SMOOTHNESS OF ORLICZ SPACES [1]). I

BY

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1. *Introduction*

Let (Ω, Σ, μ) be a measure space and L^{Φ} be an Orlicz space on it. A problem of considerable interest in L^{Φ}-space theory is to find conditions on the Young's functions Φ and Ψ such that L^{Φ} is rotund and uniformly rotund. [These also are termed strict and uniformly convex. Precise definitions will be given later.] This was needed in [12], and generally the rotundity and smoothness results, without restrictions on the measure space, are useful in many applications.

A detailed study for an important case of this problem was considered in [11], and it seems to be the only comprehensive paper on the subject. Another special case was considered in [8] whose result will be compared later in Section 4 below. The central results of [11] were obtained *if* (Ω, Σ, μ) *is a nonatomic* σ*-finite measure space.* This, however, is a restriction on the results of [11] and their use in the problems of probability theory (e.g., cf., [12], [14]) and elsewhere will be severely limited. This becomes plain if one recalls (cf., [15], p. 52) that a nonatomic (finite) measure space can be mapped in a one-to-one manner onto the Lebesgue measure space on a closed interval such that every measurable subset of the first space goes into a set of the latter, preserving measure. That the nonatomic case has special features was noted by an example in a different context in ([17], p. 40; see also Theorem 2 on p. 37). Since for $\Phi(x) = |x|^p$, $1 < p < \infty$, the space $L^{\Phi}[=L^p]$ is known to be uniformly rotund if (Ω, Σ, μ) is arbitrary, the general study is of interest both for applications and comparison.

The purpose of this paper is to present "best" conditions on the Young's functions Φ, Ψ without restrictions on (Ω, Σ, μ) such that L^{Φ} is(uniformly) rotund and smooth (Section 4). The main results of the paper are contained in Sections 3, 4 and 5. The methods of [11] which use nonatomicity and σ-finiteness so crucially, do not seem to extend to the general case. So in what follows, the differentiability of norms and certain related properties play a key role. The latter considerations have some independent interest and were found useful in deducing certain general results on the representation of functionals, and also for a probability limit theorem in Section 5. It will be seen that the results of this paper and those of [8]

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and [11] complement each other. The next section contains, as needed preliminaries, a reformulation of the representation theorem of [13], in an improved form.

2. *Reformulation of a representation theorem*

Let (Ω, Σ, μ) be a measure space subject to the (nonrestrictive) condition that μ has the finite subset property, FSP. (I.e., every measurable set of positive μ measure has a measurable subset of positive finite μ-measure. Further elaboration is given below.) Let Φ, Ψ be normalized Young's complementary functions, viz., nonnegative symmetric convex functions on the line vanishing at the origin and satisfying

$$\Phi(1)+\Psi(1)=1 \ , \quad xy \leqslant \Phi(x)+\Psi(y) \ , \quad \text{all } x, y. \tag{1}$$

The normalization here (the first part of (1)) is convenient and lends a direct comparison with the L^p-space theory, with $\Phi(x)=|x|^p/p$, $p \geqslant 1$. Let L^Φ be the set of all (equivalence classes of) measurable scalar functions f on Ω such that $N_\Phi(f)<\infty$, where

$$N_\Phi(f)=\inf\left\{ k>0, \int_\Omega \Phi\left(\frac{f}{k}\right) d\mu \leqslant \Phi(1)\right\}. \tag{2}$$

$N_\Psi(\cdot)$ and the complementary space L^Ψ are similarly defined. With (2) as norm, L^Φ $[L^\Psi]$ becomes a Banach (or B-)space, (cf., [18], [20]). Finally let $A_\Phi(\mu)$ be the class of (scalar) additive set functions G on Σ vanishing on μ-null sets, such that $\|G\|'_\Phi<\infty$, where

$$\|G\|'_\Phi=\inf\left\{k>0, I_\Phi\left(\frac{G}{k}\right) \leqslant \Phi(1)\right\}, \quad I_\Phi(G)=\sup \Sigma\Phi\left(\frac{G(A_i)}{\mu(A_i)}\right)\mu(A_i), \tag{3}$$

the supremum being taken relative to all finite disjoint collections $\{A_i\}$ in Σ of finite μ-measure. $\|G\|'_\Psi$ and $A_\Psi(\mu)$ are defined similarly.

Let $B_\Phi(\mu)$ be the class of bounded additive set functions on Σ vanishing on μ-null sets and such that the support of each ν in $B_\Phi(\mu)$ is equivalent to the support of some function f in L^Ψ satisfying $\int_\Omega \Psi(\beta f)\, d\mu<\infty$ if $\beta \leqslant 1$, and $=\infty$ if $\beta>1$. The norm of ν is the total variation, denoted by $|\nu|(\Omega)$. Define the class $\mathscr{A}_\Phi(\mu)=A_\Phi(\mu) \oplus B_\Phi(\mu)$, as the direct sum, with norm

$$\|G\|_\Phi=\|G_1\|'_\Phi+|\nu_1|(\Omega), \ G=G_1+\nu_1 \in \mathscr{A}_\Phi(\mu). \tag{3'}$$

$\mathscr{A}_\Psi(\mu)$ is defined similarly.

It should be mentioned that all these set functions G's in $\mathscr{A}_\Phi(\mu)$ (and $\mathscr{A}_\Psi(\mu)$) and μ, are assumed to have been defined initially on the same ring of sets $\mathscr{R}$ and then are extended in the usual way. This and its importance was discussed and stressed in [19]. A brief description (following [10]) may be instructive. Let $\mathscr{R}$ be the class of sets in Σ such that each set has a finite measure relative to all set functions in $\mathscr{A}_\Phi(\mu)$, $\mathscr{A}_\Psi(\mu)$ and μ. Then $\mathscr{R}$ is a ring. Let Σ_1 be the collection of sets E of Ω, such that $E \cap F \in \mathscr{R}$ for all $F \in \mathscr{R}$. Then Σ_1 is a field, and let Σ' be the σ-field generated by Σ_1. Define G' such that $E \in \Sigma', G'_{\pm}(E) = \sup G_{\pm}(E \cap F)$, for $F \in \mathscr{R}$, where $G_+(G_-)$ denotes the positive (negative) part of G. If $G' = G'_+ - G'_-$, (and similarly for μ) then these new set functions μ', G' on Σ' have the same properties as the old ones. It is these new ones that are of interest in the sequel. Dropping the primes, the old symbols will be used hereafter in the new sense. [I have briefly mentioned this in [13], referring to [10] and [19], but the above elaboration would have been more helpful.] This assumption removes certain uninteresting and essentially trivial cases without being restrictive. This formulation will be referred to as the FSP.

It was shown in [13], that $\mathscr{A}_{\Phi}(\mu)$, $\mathscr{A}_{\Psi}(\mu)$ are B-spaces with (3′) as norm. Let $M^{\Phi}[M^{\Psi}]$ denote the closed subspace of $L^{\Phi}[L^{\Psi}]$ determined by the μ-simple functions. [As usual $f \in L^{\Phi}$ means that f is any member of its equivalence class.] So $M^{\Phi} = \{f\colon \int_{\Omega} \Phi(kf)\, d\mu < \infty, \text{ all } k\}$.

The representation theorem of [13], takes the following form.

Theorem 1. *Let Φ, Ψ be normalized Young's complementary functions and L^{Φ}, $\mathscr{A}_{\Psi}(\mu)$ be the Orlicz space and the space of set functions defined above. Then for every $F \in (L^{\Phi})^*$, the conjugate space of L^{Φ}, there exists a unique G in $\mathscr{A}_{\Psi}(\mu)$ such that*

$$F(f) = \int_{\Omega} f dG \ , \quad f \in L^{\Phi}, \tag{4}$$

and

$$\|F\| = \|G\|_{\Psi}. \tag{5}$$

In particular, if $F \in (M^{\Phi})^*$, conjugate of M^{Φ}, then there exists a μ-unique "quasi-function" (i.e., one that is equivalent to a measurable function on sets of finite μ-measure) g^* in L^{Ψ} such that, for all f in M^{Φ},

$$F(f) = \int_{\Omega} f g^* \, d\mu \ , \quad \|F\| = N_{\Psi}(g^*). \tag{6}$$

g^* is measurable if and only if either (i) *$M^{\Psi} = L^{\Psi}$ or* (ii) *μ is localizable* (*or σ-finite*). *An exactly similar result holds if Φ and Ψ are interchanged throughout the above statement.*

This result without normalization (and omitting the last line) was proved in ($[13]_{\text{II}}$, Theorems 3 and 4). Except for obvious modifications (where 1 should be replaced by $\Phi(1)$ or $\Psi(1)$ appropriately) that proof applies here verbatim. The present interest is the equality (5) and the last statement about L^{Ψ}. This is further illustrated by the following result whose proof again is the same as that of ([13], Theorem 5).

Corollary 1.1. *If Φ, Ψ are normalized Young's functions such that $M^{\Phi} = L^{\Phi}$ and $M^{\Psi} = L^{\Psi}$, then L^{Φ} is isometrically equivalent to its second conjugate $(L^{\Phi})^{**}$, (i.e., L^{Φ}, L^{Ψ} are reflexive).*

The point here is that, in the past, there was only the topological equivalence between L^{Φ} and $(L^{\Phi})^{**}$. But now there is the isometric equivalence. This clarifies the remarks in ([7], p. 126, p. 224) on relations between various norms. The usefulness of the normalization was stressed in [20] and it will be exploited here. Finally, it will be noted that, if $M^{\Phi}=L^{\Phi}$ and $\Psi(\cdot)$ is continuous, then $\frac{\Phi(x)}{x}\uparrow\infty$ or $\downarrow 0$ as $x\uparrow\infty$ or $x\downarrow 0$. Since $\Phi'(\cdot)$, the derivative, exists a.e., it may be assumed, in the case when $\Psi(\cdot)$ is continuous, to exist everywhere (i.e., $\Phi'(\cdot)$ is continuous) by a redefinition, (e.g., joining the discontinuities with straight line segments; cf., [20], p. 25 and [7], p. 6 ff). This will be *assumed* together with the normalization of Φ, Ψ in the rest of the paper, whenever $M^{\Phi}=L^{\Phi}$, and $M^{\Psi}=L^{\Psi}$ (or Ψ continuous).

3. *Differentiability of norms*

If $\mathscr{X}$ is a B-space and $x_0\in\mathscr{X}$, then the norm $\|\cdot\|$ is said to be *weakly (or Gateaux) differentiable* at x_0 if

$$(*)\qquad \lim_{t\to 0}\frac{\|x_0+tx\|-\|x_0\|}{t}$$

exists for each x in $\mathscr{X}$, and is *strongly (or Fréchet) differentiable* if the limit in (*) is uniform in x on the unit sphere $S=\{x:\|x\|=1\}$ of $\mathscr{X}$. The norm is *uniformly strongly differentiable* if the limit in (*) is uniform in both x_0, x for $x_0, x\in S$, (cf., [3] and [2]). Further classifications were introduced in [2], but the above definitions will suffice here. In this section, conditions for differentiability of the norm (2) for L^{Φ} (and L^{Ψ}) will be obtained without restricting the measure space.

To begin with, the weak differentiability of the norm is easy to settle, and will be useful in other computations. It is convenient to extend the definitions of $\Phi'(\cdot)$ and $\Psi'(\cdot)$ to negative values by setting $\Phi'(-x)=-\Phi'(x)$ and $\Psi'(-x)=-\Psi'(x)$, for $x\geqslant 0$.

Proposition 1. *Let Φ and Ψ be continuous Young's functions and M^{Φ} be the subspace of L^{Φ} determined by the μ-simple functions. Then the norm functional $N_{\Phi}(\cdot)$ is weakly differentiable at every point of M^{Φ}, except at the origin. Moreover if Φ' is continuous, the weak derivative $G(f_0;\cdot)$ at $f_0\in M^{\Phi}\cap S^{\Phi}$, where S^{Φ} is the unit sphere of L^{Φ}, is given by*

$$(7)\qquad G(f_0;f)=\int_{\Omega} f\Phi'(f_0)\,d\mu\ ,\quad f\in M^{\Phi}\cap S^{\Phi}.$$

Proof. First note that for $f\in M^{\Phi}$, $\int_{\Omega}\Phi(f/k)\,d\mu$ exists for all $k>0$ and tends to 0 or $+\infty$ according as $k\to+\infty$ or 0. This is a simple consequence of the fact that simple functions are dense in M^{Φ} and that Φ is continuous. Next note that $\Phi'(f)\in L^{\Psi}$ even if $\Phi'(\cdot)$ is a right (or left) derivative of Φ. This fact was proved for a finite nonatomic measure space in ([7], p. 73), and the same proof applies for the general case with the following modification. Since $f \in M^{\Phi}$, there is a sequence of simple functions $\{f_n\}$ such that $f_{n_i} \to f$ pointwise and in norm. With $\{f_{n_i}\}$ for the bounded sequence used in the proof of [7], the same holds here. [In fact, if $f \in L^{\Phi}$, then $\Phi'(f)\chi_E \in L^{\Psi}$ for every set E of finite μ-measure by the proof of [7], and the general case follows with the procedure used in the proof of ([13], Lemma 1).]

Now let $k(t) = N_{\Phi}(f_0 + tf)$, for $f_0, f \in M^{\Phi} \cap S^{\Phi}$. Then by what precedes and ([20], p. 175), it results that

$$\Phi(1) = \int_{\Omega} \Phi\left(\frac{f_0 + tf}{k(t)}\right) d\mu. \tag{8}$$

On the other hand for each f_0, f if $F(t, k) = \Phi\left(\frac{f_0 + tf}{k}\right)$, then

$$dF = -\Phi'\left(\frac{f_0 + tf}{k}\right)\frac{f_0 + tf}{k^2}\, dk + \Phi'\left(\frac{f_0 + tf}{k}\right)\frac{f}{k}\, dt \tag{9}$$

which exists, whenever $f_0 + tf \neq 0$, by elementary differentiation, (and set it equal to zero when $f_0 + tf = 0$). For $|t| < 1$, $k(t) \geqslant \alpha > 0$ and hence the right side of (9) is dominated by $2\Phi'\left(\frac{|f_0| + |f|}{\alpha}\right)\frac{(|f_0| + |f|)}{\alpha^2}$ which is integrable by Hölder inequality and the preceding paragraph. This permits the interchange of integral and differential in (8), for $|t| < 1$, so that

$$0 = -\frac{1}{k^2}\left(\int_{\Omega} \Phi'\left(\frac{f_0 + tf}{k}\right)(f_0 + tf)\, d\mu\right) dk + \frac{1}{k}\left(\int_{\Omega} \Phi'\left(\frac{f_0 + tf}{k}\right) f\, d\mu\right) dt.$$

From this it follows that dk/dt exists, and noting that $k(0) = 1$, one gets

$$\frac{d}{dt}\left[N_{\Phi}(f_0 + tf)\right]_{t=0} = \frac{\int_{\Omega} f\Phi'(f_0)\, d\mu}{\int_{\Omega} f_0\Phi'(f_0)\, d\mu}, \tag{10}$$

since the denominator is non-negative, $\leqslant 1$ by Hölder inequality, and zero if and only if $f_0\Phi'(f_0) = 0$, a.e. This last possibility is ruled out since $f_0 \in M^{\Phi} \cap S^{\Phi}$. If moreover, $\Phi'(\cdot)$ is continuous, then by ([20], p. 175) there is equality in Hölder inequality and thus the denominator in (10) is 1. Consequently (10) reduces to (7), completing the proof.

Remark. The above proof is based on the arguments of ([9], p. 404), and the nonatomic case of ([7], p. 188). A precise comparison of the latter will be given after the next result.

Now the conditions for strong differentiability can be given in

Theorem 2. *Let Φ, Ψ be continuous Young's functions, $\Phi'(\cdot)$ be continuous, and $M^{\Phi} \subset L^{\Phi}$ be as above. Suppose $\Psi(\cdot)$ is such that*

$$\int_{\Omega} \Psi\left(\frac{f}{N_{\Psi}(f)}\right) d\mu = \Psi(1)$$

for $f \in L^{\Psi}$. *Then the norm functional* $N_{\Phi}(\cdot)$ *is strongly differentiable at every point of* M^{Φ} *except at the origin whenever either* (i) μ *is localizable or* (ii) $M^{\Psi}=L^{\Psi}$ (*and* μ *is arbitrary*). *Moreover, the strong derivative at* f_0 *in* $M^{\Phi} \cap S^{\Phi}$ *is the same as the weak derivative and is given by*

$$G(f_0;f)=\int_{\Omega} f\Phi'(f_0)\,d\mu\ ,\quad f\in M^{\Phi}\cap S^{\Phi}. \tag{7'}$$

Remark. The condition on Ψ, $\Psi(1)=\int_{\Omega}\Psi\left(\frac{f}{N_{\Psi}(f)}\right)d\mu$ is automatic in case (ii) and, in case (i), if the integral exists for $N_{\Psi}(f)$ replaced by some $k<N_{\Psi}(f)$.

Proof. If $f_0, f\in M^{\Phi}\cap S^{\Phi}$, it is to be shown that $\frac{N_{\Phi}(f_0+tf)-1}{t}\to G(f_0;f)$ uniformly in f as $t\to 0$. Only the uniformity needs to be shown, because of the above proposition. However, this is nontrivial and is proved as follows.

For f_0 in $M^{\Phi}\cap S^{\Phi}$, by a form of the Hahn-Banach theorem ([18], p. 146, Theorem 5) there exists an F_0 in $(M^{\Phi})^*$ such that $F_0(f_0)=N_{\Phi}(f_0)=1$, and $\|F_0\|=1$. By Theorem 1 (cf. (6)) there is a unique g_0 in L^{Ψ}, such that

$$1=F_0(f_0)=\int_{\Omega} f_0g_0\,d\mu\ ,\quad N_{\Psi}(g)=\|F_0\|=1, \tag{11}$$

where g_0 is actually a measurable function (not merely a quasi-function), under (i) or (ii) above. For the proof here it is necessary to show that the F_0 is unique. If F_1 in $(M^{\Phi})^*$ is another element with the same property, then $F_2=\frac{1}{2}(F_0+F_1)\in(M^{\Phi})^*$ and $\|F_2\|\leqslant 1$. Since $1=N_{\Phi}(f_0)=F_2(f_0)$, one must have $\|F_2\|=1$ and as in (11) there exist g_1, g_2 in L^{Ψ} such that

$$1=F_2(f_0)=\tfrac{1}{2}\int_{\Omega} f_0(g_0+g_1)\,d\mu=\int_{\Omega} f_0g_2\,d\mu,$$

and by uniqueness of this presentation and isometry one has $g_2=\frac{1}{2}(g_0+g_1)$ a.e., and $N_{\Psi}(g_2)=1=\frac{1}{2}(N_{\Psi}(g_0)+N_{\Psi}(g_1))$. Now using the condition on $\Psi(\cdot)$, it follows that

$$\Psi(1)=\int_{\Omega}\Psi(g_2)\,d\mu=\int_{\Omega}\Psi\left(\frac{g_0+g_1}{2}\right)d\mu<\tfrac{1}{2}\int_{\Omega}[\Psi(g_0)+\Psi(g_1)]\,d\mu=\Psi(1),$$

since $\frac{\Psi(x)}{x}\uparrow\infty$ as $x\uparrow\infty$, on account of $\Phi'(x)\uparrow\infty$. The contradiction of the above line shows that $F_0=F_1$. Moreover from (11) since $\Phi'(\cdot)$ is continuous, and the fact that (Hölder) equality holds there, one has (cf. [20], p. 175),

$$g_0=\Phi'(f_0),\ \text{a.e.}\ ,\quad \left(=\Phi'\left(\frac{f_0}{N_{\Phi}(f)}\right)\right). \tag{12}$$

From the preceding paragraph, after a rearrangement of terms, and (7),

$$\left\{\begin{aligned}\frac{N_{\Phi}(f_0+tf)-1}{t}-G(f_0;f)&=\frac{1}{t}\int_{\Omega} f_0\left[\Phi'\left(\frac{f_0+tf}{N_{\Phi}(f_0+tf)}\right)-\Phi'(f_0)\right]d\mu\ +\\ &\int_{\Omega} f\left[\Phi'\left(\frac{f_0+tf}{N_{\Phi}(f_0+tf)}\right)-\Phi'(f_0)\right]d\mu.\end{aligned}\right. \tag{13}$$

By Proposition 1, for each f_0, f in $M^\Phi \cap S^\Phi$, the left side tends to 0 as $t \to 0$, so that the right side must tend to 0. Since $\Phi'(\cdot)$ is continuous and the norm $N_\Phi(\cdot)$ is absolutely continuous on M^Φ, (cf., [8], p. 58; and [7], p. 87), it follows by the dominated convergence theorem that the second term on the right of (13) tends to zero, uniformly in $f \in S^\Phi$. Consequently one must have

$$\lim_{t\to 0} \frac{1}{t} \int_\Omega f_0 \left[\Phi'\left(\frac{f_0+tf}{N_\Phi(f_0+tf)}\right) - \Phi'(f_0)\right] d\mu = 0. \tag{14}$$

Now writing $tf=f'$, it is clear that $|t|=N_\Phi(f')$, and (14) implies

$$\lim_{N_\Phi(f')\to 0} \frac{1}{N_\Phi(f')} \int_\Omega f_0 \left[\Phi'\left(\frac{f_0+f'}{N_\Phi(f_0+f')}\right) - \Phi'(f_0)\right] d\mu = 0\ ,\ f_0 \in M^\Phi \cap S^\Phi. \tag{15}$$

(15) is evidently equivalent to the statement that (14), and hence the right side of (13), tends to 0 uniformly in f. This completes the proof.

Corollary 2.1. *Let Φ, Ψ be such that $M^\Phi = L^\Phi$ and $M^\Psi = L^\Psi$, (i.e., L^Φ, and L^Ψ are reflexive). Then the norm in L^Φ (and L^Ψ) is strongly differentiable at every point except at the origin.*

This follows from the theorem since $\Phi'(\cdot)$ and $\Psi'(\cdot)$ are continuous (by the normalizations, cf. end of Section 2) and that the following holds:

$$\Phi(1) = \int_\Omega \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \quad \text{and} \quad \Psi(1) = \int_\Omega \Psi\left(\frac{f}{N_\Psi(f)}\right) d\mu.$$

Remark. Using as counterexamples the L^1 and L^∞, it can be seen that the conditions given on Φ, Ψ cannot be improved appreciably in the above results. Also the result of case (ii) of the theorem was proved in [7], for finite nonatomic measures in a different way. The present proof, based on the Hahn-Banach theorem, seems more direct.

Finally conditions for uniform strong differentiability of norms will be given, as it will be important for later work. The following useful inequality on Young's functions Φ, Ψ, discovered by Milnes, will be needed. Its proof may be found in his paper ([11], p. 1473).

Lemma. (Milnes [11]). *Let Φ, Ψ be Young's functions, such that $\Phi(2x) \leqslant C\Phi(x)$ for $x \geqslant 0$, and a $0 < C < \infty$, and $\Psi(\cdot)$ is continuous. If for each $0 < \varepsilon < 1$, there exist a $\delta(\varepsilon) > 0$ and a $k_\varepsilon > 1$, ($\delta_\varepsilon \to 0$ as $\varepsilon \to 0$) such that* (i) $\dfrac{\Phi'(u)}{\Phi'((1-\varepsilon)u)} \geqslant k_\varepsilon$ *and* (ii) $|u'-u| \geqslant \delta_\varepsilon \quad u > 0$, *then one has*

$$\Psi(v') \geqslant \Psi(v) + \Psi'(v)\,(v'-v) + L_\varepsilon \Phi(|u'-u|), \tag{16}$$

for some $L_\varepsilon > 0$, which depends only on ε, where $v = \Phi'(u)$ and $v' = \Phi'(u')$.

Thus prepared the following result can be established.

Theorem 3. *Let Φ, Ψ be Young's functions such that $M^\Phi = L^\Phi$ and $M^\Psi = L^\Psi$, and that Φ' and Ψ' are continuous. Then the norm $N_\Phi(\cdot)$ in L^Φ*

is uniformly strongly differentiable if and only if

$$\lim_{N_\Phi(f)\to 0} \frac{1}{N_\Phi(f)} \int_\Omega f_0\left[\Phi'\left(\frac{f_0+f}{N_\Phi(f_0+f)}\right) - \Phi'(f_0)\right] d\mu = 0 \tag{17}$$

uniformly in $f_0 \in S^\Phi$. *A sufficient condition for* (17) *to hold* (*i.e. for uniform strong differentiability of* $N_\Phi(\cdot)$) *is that for all* $u>0$ *and* $0<\varepsilon<1$ *there exist* $1<k_\varepsilon<C<\infty$, *such that,*

$$\Psi'((1+\varepsilon)u) \geqslant k_\varepsilon \Psi'(u)\ ,\quad \Psi(2u) \leqslant C\Psi(u). \tag{18}$$

Proof. The first part is immediate. For, if (17) holds, then the integral in (17) also tends to zero uniformly in $(f_0, f) \in S^\Phi \times S^\Phi$, so that the right side of (13) tends to zero uniformly in f_0, f. Conversely, if $N_\Phi(\cdot)$ has the latter property, then the right side of (13) must tend to zero uniformly in f_0 and f. It follows that (17) holds in that case. However it is more difficult to show (and the proof is long) that (18) implies (17), or the uniform strong differentiability of $N_\Phi(\cdot)$. This will be demonstrated now.

Since $N_\Phi(f_0+tf)$ is convex in t, it is well-known (cf., [20], p. 24) that it can be expressed as an indefinite integral of its derivative, on $[-1, 1]$. Since the (weak) derivative $G(f_0;\cdot)$ of $N_\Phi(\cdot)$ exists at $f_0 \in S^\Phi$, it follows after a slight computation (given in [2], p. 301) that, for all $f \in S^\Phi$ and $|t|<1$,

$$(19)\quad \left\{\begin{aligned} \left|\frac{N_\Phi(f_0+tf)-1}{t} - G(f_0;f)\right| &\leqslant \int_0^1 |G(f_0+t\alpha f; f) - G(f_0;f)|\, d\alpha \cdot \\ &\leqslant \int_0^1 \|G(f_0+t\alpha f;\cdot) - G(f_0;\cdot)\|\, d\alpha \\ &\qquad \text{by Hölder inequality,} \\ &= \int_0^1 \left\|G\left(\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)};\cdot\right) - G(f_0;\cdot)\right\| d\alpha \\ &\qquad \text{since } G(f_0;\cdot)=G(\beta f_0;\cdot), \text{ all } \beta>0, \\ &= \int_0^1 N_\Psi\left[\Phi'\left(\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)}\right) - \Phi'(f_0)\right] d\alpha. \end{aligned}\right.$$

The last equality follows from the fact that $G(f_0;\cdot)$ is a bounded linear functional in L^Φ, and then using Theorem 1.

It will now be shown that the right side of (19) tends to zero uniformly in $(f_0, f) \in S^\Phi \times S^\Phi$ as $t \to 0$. For this it suffices to show that for each α, the integrand tends to zero as $t \to 0$. This is equivalent to showing, in view of $M^\Psi = L^\Psi$, that

$$\lim_{t\to 0} \int_\Omega \Psi\left[\Phi'\left(\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)}\right) - \Phi'(f_0)\right] d\mu = 0. \tag{20}$$

In fact, if $\{f_n\} \subset L^\Psi$, then $N_\Psi(f_n) \to 0$, as $n \to \infty$, if and only if $\int_\Omega \Psi(kf_n)\, d\mu \to 0$ as $n \to \infty$ for each $k \geqslant 1$. This is known and is proved by a simple argument. Indeed, if $N_\Psi(f_n)\to 0$, then let n be large enough such that $kN_\Psi(f_n)\leqslant 1$, and hence, by convexity, $(0<\Psi(1)<1)$,

$$\int_\Omega \Psi(kf_n)\,d\mu \leqslant k\,N_\Psi(f_n)\int_\Omega \Psi\left(\frac{f_n}{N_\Psi(f_n)}\right)d\mu = k\,\Psi(1)\,N_\Psi(f_n)\to 0.$$

Conversely, if $k\geqslant 1$ is arbitrary, for every h in S^Φ, $N_\Phi(h/k)\leqslant 1/k$, so that, by Young's inequality (cf. (1)), for large n satisfying $\int_\Omega \Psi(kf_n)\,d\mu\leqslant 1/k$,

$$\int_\Omega |f_n h|\,d\mu \leqslant \int_\Omega \Psi(kf_n)\,d\mu + \frac{1}{k}\leqslant \frac{2}{k}.$$

Consequently the Orlicz norm $\|f_n\|_\Psi\leqslant 2/k$ and by ([20], p. 174)

$$\Psi(1)\,N_\Psi(f_n)\leqslant \|f_n\|_\Psi\leqslant 2/k.$$

Since $0<\Psi(1)<1$ and k is arbitrary, $N_\Psi(f_n)\to 0$, as was to be shown.

Now with (18), it will be shown that (20), or $\int_0^1\int_\Omega \Psi[\quad]\,d\mu\,d\alpha,\to 0$ as $t\to 0$, holds. If $0<\varepsilon<1$ and (18) holds, then for an $\eta(\varepsilon)>0$ ($\to 0$ as $\varepsilon\to 0$), there is an $L_\eta>0$, such that

$$\Phi(u')-\Phi(u)\geqslant \Phi'(u)\,(u'-u)+L_\eta\Psi(|v'-v|) \tag{21}$$

whenever $|v'-v|\geqslant \eta v>0$ and $v=\Phi'(u)>0$, $v'=\Phi'(u')>0$. In (20), first consider $f_0\geqslant 0$, $f\geqslant 0$, a.e., and $0<t<1$. Let $u=f_0$, $u' = \dfrac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)}$, and $\Omega_1=\{\omega;\,|v'-v|\geqslant \eta v>0\}$, where v' and v are the ω-functions defined in terms of u' and u above. Hence

$$\int_{\Omega_1}\Psi[v'-v]\,d\mu \leqslant \frac{1}{L_\eta}\int_{\Omega_1}[\Phi(u')-\Phi(u)-\Phi'(u)(u'-u)]\,d\mu. \tag{22}$$

But also for any x', $x\geqslant 0$, it is clear that the convex function $\Phi(\cdot)$ satisfies $\Phi(x')\geqslant \Phi(x)+\Phi'(x)\,(x'-x)$. So integrating this on $\Omega-\Omega_1$ with the above u and u' for x and x', and adding the result to (22), one has

$$\begin{aligned}\int_{\Omega_1}\Psi[v'-v]\,d\mu &\leqslant \frac{1}{L_\eta}\int_\Omega[\Phi(u')-\Phi(u)+\Phi'(u)(u-u')]\,d\mu\\ &= \frac{1}{L_\eta}\int_\Omega \Phi'(u)(u-u')\,d\mu, \text{ since } u,\,u'\in S^\Phi,\\ &\leqslant \frac{1}{L_\eta}N_\Phi\left(f_0-\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)}\right), \text{ since } \Phi'(f_0)\in S^\Psi,\\ &\leqslant \frac{1}{L_\eta}\frac{2|t|\alpha}{1-|t|}, \text{ by ([2], p. 301, eq. (2))}\end{aligned}$$

which tends to zero as $t\to 0$ uniformly in $f_0, f\in S^\Phi$. The same holds if $t<0$, by considering $f\leqslant 0$ a.e. On $\Omega-\Omega_1$ the result is simpler, since $|v'-v|\leqslant \eta v$, $0<\eta<1$,

$$\int_{\Omega-\Omega_1}\Psi(v'-v)\,d\mu\leqslant \int_\Omega \Psi(\eta v)\,d\mu\leqslant \eta\Psi(1),$$

which $\to 0$, for f_0, f in S^Φ, since η is arbitrary. Hence (20) is proved. Consequently

$$\int_0^1 \int_\Omega \Psi(v'-v)\,d\mu\,d\alpha \leqslant \frac{1}{L_\eta}\frac{|t|}{1-|t|} + \eta\Psi(1),$$

so that (19) $\to 0$ as $t \to 0$ uniformly in f_0, f in S^Φ, and thus the result is true in this case.

The general case, that f_0, f are arbitrary in S^Φ, is reduced to the special one treated above by a standard trick (e.g. cf., [11], p. 1480). Let $f_t{}^0$ and f_t' be defined as (with notations introduced above (22)),

$$f_t{}^0 = \begin{cases} |v| & \text{if } v+v' \text{ has the sign of } v \\ 0 & \text{otherwise,} \end{cases}$$

$$f'_t = \begin{cases} |v'| & \text{if } v+v' \text{ has the sign of } v' \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leqslant f_t{}^0 \leqslant |v|$, $0 \leqslant f_t' \leqslant |v'|$, a.e., and $|v+v'| \leqslant f_t{}^0 + f_t'$, and (as most easily seen by drawing a picture) $|v'-v| \leqslant 2|f_t{}^0 - f_t'|$, a.e. It is also clear that $N_\Psi(f_t{}^0) \leqslant 1$, $N_\Psi(f_t') \leqslant 1$, and $f_t{}^0 \to \Phi'(f_0)$, $f_t' \to \Phi'(f_0)$, a.e. Thus writing $s_t = \Psi'(f_t{}^0)$, $s_t' = \Psi'(f_t')$, one notes that $N_\Phi(s_t) \leqslant 1$, $N_\Phi(s'_t) \leqslant 1$, and

$$\int_0^1 \int_\Omega \Psi[v'-v]\,d\mu\,d\alpha \leqslant \int_0^1 \int_\Omega \Psi[2(f_t{}^0 - f'_t)]\,d\mu\,d\alpha \leqslant C \int_0^1 \int_\Omega \Psi[f_t{}^0 - f'_t]\,d\mu\,d\alpha,$$

and since $f_t{}^0$, f_t' and hence s_t, s_t' satisfy the hypothesis of the special case, where one uses the fact that $\int_\Omega \Phi(s_t)\,d\mu \to \Phi(1)$, $\int_\Omega \Phi(s_t')\,d\mu \to \Phi(1)$, as $t \to 0$; (since $N_\Phi(s_t) \to 1$, $N_\Phi(s_t') \to 1$ and $N_\Phi(\cdot)$ is absolutely continuous). Thus the result is valid in the general case as well. The proof of the theorem is therefore complete.

Remarks. 1. For this result, condition (18) is "best" will become clear with the work of the next section. From the above proof, it is also clear that the necessary and sufficient condition for uniform strong differentiability of $N_\Phi(\cdot)$ at every point except the origin is that $\left|G\left(\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)}; f\right) - G(f_0; f)\right| < \varepsilon$ whenever $N_\Phi\left(\frac{f_0+t\alpha f}{N_\Phi(f_0+t\alpha f)} - f\right) < \delta(\varepsilon)$, for all f_0, f in S^Φ. General conditions were given, for this to hold (in terms of the spherical image map), by Cudia, [2], for arbitrary B-spaces. However, it is not easy to translate them to the present case.

(To be continued)