Operator splitting for delay equations

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Abstract

Operator splitting methods are widely used for partial differential equations. Up until now, they have not been used for delay differential equations. In this paper we introduce splitting methods for delay equations in an abstract setting. We then prove the convergence of the method and discuss the results of some numerical experiments.

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1. Introduction

Operator splitting methods have been widely used for the numerical solution of various types of evolution equations, see e.g. [1–5]. To explain the situation we consider an abstract evolution equation

\[
\begin{align*}
\dot{u}(t) &= (A + B)u(t), \quad t \geq 0, \\
 u(0) &= x \in X
\end{align*}
\]

on some Banach space $X$, where the operators $A$ and $B$ are the generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$, respectively. Clearly, in concrete cases the operators $A$ and $B$ will be differential operators. Application of, e.g., the \textit{sequential splitting} means that the solution $u(t)$ of the evolution equation (ACP) is approximated by

\[
u_{\text{sq}}^n(t) := [S(t/n)T(t/n)]^n x, \quad n \in \mathbb{N}.
\] (1)

An application of the Trotter Product Theorem yields convergence as $n \to \infty$.

For partial differential equations with delay, however, this method has not yet been applied. The aim of this paper is to fill this gap. We will show how operator splitting can be used to solve numerically a quite general class of delay equations. These are given in the following form (see e.g. [6]):

\[
\begin{align*}
\dot{u}(t) &= Cu(t) + \Phi u_t, \quad t \geq 0, \\
 u(0) &= x \in X, \\
 u_0 &= f \in L^1([-1, 0], X)
\end{align*}
\] (DE)

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on the Banach space $X$, where $(C, D(C))$ is the generator of a strongly continuous semigroup on $X$, and $\Phi: L^1([-1, 0], X) \to X$ is a bounded and linear operator. The history function $u_t$ is defined by $u_t(\sigma) := u(t + \sigma)$ for $\sigma \in [-1, 0]$. This equation can be solved using additive perturbation theory on a product space $X \times L^1([-1, 0], X)$. This opens a way for an operator splitting approach.

The paper is organized in the following way. In Section 2 we collect the basic facts on operator splitting needed for the following discussion. Section 3 shows how delay equations can be formulated as abstract Cauchy problems, thus opening the way for an application of splitting methods. Section 4 contains the main result, namely, the convergence of the splitting applied to delay equations. In Section 5, finally, we present some numerical examples. For those who are not familiar with the operator semigroup theory, we give a short introduction in the Appendix.

2. Operator splitting

Operator splitting methods have been introduced in [7,8], their application to abstract Cauchy problems has been studied in e.g. [9,10,4], while the stiff case was analysed in [11]. They are usually applied when Eq. (ACP) cannot be solved numerically fast and accurate enough, but the sub-problems corresponding to the operators $A$ and $B$ can be treated easily. From a physical point of view, formula (1) means that the sub-processes do not act at the same time, but one after the other in a certain time step $t/n$.

From now on we assume the following.

**General Assumptions.** (a) $(A, D(A))$ generates the strongly continuous semigroup $(T(t))_{t \geq 0}$ on the Banach space $X$, 
(b) $(B, D(B))$ generates the strongly continuous semigroup $(S(t))_{t \geq 0}$ on $X$.

We have already introduced the sequential splitting (1) in the introduction. Another type of operator splitting is the Strang splitting, where the split solution of (ACP) is defined as:

$$u^n_{\text{spl}}(t) := [T(t/2n)S(t/n)T(t/2n)]^n x, \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$  \hspace{1cm} (2)

The important difference between formulae (1) and (2) is that the sequential splitting is consistent of first order, while the Strang splitting is consistent of second order (see [9,4]).

**Definition 2.1.** The split solution $u^n_{\text{spl}}(t)$ is called convergent if

$$\lim_{n \to \infty} u^n_{\text{spl}}(t) = u(t) \quad \text{for all } x \in X$$

and uniformly for $t$ in compact intervals, where $u(t)$ is the solution of (ACP) at time $t$.

The convergence of the sequential splitting and the Strang splitting is a consequence of the Chernoff Theorem (see [12], and in [13], Chapter III, Cor. 5.3).

**Theorem 2.2 (Chernoff).** Consider a function $F: \mathbb{R}^+ \to \mathcal{L}(X)$ satisfying $F(0) = I$ and the stability condition

$$\|F(t/n)^n\| \leq Me^{\omega t} \quad \text{for all } t \geq 0, n \in \mathbb{N},$$  \hspace{1cm} (3)

and some constants $M \geq 1, \omega \in \mathbb{R}$. Assume that

$$Gx := \lim_{t \downarrow 0} \frac{F(t)x - x}{t}$$

exists for all $x \in D \subset X$, where $D$ and $(\lambda_0 - G)D$ are dense sub-spaces in $X$ for some $\lambda_0 > \omega$. Then the closure $\overline{G}$ of $(G, D(G))$ generates a strongly continuous semigroup $(U(t))_{t \geq 0}$ given by

$$U(t)x = \lim_{n \to \infty} [F(t/n)^n]x$$

for all $x \in X$ and uniformly for $t \in [0, t_0]$. Moreover, $(U(t))_{t \geq 0}$ satisfies the estimate $\|U(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.  


The stability condition (3) is crucial for obtaining the convergence of the splitting method. If we put \( F(t) := S(t)T(t) \), it takes the form
\[
\| [S(t/n)T(t/n)]^n \| \leq Me^{\omega t} \quad \text{for all } t \geq 0, n \in \mathbb{N},
\]
for constants \( M \geq 1, \omega \in \mathbb{R} \). We now collect some consequences.

**Lemma 2.3.** In the situation of the General Assumptions, let us assume the stability condition (4). Then the following holds.

(i) There exist a constant \( \omega_1 \in \mathbb{R} \) such that
\[
\| [S(t/n)T(t/n)]^{n-1} \| \leq Me^{\omega_1 t} \quad \text{for all } t \geq 0, n \in \mathbb{N}.
\]

(ii) There exist constants \( M_2 \geq 1, \omega_2 \in \mathbb{R} \) such that
\[
\| [T(t/n)S(t/n)]^n \| \leq M_2 e^{\omega_2 t} \quad \text{for all } t \geq 0, n \in \mathbb{N}.
\]

(iii) There exist constants \( M_3 \geq 1, \omega_3 \in \mathbb{R} \) such that
\[
\| [T(t/2n)S(t/n)T(t/2n)]^n \| \leq M_3 e^{\omega_3 t} \quad \text{for all } t \geq 0, n \in \mathbb{N}.
\]

**Proof.** The validity of (i) follows from (4) since
\[
\| [S(t)T(t)]^n \| \leq M e^{n\omega t}
\]
for \( M \geq 1 \) and \( \omega \in \mathbb{R} \) implies
\[
\| [S(t)T(t)]^{n-1} \| \leq Me^{(n-1)\omega t} \leq M e^{n \max\{0, \omega\} t},
\]
i.e.
\[
\| [S(t/n)T(t/n)]^{n-1} \| \leq Me^{\omega_1 t} \quad \text{for all } t \geq 0, n \in \mathbb{N},
\]
where \( \omega_1 := \max\{0, \omega\} \).

In order to prove (ii), we use (i) to obtain
\[
\| [T(t/n)S(t/n)]^n \| \leq \| T(t/n) \| \| [S(t/n)T(t/n)]^{n-1} \| \| S(t/n) \| \leq M_2 e^{\omega_2 t}.
\]
The validity of (iii) equals follows from (i), since
\[
\| [T(t/2n)S(t/n)T(t/2n)]^n \| = \| T(t/2n)[S(t/n)T(t/n)]^{n-1}S(t/n)T(t/2n)\| \leq \| T(t/2n) \| \| [S(t/n)T(t/n)]^{n-1} \| \| S(t/n) \| \| T(t/2n) \| \leq M_3 e^{\omega_3 t}
\]
for some constants \( M_3 \geq 1 \) and \( \omega_3 \in \mathbb{R} \). \( \Box \)

Now, from Theorem 2.2 one can directly infer the following theorem (see e.g. in [13], Chapter III, Cor. 5.8).

**Theorem 2.4 (Trotter Product Formula).** Let \( (T(t))_{t \geq 0} \) and \( (S(t))_{t \geq 0} \) be strongly continuous semigroups on a Banach space \( X \) with generators \( \{A, D(A)\} \) or \( \{B, D(B)\} \), respectively, satisfying the stability condition (4). Consider the sum \( A + B \) on \( D := D(A) \cap D(B) \), and assume that \( D \) and \( (\lambda_0 - (A + B))D \) are dense in \( X \) for some \( \lambda_0 > \omega \). Then \( G := A + B \) generates a strongly continuous semigroup \( (U(t))_{t \geq 0} \) given by the Trotter product formula, i.e.
\[
U(t)x = \lim_{n \to \infty} [S(t/n)T(t/n)]^n x, \quad x \in X,
\]
uniformly for \( t \) in compact intervals.

Since in our applications the sum of \( A \) and \( B \) is already a generator on the intersection of the respective domains, we can restrict our discussion to this case. Then the denseness conditions in Theorems 2.2 and 2.4 are automatically fulfilled and the closure of \( A + B \) coincides with \( A + B \). Besides the General Assumptions (a) and (b) we thus assume the following, too.
**General Assumptions.** (c) The sum \( A + B \) defined on \( D(A + B) := D(A) \cap D(B) \) generates a strongly continuous semigroup \( (U(t))_{t \geq 0} \) on the Banach space \( X \).

The following corollaries are now immediate consequences of Theorem 2.2.

**Corollary 2.5.** Under the General Assumptions the sequential splitting is convergent if and only if the stability condition (4) is satisfied.

**Corollary 2.6.** Under the General Assumptions, the sequential splitting with the reverse order of the operators is convergent, i.e.,

\[
\lim_{n \to \infty} [T(t/n)S(t/n)]^nx = u(t), \quad x \in X,
\]

if and only if the stability condition (4) is satisfied.

**Proof.** By Lemma 2.3 (ii), there exist constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that the stability condition of Theorem 2.2 is satisfied with \( F(t) := T(t)S(t) \). The consistency criterion follows from

\[
\lim_{t \downarrow 0} \frac{F(t)x - x}{t} = \lim_{t \downarrow 0} \frac{T(t)S(t)x - x}{t} = \lim_{t \downarrow 0} T(t)\frac{S(t)x - x}{t} + \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = (A + B)x
\]

for all \( x \in D(A) \cap D(B) \). \( \Box \)

**Corollary 2.7.** Under the General Assumptions, the Strang splitting (2) is convergent if and only if the stability condition (4) is satisfied.

**Proof.** Similarly to the proof of Corollary 2.6 we apply Theorem 2.2 to

\[
F(t) := T(t/2)S(t)T(t/2).
\]

We note that the result of the convergence and the consistency of the Strang splitting can also be found in [4].

**Remark 2.8.** Lemma 2.3 means that the stability conditions are equivalent for the operators \( F_1(t) := S(t)T(t) \) and \( F_2(t) := T(t)S(t) \). Therefore, the convergence of the Strang splitting remains valid also in the case of the reverse order of operators, i.e., for

\[
u_n^u(t) = [T(t/2n)S(t/n)T(t/2n)]^nx.
\]

### 3. Delay equations as abstract Cauchy problems

Physical processes often depend on a former state of the system. Such processes can be described by *delay differential equations*, which contain a term depending on the *history function* (see [6]). These differential equations cannot be written as an abstract Cauchy problem on the original state space \( X \). However, if an appropriate function space (called *history* or *phase space*) is chosen, the solution of delay differential equations can be obtained by an operator semigroup on this space. We briefly show how the abstract delay equation (DE) can be formulated as an abstract Cauchy problem on the appropriate phase space. For a systematic treatment we refer to the monograph [6].

Let us define the product space \( \mathcal{E} := X \times L^1([-1, 0], X) \), and the new unknown function as

\[
t \mapsto \Phi(t) = \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}.
\]

Then (DE) can be written as an *abstract Cauchy problem* on the space \( \mathcal{E} \) in the following way:

\[
\begin{align*}
\dot{\Phi}(t) &= G\Phi(t), \quad t \geq 0, \\
\Phi(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{E},
\end{align*}
\]
where the operator \((\mathcal{G}, D(\mathcal{G}))\) is given by the matrix

\[
\mathcal{G} := \begin{pmatrix} C & \phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}
\] (8)

on the domain

\[
D(\mathcal{G}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(C) \times W^{1,1}([-1, 0], X) : f(0) = x \right\}.
\]

It can be shown (see [6], Cor. 3.5, Prop. 3.9) that the delay (DE) and the abstract Cauchy problem \((ACP)\) are equivalent in the following sense. Every classical solution of the delay equation (DE) yields a classical solution of the abstract Cauchy problem \((ACP)\) on \(\mathcal{E}\). Furthermore, for every classical solution \(\mathcal{U}\) of \((ACP)\), the function

\[
t \mapsto u(t) := \begin{cases} (\pi_1 \circ \mathcal{U})(t), & \text{if } t \geq 0, \\ f(t), & \text{if } t \in [-1, 0) \end{cases}
\]

is a classical solution of (DE), and \((\pi_2 \circ \mathcal{U})(t) = u_t\) for all \(t \geq 0\), where \(\pi_1\) and \(\pi_2\) denote the canonical projections from \(\mathcal{E}\) onto \(X\) and \(L^1([-1, 0], X)\), respectively.

Due to the equivalence of the delay equation (DE) and the abstract Cauchy problem \((ACP)\), the delay equation is well posed if and only if the operator \((\mathcal{G}, D(\mathcal{G}))\) generates a strongly continuous semigroup on the space \(\mathcal{E}\).

From the application of the bounded perturbation theorem (see e.g. [13], Chapter III., Thm. 1.3) and the discussion in Section 3.3.2. of [6], we directly obtain the following result.

**Proposition 3.1.** Let \((C, D(C))\) be the generator of a strongly continuous semigroup \((V(t))_{t \geq 0}\) on \(X\), and \(\Phi: L^1([-1, 0], X) \rightarrow X\) a bounded operator. Then the operator \((\mathcal{G}, D(\mathcal{G}))\) generates a strongly continuous semigroup on the space \(\mathcal{E}\) and so the delay equation \((ACP)\) is well posed.

In what follows, we apply the results of the previous section, i.e. can investigate the operator splitting procedures for delay equations through the abstract Cauchy problem \((ACP)\) associated to the operator \((\mathcal{G}, D(\mathcal{G}))\) on the Banach space \(\mathcal{E}\).

### 4. Application of operator splittings to delay equations

From now on let us assume the following.

**Assumptions 4.1.** (a) The operator \((C, D(C))\) generates a strongly continuous semigroup \((V(t))_{t \geq 0}\) on \(X\). By rescaling (see e.g. in [13], Chapter II, Lemma 3.10), we can assume without loss of generality that \((V(t))_{t \geq 0}\) is a contraction semigroup, i.e. \(\|V(t)\| \leq 1\) for all \(t \geq 0\).

(b) The delay operator \(\Phi: L^1([-1, 0], X) \rightarrow X\) is bounded.

Since the delay operator \(\Phi\) is bounded, the delay equation (DE) is well posed by **Proposition 3.1**.

In order to apply an operator splitting procedure, let us split the operator in \((ACP)\) as \(\mathcal{G} = \mathcal{A} + \mathcal{B}\), where the sub-operators have the form

\[
\mathcal{A} := \begin{pmatrix} C & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \quad D(\mathcal{A}) := D(\mathcal{G}),
\]

\[
\mathcal{B} := \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix}, \quad D(\mathcal{B}) := \mathcal{E}.
\] (9)

Since \(C\) is a generator and \(\Phi\) is bounded, the operators \(\mathcal{A}\) and \(\mathcal{B}\) generate the strongly continuous semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\), respectively. It is shown in [6] (Section 3.3.2, Thm. 3.25.) that \((T(t))_{t \geq 0}\) is given by

\[
T(t) := \begin{pmatrix} V(t) & 0 \\ V_t & T_0(t) \end{pmatrix},
\] (10)
Under the By (1) and of the sequential and Strang splitting, the split solutions of the delay equation with initial value $v(t)$, we only have to show that the stability condition is fulfilled. Since all norms are equivalent on the Banach space $X$, we can now state our main result.

**Theorem 4.2.** Under the Assumption 4.1, the sequential splitting (13) applied to the delay equation (DE) is convergent.

**Proof.** By Corollary 2.5, we only have to show that the stability condition (4) is fulfilled. Since all norms are equivalent on the Banach space $X$, we choose the maximum norm and apply the very rough estimate

$$
\|S(t/n)T(t/n)\|^n \leq \|S(t/n)\|^n \|T(t/n)\|^n.
$$

It now suffices to compute the norm of the semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$.

$$
\|T(t)\| = \sup_{\|x\|, \|f\| \leq 1} \|T(t)x\| = \sup_{\|x\|, \|f\| \leq 1} \left\| \begin{pmatrix} V(t)x & V(t)x + T_0(t)f \end{pmatrix} \right\| = \sup_{\|x\|, \|f\| \leq 1} \max \{\|V(t)x\|, \|V(t)x + T_0(t)f\|_{L^1}\} \leq \max \{1, 1 + t\} = 1 + t.
$$

$$
\|S(t)\| = \sup_{\|x\|, \|f\| \leq 1} \|S(t)x\| = \sup_{\|x\|, \|f\| \leq 1} \left\| \begin{pmatrix} x + t \Phi f \end{pmatrix} \right\| = \sup_{\|x\|, \|f\| \leq 1} \max \{\|x + t \Phi f\|, \|f\|\} \leq \max \{1 + t\|\Phi\|, 1\} = 1 + t\|\Phi\|.
$$

Here we used that $\|V(t)\|_{L^1} \leq t$, since

$$
\|V(t)\|_{L^1} = \int_{-1}^{1} \|V(t, \sigma)x\|d\sigma = \int_{-1}^{0} \|V(t + \sigma)x\|d\sigma = \int_{-t}^{0} \|V(t + \sigma)x\|d\sigma = \int_{0}^{t} \|V(t)\|\|x\|d\sigma \leq t\|x\|,
$$

by the contractivity of $(V(t))_{t \geq 0}$.

From (16) and (15) we draw the conclusion

$$
\|S(t/n)T(t/n)\|^n \leq \left(1 + \frac{t}{n}\|\Phi\|\right)^n \left(1 + \frac{t}{n}\right)^n \leq ce^{\left(1 + \|\Phi\|\right)t}.
$$
Hence, there exist $M := c$ and $\omega := 1 + \|\Phi\|$ for which the stability condition (4) holds. This proves that the sequential splitting is convergent. \(\square\)

By Corollary 2.7 we immediately obtain the following.

**Corollary 4.3.** Under the Assumption 4.1 the Strang splitting (14) applied to delay equations is convergent, as well.

5. Numerical experiments

In this section we describe our numerical scheme for solving the abstract delay equation (DE) by sequential splitting. We present three examples and their numerical solutions.

5.1. Description of the numerical scheme

Applying the sequential splitting (13), the split solution of the abstract Cauchy problem (ACP) can be determined at the time levels $k\tau$, $k = 1, \ldots, K$ (for some $K \in \mathbb{N}$) as follows:

$$U_{SQ}(k\tau) = M(\tau)^k(x^k),$$

where $\tau \in \mathbb{R}^+$ is fixed and called the splitting time step. For the sake of simplicity, let us assume that $1/\tau \in \mathbb{N}$. By the definition (9) of the sub-operators, the operator $M(\tau)$ has the form

$$M(\tau) := S(\tau)T(\tau) = \begin{pmatrix} V(\tau) + \tau \Phi V_t & \tau \Phi T_0(\tau) \\ V_t & T_0(\tau) \end{pmatrix}.$$

For the initial value $\left(\begin{smallmatrix} x \\ V_t \end{smallmatrix}\right) \in E$, the split solution after the first time step can be written as

$$U_{SQ}(\tau) = \begin{pmatrix} V(\tau)x + \tau \Phi(V_t x + T_0(\tau)f) \\ V_t x + T_0(\tau)f \end{pmatrix}.$$

The split solution after the $k^{th}$ time step can be computed by the iteration

$$U_{SQ}(k\tau) = \begin{pmatrix} x_k \\ f_k \end{pmatrix} = \begin{pmatrix} V(\tau)x_{k-1} + \tau \Phi f_k \\ V_t x_{k-1} + T_0(\tau)f_{k-1} \end{pmatrix},$$

for $k = 1, \ldots, K$, where $x_0 := x$ and $f_0 := f$, where $x$ and $f$ are the initial values in (DE). Since we showed in Theorem 4.2 that the sequential splitting applied to the delay equation is convergent, the values of $x_k$ and $f_k$ are the approximations of the solution $u(t)$ and the history function $u_t$ of (DE), respectively, at the time $t = k\tau$.

In order to derive the form of the split solution, we have to compute the terms in (17).

$$f_k = V_t x_{k-1} + T_0(\tau)f_{k-1} = V_t x_{k-1} + T_0(\tau)(V_t x_{k-2} + T_0(\tau)f_{k-2})$$

$$= \cdots = V_t x_{k-1} + T_0(\tau)V_t x_{k-2} + T_0(\tau)(T_0(\tau)V_t x_{k-3}) + T_0(\tau)^2(T_0(\tau)V_t x_{k-4}) + \cdots$$

$$+ T_0(\tau)^{k-2}(T_0(\tau)V_t x_0) + T_0(\tau)^k f_0$$

$$= V_t x_{k-1} + T_0(\tau)^k f + \sum_{n=0}^{k-2} T_0(\tau)^n(T_0(\tau)V_t x_{k-2-n}).$$

From the definition of the terms $T_0(t)$ and $V_t$ it follows that for $\sigma \in [a, 0]$:

$$(T_0(\tau)^n f)(\sigma) = \begin{cases} f(n\tau + \sigma), & \text{if } \sigma \in [a, -n\tau], \\ 0, & \text{else}, \end{cases}$$

$$(T_0(\tau)V_t x)(\sigma) = \begin{cases} V(t + s + \sigma), & \text{if } \sigma \in [-(t + s), -t], \\ 0, & \text{else}. \end{cases}$$
Hence, if $k\tau < 1$ we can write
\[
T_0(\tau)^n(T_0(\tau)V_{\tau}x_{k-2-n})(\sigma) = \begin{cases} (T_0(\tau)V_{\tau}x_{k-2-n})(n\tau + \sigma), & \sigma \in [-1, -n\tau], \\ 0, & \text{else,} \end{cases}
\]
\[
= \begin{cases} V(2\tau + n\tau + \sigma)x_{k-2-n}, & \sigma \in [-1, -n\tau], n\tau + \sigma \in [-2\tau, -\tau), \\ 0, & \text{else,} \end{cases}
\]
\[
= \begin{cases} V(2\tau + n\tau + \sigma)x_{k-2-n}, & \sigma \in -(n+2)\tau, -(n+1)\tau), \\ 0, & \text{else.} \end{cases}
\]

Therefore, $x_k$ is given by
\[
x_k = V(\tau)x_{k-1} + \tau \Phi \left( \frac{V(\tau + \sigma)x_{k-1} + f(k\tau + \sigma) + \sum_{n=0}^{k-2} V((n+2)\tau + \sigma)x_{k-2-n}}{\sigma \in [-1, 0] \cup \sigma \geq 1 \cup \sigma \in [-1, \max_{n}]} \right), \quad (18)
\]
where $\max_1 := \max\{-1, -k\tau\}$, $\max_2 := \max\{-(n+2)\tau, -1\}$, $\max_3 := \max\{-(n+1)\tau, -1\}$, and
\[
k^* := \begin{cases} k, & \text{if } k \leq 1/\tau \\ 1/\tau, & \text{if } k > 1/\tau, \end{cases}
\]
for $k = 1, \ldots, K$. The underbrackets in (18) denote that the corresponding term vanishes if $\sigma$ does not belong to the labelled interval.

5.2. Examples

In order to illustrate numerically the convergence of the splitting procedures applied to delay equations, let us consider the following three examples.

Example 5.1 (Bounded $\Phi$ with Exact Solution). Assume $X := \mathbb{R}$, $B := b \in \mathbb{R}$ and
\[
\begin{align*}
\dot{u}(t) &= bu(t) + \int_{-1}^{-\varepsilon} \mu(\sigma)u(t + \sigma)\,d\sigma, \quad t \geq 0, \\
u(0) &= x \in \mathbb{R}, \\
u_0 &= f \in L^1([1-1, 0], \mathbb{R}),
\end{align*} \quad (19)
\]
for some $\varepsilon \in (-1, 0)$, $\mu \in L^\infty([-1, 0])$. In this case the delay operator $\Phi$ is defined by
\[
\Phi g := \int_{-1}^{-\varepsilon} \mu(\sigma)g(\sigma)\,d\sigma \quad (20)
\]
for all $g \in L^1([-1, 0], \mathbb{R})$, and $\Phi$ is bounded. Let us choose the initial values as $x := 1$ and $f(\sigma) := 1 - \sigma$ for $\sigma \in [-1, 0)$, $\mu(\sigma) = 1$ for $\sigma \in [-1, -\varepsilon)$, and $b := -1$. As we will see, the exact solution of the delay equation can be computed by the variation of constants formula in this case.

Example 5.2 (Bounded $\Phi$ without Variation of Constants Formula). Let us consider the same setting as in Example 5.1 but with $\varepsilon := 0$ in the definition (20) of the delay operator. The operator $\Phi$ remains bounded in this case, however we will see that the exact solution cannot be computed using the variation of constants formula.

Example 5.3 (Unbounded $\Phi$ with Exact Solution). Let us consider $X := \mathbb{R}$, $B := b \in \mathbb{R}$ and
\[
\begin{align*}
\dot{u}(t) &= bu(t) + u(t - 1), \quad t \geq 0, \\
u(0) &= x \in \mathbb{R}, \\
u_0 &= f \in L^1([-1, 0], \mathbb{R}).
\end{align*} \quad (21)
\]
The delay operator in this case is
\[
\Phi g := g(-1) \quad (22)
\]
for all \( g \in W^{1,1}([-1, 0], \mathbb{R}) \). Now \( \Phi \) is unbounded on \( L^1([-1, 0], \mathbb{R}) \). Let us choose the initial values again as \( x := 1 \) and \( f(\sigma) := 1 - \sigma \) for \( \sigma \in [-1, 0] \), and \( b := -1 \). We will show that the exact solution can be computed in this case, as well.

**Remark 5.4.** In Section 3.3.2, Theorem 3.26 of [6] it is shown that the abstract Cauchy problem \((ACP)\) is well posed for a larger class of delay operators than that is stated in Proposition 3.1. This result and the numerical experiments suggest that we should formulate a result on the convergence also in the case of the delay operator defined in (22). This is the subject of our forthcoming work.

A direct calculation shows the validity of the following result.

**Proposition 5.5.** The exact solution of the delay equation \((DE)\) is given by the “variation of constants” formula

\[
 u(t) = V(t)x + \int_0^t V(t-s) \Phi u_s ds,
\]

where \( x \) is the initial condition in \((DE)\).

Due to Proposition 5.5, the exact solutions are given by:

**Example 5.1:**

\[
 u(t) = V(t)x + \int_0^t V(t-s) \int_{-1}^{-\epsilon} \mu(s)u(s + \sigma)d\sigma ds.
\]

**Example 5.2:**

\[
 u(t) = V(t)x + \int_0^t V(t-s) \int_{-1}^{0} \mu(s)u(s + \sigma)d\sigma ds.
\]

**Example 5.3:**

\[
 u(t) = V(t)x + \int_0^t V(t-s)u(s-1)ds.
\]

Since formula (25) leads to an implicit form, we cannot compute the exact solution explicitly in the case of Example 5.2. In the case of the other two examples, the exact solutions can be determined piecewise in the following way:

**Example 5.1:** \( t \in [r\epsilon, (r + 1)\epsilon) \), \( r = 1, \ldots, \frac{K\tau}{\epsilon} \), \( (27) \)

\[
 u^{(r+1)}(t) = V(t-r)u^{(r)}(r\epsilon) + \int_{r\epsilon}^t V(t-s) \int_{-1}^{-\epsilon} \mu(s)u^{(r)}(s + \sigma)d\sigma ds.
\]

**Example 5.3:** \( t \in [r, r+1) \), \( r = 1, \ldots, K\tau \), \( (28) \)

\[
 u^{(r+1)}(t) = V(t-r)u^{(r)}(r) + \int_r^t V(t-s)u^{(r)}(s-1)ds.
\]

Since formulae (27) and (28) can be numerically computed, we can compare the split solutions and the exact solutions in our numerical experiments.

5.3. **Numerical results on convergence**

The convergence of the operator splittings applied to delay equations can be demonstrated in numerical experiments by using different values of the splitting time step \( \tau \). In case of decreasing values of \( \tau \), the split solutions should approximate the exact solution better.

In Figs. 1–3 the above stated behaviour can be seen for the sequential splitting for Examples 5.1–5.3, respectively: as the values of \( \tau \) are decreasing, the corresponding split solutions converge to the exact solution of the problem.
Fig. 1. Split solutions of Example 5.1 in the case of the sequential splitting, for $\varepsilon = 0.1$.

Fig. 2. Split solutions of Example 5.2 in the case of the sequential splitting.

Fig. 3. Split solutions of Example 5.3 in the case of the sequential splitting.

Fig. 4 and 5 show that the relative error $E_{\text{rel}}(k\tau)$, which is defined by

$$E_{\text{rel}}(k\tau) := \frac{\|u_{\text{spl}}(k\tau) - u(k\tau)\|}{\|u(k\tau)\|},$$

converges to zero as $\tau \to 0$ for Examples 5.1 and 5.3, respectively. Due to the above definition of the relative error, this behaviour also shows the convergence.

We note that the numerical method presented in Section 5.1 can also be applied when the operator $B$ is a partial differential operator, for instance. In that case, however, the semigroup $(V(t))_{t \geq 0}$ has to be approximated by a numerical scheme, as well. In the proof of the convergence we thus also have to take into account the effect of this numerical scheme. This is another subject of our future work.
6. Summary

Operator splittings are well-known methods for solving evolution equations, but they have not yet been applied to differential equations containing a delay term. We did this for abstract delay equations and showed that the application of the sequential splitting and the Strang splitting results in convergent time-discretization methods for solving delay equations numerically.

At the end of the paper we tested our results numerically for three test delay equations. Among our future plans there is the generalization of our result to unbounded delay operators and approximated semigroups.

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Appendix

A.1. Basic notions of operator semigroup theory

Since we apply some notions and results from operator semigroup theory in our approach, we collect them here. For a detailed introduction we refer the reader to [6,13–16].

From now on let $X$ be a Banach space with the norm $\| \cdot \|$. Let us denote by $\mathcal{L}(X)$ the space of all bounded linear operators on $X$ endowed with the operator norm, which we denote again by $\| \cdot \|$. The identity operator on $X$ is denoted by $I$.

**Definition A.1.** A family $(T(t))_{t \geq 0}$ of bounded linear operators on a Banach space $X$ is called strongly continuous semigroup (or $C_0$-semigroup) if:
(a) $T(0) = I$,
(b) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$,
(c) The orbit maps $\mathbb{R}^+ \ni t \to T(t)x \in X$ are continuous for every $x \in X$.

**Proposition A.2.** For every strongly continuous semigroup $(T(t))_{t \geq 0}$ there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$ 

**Definition A.3.** Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on the Banach space $X$. The generator $(A, D(A))$ of $T(t)$ is the operator

$$Ax := \lim_{h \to 0} \frac{T(h)x - x}{h}$$

defined on the domain

$$D(A) := \left\{ x \in X : \lim_{h \to 0} \frac{T(h)x - x}{h} \text{ exists} \right\}.$$

**Theorem A.4.** The generator $A$ of a strongly continuous semigroup is a closed and densely defined linear operator which defines the semigroup uniquely.

**Definition A.5.** Let $X$ be a Banach space, $A : D(A) \subseteq X \to X$ a linear operator, and $x \in X$. The initial value problem

\[
\begin{align*}
\dot{u}(t) &= Au(t) \quad t \geq 0, \\
\quad u(0) &= x \tag{29}
\end{align*}
\]

is called *abstract Cauchy problem* associated to $(A, D(A))$ with the initial value $x \in X$.

The following two different concepts of a solution for (29) are of major importance.

**Definition A.6.** (i) A function $u : \mathbb{R}^+ \to X$ is called a *classical solution* of (29) if $u$ is continuously differentiable, $u(t) \in D(A)$ for all $t \geq 0$, and (29) holds.

(ii) A continuous function $u : \mathbb{R}^+ \to X$ is called a *mild solution* of (29) if $\int_0^t u(s)ds \in D(A)$ for all $t \geq 0$ and

$$u(t) = x + A\int_0^t u(s)ds \quad \text{for all } t \geq 0. \tag{30}$$

If $(A, D(A))$ is a generator we can easily characterize classical and mild solutions.

**Proposition A.7.** Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$. Then the following holds.

(i) For every $x \in D(A)$ the function

$$u : t \mapsto u(t) := T(t)x$$

is the unique classical solution of (29) with the initial value $x$.

(ii) For every $x \in X$ the function

$$u : t \mapsto u(t) := T(t)x$$

is the unique mild solution of (29) with the initial value $x$.

**Definition A.8.** The abstract Cauchy problem (29) is said to be well posed if

- the domain $D(A)$ is dense in $X$,
- for every $x \in D(A)$ there exists a unique classical solution $u$ of (29),
- for every sequence \((x_n)_{n \in \mathbb{N}}\) in \(D(A)\) one has

\[
\lim_{n \to \infty} x_n = 0 \Rightarrow \lim_{n \to \infty} u_{x_n}(t) = 0
\]

uniformly for all \(t\) in compact intervals.

**Theorem A.9.** For a closed operator \(A : D(A) \subseteq X \to X\), the associated abstract Cauchy problem (29) is well posed if and only if \((A, D(A))\) generates a strongly continuous semigroup on \(X\).

**Definition A.10.** Throughout the paper we use the function spaces defined as follows.

\[
L^1([-1, 0], X) := \{ f : [-1, 0] \to X : \int_{-1}^{0} \| f(\sigma) \| d\sigma < \infty \},
\]

\[
W^{1,1}([-1, 0], X) := \{ f : [-1, 0] \to X : f \text{ is absolutely continuous and } f' \in L^1([-1, 0], X) \}.
\]

**References**


