Piercing Balls Sitting on a Table by a Vertical Line

HIROSHI MAEHARA AND AI OSHIRO

Let $F_n$ be a family of disjoint $n$ balls all sitting on a fixed horizontal table $T$. Let $\ell$ denote a vertical line that meets $T$. We prove that if $\ell$ meets $2k + 1$ balls in $F_n$, then the radius of the smallest ball among the $2k + 1$ balls is at most $(2 - \sqrt{3})^k$ times the radius of the biggest ball among the $2k + 1$ balls. Using this result we prove that for any $F_n$ the average number of balls an $\ell$ meets is at most $\log n + o(1)$. A similar result for a two-dimensional version is also given together with a lower bound of the least upper bound.

© 2000 Academic Press

1. Introduction

By a table $T$ we mean a region in the $xy$-plane in $R^3$ bounded by a simple closed curve. A ball $B$ is said to be sitting on a table $T$ if $B$ is contained in the upper half-space $z \geq 0$ and $B$ is tangent to the $xy$-plane at a point on $T$. A line parallel to the $z$-axis is called a vertical line.

**Theorem 1.** Suppose that a vertical line $\ell$ meets $2k + 1$ balls that are mutually nonoverlapping and all sitting on a table $T$. Then the radius of the smallest ball among the $2k + 1$ balls is less than or equal to $(2 - \sqrt{3})^k$ times the radius of the biggest ball among them. This bound is sharp.

Let $F$ be a family of mutually nonoverlapping $n$ balls all sitting on a fixed table $T$. A vertical line that passes through $p \in T$ is denoted by $\ell_p$. The number of balls in $F$ that $\ell_p$ meets is called the piercing number of $F$ at $p$, and denoted by $h(p, F)$. The average piercing number $\overline{h}(F)$ is defined by

$$\overline{h}(F) = \frac{1}{\text{area}(T)} \int_T h(p, F) dp,$$

where $\text{area}(T)$ denotes the area of $T$.

**Theorem 2.** Let $F$ be a set of mutually nonoverlapping $n$ balls all sitting on a fixed table $T$. Then $\overline{h}(F) < \log n + o(1)$.

This result is applied in [6] to estimate the number of balls sitting on a table forming a certain configuration.

How about a two-dimensional version of this theorem? A family $\mathcal{D}$ of nonoverlapping disks in $R^2$ is simply called a family of coins in $R^2$. A support $L$ is a line-segment on the $x$-axis. A coin $D$ is said to be sitting on a support $L$ if $D$ is contained in the ‘upper’ half-plane $y \geq 0$ and tangent to the $x$-axis at a point on $L$. For an $x \in L$, let $\ell_x$ denote the line through $x$ and parallel to the $y$-axis. Let $h(x, \mathcal{D})$ denote the number of coins $\ell_x$ meets. Then the average piercing number $\overline{h}(\mathcal{D})$ is defined by

$$\overline{h}(\mathcal{D}) = \frac{1}{|L|} \int_L h(x, \mathcal{D}) dx,$$

where $|L|$ denotes the length of $L$. Then, we prove the following.

**Theorem 3.** Let $\mathcal{D}$ be a family of $n$ coins in $R^2$ sitting on a fixed support $L$. Then $\overline{h}(\mathcal{D}) < 2 \log n + o(1)$.
Theorem 4. For any $n$, there is a family $\mathcal{D}$ of $n$ coins sitting on a fixed support $L$ such that

$$
\frac{c \log n}{\log \log n} < \bar{h}(\mathcal{D}),
$$

where $c$ is a positive constant.

Problem 1. Improve the bounds in Theorems 2–4.

Problem 2. In the three-dimensional version, find a result like Theorem 4.

In view of a set of coins (not necessarily sitting on a support), the following result is proved by Alon et al. [1]. For any set of $n$ unit coins in the plane, there is a direction $\alpha$ such that every line with direction $\alpha$ intersects at most $O(\sqrt{n \log n})$ coins. They also show that this bound is sharp. On the other hand, there is a family of $n$ coins (of different sizes) such that for any direction $\alpha$ in the plane, there is a line with this direction that intersects at least $n - 1$ coins (see Theorem 1 of [5]).

2. Proof of Theorem 1

Lemma 1. Suppose that a vertical line $\ell$ meets three balls that are mutually nonoverlapping and sitting on a horizontal table $T$. Then the radius of the smallest ball among the three is at most $(2 - \sqrt{3})$ times the radius of the biggest ball.

Proof. Let $B_0, B_1, B_2$ be three balls mutually nonoverlapping of radii $r_0, r_1, r_2$ ($r_0 \geq r_1 \geq r_2$), respectively, all sitting on the table $T$. Let $p_i$ be the contact point of $B_i$ with the table $T$, and $D_i$ be the disk obtained by projecting $B_i$ orthogonally into $T$. Then $D_i$ has the center $p_i$ and has the same radius as $B_i$. Notice that since the vertical line $\ell$ meets all $B_0, B_1, B_2, D_0 \cap D_1 \cap D_2$ is nonempty, and $p_0, p_1$ lie outside $D_2, p_0$ lies outside $D_1$. Note that, if $D_0 \cap D_1 \cap D_2$ contains more than one point, then $B_1$ is bigger than $B_2$. Let us shrink $B_1$ by keeping the contact point $p_1$ fixed on $T$ until $D_0 \cap D_1 \cap D_2$ becomes a single point $q$. (This is indeed possible.) Denote the radius of the new $B_1$ by the same symbol $r_1$. Then $r_2 \leq r_1 \leq r_0$. When $B_i, B_j$ are tangent to each other, the distance between their centers is equal to $r_i + r_j$, and the difference in ‘heights’ is equal to $|r_i - r_j|$. Hence, generally, we have

$$
\frac{p_i p_j}{p_i p_j} \geq (r_i + r_j)^2 - (r_i - r_j)^2 = 4r_i r_j,
$$

and hence $p_i p_j$ is greater than or equal to $2\sqrt{r_i r_j}$.

Let $\alpha = \angle p_0 q p_1, \beta = \angle p_1 q p_2, \gamma = \angle p_2 q p_0$. First, suppose that one of $\alpha, \beta, \gamma$, say $\gamma$, is less than $\pi/2$. Then we have

$$
r_2^2 + r_0^2 \geq p_0 q^2 + q p_0^2 > p_0 p_2^2 \geq 4r_0 r_2.
$$

Hence $r_2^2 - 4r_0 r_2 + r_0^2 > 0$, and hence $r_2 < 2r_0 - \sqrt{4r_0^2 - r_0^2} = (2 - \sqrt{3})r_0$. Similarly, if $\alpha < \pi/2$ or $\beta < \pi/2$, then we have $r_2 \leq r_1 < (2 - \sqrt{3})r_0$ or $r_2 < (2 - \sqrt{3})r_0$.

Now, suppose that all $\alpha, \beta, \gamma$ are greater than or equal to $\pi/2$. Then $\alpha + \beta + \gamma = 2\pi$, and since $q p_i \leq r_i, i = 0, 1, 2$, we have

$$
r_0^2 + r_1^2 - 2r_0 r_1 \cos \alpha \geq p_0 p_1^2 \geq 4r_0 r_1,
$$

$$
r_1^2 + r_2^2 - 2r_1 r_2 \cos \beta \geq p_1 p_2^2 \geq 4r_1 r_2.
$$
Therefore following way:

Then, by Lemma 1, Hence we have the lemma.

\[
\begin{align*}
  r_1^2 - 2r_0(2 + \cos \alpha)r_1 + r_0^2 & \geq 0, \\
  r_2^2 - 2r_1(2 + \cos \beta)r_2 + r_1^2 & \geq 0.
\end{align*}
\]

Thus \( r_2 \) is less than or equal to

\[
  r_0(2 + \cos \alpha - \sqrt{(2 + \cos \alpha)^2 - 1}) (2 + \cos \beta - \sqrt{(2 + \cos \beta)^2 - 1})
\]

where \( \pi/2 \leq \alpha, \pi/2 \leq \beta, \alpha + \beta = 2\pi - \gamma \leq 3\pi/2 \). When does \( (1) \) take its maximum value? Since \( g(\beta) := 2 + \cos \beta - \sqrt{(2 + \cos \beta)^2 - 1} \) is monotone increasing (because \( g'(\beta) > 0 \) for \( \pi/2 < \beta < \pi \)), \( (1) \) takes its maximum value when \( \beta = 3\pi/2 - \alpha \). In this case, \( \cos \beta = -\sin \alpha \), and hence it is enough to consider the maximum value of

\[
  f(\alpha) := r_0(2 + \cos \alpha - \sqrt{(2 + \cos \alpha)^2 - 1}) (2 - \sin \alpha - \sqrt{(2 - \sin \alpha)^2 - 1}).
\]

The graph of \( y = f(\alpha) \) for \( \pi/2 \leq \alpha \leq \pi \) is given in Figure 1. Hence, the maximum value of \( f(\alpha) \) is attained when \( \alpha = \pi/2 \) or \( \alpha = \pi \), and the maximum value is

\[
  f(\pi/2) = f(\pi) = r_0(2 - \sqrt{3}).
\]

Hence we have the lemma.

**Proof of Theorem 1.** Suppose that a vertical line \( \ell \) meets \((2k+1)\) balls \( B_i \) with radius \( r_i, i = 0, 1, 2, \ldots, 2k \), all sitting on \( T \). Suppose \( r_0 \geq r_1 \geq \cdots \geq r_{2k} \), and let \( \rho = 2 - \sqrt{3} \). Then, by Lemma 1,

\[
  r_{2k} \leq \rho r_{2k-2}, r_{2k-2} \leq \rho r_{2k-4}, \ldots, r_2 \leq \rho r_0.
\]

Hence \( r_{2k} \leq \rho^k r_0 \).

To see that the bound is sharp, now let \( B(i), i = 0, 1, 2, \ldots, 2k \), be the balls defined in the following way:

<table>
<thead>
<tr>
<th>Ball</th>
<th>Center</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B(4j) )</td>
<td>( (\rho^{2j}, 0, \rho^{2j}) )</td>
<td>( \rho^{2j} )</td>
</tr>
<tr>
<td>( B(4j + 1) )</td>
<td>( (-\rho^{2j}, 0, \rho^{2j}) )</td>
<td>( \rho^{2j} )</td>
</tr>
<tr>
<td>( B(4j + 2) )</td>
<td>( (0, \rho^{2j+1}, \rho^{2j+1}) )</td>
<td>( \rho^{2j+1} )</td>
</tr>
<tr>
<td>( B(4j + 3) )</td>
<td>( (0, -\rho^{2j+1}, \rho^{2j+1}) )</td>
<td>( \rho^{2j+1} )</td>
</tr>
</tbody>
</table>

\( \rho = 2 - \sqrt{3} \).
Then these \((2k + 1)\) balls are nonoverlapping, all tangent to the \(z\)-axis and the \(xy\)-plane. The radius of the smallest ball is \((2 - \sqrt{3})^k\) times the radius of the biggest ball. \(\square\)

3. Proof of Theorem 2

Let \(\varphi\) denote the orthogonal projection of \(\mathbb{R}^3\) into the \(xy\)-plane. For each \(B \in \mathcal{F}\), define \(\chi_B : T \to \{0, 1\}\) by

\[
\chi_B(p) = \begin{cases} 
1 & \text{if } p \in \varphi(B) \\
0 & \text{if } p \notin \varphi(B).
\end{cases}
\]

Then

\[
h(p, \mathcal{F}) = \sum_{B \in \mathcal{F}} \chi_B(p).
\]

Let \(d\) be the diameter of \(T\). If the radius of \(B \in \mathcal{F}\) is greater than \(d\), then we can shrink the ball \(B\) without changing the values \(\chi_B(p)\) \((p \in T)\) until its radius becomes \(d\). Hence, we may assume that every ball in \(\mathcal{F}\) has radius \(\leq d\). Let \(\mathcal{F}_1\) be the subfamily of \(\mathcal{F}\) consisting of those balls whose radii are at most \(d/\sqrt{n}\). Then, for large \(n\),

\[
\int_T h(p, \mathcal{F}_1)dp \leq (\frac{1}{4} \log n) \text{area}(T) \quad (2)
\]

To see this, suppose that (2) does not hold. Then, since

\[
\int_T h(p, \mathcal{F}_1)dp = \int_T \sum_{B \in \mathcal{F}_1} \chi_B(p)dp = \sum_{B \in \mathcal{F}_1} \int_T \chi_B(p)dp
\]

\[
\leq \sum_{B \in \mathcal{F}_1} \pi \left(\frac{d}{\sqrt{n}}\right)^2 = (\#\mathcal{F}_1)\pi \frac{d^2}{n},
\]

where \(\#\mathcal{F}_1\) is the cardinality of \(\mathcal{F}_1\), we have

\[
\frac{\text{area}(T)}{5\pi d^2} n \log n < \#\mathcal{F}_1 \leq n,
\]

a contradiction since \(n\) is large. Hence (2) holds.

Let \(\mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1\). Then no vertical line meets more than \(\frac{4}{5} \log n\) balls in \(\mathcal{F}_2\). To see this, suppose that there is a vertical line \(\ell\) that meets more than \(\frac{4}{5} \log n\) balls of \(\mathcal{F}_2\). Then, by Theorem 1, the radius of the smallest ball the line \(\ell\) meets is at most

\[
d(2 - \sqrt{3})^{(2/5) \log n} = d(2 + \sqrt{3})^{-(2/5) \log n} = \frac{d}{n^{(2/5) \log(2 + \sqrt{3})}} < \frac{d}{\sqrt{n}}
\]

(because \(\frac{2}{5} \log(2 + \sqrt{3}) = 0.5267\ldots\)), a contradiction. Hence, no vertical line meets more than \(\frac{4}{5} \log n\) balls of \(\mathcal{F}_2\). Therefore,

\[
\int_T h(p, \mathcal{F}_2)dp < (\frac{1}{4} \log n) \text{area}(T) \quad (3)
\]

From (2) and (3), we have the theorem. \(\square\)
4. PROOF OF THEOREM 3

LEMMA 2. Suppose that y-axis meets $3k + 1$ coins sitting on a support $L$. Then the radius of the smallest coins is less than or equal to $(\sqrt{2} - 1)^{2k}$ times the radius of the biggest coin.

PROOF. First, consider the case $k = 1$. Suppose that the y-axis meets four coins $D_0, D_1, D_2, D_3$ of radii $r_0 \geq r_1 \geq r_2 \geq r_3$, sitting on a support $L$. How large $r_3$ can be for a fixed $r_0$? It is easy to see that $r_3$ becomes the largest when $D_0, D_1$ are of the same size, and $D_2, D_3$ are of the same size, all tangent to the y-axis as in Figure 2. In this case, it is not difficult to see that $r_2 = r_3 = (\sqrt{2} - 1)^2 r_0$. Hence $r_3 \leq (\sqrt{2} - 1)^2 r_0$.

Now the general case. Suppose that y-axis meets $3k + 1$ coins of radii $r_0 \geq r_1 \geq \cdots \geq r_{3k}$ all sitting on a support $L$, and let $\sigma = (\sqrt{2} - 1)^2$. Then

$$r_{3k} \leq \sigma r_{3k-3}, \; r_{3k-3} \leq \sigma r_{3k-6}, \ldots, \; r_3 \leq \sigma r_0.$$ 

Hence $r_{3k} \leq \sigma^k r_0 = (\sqrt{2} - 1)^{2k} r_0$. □

PROOF OF THEOREM 3. The proof is similar to that of Theorem 2. We may suppose that $|L| = 1$ and that every coin in $D$ has radius $\leq 1$. Let $D_1$ be the subfamily of $D$ consisting of those coins with radii at most $1/n$. Then, for large $n$,

$$\int_L h(x, D_1)dx \leq 1.$$ 

Let $D_2 = D - D_1$. Then similarly to the proof of Theorem 2, it follows that no line parallel to the y-axis meets more than $(1.71) \log n$ coins in $D_2$, where $1.70 \approx 3/(2 \log(\sqrt{2} + 1))$. Hence

$$\int_L h(x, D_2)dx \leq (1.71) \log n.$$ 

Therefore we have Theorem 3. □

5. PROOF OF THEOREM 4

Let $\psi$ denote the orthogonal projection of $R^2$ into the x-axis. The length of the line-segment $X = \overline{pq}$ is denoted by $|X|$, or by the same notation $\overline{pq}$.
LEMMA 3. Let \( D_0, D_1, D_2 \) be three coins with radius \( r_0, r_1, r_2 \) \((r_0 > r_1 > r_2)\), all sitting on a support \( L \). Let \( p_i \) be the contact point of \( D_i \) with \( L \). Suppose that \( D_0, D_1, D_2 \) are tangent to each other. Then

\[
\frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_0}}
\]

and

\[
|\psi(D_1) \cap \psi(D_2)| = \frac{r_1 r_2}{r_0}.
\]

PROOF. Regarding the \( x \)-axis as a circle of infinite radius, the first equality follows easily from the so-called Soddy’s formula (see, e.g., Coxeter [3]). This equality is also presented in [4]. We show the second equality.

\[
|\psi(D_1) \cap \psi(D_2)| = r_1 + r_2 - \frac{p_1 p_2}{r_1 r_2} = r_1 + r_2 - 2\sqrt{r_1 r_2} = (\sqrt{r_1} - \sqrt{r_2})^2 = \left(\sqrt{\frac{1}{r_1}} - \frac{1}{\sqrt{r_0} + 1/\sqrt{r_1}}\right)^2 = \left(\frac{\sqrt{r_1}}{\sqrt{r_0} + 1/\sqrt{r_1}}\right)^2 = \frac{r_1 r_2}{r_0}.
\]

Let us denote by \( r(D) \) the radius of a coin \( D \). A chain is a sequence \( D_1 D_2 \ldots D_n \) of coins such that each consecutive coins are tangent to each other.

LEMMA 4. Let \( D_0, D_1 \) be two coins tangent to each other, both sitting on a support \( L \), and \( r(D_0) > r(D_1) \). Let \( D_1 D_2 \ldots D_n \) be a chain with \( r(D_1) > r(D_2) > \cdots > r(D_n) \) such that each \( D_i \) is tangent to both \( D_0, L \). Let \( p_i \) be the contact point of \( D_i \) with \( L \). Then

\[
\frac{2 r(D_0)}{n} < \frac{2 r(D_0)}{n}.
\]

PROOF. We may suppose that \( r(D_0) = 1 \). First consider the special case when the contact point \( p_1 \) is an endpoint of the line-segment \( \psi(D_0) \). Then it follows easily that \( r(D_1) = \frac{1}{2} \). Then, by applying Lemma 3, we have \( r(D_k) = 1/(k+1)^2 \) \( k = 2, \ldots, n \). Since \( |\psi(D_k) \cap \psi(D_{k+1})| = r(D_k)r(D_{k+1}) \) by Lemma 3, we have

\[
\frac{p_k p_{k+1}}{r_k r_{k+1}} = r(D_k) + r(D_{k+1}) - |\psi(D_k) \cap \psi(D_{k+1})| = \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} = \frac{2}{(k+1)(k+2)} - \frac{1}{k+1} = \frac{1}{k+1} - \frac{2}{k+2}.
\]

Hence

\[
\frac{p_1 p_n}{r_1 r_n} = \sum_{k=1}^{n-1} \frac{p_k p_{k+1}}{r_k r_{k+1}} = \sum_{k=1}^{n-1} \left(2 \frac{2}{k+1} - \frac{2}{k+2}\right) = 1 - \frac{2}{n+1}.
\]

Thus, if \( p_1 \) is an endpoint of \( \psi(D_0) \), then

\[
\frac{p_n p_0}{r_n r_0} = \frac{p_1 p_0}{r_1 r_0} = \frac{2}{n+1}.
\]

If \( p_1 \) is interior to \( \psi(D_0) \), then clearly \( \frac{p_n p_0}{r_n r_0} < 2/(n+1) \). If \( p_1 \) is exterior to \( \psi(D_0) \), then considering the subchain \( D_2 D_3 \ldots D_n \) of \( n-1 \) coins, we can see that \( \frac{p_n p_0}{r_n r_0} < 2/n \). \( \square \)
Let $A, B$ be two coins tangent to each other, both sitting on a support $L$. Then the curvilinear triangle formed by $A, B, L$ is called the hole $ABL$. Let $C$ be the largest coin that can be inscribed in the hole $ABL$. The two-tail-chain of order $n$ inscribed in the hole $ABL$ is a sequence of coins

$$
\Gamma(n) = A_n A_{n-1} \ldots A_1 C B_1 B_2 \ldots B_n
$$

such that each $A_j$ is tangent to both $A, L$ and each $B_i$ is tangent to both $B, L$. Note that

$$
r(A_n) < r(A_{n-1}) < \cdots < r(A_1) < r(C) > r(B_1) > r(B_2) > \cdots > r(B_n).
$$

Let $a_n, b_n$ be the contact points of $A_n$ with $L$ and $B_n$ with $L$. Then the line-segment $a_n b_n$ is called the span of $\Gamma(n)$, and its length (also called the span of $\Gamma(n)$) is denoted by $\lambda(\Gamma(n))$. Let us say $A$ covers the center of $B$ if $\psi(A)$ contains the contact point of $B, L$.

**Lemma 5.** Let $A, B$ be two tangent coins sitting on a support $L$ with $r(A) \geq r(B)$. Let $\Gamma(n)$ be the two-tail-chain of order $n$ inscribed in the hole $ABL$. Suppose that $A$ does not cover the center of $B$. Then, no coin in $\Gamma(n)$ covers the center of other coin in $\Gamma(n)$, and

$$
\lambda(\Gamma) > \frac{ab}{n} \left(1 - \frac{4}{n}\right),
$$

where $a, b$ are the contact points of $A, L$ and of $B, L$, respectively.

**Proof.** Let $\Gamma(n) = A_n \ldots A_1 C B_1 \ldots B_n$. Suppose that, a coin in $\Gamma(n)$, say $A_j$, covers the center of $A_{j+1}$. Then $r(A_{j+1}) \leq r(A_j)/4$, which implies that $r(A) \leq r(A_j)$, a contradiction. Hence, no coin in $\Gamma(n)$ can cover the center of another coin in $\Gamma(n)$. Now, applying the above lemma to the chain $C A_1 \ldots A_n$ of $n+1$ coins and to the chain $C B_1 \ldots B_n$ of $n+1$ coins, we have

$$
\frac{2r(A)}{n} < \frac{2ab}{n}, \quad \frac{2r(B)}{n} < \frac{2ab}{n}.
$$

Hence $\lambda(\Gamma(n)) = \frac{ab}{n} - \frac{2ab}{n} > \frac{ab}{n} \left(1 - \frac{4}{n}\right).$ \hfill $\Box$

**Proof of Theorem 4.** Let $L = p \overline{p}$ be a support of unit length. Let $k \geq 6$ be a fixed integer. For a positive integer $m$, let us construct a family $D$ of coins all sitting on $L$ in the following way.

1. Let $A, B$ be two tangent coins of the same radius, sitting on $L$ at $p$ and at $q$, respectively.
2. Inscribe in the hole $ABL$ the two-tail-chain $\Gamma_1(k)$ of order $k$.
3. Under the chain $\Gamma_1(k)$, $2k$ holes appear. In these $2k$ holes, inscribe the two-tail-chains $\Gamma_1(2^2k), \ldots, \Gamma_{2k}(2^2k)$ of order $2^2k$.
4. Under each chain of order $2^2k$, $2 \cdot 2^2k$ holes appear. In each of these holes, inscribe the two-tail-chain of order $3^2k$.
5. Repeat the same process till the two-tail-chains of order $m^2k$ are inscribed.

Let $D$ be the family of coins obtained in this way. How many coins does $D$ have? Let us make a table:

<table>
<thead>
<tr>
<th>Order of two-tail-chains</th>
<th>Number of chains</th>
<th>Number of holes under a chain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^k$</td>
<td>1</td>
<td>$2^k$</td>
</tr>
<tr>
<td>$2^2k$</td>
<td>$2^k$</td>
<td>$2 \cdot 2^2k$</td>
</tr>
<tr>
<td>$3^2k$</td>
<td>$2 \cdot 2^2k$</td>
<td>$2 \cdot 3^2k$</td>
</tr>
<tr>
<td>$4^2k$</td>
<td>$2 \cdot 3^2k$</td>
<td>$2 \cdot 4^2k$</td>
</tr>
<tr>
<td>$5^2k$</td>
<td>$2 \cdot 4^2k$</td>
<td>$2 \cdot 5^2k$</td>
</tr>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$m^2k$</td>
<td>$2^{m-1}(m-1)!2^{m-1}$</td>
<td>$2 \cdot m^2k$</td>
</tr>
</tbody>
</table>
Denote by $N$ the number of coins in $D$. Then

$$N = 2 + (2k + 1) + 2(2 \cdot 2^2 k + 1) + \cdots + 2^{m-1}((m - 1)!2^{m-1}(2 \cdot m^2k + 1),$$

and it is not difficult to see that $N$ is less than the double of the last term of the right-hand side. Hence,

$$2^{m-1}((m - 1)!2^{m-1}(2 \cdot m^2k + 1) < N < 2 \cdot 2^{m-1}((m - 1)!2^{m-1}(2 \cdot m^2k + 1).$$

Since $m! = \sqrt{2\pi m}(m/e)^{m}e^{\theta/12m}$ ($0 < \theta < 1$) by Stirling’s formula (see, e.g., Artin [2, p. 24]),

$$2^{m-1}((m - 1)!2^{m-1}(2 \cdot m^2k + 1) = 2^{m}(m!)^2k^m \left(1 + \frac{1}{2m^2k}\right)$$

$$= 2\pi m(2k)\left(\frac{m}{e}\right)^{2m}e^{\theta/12m}\left(1 + \frac{1}{2m^2k}\right).$$

Hence

$$m^{2m} < N < 2 \cdot 2^{m}(m!)^2k^m \left(1 + \frac{1}{2m^2k}\right) < 5\pi m(2k)^m\left(\frac{m}{e}\right)^{2m}.$$ 

Thus, $m^{2m} < N < m^{3m}$ for $m \geq k$, and hence $2m \log m < \log N < 3m \log m$. Thus

$$2m < \frac{\log N}{\log m} < 3m,$$

and $m < \log N$. Since $\log N < 3m \log m < m^2$, we have $\log \log N < 2 \log m$. Therefore

$$\frac{\log N}{3 \log \log N} < m < \frac{\log N}{\log \log N}.$$ 

Next, let us consider the sum of the spans of the two-tail-chains of order $m^2k$. By Lemma 5, $\lambda(\Gamma_1(k)) > 1 - 4/k$. The span of $\Gamma_1(k)$ is divided into $2k$ intervals by $2k - 1$ contact points inside the span, and each $\Gamma_j(2^2k)$ covers more than $1 - 4/(2^2k)$ part of one interval. Hence the sum of the spans of the chains of order $2^2k$ is greater than $(1 - 4/(1^2k))(1 - 4/(2^2k))$. Similarly, it follows that the sum of the spans of chains of the order $j^2k$ is greater than

$$s(j) := \left(1 - \frac{4}{1^2k}\right)\left(1 - \frac{4}{2^2k}\right)\cdots\left(1 - \frac{4}{j^2k}\right).$$

Since $k \geq 6$, using the inequality $\log(1 - t) > -t - t^2$ for $0 < t < 0.69$, we have

$$\log s(m) = \sum_{j=1}^{m} \log(1 - 4/(j^2k)) > -4 \sum_{j=1}^{\infty} (1/(j^2k)) - 16 \sum_{j=1}^{\infty} (1/(j^4k^2)).$$

Since $\sum_{j=1}^{\infty} (1/j^2) = \pi^2/6$, $\sum_{j=1}^{\infty} (1/j^4) = \pi^4/90$,

$$\log s(m) > -4\pi^2/6k + -16\pi^4/90k^2.$$ 

Thus, letting $k = 6$, we have $\log s(m) > -1.5777$, and hence $s(m) > e^{-1.5777} = 0.2064$. Note that if $x \in L$ is covered by a chain of order $m^2k$, then $h(x, D) = m + 1$. Hence $\bar{h}(D) > (m + 1)s(m) > (0.2064)m$. 
Now, if \( n \) is sufficiently large, then, letting \( m = \lfloor (\log n)/(3 \log \log n) \rfloor \), and applying the above construction with \( k = 6 \), make a family \( D \). If the number of coins in \( D \) is less than \( n \), then add very tiny coins until the number of coins in \( D \) becomes \( n \). Then, we have

\[
\overline{h}(D) > (0.2064)m > (0.2064) \frac{\log N}{3 \log \log N} > \frac{(0.2064)m}{3} > \frac{c \log n}{\log \log n},
\]

for some constant \( c > 0 \).

REFERENCES


*Received 26 March 1999 and accepted in revised form 14 October 1999*

HIROSHI MAEHARA AND AI OSHIRO

College of Education,
Ryukyu University,
Okinawa,
Japan