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Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption

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Abstract

We consider the Cauchy problem for the damped wave equation with absorption

$$u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N.$$

The behavior of u as $t \to \infty$ is expected to be the same as that for the corresponding heat equation

$$\phi_t - \Delta\phi + |\phi|^{p-1}\phi = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N,$$

which has the similarity solution $w_a(t,x)$ with the form $t^{-1/(p-1)} f(x/\sqrt{t})$ depending on $a = \lim_{|x|\to\infty} |x|^{2/(p-1)} f(x) \geqslant 0$ provided that p is less than the Fujita exponent $p_c(N) := 1 + 2/N$. In this paper, as a first step, if $1 < p < p_c(N)$ and the data $(u_0, u_1)(x)$ decays exponentially as $|x| \to \infty$ without smallness condition, the solution is shown to decay with rates as $t \to \infty$,

$$\left(\|u(t)\|_{L^2}, \|u(t)\|_{L^{p+1}}, \|\nabla u(t)\|_{L^2}\right) = O\left(t^{-\frac{1}{p-1}+\frac{N}{4}}, t^{-\frac{1}{p-1}+\frac{N}{2(p+1)}}, t^{-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4}}\right), \qquad (*)$$

those of which seem to be reasonable, because the similarity solution $w_a(t,x)$ has the same decay rates as $(*)$. For the proof, the weighted L^2-energy method will be employed with suitable weight, similar to that in Todorova and Yordanov [Y. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001) 464–489].

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1. Introduction

We consider the Cauchy problem for the semilinear damped wave equation with absorption:

$$u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.1}$$

$$(u, u_t)(0,x) = (u_0, u_1)(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

where $p > 1$. When $(u_0, u_1) \in H^1 \times L^2$ and

$$1 < p < \frac{N+2}{N-2} \quad (N \geqslant 3), \qquad 1 < p < \infty \quad (N = 1, 2), \tag{1.3}$$

there exists a unique solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ (Strauss [31], Ginibre and Velo [9], Brenner [1], Matsumura [24], Kawashima et al. [21], etc.). In [21] it is shown that, for $1 \leqslant N \leqslant 3$, if

$$1 + \frac{4}{N} < p < \frac{N+2}{N-2} \quad (N \geqslant 3), \qquad 1 + \frac{4}{N} < p < \infty \quad (N = 1, 2),$$

then the solution $u(t,x)$ decays as

$$\big\|u(t,\cdot)\big\|_{L^2} = O\big(t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{2})}\big), \quad \text{when } (u_0,u_1) \in \big(H^1 \cap L^r\big)\big(\mathbf{R}^N\big) \times \big(L^2 \cap L^r\big)\big(\mathbf{R}^N\big)$$
$$(1 \leqslant r \leqslant 2)$$

(for more details refer to [21]), whose rate is the same as that of solutions to the linear heat equation. Based on [21], Karch [20] showed that the Gauss kernel is an asymptotic profile when $p > 1 + 4/N$ with $1 \leqslant N \leqslant 3$. Very recently, in Hayashi et al. [11] the asymptotic profile of u for $p > 1 + 2/N$ with $N = 1$ has been shown to be the Gauss kernel.

On the other hand, the global existence and blow-up of small weak solutions to the damped wave equation with the forcing term

$$u_{tt} - \Delta u + u_t = |u|^p \tag{1.4}$$

with (1.2) have been also investigated. Todorova and Yordanov [32] have shown that

$$p_c(N) := 1 + \frac{2}{N} \tag{1.5}$$

is the critical exponent, which is called the Fujita exponent named after Fujita [6], in any dimensional space satisfying $p < 1 + 2/N$ $(N \geqslant 3)$ and $p < \infty$ $(N = 1, 2)$. Refer to Zhang [33] for the blow-up in the critical case, and Li and Zhou [22], Nishihara [27] for the blow-up time. See also Ikehata et al. [18], Ikehata and Tanizawa [17], Ono [29,30], Gallay and Raugel [7,8], Karch [20], and references therein for the global existence and its profile. Recently, the first author has shown in [26] that the linear damped wave equation is approximated by the corresponding heat equation in 3-dimensional space. He has precisely derived the L^p–L^q estimate on the difference of each solution. See also Marcati and Nishihara [23] in 1-dimensional space, Hosono and Ogawa [15] in 2-dimensional space and Narazaki [25] in general space dimension and Ikehata [16], Ikehata and Nishihara [19], Chill and Haraux [3] in the abstract setting. These are applied to the semilinear problem (1.4) with (1.2). Note that the basic estimates on the solution to the linear damped wave equation were obtained by Matsumura [24].

Summing up these results, the solution to the damped wave equation (1.4) is expected to have similar behavior as $t \to \infty$ as that to the corresponding heat equation

$$\phi_t - \Delta\phi = |\phi|^p. \tag{1.6}$$

In this paper, we consider decay properties of solutions $u(t,x)$ to (1.1)–(1.2) when

$$1 < p < p_c(N) = 1 + \frac{2}{N}, \tag{1.7}$$

whose decay rates should be related to the Cauchy problem for the semilinear heat equation

$$\phi_t - \Delta\phi + |\phi|^{p-1}\phi = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.8}$$

$$\phi(0,x) = \phi_0(x), \quad x \in \mathbf{R}^N. \tag{1.9}$$

For any $p > 1$, (1.8) has a solution $w^*(t,x) := ((p-1)t)^{-1/(p-1)}$. For p satisfying (1.7), it was proved by Brezis et al. [2] that there exists a family of positive self-similar solutions $w_a(t,x)$ such that

$$\lim_{|x|\to\infty} |x|^{2/(p-1)} w_a(t,x) =: a \geqslant 0$$

exists. The solution $w_a(t,x)$ has the form

$$w_a(t,x) = t^{-\frac{1}{p-1}} f\left(\frac{x}{\sqrt{t}}\right) \tag{1.10}$$

with

$$-\Delta f - \frac{y \cdot \nabla f}{2} + |f|^{p-1} f = \frac{1}{p-1} f. \tag{1.11}$$

We recall some results on the asymptotic behavior of solutions to (1.8)–(1.9) with (1.7). Gmira and Véron [10] showed that if $\phi_0 \geqslant 0$, $\phi_0 \in L^1(\mathbf{R}^N)$ and $\lim_{|x|\to\infty} |x|^{2/(p-1)}\phi_0(x) = +\infty$, then

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \big(\phi(t,\cdot) - w^*(t,\cdot)\big) = 0, \quad \text{uniformly on } \big\{x \in \mathbf{R}^N;\ |x| \leqslant C\sqrt{t}\,\big\}.$$

Escobedo and Kavian [4] proved that if

$$\phi_0 \neq 0, \quad 0 \leqslant \phi_0(x) \leqslant Ce^{-\beta|x|^2} \quad \text{for some } \beta > 0,\ C > 0, \tag{1.12}$$

then

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \big\|\phi(t,\cdot) - w_0(t,\cdot)\big\|_{L^\infty} = 0. \tag{1.13}$$

Note that $w_0(t,x)$ decays exponentially as $|x| \to +\infty$. When

$$\phi_0 \in L^1\big(\mathbf{R}^N\big), \quad \phi_0 \neq 0, \quad \lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} \phi_0(x) =: a \geqslant 0, \tag{1.14}$$

it has been proved by Escobedo et al. [5] that, depending on $a \geqslant 0$, the positive similarity solution $w_a(t,x)$ is uniquely determined and

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \big\|\phi(t,\cdot) - w_a(t,\cdot)\big\|_{L^\infty} = 0. \tag{1.15}$$

From the observation in the above, our conjecture is that the solution $u(t,x)$ to (1.1)–(1.2) also satisfies (1.15) if the data (u_0,u_1) satisfy the condition corresponding to (1.14).

In this paper, corresponding to Escobedo and Kavian [4], we show the decay properties of the solution u to (1.1)–(1.2), provided that

$$\bigl|u_0(x)\bigr|, \bigl|u_1(x)\bigr| \leqslant Ce^{-\beta|x|^2} \quad \text{for } \beta > 0,\ C > 0, \tag{1.16}$$

but no smallness condition is assumed. To apply the weighted L^2-energy method, we assume that

$$I_0^2 := \int\limits_{\mathbf{R}^N} e^{\beta|x|^2}\bigl(u_1^2 + |\nabla u_0|^2 + u_0^2\bigr)(x)\,dx < +\infty \quad \text{for some } \beta > 0 \tag{1.17}$$

in stead of (1.16). Denoting the solution space $X(0,T)$ by

$$X(0,T) = C\bigl([0,T); H^1\bigl(\mathbf{R}^N\bigr)\bigr) \cap C^1\bigl([0,T); L^2\bigl(\mathbf{R}^N\bigr)\bigr),$$

we have our main theorem.

Theorem 1.1. *Assume that* $1 < p < 1 + 2/(N-2)$ $(N \geqslant 3)$, $1 < p < \infty$ $(N = 1,2)$ *and that* $(u_0,u_1) \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ *with* (1.17). *Then the solution* $u(t,x) \in X(0,\infty)$ *to* (1.1)–(1.2) *uniquely exists, which satisfies for* $t \geqslant 0$,

$$\bigl\|u(t,\cdot)\bigr\|_{L^2} \leqslant CI_0(1+t)^{-\frac{1}{p-1}+\frac{N}{4}}, \qquad \bigl\|u(t,\cdot)\bigr\|_{L^{p+1}} \leqslant CI_0(1+t)^{-\frac{1}{p-1}+\frac{N}{2(p+1)}}, \tag{1.18}$$

$$\bigl\|\nabla u(t,\cdot)\bigr\|_{L^2} + \bigl\|u_t(t,\cdot)\bigr\|_{L^2} \leqslant CI_0(1+t)^{-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4}} \tag{1.19}$$

for some positive constant C provided that $1 < p \leqslant 1 + 4/N$.

Remark 1.1. In the supercritical case $p > p_c(N)$, the asymptotic profile of the solution u is expected to be the Gauss kernel $G(t,x)$, whose L^r-norm $(1 < r \leqslant \infty)$ decays as $\|G(t,\cdot)\|_{L^r} = O(t^{-\frac{N}{2}(1-\frac{1}{r})})$. Hence, the decay rates (1.18)–(1.19) are less sharp, and so the subcritical case (1.7) is mainly kept in mind.

Remark 1.2. The L^r-norm $(1 \leqslant r \leqslant \infty)$ of the similarity solution w_a decays as

$$\bigl\|w_a(t,\cdot)\bigr\|_{L^r} = t^{-\frac{1}{p-1}}\Biggl(\int\limits_{\mathbf{R}^N} t^{N/2} \cdot \left|f\left(\frac{x}{\sqrt{t}}\right)\right|^r \frac{dx}{t^{N/2}}\Biggr)^{1/r} = Ct^{-\frac{1}{p-1}+\frac{N}{2r}}. \tag{1.20}$$

Hence, in the subcritical case the decay rates (1.18)–(1.19) are sharp in the L^2-sense. However, compared to our goal (1.15), the results are dissatisfactory. We have

$$\bigl\|u(t,\cdot)\bigr\|_{L^s} = O\bigl(t^{-\frac{1}{p-1}+\frac{1}{s}}\bigr) \begin{cases} 1 \leqslant s \leqslant \infty & \text{if } N = 1, \\ 1 \leqslant s < \infty & \text{if } N = 2, \\ 1 \leqslant s \leqslant \frac{2N}{N-2} & \text{if } N \geqslant 3, \end{cases} \tag{1.21}$$

applying the Sobolev inequality $(N = 1)$ and the Gagliardo and Nirenberg inequality $(N \geqslant 2)$ to (1.18)–(1.19) and (2.9) in the next section, which will be derived after stating Theorem 2.1. Note that Hayashi et al. [13] have recently obtained (1.15) for $p_c(N) - \varepsilon < p < p_c(N)$ (ε is a small positive constant) with $N = 1$ and the small data with suitable positivity (see Hayashi et al. [12] in the critical case). See also their quite recent paper [14] for large data.

Notations. By C_i, c_i $(i = 0, 1, 2, \ldots)$ or simply C we denote several generic constants. The constant depending on $a, b, \ldots$ is denoted by $C(a, b, \ldots)$. The Lebesgue space $L^q(\mathbf{R}^N)$ (respectively Sobolev space $H^m(\mathbf{R}^N)$) were already used with its norm

$$\|f\|_{L^q(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} \left|f(x)\right|^q dx\right)^{1/q} \quad \left(\text{respectively } \|f\|_m = \left(\sum_{k=0}^{m} \left\|\partial_x^k f\right\|_{L^2(\mathbf{R}^N)}\right)^{1/2}\right).$$

In particular, $\|f\| := \|f\|_{L^2(\mathbf{R}^N)} = \|f\|_0$. The space $\mathbf{R}^N$ of $L^q(\mathbf{R}^N)$ or the integrand $\mathbf{R}^N$ will be often abbreviated. For brevity, $\|f(t,\cdot)\|_{L^q} = (\int |f(t,x)|^q\,dx)^{1/q}$ will be written simply by $\|f(t)\|_{L^q}$, etc.

2. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 applying the weighted L^2-energy method. The weight function is chosen as

$$e^{\psi(t,x)}, \quad \text{with } \psi(t,x) = \frac{a|x|^2}{4(t+t_0)} \ (0 < a < 1, t_0 \geqslant 1) \tag{2.1}$$

(later a is determined as $1/4$), which is a modification of the weight introduced in Todorova and Yordanov [32]. See also Ikehata and Tanizawa [17]. The weight function ψ satisfies

$$\begin{cases} \nabla\psi = \frac{ax}{2(t+t_0)}, \quad |\nabla\psi|^2 = \frac{a^2|x|^2}{4(t+t_0)^2}, \\ \psi_t = -\frac{a|x|^2}{4(t+t_0)^2} < 0, \quad \frac{|\nabla\psi|^2}{\psi_t} = -a. \end{cases} \tag{2.2}$$

For the interval $I = [\tau, \tau + t_1]$, $t_1 > 0$, and any fixed $M > 0$, we adopt the solution space

$$X_M(I) = \left\{ \begin{matrix} u \in C(I; H^1) \cap C^1(I; L^2),\ e^{\psi(t,\cdot)}(u_t, \nabla u, u)(t,\cdot) \in L^2 \\ \text{with } \sup_{t\in I} E_\psi(t;u)^{1/2} \leqslant M \end{matrix} \right\}, \tag{2.3}$$

where

$$E_\psi(t;u) = \int e^{2\psi(t,x)}\big(|u_t|^2 + |\nabla u|^2 + u^2\big)(t,x)\,dx. \tag{2.4}$$

Also, denote

$$E_\psi\big(\tau; u_0^\tau, u_1^\tau\big) = \int e^{2\psi(\tau,x)}\big(\left|u_1^\tau\right|^2 + \left|\nabla u_0^\tau\right|^2 + \left|u_0^\tau\right|^2\big)(x)\,dx. \tag{2.5}$$

Clearly, $E_\psi(0; u_0, u_1) < M$ for suitable $t_1 \geqslant 1$ and $M > 0$ by (1.17). The global existence theorem for (1.1)–(1.2) is well known. However, we need that the solution u remains in $X_M(I)$ provided that $E_\psi(0; u_0, u_1) < M$. Hence we prepare the local existence theorem in $X_M(I)$ for

$$\begin{cases} u_{tt} - \Delta u + u_t + |u|^{p-1}u = 0, \quad t > \tau,\ x \in \mathbf{R}^N, \\ (u, u_t)(\tau, x) = (u_0^\tau, u_1^\tau)(x), \quad x \in \mathbf{R}^N. \end{cases} \tag{2.6$_\tau$}$$

Proposition 2.1. *Let $N \geqslant 1$ and $1 < p < N/(N-2)$ $(N \geqslant 3)$, $1 < p < \infty$ $(N = 1, 2)$. For any $M > 0$ and some constant $C_1 > 0$, if $(u_0^\tau, u_1^\tau) \in H^1 \times L^2$ satisfies $E_\psi(\tau; u_0^\tau, u_1^\tau)^{1/2} \leqslant M$, then there exists a time $t_1 = t_1(M)$ depending only on M such that the Cauchy problem* (2.6)$_\tau$ *has a unique solution $u(t,x)$ in $X_{2C_1M}(\tau, \tau + t_1)$.*

The sketch of the proof will be given in Appendix A. The local solution $u(t,x)\in X_M([0,T])$ satisfies the following a priori estimates.

Proposition 2.2. *Let p satisfy the conditions in Proposition 2.1 and*

$$\alpha(p) := \frac{1}{p-1} - \frac{N}{4} \geqslant 0. \tag{2.7}$$

Then the solution $u(t,x)\in X_M([0,T])$ to (1.1)–(1.2) satisfies the estimates:

$$\begin{aligned}
&\int_{\mathbf{R}^N} e^{2\psi(t,x)}\big(|u_t|^2+|\nabla u|^2+u^2+|u|^{p+1}\big)(t,x)\,dx \\
&\qquad + \int_0^t\int_{\mathbf{R}^N} e^{2\psi(\tau,x)}\big(|u_t|^2+|\nabla u|^2+|\nabla\psi|^2u^2+|u|^{p+1}\big)(\tau,x)\,dx\,d\tau \\
&\quad \leqslant C_0\int_{\mathbf{R}^N} e^{2\psi(0,x)}\big(|u_1|^2+|\nabla u_0|^2+u_0^2+|u_0|^{p+1}\big)(x)\,dx =: C_0\bar{E}_\psi(0;u_0,u_1),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&(t+t_0)^{2\alpha(p)}\int_{\mathbf{R}^N} e^{2\psi(t,x)}\big(|u_t|^2+|\nabla u|^2+u^2+|u|^{p+1}\big)(t,x)\,dx \\
&\qquad + (t+t_0)^{-\varepsilon}\int_0^t(\tau+t_0)^{2\alpha(p)+\varepsilon}\int_{\mathbf{R}^N} e^{2\psi(\tau,x)}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)(\tau,x)\,dx\,d\tau \\
&\quad \leqslant C_\varepsilon\big(1+\bar{E}_\psi(0;u_0,u_1)\big)
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&(t+t_0)^{2\alpha(p)+1}\int_{\mathbf{R}^N} e^{2\psi(t,x)}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)(t,x)\,dx \\
&\qquad + (t+t_0)^{-\varepsilon}\int_0^t(\tau+t_0)^{2\alpha(p)+1+\varepsilon}\int_{\mathbf{R}^N} e^{2\psi(\tau,x)}|u_t|^2(\tau,x)\,dx\,d\tau \\
&\quad \leqslant C_\varepsilon\big(1+\bar{E}_\psi(0;u_0,u_1)\big)
\end{aligned} \tag{2.10}$$

for some $t_0\geqslant 1$ and any fixed $\varepsilon>0$ with $C_\varepsilon\to\infty$ as $t\to\infty$.

Propositions 2.1 and 2.2 imply the global existence theorem in $X_M(0,\infty)$. In particular, the estimate (2.8) and the Gagliardo and Nirenberg inequality play a role to extend the local solution to the global one.

Lemma 2.1. *Let the exponents s,q,r $(1\leqslant s,q,r\leqslant\infty)$ and $\sigma\in[0,1]$ satisfy*

$$\frac{1}{s} = \sigma\left(\frac{1}{r}-\frac{1}{N}\right)+(1-\sigma)\frac{1}{q},$$

with $r\leqslant N$ except for the case $(s,r)=(\infty,N)$ when $N\geqslant 2$. Then it holds that

$$\|u\|_{L^s}\leqslant C\|u\|_{L^q}^{1-\sigma}\|\nabla u\|_{L^r}^{\sigma},\quad u\in L^q,\ \nabla u\in L^r,$$

for $C=C(s,q,r,N)$.

Applying Lemma 2.1 to the local solution on $[0, T]$, we have

$$\left(\int e^{2\psi(t,x)}\big|u(t,x)\big|^{p+1}\,dx\right)^{1/(p+1)} \leqslant C\left(\int e^{\frac{2}{p+1}\psi(t,x)}u(t,x)^2\,dx\right)^{\frac{1}{2}\cdot(1-\sigma)}\left(\int e^{2\psi(t,x)}\big(|\nabla u|^2+u^2\big)(t,x)\,dx\right)^{\frac{1}{2}\cdot\sigma} \tag{2.11}$$

for $\sigma = N(1/2 - 1/(p+1))$ ($\leqslant 1$ when $p \leqslant 1 + 4/(N-2)$). In fact, since $f := e^{2\psi/(p+1)}u$ satisfies

$$\nabla f = e^{\frac{2}{p+1}\psi}\left(\nabla u + \frac{1}{p+1}\frac{x}{t+t_0}u\right),$$

the inequality

$$\|f\|_{L^{p+1}} \leqslant C\|f\|^{1-\sigma}\|\nabla f\|^{\sigma}$$

implies (2.11). From (2.12) and (2.8),

$$E_\psi(t;u) \leqslant \bar{E}_\psi(t;u) \leqslant C_1^2\bar{E}_\psi(0;u_0,u_1) \leqslant C_1^2\big(E_\psi(0;u_0,u_1) + CE_\psi(0;u_0,u_1)^{\frac{p+1}{2}}\big),$$

where

$$\bar{E}_\psi(t;u) = \int e^{2\psi(t,x)}\big(|u_t|^2 + |\nabla u|^2 + u^2 + |u|^{p+1}\big)(t,x)\,dx. \tag{2.12}$$

Hence, for a given data (u_0, u_1) take $M > 0$ so that

$$C_1^2\big(E_\psi(0;u_0,u_1) + CE_\psi(0;u_0,u_1)^{\frac{p+1}{2}}\big) < M^2,$$

then $E_\psi(T;u) < M^2$, which allows the local solution to extend beyond the time T.

Theorem 2.1. *Let p satisfy the condition in Proposition* 2.2. *If $E_\psi(0;u_0,u_1) < +\infty$, then the Cauchy problem* (1.1)–(1.2) *has a unique global solution satisfying* (2.8)–(2.10) *for any $t \geqslant 0$.*

Theorem 1.1 is a direct consequence of Theorem 2.1 and (1.21) is derived as follows. By (2.9) the L^1-norm of $u(t,x)$ is estimated as

$$\big\|u(t)\big\|_{L^1} \leqslant \left(\int e^{-2\psi(t,x)}\,dx\right)^{1/2}\left(\int e^{-2\psi(t,x)}u(t,x)^2\,dx\right)^{1/2} \leqslant C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2}}.$$

Also, the L^∞-norm for $N = 1$ follows from the Sobolev inequality and (1.18)–(1.19):

$$\big\|u(t,\cdot)\big\|_{L^\infty} \leqslant \big\|u(t,\cdot)\big\|^{1/2}\big\|\nabla u(t,\cdot)\big\|^{1/2} \leqslant C(t+t_0)^{-\frac{1}{p-1}}.$$

In Lemma 2.1, if $(s,q,r,N) = (s,2,2,2)$, $2 < s < \infty$, then $\sigma = 1 - 2/s < 1$ and

$$\big\|u(t,\cdot)\big\|_{L^s} \leqslant C(t+t_0)^{(-\frac{1}{p-1}+\frac{N}{4})\cdot\frac{2}{s}+(-\frac{1}{p-1}-\frac{1}{2}+\frac{N}{4})\cdot(1-\frac{2}{s})} = C(t+t_0)^{-\frac{1}{p-1}+\frac{1}{s}}.$$

If $(q,r) = (2,2)$ and $s \leqslant \frac{2N}{N-2}$ for $N \geqslant 3$, then $\sigma \leqslant 1$ and the same estimate as above holds. Thus (1.21) is completed.

Proof of Proposition 2.2. Multiplying (1.1) by $e^{2\psi}u_t$ and $e^{2\psi}u$, we have

$$\begin{aligned} 0 &= e^{2\psi}u_t\big(u_{tt} - \Delta u + u_t + |u|^{p-1}u\big) \\ &= \frac{d}{dt}\left[e^{2\psi}\left(\frac{1}{2}\big(|u_t|^2 + |\nabla u|^2\big) + \frac{1}{p+1}|u|^{p+1}\right)\right] \\ &\quad + e^{2\psi}\left[\left(\left(1 + \frac{|\nabla\psi|^2}{\psi_t}\right) - \psi_t\right)|u_t|^2 + \frac{-2\psi_t}{p+1}|u|^{p+1}\right] \\ &\quad - \nabla\cdot\big(e^{2\psi}u_t\nabla u\big) + \frac{1}{-\psi_t}e^{2\psi}|\psi_t\nabla u - u_t\nabla\psi|^2, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} 0 &= e^{2\psi}u\big(u_{tt} - \Delta u + u_t + |u|^{p-1}u\big) \\ &= \frac{d}{dt}\left[e^{2\psi}\left(uu_t + \frac{1}{2}u^2\right)\right] + \big[e^{2\psi}\big(|\nabla u|^2 - \psi_t u^2 + |u|^{p+1}\big)\big] \\ &\quad + \big[e^{2\psi}\big(-2\psi_t uu_t - |u_t|^2 + 2u\nabla\psi\cdot\nabla u\big)\big] - \nabla\cdot\big(e^{2\psi}u\nabla u\big). \end{aligned} \tag{2.14}$$

Since we choose ψ in (2.1) with (2.2), integrating (2.13) and (2.14) over $\mathbf{R}^N$, we respectively get

$$\begin{aligned} &\frac{d}{dt}\int e^{2\psi}\left(\frac{1}{2}\big(|u_t|^2 + |\nabla u|^2\big) + \frac{1}{p+1}|u|^{p+1}\right)dx \\ &\qquad + \int e^{2\psi}\left(\left(1 - a + \frac{1}{a}|\nabla\psi|^2\right)|u_t|^2 + \frac{2}{a(p+1)}|\nabla\psi|^2|u|^{p+1}\right)dx \\ &\quad \leqslant 0 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\frac{d}{dt}\int e^{2\psi}\left(uu_t + \frac{1}{2}u^2\right)dx \\ &\qquad + \int e^{2\psi}\left(|\nabla u|^2 + \frac{1}{a}|\nabla\psi|^2u^2 + |u|^{p+1}\right)dx - \int e^{2\psi}|u_t|^2\,dx \\ &\quad \leqslant \int e^{2\psi}\left(\frac{2}{a}|\nabla\psi|^2|uu_t| + 2|\nabla\psi||u||\nabla u|\right)dx \\ &\quad \leqslant \int e^{2\psi}\left(\frac{4}{a}|\nabla\psi|^2|u_t|^2 + 2a|\nabla u|^2 + \frac{3}{4a}|\nabla\psi|^2u^2\right)dx. \end{aligned} \tag{2.16}$$

Here (2.16) is rewritten by

$$\begin{aligned} &\frac{d}{dt}\int e^{2\psi}\left(uu_t + \frac{1}{2}u^2\right)dx + \int e^{2\psi}\left((1-2a)|\nabla u|^2 + \frac{1}{4a}|\nabla\psi|^2u^2 + |u|^{p+1}\right)dx \\ &\quad - \int e^{2\psi}\left(1 + \frac{4}{a}|\nabla\psi|^2\right)|u_t|^2\,dx \leqslant 0. \end{aligned} \tag{2.17}$$

Adding (2.15) to (2.17) multiplied by ν $(0 < \nu < 1)$, we get

$$\begin{aligned} &\frac{d}{dt}\int e^{2\psi}\left\{\frac{1}{2}\big(|u_t|^2 + 2\nu uu_t + \nu u^2\big) + \frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1}\right\}dx \\ &\quad + \int e^{2\psi}\left\{\left(1 - a - \nu + \frac{1-4\nu}{a}|\nabla\psi|^2\right)|u_t|^2 + \nu(1-2a)|\nabla u|^2\right. \end{aligned}$$

$$+\frac{\nu}{4a}|\nabla\psi|^2u^2+\left(\nu+\frac{2}{a(p+1)}|\nabla\psi|^2\right)|u|^{p+1}\Bigg\}\,dx\leqslant 0. \tag{2.18}$$

We determine $a=\nu=1/4$. Then, (2.18) yields

$$\begin{aligned}&\frac{d}{dt}\tilde{E}_\psi(t;u)+H_\psi(t;u)\\&\quad:=\frac{1}{2}\frac{d}{dt}\int e^{2\psi}\left(|u_t|^2+uu_t+\frac{1}{2}u^2+|\nabla u|^2+\frac{2}{p+1}|u|^{p+1}\right)dx\\&\qquad+\frac{1}{4}\int e^{2\psi}\left(2|u_t|^2+\frac{1}{2}|\nabla u|^2+|\nabla\psi|^2u^2+|u|^{p+1}\right)dx\\&\quad\leqslant 0,\end{aligned} \tag{2.19}$$

which is the key inequality in the proof. Note that

$$\frac{1}{2}\bar{E}_\psi(t;u)\geqslant\tilde{E}_\psi(t;u)\geqslant c\bar{E}_\psi(t;u), \tag{2.20}$$

where $\bar{E}_\psi(t;u)$ is defined in (2.12). Integrating (2.19) over $[0,t]$ and using (2.20), we have (2.8).

Further, multiply (2.19) by $(t+t_0)^{2\alpha(p)+\varepsilon}$ $(0<\varepsilon<1)$, then

$$\frac{d}{dt}\big[(t+t_0)^{2\alpha(p)+\varepsilon}\tilde{E}_\psi(t;u)\big]+(t+t_0)^{2\alpha(p)+\varepsilon}\left[H_\psi(t;u)-\frac{2\alpha(p)+\varepsilon}{t+t_0}\tilde{E}_\psi(t;u)\right]\leqslant 0. \tag{2.21}$$

Making use of (2.20), we have

$$\begin{aligned}&H_\psi(t;u)-\frac{2\alpha(p)+\varepsilon}{t+t_0}\tilde{E}_\psi(t;u)\\&\quad\geqslant\left[\frac{1}{8}\int e^{2\psi}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)\,dx\right.\\&\qquad\left.-\frac{2\alpha(p)+1}{2t_0}\int e^{2\psi}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)\,dx\right]\\&\qquad+\left[\frac{1}{8}\int e^{2\psi}\big(|\nabla\psi|^2u^2+|u|^{p+1}\big)\,dx-\frac{2\alpha(p)+1}{2(t+t_0)}\int e^{2\psi}u^2\,dx\right].\end{aligned} \tag{2.22}$$

Second to the last term in (2.22) is estimated from below by

$$\frac{1}{16}\hat{E}_\psi(t;u):=\frac{1}{16}\int e^{2\psi}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)\,dx\quad\text{if } t_0\geqslant 8\big(2\alpha(p)+1\big). \tag{2.23}$$

The last term is estimated from below by

$$-C(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}}. \tag{2.24}$$

In fact, by denoting

$$\frac{2\alpha(p)+1}{2(t+t_0)}\int e^{2\psi}u^2\,dx=\int\limits_{\kappa|x|\geqslant\sqrt{t+t_0}}+\int\limits_{\kappa|x|\leqslant\sqrt{t+t_0}}=:I_1+I_2$$

with $\kappa=1/16\sqrt{(2\alpha(p)+1)}$, each term is estimated as follows:

$$I_1 \leqslant \frac{2\alpha(p)+1}{2(t+t_0)} \int\limits_{\kappa|x| \geqslant \sqrt{t+t_0}} \frac{\kappa^2|x|^2}{t+t_0} e^{2\psi} u^2\,dx$$
$$\leqslant \frac{2\kappa^2(2\alpha(p)+1)}{a^2} \int\limits_{\mathbf{R}^N} \frac{a^2|x|^2}{4(t+t_0)^2} e^{2\psi} u^2\,dx = \frac{1}{8}\int\limits_{\mathbf{R}^N} e^{2\psi}|\nabla\psi|^2 u^2\,dx$$

since $a = 1/4$ and

$$I_2 \leqslant \int\limits_{\kappa|x| \leqslant \sqrt{t+t_0}} \frac{2\alpha(p)+1}{2(t+t_0)} e^{2\psi\cdot\frac{p-1}{p+1}} \cdot e^{2\psi\cdot\frac{2}{p+1}} u^2\,dx$$
$$\leqslant \frac{1}{8}\int\limits_{\mathbf{R}^N} e^{2\psi}|u|^{p+1}\,dx + C \int\limits_{\kappa|x| \leqslant \sqrt{t+t_0}} (t+t_0)^{-\frac{p+1}{p-1}} e^{\frac{|x|^2}{8(t+t_0)}}\,dx$$
$$= \frac{1}{8}\int\limits_{\mathbf{R}^N} e^{2\psi}|u|^{p+1}\,dx + C(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}}$$

by the Young inequality with $\frac{p-1}{p+1} + \frac{2}{p+1} = 1$. Combining (2.21) with (2.22)–(2.24), we get

$$\frac{d}{dt}\big[(t+t_0)^{2\alpha(p)+\varepsilon}\tilde{E}_\psi(t;u)\big] + \frac{1}{16}(t+t_0)^{2\alpha(p)+\varepsilon}\hat{E}_\psi(t;u)$$
$$\leqslant C(t+t_0)^{2\alpha(p)+\varepsilon}\cdot(t+t_0)^{-\frac{p+1}{p-1}+\frac{N}{2}} = C(t+t_0)^{-1+\varepsilon}. \tag{2.25}$$

Hence, integrating (2.25) over $(0,t)$, $t \leqslant T$, and using (2.20), we obtain

$$(t+t_0)^{2\alpha(p)+\varepsilon}\bar{E}_\psi(t;u) + \frac{1}{16}\int_0^t (\tau+t_0)^{2\alpha(p)+\varepsilon}\hat{E}_\psi(\tau;u)\,d\tau$$
$$\leqslant t_0^{2\alpha(p)+\varepsilon}\bar{E}_\psi(0;u) + C\int_0^t (\tau+t_0)^{-1+\varepsilon}\,d\tau$$
$$\leqslant Ct_0^{2\alpha(p)+\varepsilon}\bar{E}_\psi(0;u_0,u_1) + C_\varepsilon(t+t_0)^\varepsilon. \tag{2.26}$$

Dividing (2.26) by $(t+t_0)^\varepsilon$, we reach the second desired estimate (2.9). Concerning the weight about t, we note that, if we take $\varepsilon = 0$, then the term $\log(t+t_0)$ comes out and the result become less sharp. The method to adopt $\varepsilon > 0$ instead of $\varepsilon < 0$ is seen in Nishikawa [28].

To obtain the third estimate (2.10), multiplying (2.15) with $a = 1/4$ by $(t+t_0)^{\alpha(p)+1+\varepsilon}$, we have

$$\frac{d}{dt}\left[(t+t_0)^{2\alpha(p)+1+\varepsilon}\int e^{2\psi}\left\{\frac{1}{2}\big(|u_t|^2+|\nabla u|^2\big) + \frac{1}{p+1}|u|^{p+1}\right\}dx\right]$$
$$+ \frac{3}{4}(t+t_0)^{2\alpha(p)+1+\varepsilon}\int e^{2\psi}|u_t|^2\,dx$$
$$\leqslant C\big(\alpha(p)+1\big)(t+t_0)^{2\alpha(p)+\varepsilon}\int e^{2\psi}\big(|u_t|^2+|\nabla u|^2+|u|^{p+1}\big)\,dx. \tag{2.27}$$

Integrating (2.27) over $[0,t]$ and using (2.9) just obtained, we easily show (2.10).

Thus we have completed the proof of Proposition 2.2. □

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Appendix A

We sketch the proof of Proposition 2.1. The proof of similar local existence theorem is seen in Ikehata and Tanizawa [17]. Since $\psi(t,x)$ is decreasing in t, it is enough to show the case $\tau=0$ in $(2.6)_\tau$. So, our problem is

$$\begin{cases} u_{tt}-\Delta u+u_t=-|u|^{p-1}u, & (t,x)\in[0,\infty)\times\mathbf{R}^N,\\ (u,u_t)(0,x)=(u_0,u_1)(x). \end{cases} \tag{L}$$

We show that, if $E_\psi(0;u_0,u_1)\leqslant M^2$, then there exists $t_1=t_1(M)>0$ such that (L) has a unique solution $u(t,x)$ in $X_{2C_1M}([0,t_1])$, where C_1 is some constant determined later. We construct an approximate sequence $\{u^{(n)}(t,x)\}$ as follows:

The first function $u^{(0)}(t,x)$ is a solution to

$$\begin{cases} u_{tt}-\Delta u+u_t=0,\\ (u,u_t)(0,x)=(u_0,u_1)(x), \end{cases} \tag{A.1}$$

and, iteratively, $u^{(n+1)}(t,x)$, $n=0,1,2,\dots,$ is a solution to

$$\begin{cases} u_{tt}-\Delta u+u_t=-|u^{(n)}|^{p-1}u^{(n)},\\ (u,u_t)(0,x)=(u_0,u_1)(x). \end{cases} \tag{A.2}$$

It is enough to assert the following three claims:

(i) For any $t\geqslant 0$, $E_\psi(t;u^{(0)})^{1/2}\leqslant C_1M$.
(ii) For some $t=t_1(M)>0$, if $u^{(n)}\in X_{2C_1M}([0,t_1])$, then $u^{(n+1)}\in X_{2C_1M}([0,t_1])$.
(iii) For some $t_1=t_1(M)>0$ taken to be smaller if necessary,

$$\sup_{0\leqslant t\leqslant t_1}E_\psi\big(t;u^{(n+1)}-u^{(n)}\big)\leqslant\frac{1}{4}\sup_{0\leqslant t\leqslant t_1}E_\psi\big(t;u^{(n)}-u^{(n-1)}\big).$$

Since we have $u\in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$, multiplying (A.1) by $e^{2\psi}(u_t+(1/4)u)$, we have

$$\frac{1}{2}\frac{d}{dt}\int e^{2\psi(t,x)}\left(|u_t|^2+uu_t+\frac{1}{2}u^2+|\nabla u|^2\right)dx\leqslant 0.$$

Hence, by

$$\frac{1}{2}E_\psi(t;u)\geqslant\int e^{2\psi(t,x)}\left(|u_t|^2+uu_t+\frac{1}{2}u^2+|\nabla u|^2\right)dx\geqslant cE_\psi(t;u),$$

for some constant $C_1>0$,

$$E_\psi(t;u)\leqslant C_1^2E_\psi(0;u_0,u_1)\leqslant(C_1M)^2,$$

which means (i). Next, multiplying (A.2) by $e^{2\psi}(u_t+\frac{1}{4}u)$, we have

$$E_\psi(t;u) \leqslant (C_1M)^2 + C\int_0^t \int e^{2\psi}\left|u^{(n)}\right|^p\left(|u| + |u_t|\right)dx\,d\tau$$
$$\leqslant (C_1M)^2 + \int_0^t \int e^{2\psi}\left(\left|u^{(n)}\right|^{2p} + u^2 + |u_t|^2\right)dx\,d\tau$$
$$\leqslant (C_1M)^2 + C\int_0^{t_1} E_\psi\left(\tau;u^{(n)}\right)^p d\tau + C\int_0^t E_\psi(\tau;u)\,d\tau, \quad t \leqslant t_1,$$

since $\int e^{2\psi}|u^{(n)}|^{2p}\,dx \leqslant CE_\psi(t;u^{(n)})^p$, which is obtained by similar way to (2.11). Hence the Gronwall inequality implies

$$E_\psi(t;u) \leqslant \left((C_1M)^2 + C(2C_1M)^{2p}t_1\right)e^{Ct_1} \leqslant (2C_1M)^2$$

if $0 < t_1 \ll 1$, which gives (ii). The assertion (iii) follows in the similar way as (ii).

Thus we have completed the proof of Proposition 2.1.

References

[1] P. Brenner, On space–time means and strong global solutions of nonlinear hyperbolic equations, Math. Z. 201 (1989) 45–55.
[2] H. Brezis, A. Peletier, D. Terman, A very singular solution of the heat equation with absorption, Arch. Ration. Mech. Anal. 95 (1986) 185–209.
[3] R. Chill, A. Haraux, An optimal estimate for the difference of solutions of two abstract evolution equations, J. Differential Equations 193 (2003) 385–395.
[4] M. Escobedo, O. Kavian, Asymptotic behavior of positive solutions of a nonlinear heat equation, Houston J. Math. 13 (1987) 39–50.
[5] M. Escobedo, O. Kavian, H. Matano, Large time behavior of solutions of a dissipative semilinear heat equation, Comm. Partial Differential Equations 20 (1995) 83–120.
[6] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sec. 1 13 (1966) 109–124.
[7] T. Gallay, G. Raugel, Stability of travelling waves for damped hyperbolic equation, Z. Angew. Math. Phys. 48 (1997) 451–479.
[8] T. Gallay, G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, J. Differential Equations 150 (1998) 42–97.
[9] J. Ginibre, G. Velo, The global Cauchy problem for the nonlinear Klein–Gordon equation, Math. Z. 189 (1985) 487–505.
[10] A. Gmira, L. Véron, Large time behavior of the solution of a semilinear parabolic equation in R^N, J. Differential Equations 53 (1984) 258–276.
[11] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation with super critical nonlinearity, Differential Integral Equations 17 (2004) 637–652.
[12] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation with critical nonlinearity, preprint.
[13] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation in subcritical case, J. Differential Equations 207 (2004) 161–194.
[14] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Asymptotics for nonlinear damped wave equations with large initial data, preprint.
[15] T. Hosono, T. Ogawa, Large time behavior and L^p–L^q estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004) 82–118.
[16] R. Ikehata, Diffusion phenomenon for linear dissipative wave equations in an exterior domains, J. Differential Equations 186 (2002) 633–651.
[17] R. Ikehata, K. Tanizawa, Global existence of solutions for semilinear damped wave equations in $\mathbf{R}^N$ with noncompactly supported initial data, Nonlinear Anal. 61 (2005) 1189–1208.

[18] R. Ikehata, Y. Miyaoka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\mathbf{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan 56 (2004) 365–373.
[19] R. Ikehata, K. Nishihara, Diffusion phenomenon for second order linear evolution equations, Studia Math. 158 (2003) 153–161.
[20] R. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equations, Studia Math. 143 (2000) 175–197.
[21] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995) 617–653.
[22] T.-T. Li, Y. Zhou, Breakdown of solutions to $\Box u + u_t = u^{1+\alpha}$, Discrete Contin. Dyn. Syst. 1 (1995) 503–520.
[23] P. Marcati, K. Nishihara, The L^p–L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2003) 445–469.
[24] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. Kyoto Univ. 12 (1976) 169–189.
[25] T. Narazaki, L^p–L^q estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan 56 (2004) 585–626.
[26] K. Nishihara, L^p–L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003) 631–649.
[27] K. Nishihara, L^p–L^q estimates for the 3-D damped wave equation and their application to the semilinear problem, Seminar Notes of Math. Sci., Ibaraki Univ. 6 (2003) 69–83.
[28] M. Nishikawa, Convergence rate to the traveling wave for viscous conservation laws, Funkcial. Ekvac. 41 (1998) 107–132.
[29] K. Ono, Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations, Discrete Contin. Dyn. Syst. 9 (2003) 651–662.
[30] K. Ono, Global solvability and L^p decay for the semilinear dissipative wave equations in four and five dimensions, preprint.
[31] W.A. Strauss, Nonlinear Wave Equations, CBMS Reg. Conf. Ser. Math., Amer. Math. Soc., Providence, RI, 1989.
[32] Y. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001) 464–489.
[33] Q. Zhang, A blow-up result for a nonlinear wave equation with damping: The critical case, C. R. Acad. Sci. Paris 333 (2001) 109–114.