ELSEVIER
Linear Algebra and its Applications 313 (2000) 193–201
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On a discrete nonlinear boundary value problem

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Received 15 January 2000; accepted 31 March 2000

Submitted by B.-S. Tam

Abstract

The nonlinear eigenvalue problem $\Delta^2 u_{k-1} + \lambda |u_k|^\gamma = 0,\ k = 1, 2, \ldots, n$ under the Dirichlet boundary conditions $u_0 = 0 = u_{n+1}$ is studied. An existence and uniqueness theorem is proved. Qualitative properties of solutions are also given. © 2000 Elsevier Science Inc. All rights reserved.

AMS classification: 15A18; 39A10

Keywords: Nonlinear eigenvalue problem; Jacobi matrix; Positive solution; Comparison theorem

1. Introduction

Nonlinear differential boundary value problems of the form (see e.g. [1])

$$u''(t) + \mu f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = 0 = u(1),$$

arise in steady state temperature distribution problems in a material bounded by two infinite parallel planes. By applying finite difference methods, a discrete eigenvalue problem naturally arises:

$$u_{k+1} - 2u_k + u_{k-1} + \lambda f(u_k) = 0, \quad k = 1, 2, \ldots, n,$$

$$u_0 = 0 = u_{n+1},$$

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which calls for the attention to the usual questions of existence and uniqueness of solutions. The above discrete boundary value problem can also be viewed as modelling a discrete time oscillator which is subject to nonlinear forces. More specifically, assuming that the coordinates u_k of an oscillator are sampled at discrete times $k = \dots, -2, -1, 0, 1, \dots$, it is then natural to consider $u_{k+1} - u_k \equiv \Delta u_k$ and $u_k - u_{k-1} \equiv \Delta u_{k-1}$ as the average velocities of the oscillator over the time periods $[k, k+1]$ and $[k-1, k]$, and consider $u_{k+1} - u_k - (u_k - u_{k-1}) \equiv \Delta^2 u_{k-1}$ as the average acceleration at time k. If the oscillator is subjected to a force of the form $F(u_k)$, then by Newton's law, the equation of motion is

$$\Delta^2 u_{k-1} = F(u_k).$$

Assuming that the oscillator is projected from the origin at time 0, a natural question then arises as to whether the oscillator will return to its initial state at some future time $n+1$. In case $F(u)$ is generated by an elastic spring, then by the linear Hooke's law, $F(u) = -\lambda u$, where λ is a positive proportionality constant which reflects the characteristic of the spring. Our boundary problem is then given by

$$\Delta^2 u_{k-1} + \lambda u_k = 0, \quad k = 1, 2, \dots, n, \; \lambda > 0,$$

$$u_0 = 0 = u_{n+1}.$$

This problem can be expressed in the form

$$A_n u = \lambda u,$$

where $u = \mathrm{col}(u_1, u_2, \dots, u_n)$ and

$$J_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}_{n \times n}.$$

Thus, nontrivial solutions can only be found when λ is equal to one of the eigenvalues [2]

$$\lambda_i = 4 \sin^2 \frac{i\pi}{2(n+1)}, \quad i = 1, 2, \dots, n,$$

of J_n, and are given by the nontrivial constant multiples of the corresponding eigenvectors

$$u^{(i)} = \sqrt{\frac{2}{n+1}}\, \mathrm{col}\left(\sin \frac{i\pi}{n+1}, \sin \frac{2i\pi}{n+1}, \dots, \sin \frac{ni\pi}{n+1}\right), \quad i = 1, 2, \dots, n.$$

A strange conclusion then seems to be that only certain springs could generate the right forces for the oscillator to return to the origin at time $n+1$. This is not entirely satisfactory, since intuitively we expect nontrivial solutions for any spring constant $\lambda > 0$. In this paper, we will assume a nonlinear Hooke's law that asserts $F(u) =$

$-\lambda\,|u|^{\gamma}$, where γ is any fixed number between 0 and 1. The boundary problem is now

$$\Delta^2 u_{k-1} + \lambda\,|u_k|^{\gamma} = 0, \quad k = 1, 2, \ldots, n,\ \lambda > 0,\ \gamma \in (0, 1), \tag{1}$$

$$u_0 = 0 = u_{n+1}. \tag{2}$$

We will show that for each $\lambda > 0$, there is a unique solution to our problem. Then we treat λ, γ and n as parameters of our unique solution, and derive several comparison theorems for it. We will assume throughout the rest of our paper that $\lambda > 0$, $\gamma \in (0, 1)$ and $n = 1, 2, \ldots$

We remark that it is also of interest to assume a nonlinear Hooke's law that takes on other forms. In particular, the forcing function $F(u) = -\lambda u\,|u|^{\gamma-1}$ is closely related to the one given above and the corresponding problem seems to be much more involved but interesting for future considerations.

2. Existence and uniqueness

To motivate the following, we first consider the simple case when $n = 1$ and $\lambda = 1$. Then the boundary problem (1), (2) is reduced to the single equation

$$-2u_1 + |u_1|^{\gamma} = 0.$$

One of the solutions is clearly $u_1 = 0$. We can also find a unique nontrivial positive solution

$$u_1 = \left(\frac{1}{2}\right)^{1/(1-\gamma)}.$$

When $n = 2$ and $\lambda = 1$, our boundary problem is reduced to a pair of nonlinear equations:

$$u_2 - 2u_1 + |u_1|^{\gamma} = 0,$$

$$-2u_2 + u_1 + |u_2|^{\gamma} = 0.$$

The only nontrivial solution can easily be found and is given by $u_1 = u_2 = 1$.

To simplify our presentations, we will use standard matrix notations and operations. We will also need notations for nonlinear operations defined componentwise. In particular, if x is a column vector $\mathrm{col}(x_1, \ldots, x_n)$, we write $|x|$ and x^{γ} to represent $\mathrm{col}(|x_1|, \ldots, |x_n|)$ and $\mathrm{col}(x_1^{\gamma}, \ldots, x_n^{\gamma})$, respectively. We will also write $x > 0$ to denote $x_1 > 0, \ldots, x_n > 0$. The notations $x \geqslant 0$, $x > y$, etc., where x and y can also be matrices, are similarly defined.

In terms of the notations just mentioned, our boundary problem can be written in the compact form

$$J_n u = \lambda\,|u|^{\gamma}, \tag{3}$$

where $u = \mathrm{col}(u_1, \ldots, u_n)$ and the Jacobi matrix J_n has been given before. The boundary problem (1), (2) is equivalent to (3) in the sense that $\{0, u_1, u_2, \ldots, u_n, 0\}$ is a solution of (1), (2) if, and only if, $\mathrm{col}(u_1, u_2, \ldots, u_n)$ is a solution of (3).

The Jacobi matrix is invertible, as is well known, and its inverse $J_n^{-1} = (g_{ij})$ is given by

$$g_{ij} = \begin{cases} j(n+1-i)/(n+1), & 1 \leqslant j \leqslant i \leqslant n, \\ i(n+1-j)/(n+1), & 1 \leqslant i \leqslant j \leqslant n. \end{cases}$$

Clearly, each component of J_n^{-1} is positive. As a consequence, $J_n^{-1}x > 0$ for any nonnegative and nontrivial column vector x. This implies that if we write (3) in the equivalent form

$$u = \lambda J_n^{-1} |u|^\gamma, \tag{4}$$

then we see that a nontrivial solution $u = \mathrm{col}(u_1, u_2, \ldots, u_n)$ of (3) must be positive, i.e., $u > 0$.

We first derive an existence and uniqueness theorem for the positive solutions of (4) when $\lambda = 1$.

Theorem 1. *The nonlinear system*

$$u = J_n^{-1} u^\gamma \tag{5}$$

has a unique positive solution.

Proof. Since the eigenvalues and eigenvectors of the Jacobi matrix J_n are known, it is easily checked that the eigenvalue problem

$$J_n^{-1} v = \tau v$$

has a positive eigenvalue τ and a corresponding eigenvector $v > 0$. Let

$$u_0 = \tau^{1/(1-\gamma)} \left(\frac{v}{\max v} \right)^{1/\gamma}$$

and

$$w_0 = \tau^{1/(1-\gamma)} \left(\frac{v}{\min v} \right)^{1/\gamma}.$$

Further let

$$u_{m+1} = J_n^{-1} u_m^\gamma, \quad m = 0, 1, 2, \ldots \tag{6}$$

and

$$w_{m+1} = J_n^{-1} w_m^\gamma, \quad m = 0, 1, 2, \ldots \tag{7}$$

We assert that

$$0 < u_0 \leqslant u_k \leqslant u_{k+1} \leqslant w_{k+1} \leqslant w_k \leqslant w_0$$

for any $k = 1, 2, \ldots$ The fact that $0 < u_0 \leqslant w_0$ is clear from the definitions of u_0 and w_0. Next,

$$u_1 = J_n^{-1} u_0^{\gamma} = \tau^{\gamma/(1-\gamma)} J_n^{-1} \frac{v}{\max\ v}$$
$$= \tau^{\gamma/(1-\gamma)} \tau \frac{v}{\max\ v} \geqslant \tau^{1/(1-\gamma)} \left(\frac{v}{\max\ v}\right)^{1/\gamma} = u_0.$$

Now that we have shown $u_0 \leqslant u_1$. Then $u_1 = J_n^{-1} u_0 \leqslant J_n^{-1} u_1 = u_2$. By induction, it is then clear that $u_k \leqslant u_{k+1}$ for $k = 0, 1, \ldots$ Similarly, we can show that $w_{k+1} \leqslant w_k$ for $k = 0, 1, \ldots$ Finally, $u_1 = J_n^{-1} u_0 \leqslant J_n^{-1} w_0 = w_1$ and by induction, $u_k \leqslant w_k$ for $k = 1, 2, 3, \ldots$ Our assertion is thus true.

Let u be the (positive) limit of the nondecreasing and bounded sequence $\{u_m\}$, and let w be the (positive) limit of the nonincreasing and bounded sequence $\{w_m\}$. Taking limits on both sides of (6) and (7), we see that $u = J_n^{-1} u^{\gamma}$ and $w = J_n^{-1} w^{\gamma}$, and hence they are positive solutions of (5).

In order to show uniqueness, let $u = \mathrm{col}(u_1, \ldots, u_n)$ and $w = \mathrm{col}(w_1, \ldots, w_n)$ be the two positive vectors such that $u_i < w_i$ for some i in $\{1, 2, \ldots, k\}$. We assert that there exists a positive number $\delta_0 \in (0, 1)$ such that $u \geqslant \delta_0 w$ and $u \ngeqslant \delta w$ if $\delta > \delta_0$. Indeed, take

$$\delta_0 = \min_{1 \leqslant k \leqslant n} \frac{u_k}{w_k} = \frac{u_d}{w_d},$$

which belongs to $(0, 1)$ since $u_i < w_i$. Then $u \geqslant \delta_0 w$ since

$$u_k = \frac{u_k}{w_k} w_k \geqslant \delta_0 w_k, \quad k = 1, 2, \ldots, n.$$

Furthermore, if $\delta > \delta_0$, then $u_d = \delta_0 w_d < \delta w_d$, which shows that $u \ngeqslant \delta w$.

Now let $u = \mathrm{col}(u_1, \ldots, u_n)$ and $w = \mathrm{col}(w_1, \ldots, w_n)$ be two positive solutions of (5) if $u \neq w$. We may assume without loss of generality that $u_i < w_i$ for some i in $\{1, 2, \ldots, n\}$. Let $\delta_0 \in (0, 1)$ such that $u \geqslant \delta_0 w$ and $u \ngeqslant \delta w$ if $\delta > \delta_0$. Then

$$u = J_n^{-1} u^{\gamma} \geqslant J_n^{-1} (\delta_0 w)^{\gamma} = \delta_0^{\gamma} J_n^{-1} w^{\gamma} = \delta_0^{\gamma} w.$$

But since $\delta_0^{\gamma} > \delta_0$, a contradiction is obtained. The proof is complete.

Next, note that if we let

$$u = \lambda^{1/(1-\gamma)} v,$$

then substituting it into (4), we have

$$\lambda^{1/(1-\gamma)} v = \lambda J_n^{-1} \left| \lambda^{1/(1-\gamma)} v \right|^{\gamma}$$

or

$$v = J_n^{-1} |v|^{\gamma}.$$

This shows that if v is a solution of (5), then $\lambda^{1/(1-\gamma)} v$ is a solution of (4). It is easily shown that the converse is also true. As a consequence, for each $\lambda > 0$, the nonlinear system (4) has a unique (positive) solution v, which is given by

$$v = \left(\frac{1}{\lambda}\right)^{1/(1-\gamma)} u, \tag{8}$$

where u is the unique positive solution of (5).

As an application, we remark that "symmetry" shows up in our boundary problem (1), (2). Indeed, if $u = \mathrm{col}(u_1, \dots, u_n)$ is the unique positive solution of (5), then the vector $\mathrm{col}(u_n, \dots, u_1)$ is also a positive solution of (5), as can be verified directly. Thus, $u_k = u_{n+1-k}$ for $k = 1, \dots, n$. In other words, u is a symmetric vector. □

3. Comparison theorems

The unique nontrivial solution v of (4) and the unique positive solution u of (5) depend on the parameters λ, γ and n. We will derive several results for comparing solutions corresponding to different values of the parameters. First of all, when γ and n are fixed, it is clear from (8) that when $0 < \lambda_1 < \lambda_2$, the corresponding nontrivial solutions $v(\lambda_1)$ and $v(\lambda_2)$ of (4) satisfy

$$v(\lambda_1) < v(\lambda_2).$$

Next, let $0 < \gamma_1 < \gamma_2 < 1$. We assert that the corresponding positive solutions $u(\gamma_1)$ and $u(\gamma_2)$ of (5) satisfy

$$u(\gamma_1) < u(\gamma_2).$$

To see this, recall from the proof of Theorem 1 that $u(\gamma_1) = \lim_{n\to\infty} u_m(\gamma_1)$, where

$$u_0(\gamma_1) = \tau^{1/(1-\gamma_1)} \left(\frac{v}{\max v}\right)^{1/\gamma_1},$$

$$u_{m+1}(\gamma_1) = J_n^{-1} u_m^{\gamma_1}(\gamma_1), \quad m = 0, 1, 2, \dots$$

and $u(\gamma_2) = \lim_{n\to\infty} u_m(\gamma_2)$, where

$$u_0(\gamma_2) = \tau^{1/(1-\gamma_2)} \left(\frac{v}{\max v}\right)^{1/\gamma_2},$$

$$u_{m+1}(\gamma_2) = J_n^{-1} u_m^{\gamma_2}(\gamma_2), \quad m = 0, 1, 2, \dots$$

Since $u_0(\gamma_1) < u_0(\gamma_2)$, we see that

$$u_1(\gamma_1) = J_n^{-1} u_0^{\gamma_1}(\gamma_1) \leqslant J_n^{-1} u_0^{\gamma_2}(\gamma_2) = u_1(\gamma_2)$$

and by induction that $u_m(\gamma_1) \leqslant u_m(\gamma_2)$ for $m = 1, 2, \dots$ Hence, $u(\gamma_1) \leqslant u(\gamma_2)$. To see that $u(\gamma_1) < u(\gamma_2)$, we assume to the contrary that $u_i(\gamma_1) = u_i(\gamma_1)$ for some i in $\{1, 2, \dots, n\}$. Then

$$u_{i+1}(\gamma_1) - 2u_i(\gamma_1) + u_{i-1}(\gamma_1) + u_i^{\gamma_1}(\gamma_1) = 0$$

and

$$u_{i+1}(\gamma_2) - 2u_i(\gamma_2) + u_{i-1}(\gamma_2) + u_i^{\gamma_2}(\gamma_2) = 0,$$

so that

$$0 \leqslant u_{i+1}(\gamma_2) - u_{i+1}(\gamma_1) + u_{i-1}(\gamma_2) - u_{i-1}(\gamma_1) = u_i^{\gamma_1}(\gamma_1) - u_i^{\gamma_2}(\gamma_2) \leqslant 0.$$

But then

$$u_i^{\gamma_1}(\gamma_1) = u_i^{\gamma_2}(\gamma_2),$$

which is contrary to our assumption that $\gamma_1 < \gamma_2$. Now that we have shown $u(\gamma_1) < u(\gamma_2)$ when $0 < \gamma_1 < \gamma_2 < 1$. In view of (8), we see that the corresponding nontrivial solutions $v(\gamma_1)$ and $v(\gamma_2)$ of (4) satisfy $v(\gamma_1) < v(\gamma_2)$.

Next, let n, m be positive integers such that $1 \leqslant n < m$, and let $u(n) = \mathrm{col}(u_1(n), \dots, u_n(n))$ and $u(m) = \mathrm{col}(u_1(m), \dots, u_m(m))$ be the corresponding positive solutions of (5). We assert that

$$u_k(n) < u_k(m), \quad k = 1, 2, \dots, n. \tag{9}$$

Indeed, assume to the contrary that (9) is not true. Since $u_{n+1}(n) = 0 < u_{n+1}(m)$, we see that there is an integer j in $\{1, 2, \dots, n\}$ such that

$$u_n(m) > u_m(n), u_{m-1}(m) > u_{m-1}(n), \dots, u_{j+1}(m) > u_{j+1}(n),$$

but

$$u_j(m) \leqslant u_j(n).$$

There are two cases to consider. First suppose $u_j(m) < u_j(n)$ and $u_{j+1}(m) > u_{j+1}(n)$. Then there is a positive number β and a nonpositive number α such that

$$u_{j+1}(m) = \beta + \alpha u_j(m)$$

and

$$u_{j+1}(n) = \beta + \alpha u_j(m).$$

Thus, $\{0, u_1(m), \dots, u_{j+1}(m)\}$ and $\{0, u_1(n), \dots, u_{j+1}(n)\}$ are two solutions of the boundary problem

$$\Delta^2 w_{k-1} + w_k^{\gamma} = 0, \quad k = 1, 2, \dots, j, \tag{10}$$

$$w_0 = 0, \qquad w_{j+1} = \beta + \alpha w_j. \tag{11}$$

The boundary problem (10), (11) can be written as

$$\tilde{J}_j w = w^{\gamma} + \mathrm{col}(0, \dots, 0, \beta), \tag{12}$$

where $w = \mathrm{col}(w_1, w_2, \dots, w_j)$ and

$$\tilde{J}_j = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2-\alpha \end{bmatrix}_{j \times j}.$$

It is easily shown that the matrix $\tilde{J}_j$ is invertible and its inverse has positive components [3]. As in the proof of Theorem 1, we now show that the two positive solutions $\tilde{u}(m) = \mathrm{col}(u_1(m), \ldots, u_j(m))$ and $\tilde{u}(n) = \mathrm{col}(u_1(n), \ldots, u_j(n))$ of (12) must be identical. Otherwise, we may assume without loss of generality that $\tilde{u}_i(m) > \tilde{u}_i(n)$ for some i in $\{1, 2, \ldots, j\}$. Then there exists a $\delta_0 \in (0, 1)$ such that $\tilde{u}(m) \geqslant \delta_0 \tilde{u}(n)$, but $\tilde{u}(m) \ngeqslant \delta \tilde{u}(n)$ if $\delta > \delta_0$. Thus,

$$\begin{aligned}\tilde{u}(m) &= \tilde{J}_j^{-1}\left(\tilde{u}^{\gamma}(m) + \mathrm{col}(0, \ldots, 0, \beta)\right)\\ &> \tilde{J}_j^{-1}\left(\delta_0^{\gamma}\tilde{u}^{\gamma}(n) + \delta_0^{\gamma}\ \mathrm{col}(0, \ldots, 0, \beta)\right)\\ &= \delta_0^{\gamma}\tilde{J}_j^{-1}\left(\tilde{u}^{\gamma}(n) +\ \mathrm{col}(0, \ldots, 0, \beta)\right)\\ &= \delta_0^{\gamma}\tilde{u}(n),\end{aligned}$$

which is a contradiction. Finally, the fact that $\tilde{u}(m) = \tilde{u}(n)$ is contrary to our assumption that $u_j(m) < u_j(n)$.

Next, if the case $u_j(m) = u_j(n)$ holds, then by arguments similar to those just described, we see that $u_k(m) = u_k(n)$ for $k = 1, \ldots, j$. But then,

$$\begin{aligned}u_{j+1}(m) &= 2u_j(m) - u_{j-1}(m) + u_j^{\gamma}(m)\\ &= 2u_j(n) - u_{j-1}(n) + u_j^{\gamma}(n)\\ &= u_{j+1}(n),\end{aligned}$$

which is contrary to our assumption that $u_{j+1}(m) > u_{j+1}(m)$.

We summarize the above discussions as follows.

Theorem 2. *Let* $v = v(\lambda, \gamma, n) = \mathrm{col}(v_1(\lambda, \gamma, n), \ldots, v_n(\lambda, \gamma, n))$ *be the unique nontrivial* (*positive*) *solution of* (4). *Then*

$$v(\lambda_1, \gamma, n) < v(\lambda_2, \gamma, n), \quad 0 < \lambda_1 < \lambda_2,$$

$$v(\lambda, \gamma_1, n) < v(\lambda, \gamma_2, n), \quad 0 < \gamma_1 < \gamma_2 < 1,$$

$$v_k(\lambda, \gamma, n) < v_k(\lambda, \gamma, m), \quad k = 1, 2, \ldots, n,\ 1 \leqslant n < m.$$

4. Additional properties and remarks

We have already shown that the unique nontrivial solution v of (4) is positive and symmetric. We have also shown in the proof of Theorem 1 that the unique positive solution of (5) is bounded between

$$\tau^{1/(1-\gamma)}\left(\frac{v}{\max\ v}\right)^{1/\gamma}$$

and

$$\tau^{1/(1-\gamma)}\left(\frac{v}{\min\ v}\right)^{1/\gamma},$$

where v is a positive eigenvector of J_n^{-1} and τ its corresponding positive eigenvalue.

There are a number of additional properties which may be useful. Let $u = \text{col}(u_1, \ldots, u_n)$ be the unique positive solution of (5). Since $\Delta^2 u_{k-1} = -u_k^\gamma < 0$ for $k = 1, 2, \ldots, n$, u is a strictly concave vector. It is not difficult to see that any symmetric, positive and strictly concave vector must be symmetrically decreasing. In other words, if $n = 2m$, then $u_1 < u_2 < \cdots < u_{m-1} < u_m = u_{m+1}$ and $u_{m+1} > u_{m+2} > \cdots > u_n$; if $n = 2m + 1$, then $u_1 < u_2 < \cdots < u_{m-1} < u_m$ and $u_m > u_{m+1} > u_{m+2} > \cdots > u_n$.

We can also establish an a priori bound for the unique positive solution $u = \text{col}(u_1, \ldots, u_n)$ of (5). First of all, since

$$u_k^\gamma = -u_{k-1} + 2u_k - u_{k+1}, \quad k = 1, 2, \ldots, n,$$

if we divide both sides by $u_k^{\gamma-1}$, we obtain

$$u_k^{\gamma-1} = 2 - \left(\frac{u_{k-1}}{u_k} + \frac{u_{k+1}}{u_k}\right), \quad k = 1, 2, \ldots, n.$$

Thus,

$$\begin{aligned}\sum_{k=1}^n u_k^{\gamma-1} &= 2n - \left\{\left(\frac{u_2}{u_1} + \frac{u_1}{u_2}\right) + \cdots + \left(\frac{u_n}{u_{n-1}} + \frac{u_{n-1}}{u_n}\right)\right\}\\ &\leqslant 2n - \{2 + \cdots + 2\}\\ &= 2.\end{aligned}$$

We remark that the number 2 is sharp when $n = 2$. This is due to the fact, as seen at the beginning of Section 2, that the corresponding unique solution, when $n = 2$, is given by $\text{col}(u_1, u_2) = (1, 1)$.

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