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# On $n$-fold $L(j,k)$-and circular $L(j,k)$-labelings of graphs ☆

Wensong Lin *, Pu Zhang

*Department of Mathematics, Southeast University, Nanjing 210096, PR China*

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ABSTRACT

We initiate research on the multiple distance 2 labeling of graphs in this paper.

Let $n, j, k$ be positive integers. An *$n$-fold $L(j,k)$-labeling* of a graph $G$ is an assignment $f$ of sets of nonnegative integers of order $n$ to the vertices of $G$ such that, for any two vertices $u, v$ and any two integers $a \in f(u)$, $b \in f(v)$, $|a-b| \geq j$ if $uv \in E(G)$, and $|a-b| \geq k$ if $u$ and $v$ are distance 2 apart. The span of $f$ is the absolute difference between the maximum and minimum integers used by $f$. The $n$-fold $L(j,k)$-labeling number of $G$ is the minimum span over all $n$-fold $L(j,k)$-labelings of $G$.

Let $n, j, k$ and $m$ be positive integers. An $n$-fold circular $m$-$L(j,k)$-labeling of a graph $G$ is an assignment $f$ of subsets of $\{0, 1, \ldots, m-1\}$ of order $n$ to the vertices of $G$ such that, for any two vertices $u, v$ and any two integers $a \in f(u)$, $b \in f(v)$, $\min\{|a-b|, m-|a-b|\} \geq j$ if $uv \in E(G)$, and $\min\{|a-b|, m-|a-b|\} \geq k$ if $u$ and $v$ are distance 2 apart. The minimum $m$ such that $G$ has an $n$-fold circular $m$-$L(j,k)$-labeling is called the $n$-fold circular $L(j,k)$-labeling number of $G$.

We investigate the basic properties of $n$-fold $L(j,k)$-labelings and circular $L(j,k)$-labelings of graphs. The $n$-fold circular $L(j,k)$-labeling numbers of trees, and the hexagonal and $p$-dimensional square lattices are determined. The upper and lower bounds for the $n$-fold $L(j,k)$-labeling numbers of trees are obtained. In most cases, these bounds are attainable. In particular, when $k=1$ both the lower and the upper bounds are sharp. In many cases, the $n$-fold $L(j,k)$-labeling numbers of the hexagonal and $p$-dimensional square lattices are determined. In other cases, upper and lower bounds are provided. In particular, we obtain the exact values of the $n$-fold $L(j,1)$-labeling numbers of the hexagonal and $p$-dimensional square lattices.



## 1. Introduction

Motivated from the channel assignment problem introduced by Hale [13], Griggs and Yeh [12] were the first to propose and study the $L(2,1)$-labelings of graphs. Since then the $L(2,1)$-labelings and the general case $L(j,k)$-labelings of graphs have been studied extensively; refer to the surveys [2,25,10]. In Griggs and Yeh's model, vertices of the graph represent transmitters, and the label of a vertex represents the radio channel assigned to the corresponding transmitter. Each transmitter is assigned exactly one radio channel. However, in practice, each transmitter may demand more than one radio channel. From this case, the multiple $L(j,k)$-labeling of a graph arises. In this paper, we assume that each transmitter demands the same number of channels. For a positive integer $n$, the $n$-fold $L(j,k)$-labeling of a graph $G$ is defined as follows.

For two sets of nonnegative integers $I$ and $I'$, the distance between $I$ and $I'$, $d(I,I')$, is defined as $\min\{|i-i'| : i \in I, i' \in I'\}$. Let $n, j, k$ be positive integers. An *$n$-fold $L(j,k)$-labeling* of a graph $G$ is an assignment $f$ of sets of nonnegative integers of

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* Corresponding author.
*E-mail address:* wslin@seu.edu.cn (W. Lin).

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order $n$ to the vertices of $G$ such that, for any two vertices $u$ and $v$, $d(f(u), f(v)) \geq j$ if $uv \in E(G)$ (this is called the distance 1 condition), and $d(f(u), f(v)) \geq k$ if $u$ and $v$ are distance 2 apart (this is called the distance 2 condition). Given a graph $G$, for an $n$-fold $L(j, k)$-labeling $f$ of $G$, the images of $f$ are called *label sets* and the numbers used by $f$ are called *labels*, and we define the *span* of $f$, span($f$), to be the absolute difference between the maximum and minimum numbers used by $f$. Without loss of generality we shall assume that the minimum number used in an $n$-fold $L(j, k)$-labeling $f$ of $G$ is always 0, implying that the span of the labeling is the maximum number assigned to a vertex under $f$. The *$n$-fold $L(j, k)$-labeling number* of $G$, denoted by $\lambda^n_{j,k}(G)$, is the minimum span over all $n$-fold $L(j, k)$-labelings of $G$. The onefold $L(j, k)$-labeling number of $G$, $\lambda^1_{j,k}(G)$, is equivalent to the *$L(j, k)$-labeling number* of $G$, $\lambda_{j,k}(G)$, which has been studied extensively [2,7,8,16,15].

A useful approach in investigating the $n$-fold $L(j, k)$-labeling of a graph $G$ is to consider the circular $L(j, k)$-labeling of $G$ which we define below. Suppose $m$ is a positive integer. Let $S(m)$ denote a circle of circumference $m$. We fix a point on $S(m)$ and label it with 0. We label each point on $S(m)$ with a real number $x \in [0, m)$ according to the length of the arc from 0 along the clockwise direction on $S(m)$ to this point. For any $r \in \mathbb{R}$, $[r]_m \in [0, m)$ denotes the remainder of $r$ upon division of $m$.

In this paper, we are interested in the integer points $0, 1, \ldots, m-1$ on the circle $S(m)$. Let $l$ be an integer and $n$ a positive integer. We use $S^n_m(l)$ to denote the set of $n$ consecutive integer points $[l]_m, [l+1]_m, \ldots, [l+n-1]_m$ on $S(m)$. Let $a$ and $b$ be two integers with $0 \leq a, b < m$. We use $[a, b]_m$ to denote the set of integer points $a, a+1, \ldots, b$ on $S(m)$, where additions are taken modulo $m$. Let $(a, b)_m$ denote the set of integer points $a+1, a+2, \ldots, b-1$ on $S(m)$. $[a, b)_m$ and $(a, b]_m$ are defined similarly. We call $[a, b]_m$ ($(a, b)_m$) a *closed interval* (*an open interval*) of $S(m)$. A sequence $(a_0, a_1, \ldots, a_s)$ of different points on $S(m)$ is said to be in *cyclic order* if the sets $(a_0, a_1)_m, (a_1, a_2)_m, \ldots, (a_{s-1}, a_s)_m, (a_s, a_0)_m$ are pairwise disjoint.

The *circular distance* of two numbers (or two points on $S(m)$) $a$ and $b$ with $0 \leq a, b < m$ on $S(m)$, denoted by $|a-b|_m$, is defined as $\min\{|a-b|, m-|a-b|\}$. Suppose $S$ and $Q$ are two sets of points on $S(m)$. The *circular distance* between $P$ and $Q$, denoted by $d_m(P, Q)$, is defined as $\min\{|p-q|_m : p \in P, q \in Q\}$.

Let $n, j, k$ and $m$ be positive integers. An *$n$-fold circular $m$-$L(j, k)$-labeling* of a graph $G$ is an assignment $f$ of subsets of $\{0, 1, \ldots, m-1\}$ of order $n$ to the vertices of $G$ such that, for any two vertices $u$ and $v$, $d_m(f(u), f(v)) \geq j$ if $uv \in E(G)$, and $d_m(f(u), f(v)) \geq k$ if $u$ and $v$ are distance 2 apart. The minimum $m$ such that $G$ has an $n$-fold circular $m$-$L(j, k)$-labeling is called the *$n$-fold circular $L(j, k)$-labeling number* of $G$ and is denoted by $\sigma^n_{j,k}(G)$. The onefold circular $L(j, k)$-labeling number of $G$, $\sigma^1_{j,k}(G)$, is equivalent to the *circular $L(j, k)$-labeling number* of $G$, which has been investigated in many papers; see [23,19,24,20,17,16,21].

In an $n$-fold $L(j, k)$-labeling $f$ of a graph $G$, if each vertex of $G$ receives $n$ consecutive integers (that is for each vertex $u$ of $G$ there is some integer $t \geq 0$ such that $f(u) = \{t, t+1, \ldots, t+n-1\}$), then we get an *$n$-fold consecutive $L(j, k)$-labeling* of $G$. The *$n$-fold consecutive $L(j, k)$-labeling number* of $G$, denoted by $\bar{\lambda}^n_{j,k}(G)$, is the minimum span over all $n$-fold consecutive $L(j, k)$-labelings of $G$. We have $\lambda^n_{j,k}(G) \leq \bar{\lambda}^n_{j,k}(G)$ for any graph $G$.

In an $n$-fold circular $m$-$L(j, k)$-labeling $f$ of a graph $G$, if each vertex of $G$ receives $n$ consecutive integer points on $S(m)$ (that is for each vertex $u$ of $G$ there is some integer $t \in [0, m-1]$ such that $f(u) = S^n_m(t)$), then we get an *$n$-fold consecutive circular $m$-$L(j, k)$-labeling* of $G$. The minimum $m$ such that $G$ has an $n$-fold consecutive circular $m$-$L(j, k)$-labeling is called the *$n$-fold consecutive circular $L(j, k)$-labeling number* of $G$ and is denoted by $\bar{\sigma}^n_{j,k}(G)$. It follows that $\sigma^n_{j,k}(G) \leq \bar{\sigma}^n_{j,k}(G)$ for any graph $G$.

Throughout this paper $j$, $k$ and $n$ are positive integers with $j \geq k$. Let $a$ and $b$ be two integers with $a \leq b$. We denote by $[a, b]$ the set of integers $a, a+1, \ldots, b$.

In the next section, we investigate basic properties of $n$-fold $L(j, k)$-labelings and circular $L(j, k)$-labelings of graphs. In particular, we establish the relationship between $\lambda^n_{j,k}(G)$ and $\lambda_{j+n-1,k+n-1}(G)$, and the relationship between $\sigma^n_{j,k}(G)$ and $\sigma_{j+n-1,k+n-1}(G)$. In Section 3, we present a class of graphs with $\sigma^n_{j,1}(G) < \bar{\sigma}^n_{j,1}(G)$ and $\lambda^n_{j,1}(G) < \bar{\lambda}^n_{j,1}(G)$. In Section 4, we determine $\sigma^n_{j,k}(T)$ for any tree $T$ and provide upper and lower bounds for $\lambda^n_{j,k}(T)$. The upper bound is attainable in many cases and is sharp in the case $k = 1$. The $n$-fold circular $L(j, k)$-labeling numbers and $n$-fold $L(j, 1)$-labeling numbers of the hexagonal and $p$-dimensional square lattices are determined in Sections 5 and 6, respectively. We mention that the $n$-fold $L(j, 1)$-labeling number and $n$-fold circular $L(j, 1)$-labeling number of a triangular lattice were investigated in [26].

## 2. Some basic properties and notation

Let $f$ be an $n$-fold consecutive $L(j, k)$-labeling of a graph $G$ with span $\lambda$. We may obtain an $L(j+n-1, k+n-1)$-labeling $f'$ of $G$ with span $\lambda - n + 1$ by letting $f'(u) = t$ if $f(u) = [t, t+n-1]$ for each vertex $u$ of $G$. Conversely, we may obtain an $n$-fold consecutive $L(j, k)$-labeling of a graph $G$ with span $\lambda$ from an $L(j+n-1, k+n-1)$-labeling of $G$ with span $\lambda - n + 1$. In a similar way, one can find a one-to-one correspondence between an $n$-fold consecutive circular $m$-$L(j, k)$-labeling and a circular $m$-$L(j+n-1, k+n-1)$-labeling of a graph $G$. Therefore we have the following lemma.

**Lemma 2.1.** *For any graph $G$,*

$$\lambda^n_{j,k}(G) \leq \bar{\lambda}^n_{j,k}(G) = \lambda_{j+n-1,k+n-1}(G) + n - 1,$$

*and*

$$\sigma^n_{j,k}(G) \leq \bar{\sigma}^n_{j,k}(G) = \sigma_{j+n-1,k+n-1}(G).$$

In [23] van den Heuvel, Leese, and Shepherd noted that, for any graph $G$, if $j \geq k$ then

$$\lambda_{j,k}(G) + 1 \leq \sigma_{j,k}(G) \leq \lambda_{j,k}(G) + j. \quad (1)$$

We point out that an $n$-fold circular $m$-$L(j, k)$-labeling of a graph is an $n$-fold $L(j, k)$-labeling with span $m - 1$, and an $n$-fold $L(j, k)$-labeling with span $\lambda$ is an $n$-fold circular $(\lambda + j)$-$L(j, k)$-labeling. Thus we have

$$\lambda^n_{j,k}(G) + 1 \leq \sigma^n_{j,k}(G) \leq \lambda^n_{j,k}(G) + j \quad (2)$$

By Lemma 2.1 and formula (2),

$$\lambda^n_{j,k}(G) \leq \min \{ \lambda_{j+n-1,k+n-1}(G) + n - 1,\ \sigma_{j+n-1,k+n-1}(G) - 1\}. \quad (3)$$

Let $G$ and $H$ be two graphs. The *lexicographic product* of $G$ and $H$ is the graph $G[H]$ with vertex set $V(G) \times V(H)$ and two vertices $(u, x)$ and $(v, y)$ are adjacent in $G[H]$ if and only if $uv \in E(G)$ or $u = v$ with $xy \in E(H)$. Let $\overline{K}_n$ denote the empty graph on $n$ vertices. If $G$ is nontrivial and nonempty, then it is clear that an $n$-fold $L(j, 1)$-labeling of a graph $G$ corresponds to an $L(j, 1)$-labeling of $G[\overline{K}_n]$, and so $\lambda^n_{j,1}(G) = \lambda_{j,1}(G[\overline{K}_n])$. Similarly, an $n$-fold circular $m$-$L(j, 1)$-labeling of a graph $G$ corresponds to a circular $m$-$L(j, 1)$-labeling of $G[\overline{K}_n]$, and so $\sigma^n_{j,1}(G) = \sigma_{j,1}(G[\overline{K}_n])$.

Let $G$ be a graph and $u$ a vertex of $G$. We call a vertex $v$ a *d-neighbor* of $u$ in $G$ if $d_G(u, v) = d$.

Let $f$ be an $n$-fold $L(j, k)$-labeling of $G$ with span $\lambda$. For any vertex $u$ of $G$ and a positive integer $t \geq 2$, we define the *t-closure* of $f(u)$, denoted by $[f(u)]^t$, as

$$[f(u)]^t = \{h \pm i : \text{for each } h \in f(u) \text{ and } i = 0, 1, \ldots, t - 1\} \cap [0, \lambda].$$

It follows that $[f(u)]^t$ is the disjoint union of the sets $[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p]$ for some positive integer $p$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$ and $|[b_i, a_{i+1}]| \geq 3$ for $i = 1, 2, \ldots, p - 1$. It is clear that, for each $i \in [2, p - 1]$, there are at least $2(t - 1)$ labels in $[a_i, b_i] \setminus f(u)$. All labels in the $j$-closure of $f(u)$ are forbidden for any neighbor of $u$ and all labels in the $k$-closure of $f(u)$ are forbidden for any 2-neighbor of $u$.

Let $u$ be a vertex and $[f(u)]^t$ be the disjoint union of the sets $[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p]$ with $a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$ and $|[b_i, a_{i+1}]| \geq 3$ for $i = 1, 2, \ldots, p - 1$. The *net t-closure* of $f(u)$, denoted by $[\widehat{f(u)}]^t$, is defined as the union of the sets $[\min f(u), b_1 - t + 1], [a_2 + t - 1, b_2 - t + 1], \ldots, [a_p + t - 1, \max f(u)]$. Clearly $f(u) \subseteq [\widehat{f(u)}]^t$.

Let $f$ be an $n$-fold circular $m$-$L(j, k)$-labeling of $G$. For any vertex $u$ of $G$ and a positive integer $t \geq 2$, we define the *circular t-closure* of $f(u)$, denoted by $[f(u)]^t_m$, as

$$[f(u)]^t_m = \{[h \pm i]_m : \text{for each } h \in f(u) \text{ and } i = 0, 1, \ldots, t - 1\}.$$

It follows that $[f(u)]^t_m$ is the disjoint union of the sets $[a_0, b_0]_m, [a_1, b_1]_m, \ldots, [a_{p-1}, b_{p-1}]_m$ for some positive integer $p$ with $(a_0, a_1, \ldots, a_{p-1})$ in cyclic order on $S(m)$ and $|[b_i, a_{i+1}]_m| \geq 3$ for $i = 0, 1, \ldots, p - 1$ (where "+" in the subscript is taken modulo $p$). For each $i$, there are at least $2(t-1)$ labels in $[a_i, b_i]_m \setminus f(u)$. All labels in the circular $j$-closure of $f(u)$ are forbidden for any neighbor of $u$ and all labels in the circular $k$-closure of $f(u)$ are forbidden for any 2-neighbor of $u$.

Let $u$ be a vertex and $[f(u)]^t_m$ be the disjoint union of the sets $[a_0, b_0]_m, [a_1, b_1]_m, \ldots, [a_{p-1}, b_{p-1}]_m$ (where $(a_0, a_1, \ldots, a_{p-1})$ is in cyclic order) on $S(m)$. The *net circular t-closure* of $f(u)$, denoted by $[\widehat{f(u)}]^t_m$, is defined as the union of the sets $[a_0 + t - 1, b_0 - t + 1]_m, [a_1 + t - 1, b_1 - t + 1]_m, \ldots, [a_{p-1} + t - 1, b_{p-1} - t + 1]_m$. Clearly $f(u) \subseteq [\widehat{f(u)}]^t_m$.

## 3. Graphs $G$ with $\bar{\sigma}^n_{j,k}(G) > \sigma^n_{j,k}(G)$ and $\bar{\lambda}^n_{j,k}(G) > \lambda^n_{j,k}(G)$

In general $\lambda^n_{j,k}(G)$ (respectively, $\sigma^n_{j,k}(G)$) is not equal to $\bar{\lambda}^n_{j,k}(G)$ (respectively, $\bar{\sigma}^n_{j,k}(G)$), though in many cases as we shall see later they are equal to each other.

To illustrate, consider $K_p$, the complete graph on $p$ vertices. Since any two vertices of $K_p$ are adjacent, in an $n$-fold $L(j, k)$-labeling (respectively, $n$-fold circular $L(j, k)$-labeling) of $G$, any two vertices of $K_p$ have label sets that are at distance (respectively, at circular distance) at least $j$. It follows that $\sigma^n_{j,k}(K_p) = \lambda^n_{j,k}(K_p) + j = (n + j - 1)p$.

Let $H$ be the graph formed by adding a pendant vertex to each of the vertices of $K_5$. Since $K_5$ is a subgraph of $H$, $\sigma^3_{5,2}(H) \geq \sigma^3_{5,2}(K_5) = 35$ and $\lambda^3_{5,2}(H) \geq \lambda^3_{5,2}(K_5) = 30$. On the other hand, Fig. 1 gives a threefold $L(5, 2)$-labeling of $H$ with span 30, which is also a threefold circular 35-$L(5, 2)$-labeling of $H$. Therefore $\sigma^3_{5,2}(H) = 35$ and $\lambda^3_{5,2}(H) = 30$.

Since in an optimal threefold $L(5, 2)$-labeling (respectively, circular $L(5, 2)$-labeling) of $K_5$ the labeling scheme is unique up to the permutation of vertices, it follows that $\bar{\sigma}^3_{5,2}(H) > 35$ and $\bar{\lambda}^3_{5,2}(H) > 30$. In fact, one can prove that $\bar{\sigma}^3_{5,2}(H) = 37$ and $\bar{\lambda}^3_{5,2}(H) = 32$. Fig. 2 indicates a threefold consecutive $L(5, 2)$-labeling of $H$ with span 32, which is also a threefold consecutive circular 37-$L(5, 2)$-labeling of $H$.

Later, we shall present a class of graphs $G_h$ and show that the difference $\bar{\sigma}^n_{j,1}(G_h) - \sigma^n_{j,1}(G_h)$ (respectively, $\bar{\lambda}^n_{j,1}(G_h) - \lambda^n_{j,1}(G_h)$) can be arbitrarily large. To obtain $\bar{\sigma}^n_{j,1}(G_h)$ and $\bar{\lambda}^n_{j,1}(G_h)$, by Lemma 2.1 it suffices to determine $\sigma_{j,k}(G_h)$ and $\lambda_{j,k}(G_h)$. In order to do so, we proceed by proving a lemma that is quite useful in determining $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ of graphs $G$ of diameter 2.

Given a graph $G$, the *path covering number* of $G$, denoted by $p_v(G)$, is the smallest number of vertex-disjoint paths covering $V(G)$. By $G^c$ we denote the complement graph of $G$. The following result was proved by Georges et al. [9].

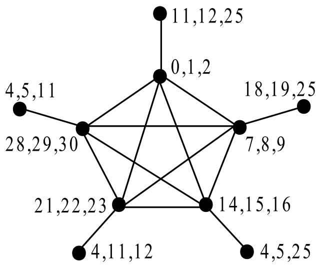


**Fig. 1.** A threefold $L(5, 2)$-labeling of $H$.

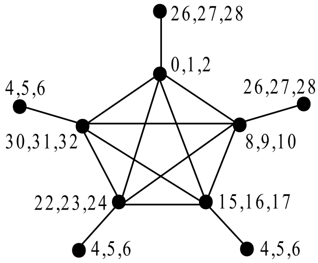


**Fig. 2.** A threefold consecutive $L(5, 2)$-labeling of $H$.

**Theorem 3.1.** *For any graph $G$,*

$$\lambda_{2,1}(G) \begin{cases} \leq |V(G)| - 1, & \text{if } p_v(G^c) = 1, \\ = |V(G)| + p_v(G^c) - 2, & \text{if } p_v(G^c) \geq 2. \end{cases}$$

We prove a similar result about $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ for graphs $G$ of diameter 2.

**Lemma 3.2.** *Let $G$ be a graph of diameter 2. If $j \leq 2k$, then*

$$\sigma_{j,k}(G) = \begin{cases} |V(G)|k, & \text{if } G^c \text{ has a Hamiltonian cycle}, \\ p_v(G^c)j + (|V(G)| - p_v(G^c))k, & \text{otherwise}; \end{cases}$$

*and*

$$\lambda_{j,k}(G) = (p_v(G^c) - 1)j + (|V(G)| - p_v(G^c))k.$$

**Proof.** Let $c = p_v(G^c)$, $p = |V(G)|$ and $m = cj + (p - c)k$. Since $G$ is of diameter 2, it is obvious that $\sigma_{j,k}(G) \geq pk$. If $G^c$ has a Hamiltonian cycle $v_0, v_1, \ldots, v_{p-1}$, then by assigning $ik$ to the vertex $v_i$ for $i = 0, 1, \ldots, p-1$ we get a circular $(pk)$-$L(j, k)$-labeling of $G$, implying $\sigma_{j,k}(G) = pk$.

Now suppose $G^c$ has no Hamiltonian cycle. Let $P_1, P_2, \ldots, P_c$ be $c$ paths in $G^c$ that form a minimum path covering of $G^c$. Denote by $p_i$ the number of vertices of $P_i$ for $1 \leq i \leq c$. For $1 \leq t \leq p_i$, let $v_{i,t}$ be the $t$th vertex along the path $P_i$. Define a function $f$ from $V(G)$ to $[0, m-1]$ as

$$f(v_{i,t}) = (i-1)j + \left[t - 1 + \sum_{s=1}^{i-1}(p_s - 1)\right]k.$$

Note that $j \leq 2k$, $f$ is obviously a circular $m$-$L(j, k)$-labeling of $G$. Thus $\sigma_{j,k}(G) \leq m$.

We next prove that $\sigma_{j,k}(G) \geq m$. Suppose $\sigma_{j,k}(G) = \sigma$ and let $f$ be a circular $\sigma$-$L(j, k)$-labeling of $G$. Since the diameter of $G$ is 2, any two vertices of $G$ receive labels on $S(\sigma)$ that are at circular distance at least $k$. With no loss of generality, we may assume the label 0 is always used. The sequence of labels used by $f$ in the increasing order is denoted by $\mathcal{F} = (f_0, f_1, \ldots, f_{p-1})$ (with $f_0 = 0$).

For any two numbers $x$ and $y$ in $[0, \sigma)$, denote by $\|y - x\|_\sigma$ the length of the arc from the point $x$ to the point $y$ along the clockwise direction on $S(\sigma)$.

A consecutive subsequence $(f_s, f_{s+1}, \ldots, f_t)$ (where additions in the subscripts are taken modulo $p$) of $\mathcal{F}$ with at least two terms is said to be *circularly-$j$-restricted* if $\|f_{i+1} - f_i\|_\sigma < j$ for $i = s, s+1, \ldots, t-1$. We also call the one-term subsequence $(f_s)$ *circularly $j$-restricted*. A circularly $j$-restricted subsequence is called maximal if it is not properly contained in any other circularly $j$-restricted subsequence. It follows that $\mathcal{F}$ is the disjoint union of maximal circularly $j$-restricted subsequences. Let $b$ be the number of disjoint maximal circularly $j$-restricted subsequences of $\mathcal{F}$. It is obvious that $\sigma \geq bj + (p-b)k$. On the other hand, it is easy to see that the sequence of vertices of $G$ corresponding to a maximal circularly $j$-restricted subsequence forms a path in $G^c$. This implies that $b \geq p_v(G^c)$. Consequently, $\sigma \geq bj + (p-b)k \geq m$, giving the result.

Through an argument similar to the one presented above, we can prove that $\lambda_{j,k}(G) = (p_v(G^c) - 1)j + (|V(G)| - p_v(G^c))k$. □

Let $h \geq 2$ be a positive integer. We define the graph $G_h$ as follows. The vertex set of $G_h$ is $\{u_0, u_1, \ldots, u_{h-1}\} \cup \{v_0, v_1, \ldots, v_{h-1}\}$. The subset $U = \{u_0, u_1, \ldots, u_{h-1}\}$ induces a clique of $G_h$ while the subset $V = \{v_0, v_1, \ldots, v_{h-1}\}$ forms an independent set of $G_h$. And for each $i = 0, 1, \ldots, h-1$, the vertex $u_i$ in $U$ is adjacent to all vertices in $V$ except the vertex $v_i$. A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set. It is clear that $G_h$ is a split graph. The problem of deciding whether $\lambda_{2,1}(G) \leq |V(G)|$ for split graphs $G$ was proved to be NP-complete in [1]. The $L(2,1)$-labeling number of $G_h$ was obtained in [18]. It is worth noting that $G_h$ and the graph $H$ defined at the beginning of this section are also matrogenic graphs defined in [22]. $L(2,1)$-labelings of such graphs were investigated in [4]. In the following, we determine $\sigma_{j,k}(G_h)$ and $\lambda_{j,k}(G_h)$.

**Theorem 3.3.**

$$\sigma_{j,k}(G_h) = \begin{cases} \lceil h/2 \rceil j + (2h - \lceil h/2 \rceil)k, & \text{if } j/k \leq 2, \\ hj + \lceil h/2 \rceil k, & \text{if } j/k \geq 2, \end{cases}$$

*and*

$$\lambda_{j,k}(G_h) = \begin{cases} (\lceil h/2 \rceil - 1)j + (2h - \lceil h/2 \rceil)k, & \text{if } j/k \leq 2, \\ (h-1)j + \lceil h/2 \rceil k, & \text{if } j/k \geq 2. \end{cases}$$

**Proof.** Due to the structure of $G_h^c$, it is not difficult to see that $p_v(G_h^c) = \lceil h/2 \rceil$. Then the case $j/k \leq 2$ follows from Lemma 3.2.

Now suppose $j/k \geq 2$. Let $m = hj + \lceil h/2 \rceil k$. The function $f$ from $V(G_h)$ to $[0, m-1]$ defined by

$$\begin{aligned} f(u_i) &= ij + \lceil i/2 \rceil k, && \text{for } i \in [0, h-1], \\ f(v_i) &= ij + (i/2 + 1)k, && \text{if } i \in [0, h-1] \text{ and } i \text{ is even}, \\ f(v_i) &= ij + \lfloor i/2 \rfloor k, && \text{if } i \in [0, h-1] \text{ and } i \text{ is odd}, \end{aligned}$$

is a circular $m$-$L(j,k)$-labeling of $G_h$. Thus, we have $\sigma_{j,k}(G_h) \leq m$.

We next prove $\sigma_{j,k}(G_h) \geq m$. Suppose $\sigma_{j,k}(G_h) = \sigma$ and let $f$ be a circular $\sigma$-$L(j,k)$-labeling of $G_h$. Since the diameter of $G_h$ is 2, any two vertices of $G$ receive labels on $S(\sigma)$ that are at circular distance at least $k$. With no loss of generality, we may assume that the label sequence $(f(u_0), f(u_1), \ldots, f(u_{h-1}))$ with $f(u_0) = 0$ is in cyclic order on $S(\sigma)$. For $i = 0, 1, \ldots, h-1$, let $l_i$ denote the number of vertices in $V$ with their labels in the open interval $(f(u_i), f(u_{i+1}))_\sigma$, where additions in the subscript are taken modulo $h$. Clearly, $\sum_{i=0}^{h-1} l_i = h$ and $0 \leq l_i \leq h$ for $i = 0, 1, \ldots, h-1$. For $l_i > 0$, let $v_{i,1}, v_{i,2}, \ldots, v_{i,l_i}$ be the $l_i$ vertices in $V$ with their labels in the open interval $(f(u_i), f(u_{i+1}))_\sigma$. Suppose $(f(u_i), f(v_{i,1}), f(v_{i,2}), \ldots, f(v_{i,l_i}), f(u_{i+1}))$ is in cyclic order on $S(\sigma)$.

As in the proof of Lemma 3.2, for any two numbers $x$ and $y$ in $[0, \sigma)$, denote by $\|y - x\|_\sigma$ the length of the arc from the point $x$ to the point $y$ along the clockwise direction on $S(\sigma)$.

If $l_i = 0$, then $\|f(u_{i+1}) - f(u_i)\|_\sigma \geq j$. If $l_i = 1$, then since $u_i v_{i,1}$ or $u_{i+1} v_{i,1}$ is an edge of $G_h$, $\|f(u_{i+1}) - f(u_i)\|_\sigma \geq j + k$. If $l_i \geq 2$, then since $u_i$ is adjacent to $v_{i,1}$ or $v_{i,2}$, it follows that $\|f(u_{i+1}) - f(u_i)\|_\sigma \geq j + (l_i - 1)k$. Thus for each $i = 0, 1, \ldots, h-1$, $\|f(u_{i+1}) - f(u_i)\|_\sigma \geq j + \lceil l_i/2 \rceil k$, implying $\sigma \geq \sum_{i=0}^{h-1} \|f(u_{i+1}) - f(u_i)\|_\sigma \geq hj + \lceil h/2 \rceil k$.

Similarly, we can prove that if $j/k \geq 2$ then $\lambda_{j,k}(G_h) = (h-1)j + \lceil h/2 \rceil k$. □

The next result follows from Theorem 3.3 and Lemma 2.1.

**Corollary 3.4.**

$$\bar{\sigma}_{j,1}^n(G_h) = \bar{\lambda}_{j,1}^n(G_h) + j = \begin{cases} 2hn + \lceil h/2 \rceil (j-1), & \text{if } j \leq n+1, \\ h(j+n-1) + \lceil h/2 \rceil n, & \text{if } j \geq n+1. \end{cases}$$

**Theorem 3.5.** *If $n \geq 2$ and $j \leq \lfloor n/2 \rfloor + 1$, then $\sigma_{j,1}^n(G_h) = \lambda_{j,1}^n(G_h) + 1 = 2hn$.*

**Proof.** Since the diameter of $G_h$ is 2, it is obvious that $\sigma_{j,1}^n(G_h) \geq 2hn$ and $\lambda_{j,1}^n(G_h) \geq 2hn - 1$.

To prove the result, it suffices to demonstrate a certain $n$-fold $L(j,1)$-labeling of $G_h$. For $i = 0, 1, \ldots, h-1$, define

$$\begin{cases} f(u_i) = [2in + \lfloor n/2 \rfloor, (2i+1)n + \lfloor n/2 \rfloor - 1], \\ f(v_i) = [2in, 2in + \lfloor n/2 \rfloor - 1] \cup [(2i+1)n + \lfloor n/2 \rfloor, (2i+2)n - 1]. \end{cases}$$

Note that $j \leq \lfloor n/2 \rfloor + 1$, it is straightforward to check that $f$ is an $n$-fold circular $(2hn)$-$L(j,k)$-labeling of $G_h$ as well as an $n$-fold $L(j,k)$-labeling of $G_h$ with span $2hn - 1$. Thus the theorem holds. □

Therefore, if $n \geq 2$ and $j \leq \lfloor n/2 \rfloor + 1$, then $\bar{\sigma}_{j,1}^n(G_h) - \sigma_{j,1}^n(G_h) = \lceil h/2 \rceil (j-1)$ and $\bar{\lambda}_{j,1}^n(G_h) - \lambda_{j,1}^n(G_h) = (\lceil h/2 \rceil - 1)(j-1)$. Thus we conclude that the differences $\bar{\sigma}_{j,1}^n(G) - \sigma_{j,1}^n(G)$ and $\bar{\lambda}_{j,1}^n(G) - \lambda_{j,1}^n(G)$ could be arbitrarily large for certain graphs $G$.

## 4. Trees

Liu and Zhu in [20], and Leese and Noble in [17] proved the following theorem.

**Theorem 4.1.** *$\sigma_{j,k}(T) = 2j + (\Delta - 1)k$ for any tree $T$ with maximum degree $\Delta$.*

We establish the $n$-fold version of this result.

**Theorem 4.2.** *For any tree $T$ with maximum degree $\Delta$,*

$$\sigma^n_{j,k}(T) = 2j + (\Delta - 1)k + (\Delta + 1)(n - 1).$$

**Proof.** By Lemma 2.1 and Theorem 4.1, $\sigma^n_{j,k}(T) \le \sigma_{j+n-1,k+n-1}(T) = 2(j+n-1)+(\Delta-1)(k+n-1) = 2j+(\Delta-1)k+(\Delta+1)(n-1)$.

Let $m$ be a positive integer, and suppose there is an $n$-fold circular $m$-$L(j,k)$-labeling $f$ of $T$. Let $u$ be a vertex of $T$ with $\Delta$ neighbors $v_1, v_2, \dots, v_\Delta$. Let $[\widehat{f(u)}]^j_m = \bigcup_{i=0}^{p_0-1}[a_{i,0}, b_{i,0}]_m$ and $[\widehat{f(v_s)}]^k_m = \bigcup_{i=0}^{p_s-1}[a_{i,s}, b_{i,s}]_m$ for $s = 1, 2, \dots, \Delta$. Then all sets $[a_{i,s}, b_{i,s}]_m$ ($s = 0, 1, \dots, \Delta$, $i = 0, 1, \dots, p_s - 1$) are pairwise disjoint and are pairwise at circular distance at least $k$ on $S(m)$. Furthermore, each $[a_{i,0}, b_{i,0}]_m$ ($i \in [0, p_0 - 1]$) is at circular distance at least $j$ on $S(m)$ from any other sets $[a_{i,s}, b_{i,s}]_m$. It follows that $m \ge (\Delta+1)n+2(j-1)+(\Delta-1)(k-1) = 2j+(\Delta-1)k+(\Delta+1)(n-1)$. Hence the theorem holds. □

Since $K_{1,\Delta}$ is a subgraph of any graph with maximum degree $\Delta$, the following corollary follows from Theorem 4.2 and formula (2).

**Corollary 4.3.** *For any graph $G$ with maximum degree $\Delta$,*

$$\sigma^n_{j,k}(G) \ge 2j + (\Delta - 1)k + (\Delta + 1)(n - 1),$$
$$\lambda^n_{j,k}(G) \ge j + (\Delta - 1)k + (\Delta + 1)(n - 1).$$

The lower bounds in Corollary 4.3 are attained by the graph $K_{1,\Delta}$ since, by Theorem 4.2, $\sigma^n_{j,k}(K_{1,\Delta}) = 2j + (\Delta - 1)k + (\Delta + 1)(n - 1)$, and it is not difficult to prove that $\lambda^n_{j,k}(K_{1,\Delta}) = j + (\Delta - 1)k + (\Delta + 1)(n - 1)$.

**Theorem 4.4** ([7]). *Let $G$ be a graph with maximum degree $\Delta$. Suppose there is a vertex with $\Delta$ neighbors, each of which has degree $\Delta$. Then,*

$$\lambda_{j,k}(G) \ge \begin{cases} 2j + (\Delta - 2)k, & \text{if } j/k \le \Delta, \\ j + 2(\Delta - 1)k, & \text{if } j/k \ge \Delta. \end{cases}$$

We extend this theorem to $n$-fold $L(j,k)$-labelings of graphs. For a vertex $u$ of a graph $G$, by $N(u)$ we denote the set of all neighbors of $u$ and by $N[u]$ the set $N(u) \cup \{u\}$.

**Theorem 4.5.** *Let $G$ be a graph with maximum degree $\Delta$. Suppose there is a vertex with $\Delta$ neighbors, each of which has degree $\Delta$. Then,*

$$\lambda^n_{j,k}(G) \ge \begin{cases} 2j + (\Delta - 2)k + (\Delta + 1)(n - 1), & \text{if } (j + n - 1)/(k + n - 1) \le \Delta, \\ j + 2(\Delta - 1)k + 2\Delta(n - 1), & \text{if } (j + n - 1)/(k + n - 1) \ge \Delta. \end{cases}$$

**Proof.** Let $u$ be a vertex of $G$ with $\Delta$ neighbors $v_1, v_2, \dots, v_\Delta$, each of which has degree $\Delta$. Let $f$ be any $n$-fold $L(j,k)$-labeling of $G$ with span $\lambda$. Let $[\widehat{f(u)}]^j = \bigcup_{i=1}^{p_0}[a_{i,0}, b_{i,0}]$ and $[\widehat{f(v_s)}]^k = \bigcup_{i=1}^{p_s}[a_{i,s}, b_{i,s}]$ for $s = 1, 2, \dots, \Delta$. Then all sets $[a_{i,s}, b_{i,s}]$ ($s = 0, 1, \dots, \Delta$, $i = 1, 2, \dots, p_s$) are pairwise at distance at least $k$. Furthermore, each $[a_{i,0}, b_{i,0}]$ ($i \in [1, p_0]$) is at distance at least $j$ from all other sets $[a_{i,s}, b_{i,s}]$ ($s = 1, 2, \dots, \Delta$, $i = 1, 2, \dots, p_s$).

Notice that if $(j+n-1)/(k+n-1) \ge \Delta$ then $2j+(\Delta-2)k+(\Delta+1)(n-1) \ge j+2(\Delta-1)k+2\Delta(n-1)$, and if $(j+n-1)/(k+n-1) \le \Delta$ then $2j+(\Delta-2)k+(\Delta+1)(n-1) \le j+2(\Delta-1)k+2\Delta(n-1)$. To prove the theorem, it suffices to prove that $\lambda \ge 2j+(\Delta-2)k+(\Delta+1)(n-1)$ or $\lambda \ge j+2(\Delta-1)k+2\Delta(n-1)$.

If $[\widehat{f(u)}]^j$ contains more than one interval, then at least $2j-2$ labels not in $f(u)$ are forbidden for any neighbor of $u$. By considering the label sets of the vertices in $N[u]$ and the distance conditions, we know that $\lambda \ge (\Delta+1)n+2j-3+(\Delta-2)(k-1) = 2j+(\Delta-2)k+(\Delta+1)(n-1)$. Therefore we may assume $[\widehat{f(u)}]^j = [a,b] \subseteq [0,\lambda]$. Note that we actually have proved that for any vertex $v$ of maximum degree, $[\widehat{f(v)}]^j$ must consist of consecutive numbers (otherwise we are done by the above argument). Thus we may assume $[\widehat{f(v_s)}]^j = [a_s, b_s]$ for $s \in [1, \Delta]$.

If $[a,b] \subseteq [j-1, \lambda-j+1]$, then the $2j-2$ labels in $[a-j+1, a-1] \cup [b+1, b+j-1]$ are forbidden for any neighbor of $u$. By considering the label sets of the vertices in $N[u]$ and the distance conditions, we know that $\lambda \ge (\Delta+1)n+2j-3+(\Delta-2)(k-1) = 2j+(\Delta-2)k+(\Delta+1)(n-1)$. Thus $[a,b]$ is not contained in $[j-1, \lambda-j+1]$. Note that we have actually proved that for any vertex $v$ of maximum degree, $[\widehat{f(v)}]^j$ is not contained in $[j-1, \lambda-j+1]$.

Now, without loss of generality, we may assume $a \le j-2$. As $|[a,b]| \ge n$, $a_s \ge j+n-1$ for $s \in [1,\Delta]$. Since $v_1, v_2, \ldots, v_\Delta$ are pairwise at distance at most 2, $a_h = \min\{a_1, a_2, \ldots, a_\Delta\} \le \lambda - [\Delta n + (\Delta-1)(k-1)] + 1$, that is $\lambda \ge a_h + \Delta n + (\Delta-1)(k-1) - 1$. From the above discussion, $[\widehat{f(v_h)}]^j = [a_h, b_h]$ is not contained in $[j-1, \lambda-j+1]$. Since $a_h \ge j+n-1$, we must have $b_h > \lambda - j + 1$. It follows that all label sets of the $\Delta$ neighbors of $v_h$ are contained in $[0, a_h - j]$, implying $a_h \ge j-1+\Delta n+(\Delta-1)(k-1)$. Therefore $\lambda \ge a_h+\Delta n+(\Delta-1)(k-1)-1 \ge j-2+2\Delta n+2(\Delta-1)(k-1) = j+2(\Delta-1)k+2\Delta(n-1)$. This completes the proof. □

The following theorem can be found in [8] and is essential in establishing the upper bounds for $n$-fold $L(j,k)$-labeling numbers of trees.

**Theorem 4.6** ([8]). *Let $T$ be any tree with maximum degree $\Delta$. Then*

$$\lambda_{j,k}(T) \le \begin{cases} 2j+(\Delta-2)k, & \text{if } j/k \le \Delta \text{ and } j \text{ is a multiple of } k, \\ j+2(\Delta-1)k, & \text{if } j/k \ge \Delta. \end{cases}$$

**Theorem 4.7.** *Let $T$ be a tree with maximum degree $\Delta$. Then*

$$\lambda^n_{j,k}(T) \le 2j+(\Delta-1)k+(\Delta+1)(n-1)-1.$$

*Furthermore,*

$$\lambda^n_{j,k}(T) \le \begin{cases} 2j+(\Delta-2)k+(\Delta+1)(n-1), & \text{if } (j+n-1)/(k+n-1) \le \Delta \text{ and } (k+n-1)|(j+n-1), \\ j+2(\Delta-1)k+2\Delta(n-1), & \text{if } (j+n-1)/(k+n-1) \ge \Delta, \end{cases}$$

*and the inequality is an equality if $T$ has a vertex with $\Delta$ neighbors of degree $\Delta$.*

**Proof.** By Theorem 4.2 and formula (3), for any tree $T$ with maximum degree $\Delta$, $\lambda^n_{j,k}(T) \le 2j+(\Delta-1)k+(\Delta+1)(n-1)-1$. By Theorem 4.6 and Lemma 2.1, if $(j+n-1)/(k+n-1) \le \Delta$ and $j+n-1$ is a multiple of $k+n-1$ then $\lambda^n_{j,k}(T) \le 2j+(\Delta-2)k+(\Delta+1)(n-1)$, and if $(j+n-1)/(k+n-1) \ge \Delta$ then $\lambda^n_{j,k}(T) \le j+2(\Delta-1)k+2\Delta(n-1)$. The last statement follows from Theorem 4.5. □

When $k=1$, both the lower and upper bounds for $n$-fold $L(j,k)$-labeling numbers of trees that we obtained so far are sharp. This is summarized as the following corollary.

**Corollary 4.8.** *For any tree $T$ with maximum degree $\Delta$,*

$$(\Delta+1)n+j-2 \le \lambda^n_{j,1}(T) \le \min\{(\Delta+1)n+2j-3, 2\Delta n+j-2\}.$$

*The lower and the upper bounds for $\lambda^n_{j,1}(T)$ are both attainable.*

Corollary 4.8 generalizes the following theorem proved by Chang et al. in [5].

**Theorem 4.9** ([5]). *For any tree $T$ with maximum degree $\Delta$,*

$$\Delta+j-1 \le \lambda_{j,1}(T) \le \min\{\Delta+2j-2, 2\Delta+j-2\}.$$

*Moreover, the lower and the upper bounds for $\lambda_{j,1}(T)$ are both attainable.*

By Corollary 4.8, $\lambda^n_{1,1}(T) = (\Delta+1)n-1$ for any tree $T$ with maximum degree $\Delta$.

**Corollary 4.10.** *For any tree $T$ with maximum degree $\Delta$,*

$$(\Delta+1)n \le \lambda^n_{2,1}(T) \le (\Delta+1)n+1.$$

Corollary 4.10 is a generalization of the result $\Delta+1 \le \lambda_{2,1}(T) \le \Delta+2$ for any tree with maximum degree $\Delta$, which was proved by Griggs and Yeh in [12]. In [6], Chang and Kuo gave a polynomial time algorithm for determining whether $\lambda_{2,1}(T) = \Delta+1$ for any tree $T$ with maximum degree $\Delta$. It was indicated in [5] that this algorithm can be modified to determine $\lambda_{j,1}(T)$ and the modified algorithm also runs in a polynomial time. A linear time algorithm for $L(2,1)$-labeling of trees was given in [14]. The authors also showed that it can be extended to a linear time algorithm for $L(j,1)$-labeling of trees with a constant $j$.

We conclude this section by asking the following two questions.

**Question 1.** For a fixed positive integer $n \ge 2$, is there a polynomial time algorithm for computing $\lambda^n_{j,1}(T)$ for any tree $T$?

**Question 2.** For positive integers $n \ge 2$, how do we characterize all trees $T$ with maximum degree $\Delta$ and $\lambda^n_{2,1}(T) = (\Delta+1)n$?

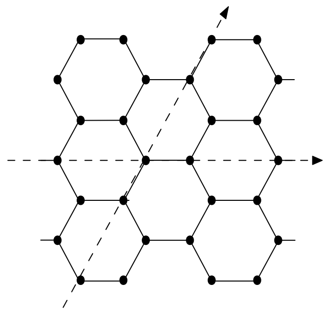

**Fig. 3.** The hexagonal lattice $\Gamma_6$.

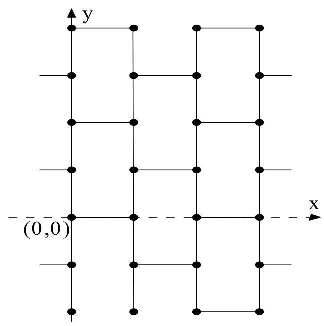


**Fig. 4.** Another drawing of $\Gamma_6$.

## 5. The hexagonal lattice

Let $\mathbf{e_1} = (1, 0)^T$, $\mathbf{e_2} = (0, 1)^T$ and $\mathbf{f} = (1/2, \sqrt{3}/2)^T$ be three vectors in the Euclidean plane. The *triangular lattice* $\Gamma_3$ is an infinite graph with vertex set $\{x\mathbf{e_1} + y\mathbf{f} : x, y \in \mathbb{Z}\}$ with two different vertices $(x_1, y_1)$, $(x_2, y_2)$ adjacent if the Euclidean distance between them is 1. The *square lattice* $\Gamma_4$ is an infinite graph with vertex set $\{x\mathbf{e_1} + y\mathbf{e_2} : x, y \in \mathbb{Z}\}$ with two different vertices $(x_1, y_1)$, $(x_2, y_2)$ adjacent if the Euclidean distance between them is 1.

The *hexagonal lattice* $\Gamma_6$ is the subgraph of $\Gamma_3$ induced by the vertex set $V(\Gamma_3) \setminus \{(x, x+3y+1) : x, y \in \mathbb{Z}\}$. If two vertices $(x_1, y_1)$ and $(x_2, y_2)$ in $\Gamma_3$ (or $\Gamma_4$, $\Gamma_6$) are adjacent, then we write the edge joining them by $(x_1, y_1)(x_2, y_2)$. One can also view the hexagonal lattice $\Gamma_6$ as a spanning subgraph of $\Gamma_4$ with edge set $E(\Gamma_4) \setminus E^*$, where $E^* = \{(x, y)(x + 1, y) : x, y \in \mathbb{Z}$ and $x + y$ is odd$\}$. Please see Figs. 3 and 4 for illustrations. We shall use the latter in the proof of the following theorem.

**Theorem 5.1.** $\sigma_{j,k}(\Gamma_6) = 2j + 2k$.

**Proof.** Since the maximum degree of $\Gamma_6$ is 3, by applying Corollary 4.3 for $n = 1$, we obtain $\sigma_{j,k}(\Gamma_6) \geq 2j+2k$. Let $m = 2j+2k$. Define a function $f$ from $V(\Gamma_6)$ to $[0, m - 1]$ as

$$f((x, y)) = \begin{cases} (yk) \bmod m, & \text{if } x + y \text{ is even,} \\ (j + k + yk) \bmod m, & \text{if } x + y \text{ is odd.} \end{cases}$$

We now show that $f$ is a circular $m$-$L(j, k)$-labeling of $\Gamma_6$. Let $(x, y)$ be any vertex of $\Gamma_6$. If $x + y$ is even, then the three neighbors of $(x, y)$ are $(x + 1, y), (x, y + 1), (x, y - 1)$. It follows that $f((x, y)) = (yk) \bmod m, f((x + 1, y)) = (j + k + yk) \bmod m, f((x, y + 1)) = (j + k + (y + 1)k) \bmod m$, and $f((x, y - 1)) = (j + k + (y - 1)k) \bmod m$. It is obvious that the circular distance between the label of $(x, y)$ and the labels of its three neighbors is at least $j$. The case where $x + y$ is odd can be shown similarly. Thus the distance 1 condition is satisfied.

All vertices at distance 2 from $(x, y)$ are $(x, y+2)$, $(x, y-2)$, $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$ and $(x-1, y-1)$. Notice that the sum of the two coordinates of each 2-neighbor of $(x, y)$ has the same parity. It is not difficult to check that the distance 2 condition is also satisfied. Thus $f$ is a circular $m$-$L(j, k)$-labeling of $\Gamma_6$, proving the theorem. $\square$

**Theorem 5.2.** $\sigma^n_{j,k}(\Gamma_6) = 2j + 2k + 4n - 4$.

**Proof.** Since the maximum degree of $\Gamma_6$ is 3, by Corollary 4.3, $\sigma^n_{j,k}(\Gamma_6) \geq 4n + 2(j - 1) + 2(k - 1)$. On the other hand, by Theorem 5.1 and Lemma 2.1, $\sigma^n_{j,k}(\Gamma_6) \leq 4n + 2(j - 1) + 2(k - 1)$. Thus the theorem holds. $\square$

The following corollary follows from Theorem 5.2 and formula (3).

**Corollary 5.3.** $\lambda^n_{j,k}(\Gamma_6) \le 2j + 2k + 4n - 5$.

**Theorem 5.4** ([3,11]).

$$\lambda_{j,k}(\Gamma_6) = \begin{cases} 3j, & \text{if } 1 \le j/k \le 5/3,\\ 5k, & \text{if } 5/3 \le j/k \le 2,\\ 2j + k, & \text{if } 2 \le j/k \le 3,\\ j + 4k, & \text{if } 3 \le j/k. \end{cases}$$

**Theorem 5.5.**

$$\lambda^n_{j,k}(\Gamma_6) \begin{cases} \in [2j + k + 4n - 4, 3j + 4n - 4], & \text{if } 1 \le (j + n - 1)/(k + n - 1) \le 5/3,\\ \in [2j + k + 4n - 4, 5k + 6n - 6], & \text{if } 5/3 \le (j + n - 1)/(k + n - 1) \le 2,\\ = 2j + k + 4n - 4, & \text{if } 2 \le (j + n - 1)/(k + n - 1) \le 3,\\ = j + 4k + 6n - 6, & \text{if } 3 \le (j + n - 1)/(k + n - 1). \end{cases}$$

**Proof.** The upper bounds follow from Theorem 5.4 and Lemma 2.1, and the lower bounds follow from Theorem 4.5. $\square$

**Corollary 5.6.**

$$\lambda^n_{j,1}(\Gamma_6) = \begin{cases} 2j + 4n - 3, & \text{if } j \le 2n + 1,\\ j + 6n - 2, & \text{if } j \ge 2n + 1. \end{cases}$$

## 6. The *p*-dimensional square lattice

Let $p \ge 2$ be an integer. The *p-dimensional square lattice* $\Gamma^p_4$ is an infinite graph with vertex set $\{(x_1, x_2, \ldots, x_p) : x_1, x_2, \ldots, x_p \in \mathbb{Z}\}$, and with two different vertices adjacent if and only if the Euclidean distance between them is 1. Clearly, if $p = 2$, then $\Gamma^2_4$ is the so called square lattice $\Gamma_4$. The $p$-dimensional square lattice is $(2p)$-regular.

Let $u = (x_1, x_2, \ldots, x_p)$ and $v = (y_1, y_2, \ldots, y_p)$ be two vertices of $\Gamma^p_4$. Then $u$ is adjacent to $v$ if and only if there is some $q \in [1, p]$ such that $|x_q - y_q| = 1$ and $x_i = y_i$ for $i \in [1, p] \setminus \{q\}$. And $u$ is distance 2 away from $v$ if and only if there are two integers $q, s \in [1, p]$ such that $|x_q - y_q| + |x_s - y_s| = 2$ and $x_i = y_i$ for $i \in [1, p] \setminus \{q, s\}$.

**Theorem 6.1.** $\sigma_{j,k}(\Gamma^p_4) = 2j + (2p - 1)k$.

**Proof.** Since $K_{1,2p}$ is a subgraph of $\Gamma^p_4$, it follows from Theorem 4.2 that $\sigma_{j,k}(\Gamma^p_4) \ge 2j + (2p - 1)k$. Let $m = 2j + (2p - 1)k$. Define a function $f$ from $V(\Gamma^p_4)$ to $[0, m - 1]$ as follows: for any vertex $u = (x_1, x_2, \ldots, x_p)$ of $\Gamma^p_4$,

$$f(u) = \left(\sum_{i=1}^{p} [j + (i - 1)k]x_i\right) \bmod m.$$

Let $u = (x_1, x_2, \ldots, x_p)$ and $v = (y_1, y_2, \ldots, y_p)$ be any two vertices of $\Gamma^p_4$. If $u$ is adjacent to $v$, then there is some $q \in [1, p]$ such that $|x_q - y_q| = 1$ and $x_i = y_i$ for $i \in [1, p] \setminus \{q\}$. It follows from the definition of $f$ that $|f(u) - f(v)|_m = j + (q - 1)k \ge j$. Thus the distance 1 condition is satisfied. If $u$ and $v$ are at distance 2, then there are two integers $q, s \in [1, p]$ such that $|x_q - y_q| + |x_s - y_s| = 2$ and $x_i = y_i$ for $i \in [1, p] \setminus \{q, s\}$. Suppose $q > s$. Then

$$k \le (q - s)k \le \left|\sum_{i=1}^{p} [j + (i - 1)k]x_i - \sum_{i=1}^{p} [j + (i - 1)k]y_i\right| \le 2j + 2(q - 1)k \le 2j + (2p - 2)k.$$

Therefore $|f(u) - f(v)|_m \ge k$. The distance 2 condition is also satisfied. Thus $f$ is a circular $m$-$L(j, k)$-labeling of $\Gamma^p_4$, and so $\sigma_{j,k}(\Gamma^p_4) \le 2j + (2p - 1)k$. This proves the theorem. $\square$

The following theorem follows from Theorem 6.1 and Corollary 4.3.

**Theorem 6.2.** $\sigma^n_{j,k}(\Gamma^p_4) = 2j + (2p - 1)k + (2p + 1)(n - 1)$.

**Theorem 6.3** ([7]).

$$\lambda_{j,k}(\Gamma^p_4) \begin{cases} \in [2j + (2p - 2)k, 2j + (2p - 1)k - 1], & \text{if } 1 \le j/k \le 2p,\\ = 2j + (2p - 2)k, & \text{if } 1 \le j/k \le 2p \text{ and } k|j, \text{ or } 2p - 1 < j/k < 2p,\\ = j + (4p - 2)k, & \text{if } j/k \ge 2p. \end{cases}$$

Theorem 6.3, Lemma 2.1 and Theorem 4.5 imply the following theorem.

**Theorem 6.4.**

$$\lambda^n_{j,k}(\Gamma^p_4)\begin{cases}\in [2j+(2p-2)k+(2p+1)(n-1),\, 2j+(2p-2)k+(2p+1)(n-1)+k-1],\\ \quad \textit{if } (j+n-1)/(k+n-1)\le 2p,\\ = 2j+(2p-2)k+(2p+1)(n-1), \quad \textit{if } 2p-1<(j+n-1)/(k+n-1)<2p,\\ \quad \textit{or } 1\le (j+n-1)/(k+n-1)\le 2p \textit{ and } (k+n-1)|(j+n-1),\\ = j+(4p-2)k+4p(n-1), \quad \textit{if } (j+n-1)/(k+n-1)\ge 2p.\end{cases}$$

**Corollary 6.5.**

$$\lambda^n_{j,1}(\Gamma^p_4)=\begin{cases}2j+(2p+1)n-3, & \textit{if } j\le (2p-1)n+1,\\ j+4pn-2, & \textit{if } j\ge (2p-1)n+1.\end{cases}$$

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