where $x_1=K(v_1)$. By virtue of Proposition 4.1 of Papageorgiou [25], we can take $v_1 \in S^1_{F(\cdot,x(\cdot))}$. By an easy application of Aumann's selection theorem (as in the proof of Theorem 22), we can find $v_2 \in S^1_{F(\cdot,x_1(\cdot))}$ such that

$$\|v_1(t)-v_2(t)\|=h(F(t,x(t)),F(t,x_1(t)))\leqslant k(t)\,\|x(t)-x_1(t)\| \text{ a.e. on } T$$

$$\Rightarrow \|v_1-v_2\|_1 \leqslant \|k\|_1\,\|x-x_1\|_\infty \leqslant \|k\|_1\,\varepsilon.$$

Suppose $v_1,...,v_n \in L^1(T,\mathbb{R}^N)$ have been chosen such that

$$\|v_{m+1}-v_m\|_1 \leqslant \|k\|_1^m\,\varepsilon, \qquad v_{m+1}\in S^1_{F(\cdot,x_m(\cdot))}, \qquad x_m=K(v_m)$$
$$\text{for} \quad m=1,2,...,n-1.$$

Let $x_n=K(v_n)$. Since $\|K\|_{\mathscr{L}}\leqslant 1$, for every $m=1,2,...,n-1$, we have

$$\|x_{m+1}-x_m\|_\infty=\|K(v_{m+1})-K(v_m)\|_\infty \leqslant \|v_{m+1}-v_m\|_1 \leqslant \|k\|_1^m\,\varepsilon.$$

Therefore we have

$$\|x_n-x\|_\infty \leqslant \sum_{m=0}^{n-1}\|x_{m+1}-x_m\|_\infty \leqslant \varepsilon\sum_{m=0}^{n-1}\|k\|_1^m \qquad (x_0=x).$$

A new application of Aumann's selection theorem, gives $v_{n+1}\in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$\|v_{n+1}(t)-v_n(t)\|=h(F(t,x_n(t)),F(t,x_{n-1}(t)))$$
$$\leqslant k(t)\,\|x_n(t)-x_{n-1}(t)\| \text{ a.e. on } T$$
$$\Rightarrow \|v_{n+1}-v_n\|_1 \leqslant \|k\|_1\,\|x_n-x_{n-1}\|_\infty \leqslant \|k\|_1^n\,\varepsilon.$$

So by induction we have generated a sequence $\{v_n\}_{n\geqslant 1}\subseteq L^1(T,\mathbb{R}^N)$ such that

$$\|v_{n+1}-v_n\|_1 \leqslant \|k\|_1^n\,\varepsilon, \qquad v_{n+1}\in S^1_{F(\cdot,x_n(\cdot))}, \qquad x_n=K(v_n), \qquad n\geqslant 1.$$

Since $\|k\|_1<1$, we see that $\{v_n\}_{n\geqslant 1}$ is Cauchy in $L^1(T,\mathbb{R}^N)$ and so $v_n\to\hat{v}$ in $L^1(T,\mathbb{R}^N)$ as $n\to\infty$. Because of Hypothesis $H(F)_{13}$ (ii), from Papageorgiou [23] we know that $x\to S^1_{F(\cdot,x(\cdot))}$ is h-continuous from $C(T,\mathbb{R}^N)$

into $L^1(T, \mathbb{R}^N)$. Therefore $\hat{v} \in S^1_{F(\cdot, x(\cdot))}$. Moreover, since $\|x_{n+1} - x_n\|_\infty \leqslant \|K\|_{\mathscr{L}} \|v_{n+1} - v_n\|_1 \leqslant \|v_{n+1} - v_n\|_1 \leqslant \|k\|_1^n \varepsilon$, we see that $\{x_n\}_{n \geqslant 1}$ is Cauchy in $C(T, \mathbb{R}^N)$ and so $x_n \to \hat{x}$ in $C(T, \mathbb{R}^N)$ as $n \to \infty$. Evidently $\hat{x} = K(\hat{v})$ and so $\hat{x} \in S$. Moreover, since for every $n \geqslant 1$

$$\|x_n - x\|_\infty \leqslant \sum_{m=0}^{n-1} \|x_{m+1} - x_m\|_\infty \leqslant \varepsilon \sum_{m=0}^{n-1} \|k\|_1^m \qquad (x_0 = x)$$

in the limit as $n \to \infty$, we have

$$\|\hat{x} - x\|_\infty \leqslant \frac{\varepsilon}{1 - \|k\|_1} \qquad (\text{recall } \|k\|_1 < 1).$$

Since $\varepsilon > 0$ was arbitrary and $\hat{x} \in S$, we infer that $S_c \subseteq \bar{S}$ the closure taken in $C(T, \mathbb{R}^N)$. So thus far, we have proved that if $x \in S_c$, we can find $\{x_n\}_{n \geqslant 1} \subseteq S$ such that $x_n \to x$ in $C(T, \mathbb{R}^N)$ as $n \to \infty$. Directly from the equation and using Hypothesis $H(F)_{13}$ (iii), we see that $\{x_n\}_{n \geqslant 1}$ is bounded in $W^{2,1}(T, \mathbb{R}^N)$ (recall that $y \to \|y''\| + \|y\|_1$ is an equivalent norm on $W^{2,1}(T, \mathbb{R}^N)$. But $W^{2,1}(T, \mathbb{R}^N)$ is embedded compactly in $W^{1,1}(T, \mathbb{R}^N)$. So we also have $x_n \to x$ in $W_0^{1,1}(T, \mathbb{R}^N)$ and this shows that $S_c \subseteq \bar{S}$, the closure taken in $W_0^{1,1}(T, \mathbb{R}^N)$. Since S_c is already closed in $W_0^{1,1}(T, \mathbb{R}^N)$, we conclude that $S_c = \bar{S}$. ∎

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