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Theorems on Closed Coverings of a Simplex and Their Applications to Cooperative Game Theory*

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New theorems of the Knaster–Kuratowski–Mazurkiewicz type are presented; they extend the related results of Scarf and Shapley. Applications to cooperative game theory are also given. © 1990 Academic Press, Inc

1. INTRODUCTION

Let N be a nonempty finite set, and let $\{e^j\}_{j \in N}$ be the unit vectors of the $(\# N)$-dimensional Euclidean space $\mathbf{R}^N$; $e^j_j = 1$ and $e^j_i = 0$ for every $i \neq j$. Denote by $\mathcal{N}$ the family of nonempty subsets of N (i.e., $\mathcal{N} := 2^N \setminus \{\varnothing\}$). Given a subset X of $\mathbf{R}^N$, denote the convex hull of X by co X, the interior of X by $\mathring{X}$, the relative interior of X by ri X, and the affine hull of X by aff X. The faces of the unit simplex are then given by $\Delta^S := \text{co}\{e^i \mid i \in S\}$ for every $S \in \mathcal{N}$. The simplex Δ^N is endowed with the relativized Euclidean topology. For each $S \in \mathcal{N}$, its characteristic vector is given by $\chi_S := \sum_{i \in S} e^i$. Given two vectors x and y in $\mathbf{R}^N$, $x \cdot y$ denotes the Euclidean inner product, and the closed line segment joining the two (i.e., co$\{x, y\}$) is denoted by $[x, y]$.

It was sixty years ago when Sperner [25] published the following:

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THEOREM 1.1 (Sperner [25]). *Let* $\{C^i\}_{i\in N}$ *be a closed covering of* Δ^N *such that* $\Delta^{N\setminus\{i\}} \cap C^i = \emptyset$ *for every* $i \in N$. *Then* $\bigcap_{i\in N} C^i \neq \emptyset$.

A year later, Knaster, Kuratowski, and Mazurkiewicz [17] published the following generalization of Theorem 1.1:

THEOREM 1.2 (Knaster *et al.* [17]). *Let* $\{C^i\}_{i\in N}$ *be a family of closed subsets of* Δ^N *such that* $\Delta^S \subset \bigcup_{i\in S} C^i$ *for every* $S \in \mathcal{N}$. *Then* $\bigcap_{i\in N} C^i \neq \emptyset$.

Actually, each of Theorems 1.1 and 1.2 is easily shown to be equivalent to Brouwer's fixed-point theorem, by using Browder's [4] technique which involves a partition of unity (see the independent work of Border [3] and Dugundji and Granas [5] for the equivalence of Theorem 1.2 and the Brouwer theorem). Fan [6] pointed out that Theorem 1.1 can be re-formulated as:

THEOREM 1.3 (Sperner [25]). *Let* $\{C^i\}_{i\in N}$ *be a closed covering of* Δ^N *such that* $\Delta^{N\setminus\{i\}} \subset C^i$ *for every* $i \in N$. *Then* $\bigcap_{i\in N} C^i \neq \emptyset$.

(To show the equivalence of Theorems 1.1 and 1.3, use the Lebesgue number.)

Let K be a finite set such that $K \supset N$, and let $A := ((a_{ij}))_{i\in N, j\in K}$ and $c := (c_i)_{i\in N}$ be a $(\#N)\times(\#K)$ real matrix and a $(\#N)\times 1$ real matrix, respectively, such that

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \in N; \\ 0 & \text{if } i, j \in N \text{ but } i \neq j; \end{cases}$$

$$c_i \geqslant 0 \qquad \text{for every } i \in N;$$

$$c_i > 0 \qquad \text{for some } i \in N.$$

Notice that $\{x \in \mathbf{R}^K_+ \mid Ax = c\} \neq \emptyset$. Theorem 1.3 is a special case of Scarf's theorem [20]:

THEOREM 1.4 (Scarf [20]). *Let* $\{C^j\}_{j\in K}$ *be a closed covering of* Δ^N *such that* $\Delta^{N\setminus\{j\}} \subset C^j$ *for every* $j \in N$. *Assume that the set* $\{x \in \mathbf{R}^K_+ \mid Ax = c\}$ *is bounded. Then there exists* $x \in \mathbf{R}^K_+$ *such that* $Ax = c$ *and* $\bigcap\{C^j \mid j \in K, x_j > 0\} \neq \emptyset$.

Scarf [19, 20] used the "path-following technique" of Lemke and Howson [18] to establish a theorem on primitive sets (Theorem 4.6 of this paper), and then used the latter theorem to prove Theorem 1.4. An alternative proof of Theorem 1.4 was made by Kannai [15]; he used the Brouwer fixed-point theorem only.

A generalization of Theorem 1.2 was made by Shapley [21]. To formulate Shapley's result we need the following:

DEFINITION 1.5. A subfamily $\mathscr{B}$ of $\mathscr{N}$ is called *balanced*, if there exists $\{\lambda_S\}_{S\in\mathscr{B}} \subset \mathbf{R}_+$ such that $\sum_{S\in\mathscr{B}: S\ni i} \lambda_S = 1$ for every $i \in N$.

THEOREM 1.6 (Shapley [21]). *Let $\{C^S\}_{S\in\mathscr{N}}$ be a family of closed subsets of Δ^N such that $\Delta^T \subset \bigcup_{S\subset T} C^S$ for every $T\in\mathscr{N}$. Then there exists a balanced family $\mathscr{B}$ such that $\bigcap_{S\in\mathscr{B}} C^S \neq \varnothing$.*

To see the relationship between the conclusions of Theorem 1.4 and of Theorem 1.6, let $\tilde{A}$ be the $(\#N)\times(\#\mathscr{N})$ matrix whose rows (columns, resp.) are indexed by $i\in N$ (by $S\in\mathscr{N}$, resp.) such that column S is precisely χ_S. The set $\{x\in\mathbf{R}_+^{\mathscr{N}} \mid \tilde{A}x=\chi_N\}$ is nonempty and bounded. Then the conclusion of Theorem 1.6 is re-formulated as: There exists $x\in\mathbf{R}_+^{\mathscr{N}}$ such that

$$\tilde{A}x=\chi_N,$$

and

$$\bigcap \{C^S \mid S\in\mathscr{N}, x_S>0\} \neq \varnothing.$$

Actually, motivated by Billera's generalization [1, 2] of Scarf's theorem [19] for nonemptiness of the core (Theorem 4.4 in this paper), Shapley [21] established a more general theorem (Theorem 1.6′ below). Define $\Pi := \mathsf{X}_{S\in\mathscr{N}} \Delta^S$.

DEFINITION 1.5′. Choose any $\pi := (\pi_S)_{S\in\mathscr{N}} \in \Pi$. A subfamily $\mathscr{B}$ of $\mathscr{N}$ is called *π-balanced*, if $\pi_N \in \operatorname{co}\{\pi_S \mid S\in\mathscr{B}\}$.

THEOREM 1.6′ (Shapley [21]). *Let $\{C^S\}_{S\in\mathscr{N}}$ be a family of closed subsets of Δ^N such that $\Delta^T \subset \bigcup_{S\subset T} C^S$ for every $T\in\mathscr{N}$. Choose any $\pi\in\Pi$. Then there exsts a π-balanced family $\mathscr{B}$ such that $\bigcap_{S\in\mathscr{B}} C^S \neq \varnothing$.*

The additional assumption in Shapley [21] that $\pi \in \operatorname{ri} \Pi$ is nonessential: For an arbitrary $\pi\in\Pi$, choose a sequence in $\operatorname{ri}\Pi$ which converges to π. Theorem 1.6 is a special case of Theorem 1.6′ in which $\pi_S = \chi_S/(\#S)$. Shapley [21] proved Theorem 1.6′ by using the "path-following technique" of Lemke and Howson [18]. Todd [26, 27] has a proof of Theorem 1.6 which makes use of the Brouwer fixed-point theorem and a sequence of simplicial partitions. Shapley [22] has a shorter proof of Theorem 1.6 using Kakutani's fixed-point theorem. Ichiishi [12] has a yet shorter proof of Theorem 1.6 using Fan's [7] coincidence theorem (see also Ichiishi [13]).

Recently Ichiishi [14] established the following theorem, which is dual to Theorem 1.6 just as Theorem 1.3 is dual to Theorem 1.2, and which is also a generalization of Theorem 1.3:

THEOREM 1.7 (Ichiishi [14]). *Let* $\{C^S\}_{S\in\mathcal{N}}$ *be a family of closed subsets of* Δ^N *such that* $\Delta^T\subset\bigcup_{S\supset N\setminus T}C^S$ *for every* $T\in\mathcal{N}$. *Then there exists a balanced family* $\mathcal{B}$ *such that* $\bigcap_{S\in\mathcal{B}}C^S\neq\varnothing$.

It was pointed out by David Schmeidler that Theorems 1.6 and 1.7 are equivalent; Schmeidler's argument is reproduced in Ichiishi [14]. Neither of Theorems 1.4 and 1.7 includes the other.

The first purpose of the present paper is to establish general theorems on closed coverings of a simplex in order to give a unified treatment of the above theorems. We prove these general theorems by using a certain geometric lemma and the following special case of Fan's [7, 9] coincidence theorem:

THEOREM 1.8 (Fan [9]). *Let X be a nonempty, compact, and convex subset of* $\mathbf{R}^N$, *and let F and G be upper semicontinuous correspondences from X to the subsets of* $\mathbf{R}^N$, *such that both $F(x)$ and $G(x)$ are nonempty, compact, and convex for each $x\in X$, and such that*

$$(\forall x\in X)\colon(\forall p\in\mathbf{R}^N\colon p\cdot x=\min\, p\cdot X)\colon$$
$$\exists u\in F(x)\colon\exists v\in G(x)\colon p\cdot u\geqslant p\cdot v.$$

Then there exists $x^*\in X$ such that $F(x^*)\cap G(x^*)\neq\varnothing$.

Other covering properties of convex sets were given, e.g., by Fan [6, 8, 10, 11] and Shih and Tan [23, 24].

The second purpose of the present paper is to clarify relationships between the above theorems on closed coverings of a simplex and certain theorems related to the core of a cooperative game without side-payments.

2. Main Results

Let K, A, c be given as in the paragraph that precedes the statement of Theorem 1.4. Denote column j of the matrix A by a^j.

THEOREM 2.1. *Assume that $c\in\Delta^N$ and $a^j\in\operatorname{aff}\Delta^N$ for every $j\in K$. Let $\{C^j\}_{j\in K}$ be a closed covering of Δ^N such that*

$$\forall T\in\mathcal{N}\setminus\{N\}\colon\Delta^T\subset\bigcup\,\{C^j\mid j\in K,\, a^j\in\Delta^T\}.$$

Then there exists a subset I of K such that $c\in\operatorname{co}\{a^j\mid j\in I\}$ and $\bigcap_{j\in I}C^j\neq\varnothing$.

Proof of Theorem 2.1. For each $x \in \Delta^N$ define $I(x) := \{j \in K \mid C^j \ni x\}$, $F(x) := \{c\}$, and $G(x) := \text{co}\{a^j \mid j \in I(x)\}$. Then the correspondences F and G from Δ^N to the subsets of aff Δ^N are upper semicontinuous with nonempty compact and convex values. Choose $x \in \Delta^N$ and $p \in \mathbf{R}^N$ such that $p \cdot x = \min p \cdot \Delta^N$. There exists a unique $S \subset N$ such that $x \in \text{ri}\, \Delta^S$. Thus we have $p \cdot y = \min p \cdot \Delta^N$ for all $y \in \Delta^S$. If $S = N$, then for all $u \in F(x)$ and all $v \in G(x)$, $p \cdot u = p \cdot v$. If $S \neq N$, then by the assumption of the present theorem there exists $j \in K$ such that $a^j \in \Delta^S$ and $x \in C^j$. For this j, $a^j \in G(x)$ and $p \cdot a^j = \min p \cdot \Delta^N \leqslant p \cdot c$. All the assumptions of Theorem 1.8 are now satisfied, so there exists $x^* \in \Delta^N$ such that $F(x^*) \cap G(x^*) \neq \emptyset$. The set $I(x^*)$ is the required set I. Q.E.D.

A generalization of Theorem 2.1 is given by:

THEOREM 2.2. *Assume that* $c \in \Delta^N$ *and that the set* $\{x \in \mathbf{R}^K \mid Ax = c\}$ *is bounded. Let* $\{C^j\}_{j \in K}$ *be a closed covering of* Δ^N *such that*

$$\forall T \in \mathcal{N} \setminus \{N\}: \Delta^T \subset \bigcup \{C^j \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists $x \in \mathbf{R}^K_+$ *such that* $Ax = c$ *and* $\bigcap \{C^j \mid x_j > 0\} \neq \emptyset$.

We shall provide two proofs of this theorem. Both proofs make use of the following claim:

Claim 2.3. *Let* $n \leqslant k$, *let* A *be an* $n \times k$ *matrix whose first* n *columns constitute the unit matrix, and let* c *be an* $n \times 1$ *nonnegative matrix. Then the following two conditions* (i) *and* (ii) *are equivalent.*

(i) *Set* $\{x \in \mathbf{R}^k_+ \mid Ax = c\}$ *is bounded; and*

(ii) $\neg \exists x \in \mathbf{R}^k_+ \setminus \{\mathbf{0}\}: Ax = \mathbf{0}$.

Moreover, for any $n \times 1$ *nonnegative, nonzero matrix* d, *any of the conditions* (i) *and* (ii) *implies the following condition* (iii).

(iii) $\neg \exists x \in \mathbf{R}^k_+: Ax = -d$.

Proof of Theorem 2.2, *Using Theorem* 2.1. Define $D := \{Ax \mid x \in \mathbf{R}^K_+, \sum_{j \in K} x_j = 1\}$; it is a convex compact subset of $\mathbf{R}^N$. By Claim 2.3(ii) and (iii), $D \cap (-\mathbf{R}^N_+) = \emptyset$. There exists, therefore, a hyperplane H which strictly separates D and $-\mathbf{R}^N_+$, in particular $\mathbf{0} \notin H$. Then for each $y \in D$ there exists a unique vector $\hat{y} \in [\mathbf{0}, y] \cap H$. Notice that $c \in D$, and $a^j \in D$, for every j (in particular, $\Delta^N \subset D$). Define $\hat{\Delta}^S := \{\hat{y} \mid y \in \Delta^S\}$, and $\hat{C}^j := \{\hat{y} \mid y \in C^j\}$. Under the assumption of Theorem 2.2,

$$\forall T \in \mathcal{N} \setminus \{N\}: \hat{\Delta}^T \subset \bigcup \{\hat{C}^j \mid j \in K, \hat{a}^j \in \hat{\Delta}^T\}.$$

By Theorem 2.1 applied to $(\hat{\Delta}^N, \{\hat{a}^j\}_{j\in K}, \hat{c}, \{\hat{C}^j\}_{j\in K})$, there exists $I \subset K$ such that $\hat{c} \in \mathrm{co}\{\hat{a}^j \mid j \in I\}$ and $\bigcap\{\hat{C}^j \mid j \in I\} \neq \emptyset$. We can now choose a suitable $x \in \mathbf{R}_+^K$ such that $x_j = 0$ for $j \in K \setminus I$, $Ax = c$, and $\bigcap\{C^j \mid x_j > 0\} \neq \emptyset$. Q.E.D.

Proof of Theorem 2.2, *Using Theorem* 1.8. Define

$$\forall j \in K\colon \hat{a}^j := a^j + \left(1 - \sum_{i \in N} a_{ij}\right) c.$$

Then, $\hat{a}^j \in \mathrm{aff}\, \Delta^N$ for every $j \in K$, and $\hat{a}^j = a^j$ if $a^j \in \mathrm{aff}\, \Delta^N$. Define for each $x \in \Delta^N$,

$$F(x) := \{c\},$$
$$G(x) := \mathrm{co}\{\hat{a}^j \mid j \in K, C^j \ni x\}.$$

As in the proof of Theorem 2.1, one can show that all the assumptions of Theorem 1.8 are satisfied, so there exists $x^* \in \Delta^N$ such that $F(x^*) \cap G(x^*) \neq \emptyset$. Define $I := \{j \in K \mid C^j \ni x^*\}$. Then there exists $\{z_j\}_{j \in I} \subset \mathbf{R}_+$ such that $c = \sum_{j \in I} z_j \hat{a}^j$. By substituting the definition of $\hat{a}^j$'s and by setting $t_j := 1 - \sum_{i \in N} a_{ij}$, one obtains

$$c = \sum_{j \in I} z_j (a^j + t_j c).$$

To sum up, there exist $z_j \in \mathbf{R}_+$, $j \in I$, not all zero, such that

$$\left(1 - \sum_{j \in I} z_j t_j\right) c = \sum_{j \in I} z_j a^j.$$

By Claim 2.3,

$$1 - \sum_{j \in I} z_j t_j > 0;$$

thus there exists $z^* \in \mathbf{R}_+^K$ such that

$$Az^* = c,$$

and

$$\bigcap \{C^j \mid z_j^* > 0\} \supset \bigcap_{j \in I} C^j \neq \emptyset. \qquad \text{Q.E.D.}$$

Now we generalize Theorem 1.7. We need the following geometric lemma:

LEMMA 2.4. *Let C be a compact, convex subset of $\mathbf{R}^N$, and let F be a finite subset of ∂C, the relative boundary of C. Choose any $c \in \operatorname{ri} \operatorname{co} F$. For each $x \in F$ choose $x' \in \partial C$ so that $c \in [x, x']$, and define $F' := \{x' \mid x \in F\}$. Then $c \in \operatorname{co} F'$.*

Proof of Lemma 2.4. There exists $\{\alpha_x\}_{x \in F} \subset \mathbf{R}_+$, $\sum_{x \in F} \alpha_x = 1$, such that $c = \sum_{x \in F} \alpha_x x$. For each $x \in F$ there exists β_x, $0 < \beta_x < 1$, such that $c = \beta_x x + (1 - \beta_x) x'$. Then $c = \sum_{x \in F} \alpha_x (c - (1 - \beta_x) x')/\beta_x$, so $((\sum_{x \in F} \alpha_x/\beta_x) - 1) c = \sum_{x \in F} (\alpha_x/\beta_x - \alpha_x) x'$; therefore $c \in \operatorname{co} F'$. Q.E.D.

THEOREM 2.5. *Assume that $c \in \operatorname{ri} \Delta^N$ and $a^j \in \operatorname{aff} \Delta^N$ for every $j \in K$. Assume also that for every $j \in K$ for which $a^j \in \partial \Delta^N$, there exists $j' \in K$ such that $a^{j'} \in \partial \Delta^N$ and $c \in [a^j, a^{j'}]$. Let $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that*

$$\forall T \in \mathcal{N} \setminus \{N\} \colon \Delta^T \subset \bigcup \{C^{j'} \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists a subset I of K such that $c \in \operatorname{co}\{a^j \mid j \in I\}$ and $\bigcap_{j \in I} C^j \neq \emptyset$.

Proof of Theorem 2.5. Define $D^j := C^{j'}$ for every j for which $a^j \in \partial \Delta^N$, and $D^j := C^j$ for all other j. All the assumptions of Theorem 2.1 are satisfied for $(\Delta^N, \{a^j\}_{j \in K}, c, \{D^j\}_{j \in K})$, so there exists a subset I of K such that $c \in \operatorname{co}\{a^j \mid j \in I\}$ and $\bigcap_{j \in I} D^j \neq \emptyset$. Define $I' := \{j' \mid j \in I\}$. By Lemma 2.4, $c \in \operatorname{co}\{a^j \mid j \in I'\}$. Moreover, $\bigcap_{j \in I'} C^j = \bigcap_{j \in I} D^j \neq \emptyset$. Q.E.D.

Using the same method and Theorem 2.2, we can prove:

THEOREM 2.6. *Assume that $c \in \operatorname{ri} \Delta^N$ and that the set $\{x \in \mathbf{R}^K \mid Ax = c\}$ is bounded. Assume also that for every $j \in K$ for which $a^j \in \partial \Delta^N$, there exists $j' \in K$ such that $a^{j'} \in \partial \Delta^N$ and $c \in [a^j, a^{j'}]$. Let $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that*

$$\forall T \in \mathcal{N} \setminus \{N\} \colon \Delta^T \subset \bigcup \{C^{j'} \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists $x \in \mathbf{R}_+^K$ such that $Ax = c$ and $\bigcap \{C^j \mid x_j > 0\} \neq \emptyset$.

3. REMARKS

The K-K-M theorem (Theorem 1.2) follows from Theorem 2.1 if we take $K = N$ and $c \in \operatorname{ri} \Delta^N$. Scarf's theorem (Theorem 1.4) for the case $c \in \operatorname{ri} \Delta^N$ follows from Theorem 2.6 if we take $C^{j'} = C^j$ for each $j \in N$ (Theorem 1.4 would be trivial if $c \in \partial \Delta^N$). Shapley's theorem (Theorem 1.6′) follows from Theorem 2.2 if we take $K = \mathcal{N}$, a^S (:= column S of the matrix A) $= \pi_S \in \Delta^S$, and $c = \pi_N$. Ichiishi's theorem (Theorem 1.7) follows from Theorem 2.6 if we take $K = \mathcal{N}$, $a^S = \chi_S$, and $c = \chi_N$.

All the results in Section 2 are valid for an arbitrary real matrix A of dimension $(\# N) \times (\# K)$, $N \subset K$, in which there are $\# N$ linearly independent columns, and c $(\neq \mathbf{0})$ is a nonnegative linear combination of those columns.

Theorems similar to those of Section 2 can be proved for a compact polyhedron instead of a simplex.

4. Core

The finite set N is now interpreted as the set of *players*, and $\mathcal{N}$ as the family of nonempty *coalitions.*

DEFINITION 4.1. A *nonside-payment game* is a function V from $\mathcal{N}$ to the subsets of $\mathbf{R}^N$ such that for every $S \in \mathcal{N}$, $V(S)$ is a cylinder; i.e., $[u, v \in \mathbf{R}^N$, $\forall i \in S$: $u_i = v_i]$ implies $[u \in V(S)$ iff $v \in V(S)]$.

The set $V(S)$, or rather its projection to $\mathbf{R}^S$, is interpreted as the set of utility allocations within S; each is made feasible by some coordination of strategies of the members of S.

DEFINITION 4.2. The *core* of a nonside-payment game V is the set $C(V)$ of all $u \in \mathbf{R}^N$ such that (a) $u \in V(N)$ and (b) it is not true that there exist $S \in \mathcal{N}$ and $u' \in V(S)$ such that $u_i < u'_i$ for every $i \in S$.

The core is a typical solution concept; condition (a) says that the utility allocation u is feasible within the grand coalition N, and condition (b) says that no coalition can improve upon u.

DEFINITION 4.3. A nonside-payment game V is called *balanced* if for every balanced subfamily $\mathcal{B}$ of $\mathcal{N}$, $\bigcap_{S \in \mathcal{B}} V(S) \subset V(N)$.

See, e.g., Ichiishi [13, Chap. 5] for further discussions of Definitions 4.1, 4.2, and 4.3. Scarf's [19] fundamental theorem for nonemptiness of the core:

THEOREM 4.4 (Scarf [19]). *Let $V: \mathcal{N} \to 2^{\mathbf{R}^N}$ be a nonside-payment game, and define $b \in \mathbf{R}^N$ by $b_i := \sup\{u_i \in \mathbf{R} \mid u \in V(\{i\})\}$ for each $i \in N$. The core of V is nonempty if* (i) *$V(S) - \mathbf{R}^N_+ = V(S)$ for every $S \in \mathcal{N}$;* (ii) *there exists $M \in \mathbf{R}$ such that for every $S \in \mathcal{N}$, $[u \in V(S) \cap [\{b\} + \mathbf{R}^N_+]]$ implies $[u_i < M$ for every $i \in S]$;* (iii) *$V(S)$ is closed in $\mathbf{R}^N$ for every $S \in \mathcal{N}$; and* (iv) *V is balanced.*

Scarf [19] established the following Theorem 4.6, and then derived from it Theorem 4.4. Let K, A, c be given as in the paragraph that precedes the statement of Theorem 1.4. Choose vectors $P := \{\pi^j\}_{j \in K}$ in $\mathbf{R}^N$ such that

$$\pi^i = (\overbrace{R_i, ..., R_i, 0}^{i}, R_i, ..., R_i) \qquad \text{if} \quad i \in N;$$

$$\pi^j \in (\Delta^N - \mathbf{R}_+^N) \cap \mathbf{R}_+^N \qquad \text{if} \quad j \in K \setminus N,$$

where $R_i > 1$ for each $i \in N$.

DEFINITION 4.5. A subset of P, $\{\pi^j\}_{j \in I}$, is called a *primitive set*, if there does not exist $\pi \in P$ such that

$$\forall i \in N\colon \pi_i > \min\{\pi_i^j \mid j \in I\}.$$

THEOREM 4.6 (Scarf [19, 20]). *If the set* $\{x \in \mathbf{R}_+^K \mid Ax = c\}$ *is bounded, then there exists* $x \in \mathbf{R}_+^K$ *such that* $Ax = c$ *and* $\{\pi^j \mid j \in K,\ x_j > 0\}$ *is a primitive set.*

Remark that the vectors π^j, $j \in K \setminus N$, can actually be chosen arbitrarily from $\mathbf{R}_+^N$, provided that the R_i, $i \in N$, are suitably re-defined. Due to arbitrariness of the finite set K (provided that it contains N), and hence the generality of matrix A compared with $\tilde{A}$ (the matrix $\tilde{A}$ was introduced in a paragraph between the statement of Theorem 1.6 and Definition 1.5′), Theorem 4.6 together with a certain nondegeneracy assumption summarizes an analytical feature of Scarf's algorithm to compute a member of the core.

It was pointed out earlier that Scarf [20] derived Theorem 1.4 from Theorem 4.6. Conversely, Theorem 4.6 can be derived from Theorem 1.4; the proof is based on the idea in Vohra [28]:

Derivation of Theorem 4.6 *from Theorem* 1.4. Define $C_1^j := \{\pi^j\} - \mathbf{R}_+^N$. Denote by F the boundary of $\bigcup_{j \in K} C_1^j$, and define for each $j \in K$,

$$C^j := \left\{ z \in \Delta^N \,\middle|\, \exists y \in C_1^j \cap F\colon z = y \middle/ \sum_{i \in N} y_i \right\}.$$

If $\pi \notin \mathring{\mathbf{R}}_+^N$ for any $\pi \in P$, then the assertion of Theorem 4.6 is trivial. Assume, therefore, that there exists $\pi \in P \cap \mathring{\mathbf{R}}_+^N$. Then $\mathbf{0}$ is in the interior of $\bigcup_{j \in K} C_1^j$, so $\{C^j\}_{j \in K}$ is a closed covering of Δ^N. Observe that $y \in F$, if $y \in \bigcup_{j \in K} C_1^j$ and $y_i \geq R_i$ for some $i \in N$. By this observation, it is easy to check $\Delta^{N \setminus \{j\}} \subset C^j$. Thus $\{C^j\}_{j \in K}$ satisfies the assumption of Theorem 1.4, so there exists $x^* \in \mathbf{R}_+^K$ such that $Ax^* = c$ and $\bigcap\{C^j \mid j \in K,\ x_j^* > 0\} \neq \emptyset$. Set $I := \{j \in K \mid x_j^* > 0\}$, choose $z^* \in \bigcap_{j \in I} C^j$, and consider $y^* \in F$ defined by

$z^* = y^*/\sum_{i \in N} y_i^*$. Then $\pi^j \geq y^*$ for all $j \in I$; so $\{\pi^j\}_{j \in I}$ is the required primitive set. Q.E.D.

Many alternative proofs of Theorem 4.4 have appeared in the literature: Shapley [21] derived Theorem 4.4 from Theorem 1.6. Keiding and Thorlund-Petersen [16] and Vohra [28] proved Theorem 4.4 using Theorem 1.2 and Kakutani's fixed-point theorem, respectively. Ichiishi [14] pointed out that the geometric insights of Keiding and Thorlund-Petersen and of Vohra can be re-formulated as Theorem 1.7. It will be shown here that Theorem 4.4 follows simply from a theorem which is weaker than Theorem 1.4 and weaker than Theorem 1.7; the proof is based on the idea in Vohra [28]:

Derivation of Theorem 4.4 *either from Theorem* 1.4 *or from Theorem* 1.7. The special case of Theorem 1.4 and of Theorem 1.7, in which $K = \mathcal{N}$, $A = \tilde{A}$, and $c = \chi_N$, will be used here. Assume without loss of generality that $b = \mathbf{0}$, and that $\mathbf{0} \in \mathring{V}(S)$ for all $S \in \mathcal{N}$. Choose two real numbers M_1 and M_2 such that $M_1 > M_2 > M$, and denote by F the boundary of the set,

$$\bigcup_{i \in N} \{u \in V(\{i\}) \mid \forall k \in N \setminus \{i\}: u_k \leq M_1\}$$

$$\cup \bigcup_{S \in \mathcal{N},\ \#S \geq 2} \{u \in V(S) \mid \forall k \in N \setminus S: u_k \leq M_2\}.$$

For each $\pi \in \Delta^N$, consider the unique point $f(\pi) \in F \cap \mathbf{R}_+^N$ defined by $\pi = f(\pi)/\sum_{i \in N} f_i(\pi)$. Define $C^S := \{\pi \in \Delta^N \mid f(\pi) \in V(S)\}$ for every $S \in \mathcal{N}$. The family $\{C^S\}_{S \in \mathcal{N}}$ is a closed covering of Δ^N, and it is easy to check $\Delta^{N \setminus \{j\}} \subset C^{\{j\}}$ for every $j \in N$. All the assumptions of Theorem 1.4 and of Theorem 1.7 are satisfied, so there exist $x^* \in \mathbf{R}_+^{\mathcal{N}}$ and $\pi^* \in \Delta^N$ such that $\tilde{A}x^* = \chi_N$ and $\pi^* \in \bigcap \{C^S \mid S \in \mathcal{N}, x_S^* > 0\}$. The point $f(\pi^*)$ will be shown to be a member of $C(V)$. The family $\mathscr{B} := \{S \in \mathcal{N} \mid x_S^* > 0\}$ is balanced and $f(\pi^*) \in \bigcap_{S \in \mathscr{B}} V(S)$. So by the balancedness assumption on V, $f(\pi^*) \in V(N)$. Consequently, $f(\pi^*) \in \{u \in F \mid \forall i \in N: u_i < M\}$, which implies that the utility allocation $f(\pi^*)$ cannot be improved upon by any coalition. Q.E.D.

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