On commutators of Marcinkiewicz integrals with rough kernel

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Received 13 February 2001
Submitted by B. Bongiorno

Abstract

In this note we prove that the higher-order commutators $\mu_{\Omega,b}^m$, $\mu_{\Omega,\lambda,b}^*$, $\mu_{\Omega,S,b}^m$ and $\mu_{\Omega,S,b}^*$ are all of bounded operators on the weighted $L^p$ spaces. These commutators are formed respectively by a BMO($\mathbb{R}^n$) function $b(x)$ and a class of rough Marcinkiewicz integral operators $\mu_{\Omega}$, $\mu_{\Omega,\lambda}^*$ and $\mu_{\Omega,S}$, which are corresponding to the Littlewood–Paley $g$-function, Littlewood–Paley $g^*_{\lambda}$-function and the Lusin area integral, respectively. The results in this paper are essential improvements and extensions of the results by Torchinsky and Wang (1990) and by Alvarez et al. (1993).

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Keywords: Marcinkiewicz integral; Littlewood–Paley function; Commutator; Rough kernel; $A_p$ weight

* The research was supported partly by National Natural Science Foundation (Grant 19971010), Doctoral Programme Foundation of Institution of Higher Education, National 973 Project of China (G 19990751), and the Grant-in-Aid for Basic Scientific Research (Grant B-10440046), Japan.

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PII: S0022-247X(02)00230-5
1. Introduction and results

It is well known that the Littlewood–Paley $g$ function is a very important tool in harmonic analysis. In 1958, Stein [10] introduced the Marcinkiewicz integral $\mu_{\Omega}$ of higher dimension corresponding to the Littlewood–Paley $g$ function and studied the $L^p$ boundedness of $\mu_{\Omega}$. Now let us give its definition. Suppose that $S^{n-1}$ is the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $h(r) \in L^\infty(\mathbb{R}^+)$ and $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero and

$$\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral of higher dimension is defined by

$$\mu_{\Omega}(f)(x) = \left( \int_0^\infty \left| \frac{F_t(x)}{t^3} \right|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h(|x-y|) f(y) \, dy.$$

If $h(r) \equiv 1$, we set $\varphi(x) = \Omega(x)|x|^{-n+1}\chi_B(x)$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$, where $B = \{x \in \mathbb{R}^n: |x| \leq 1\}$ and $\chi_B$ denotes the characterization function of $B$. Then the Marcinkiewicz integral is just the Littlewood–Paley $g$-function. That is,

$$\mu_{\Omega}(f)(x) = g(f)(x) = \left( \int_0^\infty \left| \varphi_t * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

In [10], Stein proved that if $h(r) \equiv 1$ and $\Omega$ is continuous and satisfies a $\text{Lip}_\alpha$ $(0 < \alpha \leq 1)$ condition on $S^{n-1}$, then $\mu_{\Omega}$ is of type $(p, p)$ for $1 < p \leq 2$ and of weak type $(1, 1)$. In [2], Benedek et al. proved that if $h(r) \equiv 1$ and $\Omega$ is continuously differentiable on $S^{n-1}$, then $\mu_{\Omega}$ is of type $(p, p)$ $(1 < p < \infty)$. In 1990, Torchinsky and Wang [13] considered the weighted case. They proved that if $h(r) \equiv 1$, $\Omega$ is continuous and satisfies a $\text{Lip}_\alpha$ $(0 < \alpha \leq 1)$ condition on $S^{n-1}$, then for $1 < p < \infty$ and $\omega \in A_p$, $\mu_{\Omega}$ is bounded on $L^p(\omega)$. Recently, Sakamoto and Yabuta [9] also considered the boundedness for a class of parametrized Littlewood–Paley operators on $L^p$ spaces and the Campanato space $\mathcal{E}^{\alpha,p}$ for $\Omega \in \text{Lip}_\alpha$ $(0 < \alpha \leq 1)$.

It is worth pointing out that the results mentioned above were obtained when $\Omega$ satisfies some smoothness conditions on $S^{n-1}$. Recently, the authors of [6] improved the results mentioned above proving that when $\Omega \in H^1(S^{n-1})$, $\mu_{\Omega}$ (for
1 < p < \infty), \mu_{\Omega,\lambda}^* and \mu_{\Omega,S} (for 2 \leq p < \infty) are bounded on the \(L^p\) spaces. Here \(H^1(S^{n-1})\) denotes the Hardy space on \(S^{n-1}\) (see [4] or [5] for the definition), and \(\mu_{\Omega,\lambda}^*\) and \(\mu_{\Omega,S}\) are a class of the Marcinkiewicz integral operators related respectively to the Littlewood–Paley \(g^s_{\lambda}\)-function and the Lusin area integral \(S\) defined by
\[
\mu_{\Omega,\lambda}^*(f)(x) = \left( \int_\mathbb{R}^n (\frac{t}{t+|x-y|})^{n\lambda} \left| F_t(y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2}, \quad \lambda > 1,
\]
and
\[
\mu_{\Omega,S}(f)(x) = \left( \int_{\Gamma(x)} \left| F_t(y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2},
\]
where \(\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+: |x-y| < t\}\).

To state the following results, let us recall the definition of \(A_p\) weight class. A locally integrable nonnegative function \(\omega\) is said to belong to \(A_p (1 < p < \infty)\) if there is a constant \(C > 0\) such that
\[
\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C,
\]
where \(Q\) denotes a cube in \(\mathbb{R}^n\) with its sides parallel to the coordinate axes. The smallest constant \(C\) such that the above inequality holds is called the \(A_p\) constant of \(\omega\) denoted by \(C_p(\omega)\). In [7], the authors gave the following weighted \(L^p\)-boundedness of \(\mu_{\Omega}, \mu_{\Omega,\lambda}^*,\) and \(\mu_{\Omega,S}\).

**Theorem A.** Suppose that \(\Omega \in L^q(S^{n-1}) (q > 1)\) satisfying (1.1) and \(h(r) \in L^\infty(\mathbb{R}_+)\). If \(p, q,\) and \(\omega\) satisfy one of the following conditions:

(a) \(q' < p < \infty\) and \(\omega \in A_{p/q'}\),
(b) \(1 < p < q\) and \(\omega^{1-p'} \in A_{p'/q'}\),
(c) \(1 < p < \infty\) and \(\omega^q \in A_p\),

then \(\|\mu_{\Omega}(f)\|_{p,\omega} \leq \tilde{C} \|f\|_{p,\omega}\), where \(\tilde{C}\) depends only on \(n, p, q, h, \Omega, C_{p/q'}(\omega), C_{p'/q'}(\omega^{1-p'})\) and \(C_p(\omega^q)\), respectively.

**Theorem B.** Suppose that \(\Omega \in L^q(S^{n-1}) (q > 1)\) satisfying (1.1) and \(h(r) \in L^\infty(\mathbb{R}_+)\). If \(p, q\) and \(\omega\) satisfy one of the following conditions:

(a) \(\max\{q', 2\} = \alpha < p < \infty\) and \(\omega \in A_{p/\alpha}\),
(b) \(2 < p < q\) and \(\omega^{1-(p/2)'} \in A_{p'/q'}\),
(c) $2 \leq p < \infty$ and $\omega^q' \in A_{p/2},$

then $\|\mu^*_{\omega,\lambda}\|_{p,\omega} \leq \tilde{C} \|f\|_{p,\omega}$ and $\|\mu_{\omega,\lambda}\|_{p,\omega} \leq \tilde{C} \|f\|_{p,\omega},$ where $\tilde{C}$ depends only on $n, p, q, h, \Omega$ and $C_{p/\alpha}(\omega), C_{p'/q'}(\omega^{1-(p'/2)'})$ and $C_{p/2}(\omega^q'),$ respectively.

On the other hand, in [13] Torchinsky and Wang also discussed the weighted $L^p$ boundedness of the commutator $\mu_{\Omega}.$ First let us give some definitions.

A locally integrable function $b(x)$ is said to belong $\text{BMO}(\mathbb{R}^n)$ if there is a constant $C > 0$ such that

$$\|b\|_* := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx \leq C,$$

where $Q$ denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes and $b_Q = (1/|Q|) \int_Q b(x) \, dx.$ We denote the norm of $b(x)$ in $\text{BMO}(\mathbb{R}^n)$ by $\|b\|_*.$

For $m \in \mathbb{Z}_+,$ $b(x) \in \text{BMO}(\mathbb{R}^n),$ the higher-order commutator $\mu^m_{\Omega,b}, \mu^*_{m,\omega,\lambda,b}$ and $\mu^m_{\omega,\lambda,S,b}$ are defined respectively by

$$\mu^m_{\omega,b}(f)(x) = \left( \int_0^\infty \left| F^m_{t,b}(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu^*_{m,\omega,\lambda,b}(f)(x) = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n+1} \frac{t}{t + |x-y|} \frac{n\lambda}{m} |F^m_{t,b}(y)|^2 \frac{dy \, dt}{t^{n+3}} \right)^{1/2}, \quad \lambda > 1,$$

and

$$\mu^m_{\omega,\lambda,S,b}(f)(x) = \left( \int_{\Gamma(x)} \left| F^m_{t,b}(y) \right|^2 \frac{dy \, dt}{t^{n+3}} \right)^{1/2},$$

where

$$F^m_{t,b}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)]^m h(|x-y|) f(y) \, dy$$

and $\Gamma(x) = \{(y, t) \in \mathbb{R}^n+1 : |x-y| < t\}.$

We denote simply $\mu^m_{\omega,b}$ by $\mu_{\omega,b}$ if $m = 1.$ Torchinsky and Wang [13] proved that if $h(r) \equiv 1$ and $\Omega$ is continuous and satisfies a $\text{Lip}_\alpha$ $0 < \alpha \leq 1$ condition on $S^{n-1},$ then $\mu_{\omega,b}$ is bounded on $L^p(\omega)$ $1 < p < \infty$ for $\omega \in A_p.$

In this note we shall discuss the weighted $L^p$ boundedness for the higher-order commutators $\mu^m_{\omega,b}, \mu^*_{m,\omega,\lambda,b}$ and $\mu^m_{\omega,\lambda,S,b}$ with rough kernels. We obtain the following conclusions.
Theorem 1. Suppose that $\Omega \in L^q(S^{n-1})$ ($q > 1$) satisfying (1.1), $h(r) \in L^\infty(\mathbb{R}_+)$ and $m \in \mathbb{Z}_+$, $b(x) \in \text{BMO}(\mathbb{R}^n)$. If $p$, $q$ and $\omega$ satisfy one of the following conditions:

(a) $q' < p < \infty$ and $\omega \in A_{p/q'}$,
(b) $1 < p < q$ and $\omega^{1-p'} \in A_{p'/q'}$,
(c) $1 < p < \infty$ and $\omega'^{q'} \in A_p$,

then $\|\mu_{\Omega,b}^m(f)\|_{p,\omega} \leq \tilde{C}\|f\|_{p,\omega}$, where $\tilde{C}$ depends only on $n$, $p$, $q$, $h$, $\Omega$, $b$, $m$ and $C_{p/q'}(\omega)$, $C_{p'/q'}(\omega^{1-p'})$ and $C_p(\omega'^{q'})$, respectively.

Theorem 2. Suppose that $\Omega \in L^q(S^{n-1})$ ($q > 1$) satisfying (1.1), $h(r) \in L^\infty(\mathbb{R}_+)$ and $m \in \mathbb{Z}_+$, $b(x) \in \text{BMO}(\mathbb{R}^n)$. If $p$, $q$ and $\omega$ satisfy one of the following conditions:

(a) $\max\{q', 2\} = \alpha < p < \infty$ and $\omega \in A_{p/\alpha}$,
(b) $2 < p < q$ and $\omega^{1-(p/2)'} \in A_{p'/q'}$,
(c) $2 \leq p < \infty$ and $\omega'^{q'} \in A_{p/2}$,

then $\|\mu_{\Omega,b}^{*,m}(f)\|_{p,\omega} \leq \tilde{C}\|f\|_{p,\omega}^r$ and $\|\mu_{\Omega,S,b}^m\|_{p,\omega} \leq \tilde{C}\|f\|_{p,\omega}$, where $\tilde{C}$ depends only on $n$, $p$, $q$, $h$, $\Omega$, $b$, $m$ and $C_{p/\alpha}(\omega)$, $C_{p'/q'}(\omega^{1-(p/2)'})(\omega'^{q'})$ and $C_{p/2}(\omega'^{q'})$, respectively.

Remark 1. Clearly, Theorems 1 and 2 improve and extend Torchinsky and Wang’s result [13] about the commutator $\mu_{\Omega,b}$.

Remark 2. In [1], Alvarez et al. obtained the weighted boundedness of commutators for a class of the Littlewood–Paley operators with Schwartz function kernels and nonnegative BMO functions. Obviously, the smoothness condition and the nonnegative condition assumed respectively on kernel function and BMO function have been removed from Theorems 1 and 2 in this note.

Remark 3. Since the operators we consider in this paper are not linear operators, we cannot apply directly the results in [1]. But it should be pointed that in the proof of our conclusions we use some ideas from [3] and [1].

2. Proofs of Theorems 1 and 2

Let us begin by recalling some known conclusions.
John–Nirenberg inequality [8, p. 164]. There exist constants $c_1, c_2 > 0$, depending only on the dimension $n$, such that for every $b(x) \in \text{BMO}(\mathbb{R}^n)$, every cube $Q$ in $\mathbb{R}^n$ and every $t > 0$,

$$\left| \left\{ x \in Q : |b(x) - b_Q| > t \right\} \right| \leq c_1 |Q| e^{-c_2 t/\|b\|_*}.$$  

**Lemma 1.** Let $1 < p < \infty$ and $\lambda > 0$. Then when $b(x) \in \text{BMO}(\mathbb{R}^n)$ with $\|b\|_* < \min\{c_2/\lambda, c_2(p-1)/\lambda\}$, where $c_2$ is the constant in the John–Nirenberg inequality, we have $e^{\lambda b(x)} \in A_p$.

The conclusion of Lemma 1 is known (see [8, p. 409] or [11, p. 218]).

**Remark 4.** By checking the proof of Lemma 1, it is easy to see that if $1 < p < \infty$, $\lambda > 0$ and $a(x), b(x) \in \text{BMO}(\mathbb{R}^n)$ with $\|a\|_* \leq \|b\|_* < \min\{c_2/\lambda, c_2(p-1)/\lambda\}$, then $e^{\lambda a(x)}, e^{\lambda b(x)} \in A_p$ and the $A_p$ constant of $e^{\lambda a(x)}$ satisfies

$$C_p \left( e^{\lambda a(x)} \right) \leq \begin{cases} 
(1 + \frac{c_1 \lambda}{c_2/\|a\|_* - \lambda})^2 & \text{for } 2 \leq p < \infty, \\
(1 + \frac{c_1 \lambda_0}{c_2/\|a\|_* - \lambda_0})^{2(p-1)} & \text{for } 1 < p < 2,
\end{cases}$$

(2.1)

where $\lambda_0 = \lambda/(p-1)$.

Now let us return to the proof of Theorem 1. We consider only the case (a) here. The conclusions of Theorem 1 under the conditions (b) and (c) can be proved by using the same idea as in the case (a).

We need to prove that if $q' < p < \infty$ and $\omega \in A_{p/q'}$, then

$$\left\| \mu_{\Omega,b}(f) \right\|_{p,\omega} \leq \tilde{C} \|f\|_{p,\omega}. \quad (2.2)$$

The proof of (2.2) will be finished by induction on $m$. By Theorem A we see that (2.2) holds for $m = 0$. Below we prove that if (2.2) holds for $m - 1$, then (2.2) holds also for $m$. By the elementary property of $A_p$ weight class, we may choose an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in A_{p/q'}$. Thus by the assumption of induction,

$$\left\| \mu_{\Omega,b}^{-1}(\phi) \right\|_{p,\omega^{1+\varepsilon}} \leq \tilde{C}_1 \|\phi\|_{p,\omega^{1+\varepsilon}}, \quad \text{for } \phi \in L^p(\omega^{1+\varepsilon}). \quad (2.3)$$

Now we take $\lambda = p(1+\varepsilon)/\varepsilon$. By Lemma 1 we have $e^{p(1+\varepsilon)b(x)/\varepsilon} \in A_{p/q'}$ for every $b(x) \in \text{BMO}(\mathbb{R}^n)$ with $\|b\|_* < \min\{c_2/\lambda, c_2(p/q' - 1)/\lambda\}$. Since $b(x) \in \text{BMO}(\mathbb{R}^n)$ implies that $tb(x) \in \text{BMO}(\mathbb{R}^n)$ for $|t| \leq 1$ with smaller BMO norm, we have

$$e^{p(1+\varepsilon)tb(x)/\varepsilon} \in A_{p/q'}, \quad \text{for } b(x) \in \text{BMO}, \text{ with } \|b\|_* < \eta \text{ and } |t| \leq 1,$$

(2.4)

where $\eta = \min\{c_2/\lambda, c_2(p/q' - 1)/\lambda\}$. 


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Without loss of generality we may assume that \( \|b\|_s < \eta \). Otherwise we take \( 0 < \delta < \eta \) and set \( b_0(x) = \delta b(x)/\|b\|_s \). Then \( \|b_0\|_s = \delta < \eta \) and \( \mu_{\Omega, b}^m(f)(x) = (\|b\|_s/\delta)^m \mu_{\Omega, b_0}^m(f)(x) \). Therefore, it suffices to consider \( \mu_{\Omega, b_0}^m(f)(x) \). Hence, by (2.4) and the assumption of induction we know that for any \( \theta \in [0, 2\pi] \) and \( \phi \in L^p(e^{p(1+\varepsilon)|b(x)\cos\theta|/\varepsilon}) \),

\[
\|\mu_{\Omega, b_0}^{-1}(\phi)\|_{p, e^{p(1+\varepsilon)|b(x)\cos\theta|/\varepsilon}} \leq \tilde{C}_2 \|\phi\|_{p, e^{p(1+\varepsilon)|b(x)\cos\theta|/\varepsilon}},
\]

where \( \tilde{C}_2 \) depends on \( n, p, q, h, b, \Omega \) but not on \( \theta \) and \( \phi \) by Remark 4. Applying the Stein–Weiss interpolation theorem with change of measures \([12]\) between (2.3) and (2.5), we have for any \( \theta \in [0, 2\pi] \) and \( \phi \in L^p(\omega e^{p b(x)\cos\theta}) \)

\[
\|\mu_{\Omega, b_0}^{-1}(\phi)\|_{p, \omega e^{p b\cos\theta}} \leq \tilde{C} \|\phi\|_{p, \omega e^{p b\cos\theta}},
\]

where \( \tilde{C} = \max\{\tilde{C}_1, \tilde{C}_2\} \) depending only on \( n, p, q, h, b, \omega, \Omega \) but not on \( \theta \) and \( \phi \). Denote \( F(z) = e^{z[b(x)-b(y)]}, z \in \mathbb{C} \). Then by the analyticity of \( F(z) \) on \( \mathbb{C} \) and the Cauchy integration formula, we have

\[
b(x) - b(y) = F'(0) = \frac{1}{2\pi i} \int_{|z|=1} F(z) \frac{dz}{z^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta[b(x)-b(y)]} e^{-i\theta} d\theta.
\]

By (2.7) we have

\[
\mu_{\Omega, b}^m(f)(x) = \left( \int_0^{\infty} \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(|x-y|)}{|x-y|^{n-1}} \left[ b(x) - b(y) \right]^m \right) dy \right)^{1/2} \times \left( \int_0^{2\pi} \frac{dt}{t^3} \right)^{1/2}
\]

\[
= \left( \int_0^{\infty} \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(|x-y|)}{|x-y|^{n-1}} \left[ b(x) - b(y) \right]^{m-1} \right) dy \right)^{1/2} \times \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta[b(x)-b(y)]} e^{-i\theta} d\theta \right) \left( \int_0^{2\pi} \frac{dt}{t^3} \right)^{1/2}
\]

\[
= \left( \int_0^{\infty} \left( \int_0^{2\pi} \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(|x-y|)}{|x-y|^{n-1}} \right) dy \right)^{1/2} \right)^{1/2} \left( \int_0^{2\pi} \frac{dt}{t^3} \right)^{1/2}
\]
\[ \times [b(x) - b(y)]^{m-1} e^{-b(y)e^{i\theta}} f(y)e^{b(x)e^{i\theta}} dy \]
\[ \times e^{-i\theta} d\theta \left( \frac{dt}{t^3} \right)^{1/2} \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(|x-y|)}{|x-y|^{n-1}} [b(x) - b(y)]^{m-1} \times e^{-b(y)e^{i\theta}} f(y)e^{b(x)e^{i\theta}} dy \right)^2 \frac{dt}{t^3} \right)^{1/2} d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \mu_{m-1}^{\Omega,b}(g_\theta)(x)e^{b(x)\cos \theta} d\theta, \quad (2.8) \]

where \( g_\theta(x) = f(x)e^{-b(x)e^{i\theta}} \) for \( \theta \in [0, 2\pi] \). Since \( f \in L^p(\omega) \), it is easy to check that for any \( \theta \in [0, 2\pi] \),
\[ g_\theta \in L^p(\omega e^{pb(x)\cos \theta}) \quad \text{and} \quad \|g_\theta\|_{p,\omega e^{pb \cos \theta}} = \|f\|_{p,\omega}. \quad (2.9) \]

Using Minkowski's inequality for (2.8) and noting (2.6) and (2.9), we have
\[ \|\mu_{m,\Omega,b}(f)\|_{p,\omega} \leq \left( \int_{\mathbb{R}^n} \left( \frac{1}{2\pi} \int_0^{2\pi} \mu_{m-1}^{\Omega,b}(g_\theta)(x)e^{b(x)\cos \theta} d\theta \right)^p \omega(x) dx \right)^{1/p} \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\mathbb{R}^n} [\mu_{m-1}^{\Omega,b}(g_\theta)(x)]^p \omega(x)e^{pb(x)\cos \theta} dx \right)^{1/p} d\theta \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{C} \|g_\theta\|_{p,\omega e^{pb \cos \theta}} d\theta = \tilde{C} \|f\|_{p,\omega}. \]

This is just the conclusion of Theorem 1 under the condition (a).

Finally, let us show simply the proof idea of Theorem 2. Using the method of proving Lemma 2 in [7], we may get

Lemma 2. For any nonnegative function \( \phi \), we have
\[ \int_{\mathbb{R}^n} (\mu_{\ast,m,\lambda,b}(f)(x))^2 \phi(x) dx \leq C_\lambda \int_{\mathbb{R}^n} (\mu_{\Omega,b}(f)(x))^2 M \phi(x) dx, \]

where \( M \) denotes the usual Hardy–Littlewood maximal operator.
Applying the conclusions of Theorem 1 and the same method as proving Theorem 2 in [7], we may obtain the conclusions of Theorem 2 in this note. Here we omit the details.

Acknowledgment

The authors would like to express their gratitude to the referee for his very valuable comments and suggestions.

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