A Study of Auchmuty's Error Estimate

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Abstract—We analyze the absolute error estimate of Auchmuty [1] developed for linear systems. In the Euclidean norm, this estimate and its geometrical interpretation are derived from the Kantorovich inequality. The estimate is then compared with other estimates known in the literature. A probabilistic analysis and extension of the estimate to nonlinear systems are also given. We also report on computational test results, which indicate that Auchmuty's estimate is an appropriate tool for practice. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $x^*$ be the exact solution of the linear system

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}, \quad \text{det}(A) \neq 0,$$

and let $r(x) = Ax - b$ denote the residual error for any approximate solution $x$. There are several a posteriori error estimates which exploit the residual information [2-5]. Here we recall the following estimates:

$$\frac{\|Br(x)\|}{1 + \|BA - I\|} \leq \|x - x^*\| \leq \frac{\|Br(x)\|}{1 - \|BA - I\|},$$

$$\|r(x)\| \leq \|x - x^*\| \leq \|A^{-1}\| \|r(x)\|,$$

where $I$ stands for the unit matrix, $B$ is an approximation to $A^{-1}$ satisfying $\|BA - I\| < 1$, the matrix norms are multiplicative, and the vector norms are consistent.

Estimating (1) requires the knowledge of $\|A\|$ and $\|A^{-1}\|$, while estimate (2), which is due to Aird and Lynch [3,4], requires an approximate inverse $B$ of matrix $A$. Auchmuty's estimate [1] requires neither information. Let $x \in \mathbb{R}^n$ be an arbitrary approximate solution ($r(x) \neq 0$). Then

$$\|x - x^*\|_p = c_\alpha \frac{\|r(x)\|^2}{\|A^\top r(x)\|_q}, \quad 1 \leq p \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$
holds with $1 \leq c \leq C_p(A)$, where
\[
C_p(A) = \sup_{y \neq 0} \frac{\|A^T y\|_q \|A^{-1} y\|_p}{\|y\|_2^q}.
\] (4)

Auchmuty’s estimate seems unnoticed although computational experiments indicate that the error constant $c$ is usually less than 10 in practice [6]. Such a ratio between the estimate and the estimated quantity is usually acceptable (see, e.g., [7, p. 294]).

In the sequel, we investigate the Auchmuty estimate for the Euclidean norm, which has the form
\[
\frac{\|r(x)\|^2_2}{\|A^T r(x)\|_2} \leq \|x - x^*\|_2 \leq C_2(A) \frac{\|r(x)\|^2_2}{\|A^T r(x)\|_2},
\] (5)

with
\[
C_2(A) = \sup_{y \neq 0} \frac{\|A^T y\|_2 \|A^{-1} y\|_2}{\|y\|_2^2},
\] (6)

We first show that the error estimate is a consequence of the Kantorovich inequality. This approach leads to the exact value of $C_2(A)$ and the characterization of all cases when equality appears in the upper bound of (5). Using the Greub-Rheinboldt formulation of the Kantorovich inequality, we derive the geometric interpretation of the estimate. This shows that Auchmuty’s lower estimate orthogonally projects the error vector $x - x^*$ into the subspace span ($A^T r(x)$). We also make some probability reasoning about the possible values of $c$ and $C_2(A)$ giving a better background for the numerical testing. The Auchmuty estimate is then extended to nonlinear systems of the form $F(x) = 0$. This result can be used in conjunction with the Newton and Newton-like methods. We carried out an intensive computational testing for linear systems. The results which indicate the usefulness of the estimate are evaluated in Section 6, where a practical version (formula (28)) is also suggested.

2. DERIVATION AND GEOMETRY OF THE AUCHMUTY ESTIMATE

We first show that Auchmuty’s estimate is a consequence of the Kantorovich inequality given in the following form (see, e.g., [8–10]). If $B \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ and $x \in \mathbb{R}^n$ is an arbitrary vector, then
\[
\|x\|^2_2 \leq (x^T B x) (x^T B^{-1} x) \leq \frac{1}{4} \frac{(\lambda_1 + \lambda_n)^2}{\lambda_1 \lambda_n} \|x\|^4_2.
\] (7)

The Kantorovich inequality is sharp. Let $D = U \Sigma U^T$ with orthogonal $U = [u_1, \ldots, u_n]$ and $\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let the multiplicity of $\lambda_1$ and $\lambda_n$ be $k$ and $l$, respectively. It follows from Henrici [11] that equality holds for $x \neq 0$ in the upper bound, if and only if $x = \tau U y$, where $\tau \in \mathbb{R}$ ($\tau \neq 0$) and $y = [y_1, \ldots, y_k, 0, \ldots, 0, y_{n-k+1}, \ldots, y_n]^T$ is such that
\[
\sum_{i=1}^k y_i^2 = \sum_{i=n-k+1}^n y_i^2 = \frac{1}{2}.
\] (8)

Particularly, for $x = (u_1 + u_n)/\sqrt{2}$, equality is achieved in the upper bound. Notice that for $x = u_i$ ($i = 1, \ldots, n$), equality holds in the lower bound.

Let $A = U \Sigma_A U^T$ ($\Sigma_A = \text{diag}(\sigma_1, \ldots, \sigma_n)$) be the singular value decomposition of $A$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and let $B = AA^T$. As $x^T B x - \|A^T x\|^2_2$, $x^T B^{-1} x - \|A^{-1} x\|^2_2$, and $\lambda_1 = \lambda_1(B) = \sigma_1^2 (A) = \sigma_1^2$, where $\sigma_i$ is the $i$th singular value of $A$, we can write
\[
\|x\|^2_2 \leq \|A^T x\|^2_2 \|A^{-1} x\|^2_2 \leq \frac{1}{4} \frac{(\sigma_1^2 + \sigma_n^2)^2}{\sigma_1^2 \sigma_n^2} \|x\|^4_2,
\]
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from which
\[ \|x\|_2^2 \leq \|A^T x\|_2 \|A^{-1} x\|_2 \leq \frac{1}{2} \sigma_1^2 + \sigma_n^2 \|x\|_2^2 \]  
(9)

follows. Substituting \( x \) by \( r(x) = Ax - b = A(x - x^*) \), we have
\[ \|r(x)\|_2^2 \leq \|A^T r(x)\|_2 \|x - x^*\|_2 \leq \frac{1}{2} \sigma_1^2 + \sigma_n^2 \|r(x)\|_2^2, \]
which implies the Auchmuty estimate
\[ \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2^2} \leq \frac{\|x - x^*\|_2}{\|A^T r(x)\|_2} \leq \frac{1}{2} \sigma_1^2 + \sigma_n^2 \frac{\|r(x)\|_2}{\|A^T r(x)\|_2}. \]
(10)

The upper bound is sharp for \( x = x^* + \tau V \Sigma_A^{-1} y \), where \( \tau \) and \( y \) are defined at (8). We may conclude that
\[ C_2(A) = \frac{1}{2} \left( \frac{\kappa_2(A)}{\sigma_1} + \frac{1}{\kappa_2(A)} \right). \]
(11)

Auchmuty [1] mentions that for \( p = 2 \), a weaker form of the upper bound in (5) can be obtained from the Kantorovich inequality. Here we point out that exactly the same inequality can be derived from the Kantorovich inequality and \( C_2(A) \) is equal to \( (\sigma_1^2 + \sigma_n^2)/(2\sigma_1\sigma_n) \). Observe that
\[ C_2(A) \approx \frac{1}{2} \kappa_2(A), \]
(12)

if \( \kappa_2(A) \) is large enough.

As \( r(x) = Ae \) \((e = x - x^*)\), we can write the error constant \( c \) in the form
\[ c^2 = \frac{e^T A^T A e}{(e^T A e)^2}. \]
(13)

Observe that \( c \) is invariant under the transformation \( e \rightarrow \gamma e \). So the error constant \( c \) depends only on the direction of the error vector \( e \). For later use, we introduce the notation \( c = c(A, e) \).

For the geometrical interpretation of the estimate, we need the Greub-Rheinboldt reformulation of the Kantorovich inequality [8,9].

Let \( D, E \in \mathbb{R}^{n \times n} \) be two positive definite, symmetric, and commuting matrices. Denote by \( \lambda_1 \) and \( \lambda_n \) the largest and the smallest eigenvalue of \( D \), respectively. Similarly, denote by \( \mu_1 \) and \( \mu_n \) the largest and the smallest eigenvalue of \( E \), respectively. Then
\[ (x^T D^2 x)(x^T E^2 x) \leq \frac{(\lambda_1 \mu_1 + \lambda_n \mu_n)^2}{4 \lambda_1 \lambda_n \mu_1 \mu_n} \left( x^T D E x \right)^2, \]
(14)

for all \( x \in \mathbb{R}^n \).

Let \( \cos(x, y) \) denote the cosine between the vectors \( x \) and \( y \). If \( D \) is positive definite and symmetric, then
\[ \cos(Dx, x) \geq \frac{2 \sqrt{\kappa_2(D)}}{1 + \kappa_2(D)}, \quad x \neq 0. \]
(15)

The definition of cosine and the Greub-Rheinboldt inequality (14) with \( E = I \) imply that
\[ \cos^2(Dx, x) = \frac{(x^T D x)^2}{(x^T D^2 x)(x^T x)} \geq \frac{4 \lambda_1 \lambda_n}{(\lambda_1 + \lambda_n)^2} = \frac{4 \kappa_2(D)}{(1 + \kappa_2(D))^2}. \]

Inequality (15) is sharp. Let \( D = A^T A \) and let \( A = U \Sigma_A V^T \) be again the singular value decomposition of \( A \). The lower bound is then achieved for \( x = \tau V \Sigma_A^{-1} y \), where \( \tau \) and \( y \) are...
defined at (8). We note that quantity \(2\sqrt{\kappa_2(D)/(1 + \kappa_2(D))}\) is equal to \(\cos D\), which is the cosine of operator \(D\) (see [12]). In general,

\[
\cos (A) = \inf_{x \neq 0, Ax \neq 0} \frac{x^T Ax}{\|Ax\| \|x\|}, \quad A \in \mathbb{R}^{n \times n}.
\]

(16)

We can easily recognize that the error constant \(c = c(A, e)\) can be expressed as

\[
c = c(A, e) = \frac{1}{\cos (A^T Ae, e)},
\]

(17)

where the angle \(\alpha = (A^T Ae, e)_\perp\) can vary in \([0, \cos^{-1}(2\sigma_1\sigma_n/(\sigma_1^2 + \sigma_n^2))].\) It is clear that \(c\) is maximal, if \(\alpha\) is also maximal.

We can now express Auchmuty’s estimate as follows.

**Theorem 1.** For the absolute error, the relation

\[
\frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2} = \cos (A^T Ae, e) \|e\|_2
\]

holds with

\[
1 \geq \cos (A^T Ae, e) \geq \frac{1}{C_2(A)} = \cos A^T A.
\]

(18)

(19)

So we can think that Auchmuty’s lower estimate orthogonally projects the error vector \(e\) into the subspace \(\text{span}(A^T e) = \text{span}(AT r)\). The smaller the angle \((A^T e, e)_\perp\), the better the estimate.

### 3. COMPARISON OF THE ESTIMATES

We compare estimates (1) and (5). These estimates give the inclusion intervals

\[
\left[ \frac{\|r(x)\|_2}{\|A\|_2}, \|A^{-1}\|_2 \|r(x)\|_2 \right], \quad \left[ \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2^2}, C_2(A) \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2^2} \right]
\]

for \(\|e\|_2\), respectively. The ratio of the upper and lower interval bounds are \(\kappa_2(A)\) and \(C_2(A)\), respectively. As \(C_2(A) \approx \kappa_2(A)/2\) for large \(\kappa_2(A)\), this ratio is smaller for the Auchmuty estimate.

The lower bounds satisfy

\[
\frac{\|r(x)\|_2}{\|A\|_2} \leq \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2} \leq \|x - x^*\|_2.
\]

Thus, Auchmuty’s lower estimate is a better approximation to \(\|e\|_2\) than the lower bound of estimate (1). For the upper bounds of the inclusion intervals, the relation

\[
\frac{1}{2} \|A^{-1}\|_2 \|r(x)\|_2 \leq C_2(A) \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2} \leq C_2(A) \|A^{-1}\|_2 \|r(x)\|_2
\]

holds.

The relative position of the corresponding upper bounds depends on the value of \(\|A^T r(x)\|_2\), which may lie in \([\sigma_n \|r(x)\|_2, \sigma_1 \|r(x)\|_2]\). One can easily prove that

\[
C_2(A) \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2} \geq \|A^{-1}\|_2 \|r(x)\|_2,
\]

for \(\|A^T r(x)\|_2 = \sigma_n \|r(x)\|_2\), and

\[
C_2(A) \frac{\|r(x)\|_2^2}{\|A^T r(x)\|_2} < \|A^{-1}\|_2 \|r(x)\|_2,
\]

for \(\|A^T r(x)\|_2 = \sigma_1 \|r(x)\|_2\).
Brezinski gave five error estimates using the theory of moments and interpolation [5]. The closest one of these estimates is \( e_3 = \frac{\|r(x)\|^2}{\|Ar(x)\|_2} \) for which he proved that \( e_3/\kappa_2(A) \leq \|e\|/\kappa_2(A)e_3 \). For symmetric \( A \), estimate \( e_3 \) is identical with Auchmuty's lower bound. In general, \( e_3 \) can be less or greater than the lower Auchmuty estimate. It is easy to prove that

\[
\frac{e_3}{\kappa_2(A)} \leq \frac{\|r(x)\|^2}{\|A^T r(x)\|^2} \leq \kappa_2(A)e_3.
\]

4. PROBABILISTIC ANALYSIS

We investigate the behavior of \( c \) and \( C_2(A) \) for random values of \( e \) and \( A \), respectively. We can assume that \( \|e\|_2 = 1 \), without loss of generality. Let us assume first that \( A \) is fixed and \( e \) is random on the surface of the \( n \)-dimensional unit sphere \( S_n = \{ x \in \mathbb{R}^n \mid x^T x = 1 \} \). As the random variable \( c(A, e) \) is bounded, that is, \( 1 \leq c(A, e) \leq C_2(A) \), its expected value and variance must satisfy the inequalities

\[
1 \leq E(c(A, e)) \leq C_2(A), \quad \text{Var}(c(A, e)) \leq \left( \frac{C_2(A) - 1}{2} \right)^2,
\]

respectively. Considering the fact that the extremum of \( c(A, e) \) is achieved only on a special subset of \( S_n \), we may hope that for a relatively small positive \( \xi \), the inequality \( c(A, e) \leq \xi \) (or \( \cos(A^T Ae, e) \geq \xi^{-1} \)) holds with a high probability. In such a case, the expected values and variances can be significantly smaller than the corresponding upper bounds in (20). The results of numerical testing, in which \( e \) was uniformly distributed on \( S_n \), strongly support this expectation.

If the matrix \( A \) is assumed to be random, we can use the special relationship between \( C_2(A) \) and \( \kappa_2(A) \) and known results on the condition number distribution of random matrices [13, 14]. The matrix \( A \in \mathbb{R}^{n \times n} \) is called Gaussian if its elements are independent standard normal random variables. For the condition number \( \kappa_2(A) = \|A\|_{F} / \|A^{-1}\|_{2} \), Demmel proved that

\[
P(\kappa_2(A) \geq t) \leq 2 \left( \frac{2n}{t} \right)^{n^2} - 1,
\]

if \( A \in \mathbb{R}^{n \times n} \) is a Gaussian matrix (see [13, Theorem 5.2; 14]).

This tail probability bound is proportional to \( n/t \). It is less than 1, if \( t \) exceeds about \( 5n^2 \). So for Gaussian matrices of a given order \( n \), it is very unlikely that \( \kappa_2(A) \) exceeds a rather large value of \( t \).

As \( C_2(A) \leq \kappa_2(A) \leq \kappa_2(A) \), one can easily obtain

\[
P(\xi \geq t) \leq P(C_2(A) \geq t) \leq P(\kappa_2(A) \geq t) \leq 2 \left( \frac{2n}{t} \right)^{n^2} - 1,
\]

if \( A \in \mathbb{R}^{n \times n} \) is a Gaussian matrix.

Edelman [14] proved that for Gaussian matrices \( A_n \in \mathbb{R}^{n \times n} \),

\[
E(\log(\kappa_2(A_n))) \approx \log(n) + 1.537,
\]

as \( n \to \infty \). This result indicates that \( \kappa_2(A) \) is unlikely to be large for such random matrices. From (23), we can derive, with a reasonable heuristic, that \( E(\log(C_2(A_n))) \approx \log(n) + 0.844 \) as \( n \to \infty \). Consequently, \( C_2(A_n) \) is likely to be under \( \alpha n \), where \( \alpha \) is an appropriate constant.

Denote by \( L_n \) the lower triangular part of a Gaussian matrix \( A_n \). Viswanath and Trefethen [15] recently proved that

\[
\sqrt{\kappa_2(L_n)} \to 2, \quad \text{almost surely}
\]

as \( n \to \infty \). This bound gives a rather large value for \( C_2(L_n) \approx \kappa_2(L_n)/2 \). Numerical testing up to the size \( n = 300 \) indicates that \( E(c(A, e)) \) is likely to be small for both \( A_n \) and \( L_n \).
5. THE EXTENSION OF AUCHMUTY'S ESTIMATE TO NONLINEAR SYSTEMS

We consider the nonlinear algebraic systems of the form
\[ F(x) = 0, \quad F : \mathbb{R}^n \to \mathbb{R}^n, \]
and assume that the Jacobian matrix \( F'(x^*) \) is invertible, \( F' \in C^1(\overline{S(x^*, \delta)}) \), and
\[ \|F'(x) - F'(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in \overline{S(x^*, \delta)}. \]

Here \( \overline{S(x^*, \delta)} = \{x | \|x^* - x\|_2 \leq \delta \} \) and \( \delta > 0 \). Assume that \( x \) is close enough to \( x^* \). Let \( B = F'(x)F'(x)^\top \) and apply the Kantorovich inequality (7). We obtain
\[ \|z\|_2 \leq \|F'(x)^\top z\|_2 \|F'(x)^{-1}z\|_2 \leq \frac{1}{2} \frac{\sigma_1^2 + \sigma_n^2}{\sigma_1 \sigma_n} \|z\|_2^2, \quad z \in \mathbb{R}^n, \]
where \( \sigma_1 = \sigma_1(F'(x)) \). Let \( z = F(x) \). From the Lipschitz continuity, it follows that \( F(x) = F'(x)(x - x^*) + O(\|e\|_2^2) \) and \( F'(x)^{-1}F(x) - x - x^* + O(\|e\|_2^2) \). Hence,
\[ \|F(x)\|_2^2 \leq \|F'(x)^\top F(x)\|_2 \left(\|x - x^*\|_2 + O(\|e\|_2^2)\right) \leq C_2(F'(x)) \|F(x)\|_2^2 \]
and
\[ \frac{\|F(x)\|_2^2}{\|F'(x)^\top F(x)\|_2} \leq \frac{\|x - x^*\|_2 + O(\|e\|_2^2)}{\|F'(x)^\top F(x)\|_2} \leq C_2(F'(x)) \frac{\|F(x)\|_2^2}{\|F'(x)^\top F(x)\|_2}. \]

Thus, we obtained the approximate absolute error estimate
\[ \|x - x^*\|_2 \approx c \frac{\|F(x)\|_2^2}{\|F'(x)^\top F(x)\|_2}, \quad (25) \]
where \( 1 \leq c \leq C_2(F'(x)) \).

6. NUMERICAL TESTING

For linear systems, we investigated the value of \( c(A, e) \) when \( e \) is a uniformly distributed random vector on the surface of the \( n \)-dimensional unit sphere \( S_n \). This means that the computed solution \( \hat{x} \) satisfies the perturbed equation \( A\hat{x} = b + Ae \), where \( e \in S_n \) is uniformly distributed.

The test matrices were mainly taken from the Higham collection \([7]\) (gallery in MATLAB 5.1). We selected two groups of test problems. Groups 1 and 2 consist of 42 and 8 variable size test problems (matrix families), respectively. In Group 1, the size of the matrices were chosen as \( n = 10, 20, \ldots, 300 \). This choice gives 1260 matrices in Group 1. This group consists of two subgroups, namely, matrices with relatively small and matrices with relatively high condition numbers. In Group 2, the size of the matrices were chosen as \( n = 5, 10, 15, \ldots, 50 \). Thus, we have 80 matrices in Group 2. The maximum size in Group 2 was limited by MATLAB’s built-in cond function.

The testing sequence was carried out as follows. For each matrix, we generated 2000 uniformly distributed random vectors \( e \) on \( S_n \) and calculated the values of \( c(A, e) \) by formula (13). The sample estimate of the expected value \( \overline{c}(A) = E(c(A, e)) \) and variance \( \sigma^2(A) = \text{Var}(c(A, e)) \) are denoted by \( \overline{c}(A) \) and \( \overline{\sigma^2}(A) \), respectively. For each dimension \( n \), we calculated the average of \( \overline{c}(A) \)s and \( \kappa_n(A) \)s, respectively. These averages are denoted by \( c(n) \) and \( \kappa(n) \), respectively. The reliability of the test results is about
\[ P(\|\overline{c}(A) - c(A)\| < 0.044\sigma(A)) \approx 0.95 \]
for 2000 sample elements.
The results presented in Table 1 were obtained.

Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Group 1</th>
<th>Group 2</th>
</tr>
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<tr>
<td>$\bar{c}(A)_{\text{min}}$</td>
<td>1.0015</td>
<td>1.0804</td>
</tr>
<tr>
<td>$\bar{c}(A)_{\text{max}}$</td>
<td>128.20</td>
<td>35.573</td>
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<td>$c(n)_{\text{min}}$</td>
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<td>2.8304</td>
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<tr>
<td>$c(n)_{\text{max}}$</td>
<td>13.873</td>
<td>11.856</td>
</tr>
<tr>
<td>$\kappa_2(A)_{\text{min}}$</td>
<td>1.4142</td>
<td>2.3915</td>
</tr>
<tr>
<td>$\kappa_2(A)_{\text{max}}$</td>
<td>$1.3051 \times 10^{23}$</td>
<td>$9.5911 \times 10^{145}$</td>
</tr>
<tr>
<td>$\kappa(n)_{\text{min}}$</td>
<td>$2.2393 \times 10^{16}$</td>
<td>$3.8835 \times 10^7$</td>
</tr>
<tr>
<td>$\kappa(n)_{\text{max}}$</td>
<td>$3.1824 \times 10^{21}$</td>
<td>$1.1989 \times 10^{145}$</td>
</tr>
</tbody>
</table>

The results of Group 1 testing are shown in Figures 1–3. In Figure 1, we can see that the average of $\bar{c}(A)s$ ($c(n)$) tends to increase with $n$.

This tendency is similar to the Edelman result given by (23). Graphic presentation of $\bar{c}(A)s$ and $\kappa_2(A)s$ versus test matrix families and dimension are given in Figures 2 and 3. These two pictures show that for several test problems, the $\bar{c}(A)s$ are relatively small, while the condition numbers are quite high. The weak dependence on $\kappa_2(A)$ is also indicated by the following multiple linear regression result:

$$c(A) = 5.7164 \times 10^{-2} \dim(A) + 9.0520 \times 10^{-23} \kappa_2(A),$$

(26)

where the coefficient of $\kappa_2(A)$ is not significantly different from 0 at 95% confidence level.

In Group 1, the 90th percentile of the $\bar{c}(A)s$ is 23.128, which indicates that $\bar{c}(A)$ is likely to be remain small. Those cases for which $\bar{c}(A)$ exceeded 23.128 were the cauchy, krylov, lotkin, minij, moler, psi, randsvd, and magic matrices.

![Figure 1. The average of $\bar{c}(A)s$ versus dimension.](image-url)
The results of Group 2 testing are shown in Figures 4-6. The average of $\bar{c}(A)$s again tends to increase with $n$, as shown by Figure 4. Graphic presentation of $\bar{c}(A)$s and $\kappa_2(A)$s versus test matrix families and dimension are given in Figures 5 and 6. The multiple regression result is

$$c(A) = 2.6845 \times 10^{-1} \ dim(A) + 1.0648 \times 10^{-14} \kappa_2(A),$$

where the coefficient of $\kappa_2(A)$ is not significantly different from 0 at 95% confidence level. So we can conclude again that $c(A)$ depends on $\dim(A)$ rather than $\cond(A)$.

In Group 2, the 90th percentile of $\bar{c}(A)$s is 22.482, which indicates that $\bar{c}(A)$ is likely to remain small. Those matrices for which $\bar{c}(A)$ exceeded 22.482 were the invol and ipjfact.

In most of the Group 1 and 2 cases when $\bar{c}(A)$ exceeded the 90th percentile, the singular values are concentrated roughly in two clusters, where the cluster members are of equal size in
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2

5 10 15 20 25 30 35 40 45 50

dimension

Figure 4. The average of $\hat{c}(A)$ versus dimension.

40

35

0

1

2 3 4 5 6 7 8

index of matrix family

Figure 5. The values of $\hat{c}(A)$ versus matrices and dimension.

each group. Usually the first cluster contains a few large singular values while the remaining singular values, which belong to the second cluster, are small. In the case of the moler matrix, the situation is the opposite. It has only a few small singular values of the same size, while the remaining ones are large and approximately equal. So we can think that the above singular value distribution is at least partially responsible for $\hat{c}(A)$ being high.

We can now make the following conclusions. The average of the error constant $c$ in Auchmuty’s estimate is slowly increasing with $n$, and it depends on $n$ rather than $\text{cond}(A)$. Upon the basis of the observed trend of $c(n)$ and the regression results (26),(27), the following estimate holds with a high degree of probability:

$$
\|x - x^*\|_2 \lesssim \frac{0.5 \dim(A) \|r(x)\|_2^2}{\|A^T r(x)\|_2}.
$$

(28)
REFERENCES