Suzuki groups, one-factorizations and Lüneburg planes

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Received 10 May 1994; revised 16 May 1995

Abstract

In this paper we give a method for studying a plane of order $q^2$ admitting $Sz(q)$ as a collineation group fixing an oval and acting 2-transitively on its points; we prove in particular that for $q = 8$ the dual Lüneburg plane is the unique plane with this property. We also determine all one-factorizations of the complete graph on $q^2$ vertices admitting the one-point-stabilizer of $Sz(q)$ as an automorphism group and having $q - 1$ prescribed one-factors.

1. Introduction

In 1965 Lüneburg [9] constructed a translation plane of order $q^2$ admitting the Suzuki group $Sz(q)$ as a collineation group. In the same paper he also proved that the action of $G \cong Sz(q)$ as a collineation group of an arbitrary projective plane of order $q^2$ is of three possible types (see Theorem 28.11 in [10]):

(a) $G$ fixes an antiflag $(P, l)$ and acts 2-transitively on the points of $l$ and on the lines through $P$;
(b) $G$ fixes an oval $C$ and acts 2-transitively on its points;
(c) $G$ fixes a line-oval $C^*$ and acts 2-transitively on its lines

(the 2-transitive action is the natural one everywhere; the sizes of point and line orbits in the whole plane are also specified).

The fact that (a) occurs in the Lüneburg plane was pointed out by Lüneburg himself [9]. Possibility (c) for the Lüneburg plane was first obtained as a consequence of results of Pollatsek [11]; later on Kantor [6] and Korchmáros [7] showed that possibility (b) occurs in the dual Lüneburg plane, thus proving possibility (c) again for the Lüneburg plane by duality.

* Work done within the activity of G.N.S.A.G.A. of C.N.R. with the support of the Italian Ministry for Research and Technology
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SSDI 0012-365X(95)00273-1
It seems quite natural to ask whether the Lüneburg plane of order $q^2$ and its dual plane are the unique planes on which $Sz(q)$ can act as a collineation group. The answer is positive if further assumptions are made on the plane. Liebler [8] showed that a translation plane of order $q^2$ admitting $Sz(q)$ as a collineation group must be the Lüneburg plane, see also Büttner [1, 2].

The question we want to address in this paper is whether occurrence (b) characterizes the dual Lüneburg plane of order $q^2$ (equivalently: whether occurrence (c) characterizes the Lüneburg plane of order $q^2$).

The approach we have developed is based on the possibility of describing a projective plane $\pi$ of even order possessing an oval $\mathcal{O}$ by means of the one-factorizations of certain complete graphs arising from the lines of $\pi$ (see Section 2): we were able to determine all one-factors which may occur in such one-factorizations, obtaining in particular all one-factorizations of the complete graph on $q^2$ vertices admitting the one-point-stabilizer of $Sz(q)$ as an automorphism group and having $q - 1$ prescribed one-factors, namely those arising from the involutions in the group.

Although we do not settle in general when two such one-factorizations may arise from distinct lines in the same plane, the method seems adequate for computer calculations, which we have actually performed in the smallest case $q = 8$: the dual Lüneburg plane is indeed the only plane of order 64 for which possibility (b) occurs.

2. Ovals and one-factorizations

Assume that $\pi$ is a projective plane of even order $n$ containing an oval $\mathcal{O}$ and let $\mathcal{O}'$ denote the hyperoval arising from $\mathcal{O}$. Each line $l$ of $\pi$ leads to a one-factorization of the complete graph whose vertices are the points of $\mathcal{O}'$ not lying on $l$. In fact, the lines through a point $P$ outside $\mathcal{O}'$ partition $\mathcal{O}'$ into 2-subsets. Now, if $l$ is an external line of $\mathcal{O}$, the set of such partitions, as $P$ varies on $l$, is a one-factorization of the complete graph on $\mathcal{O}'$. If $l$ is a tangent or a secant of $\mathcal{O}$ (i.e. $l$ meets $\mathcal{O}'$) the partition induced by $P \in l \setminus \mathcal{O}'$ contains $\mathcal{O}' \cap l$ as one part; the set of such partitions as $P$ varies in $l \setminus \mathcal{O}'$ yields thus a one-factorization of the complete graph on $\mathcal{O}' \setminus l$. Clearly, any collineation of $\pi$ fixing $\mathcal{O}$ and $l$ induces an automorphism of the associated one-factorization; in particular an involutory elation fixing $\mathcal{O}$ and $l$ yields a one-factor symmetry in the sense of [5] (although the one-factorization here need not be perfect).

Note that there exists a one-to-one correspondence between one-factorizations of the complete graph on $2m$ vertices and sharply transitive permutation sets of degree $2m$ consisting of the identity and $2m - 1$ involutory permutations, see [4].

Incidentally, if we assume $S$ to be a permutation set on $2m$ elements consisting of the identity and of $2m - 1$ involutions, then $S$ will be sharply transitive if and only if each involution in $S$ as well as the product of any two involutions in $S$ is fixed-point-free.

We now want to consider the above construction in a special situation. Let $\pi$ be a finite projective plane of order $q^2$ and let $G$ be a collineation group of $\pi$ which is
isomorphic to $Sz(q)$. Assume $G$ fixes an oval $\mathcal{O}$ and acts 2-transitively on its points (case (b) of Theorem 28.11 in [10]).

As we have seen, if $l_X$ is the tangent line of $\mathcal{O}$ at the point $X$, then $l_X$ defines a one-factorization of the complete graph on $\mathcal{O} \setminus \{X\}$ as follows. Each point $P \in l_X$, $P \neq X$, defines a one-factor by considering the lines through $P$ which are secant to $\mathcal{O}$.

Equivalently, we may consider the involutory permutation $j_P$ on $\mathcal{O} \setminus \{X\}$ mapping each point $Q$ to the further point of intersection of the line $PQ$ with $\mathcal{O}$; the permutation set $J_X = \{j_P; P \in l_X, P \neq X\} \cup \{\text{id}\}$ is sharply transitive on $\mathcal{O} \setminus \{X\}$. Whenever needed, we extend the action of $J_X$ to the whole of $\mathcal{O}$ by agreeing that $J_X$ fixes $X$. Since $G_X$ fixes $X$ and $l_X$ we see that $J_X$ is invariant under conjugation by $G_X$.

Consider the characteristic subgroup $V_X$ of $G_X$ consisting of the identity and of the $q - 1$ involutions in the unique Sylow 2-subgroup of $G_X$. These involutions, when considered as collineations of $\pi$, are necessarily elations, see [3], and their centers lie on $l_X$, which means $V_X \subseteq J_X$. Note that $V_X$ commutes with $J_X$ elementwise.

The action of $G_X$ on $l_X$ yields three point-orbits, namely $\{X\}$, an orbit $A$ of $q - 1$ points which are precisely the centers of the elations in $V_X$ and an orbit $A$ consisting of the remaining $q^2 - q$ points, see [10, Section 28]. It follows that for each $P \in A$ the involution $j_P$ has $q^2 - q$ conjugates under the action of $G_X$ and they all lie in $J_X$; the centralizer of $j_P$ in $G_X$ is thus precisely $V_X$ and we have $J_X = V_X \cup \{k j_P k^{-1}; k \in G_X\}$.

We shall see in the next section how all these permutation sets $J_X$ can be described.

Let $Y, l_Y$ and $J_Y$ be a point of $\mathcal{O}$ different from $X$, the tangent to $\mathcal{O}$ at $Y$ and the corresponding permutation set on $\mathcal{O}$ respectively. If $P, Q$ are points with $P \in l_X$, $P \neq X, Q \in l_Y, Q \neq Y$ and $j_P, j_Q$ denote the corresponding involutory permutations on $\mathcal{O}$, then since the line $PQ$ meets $\mathcal{O}$ in at most two points, we have $|\text{Fix}(j_P j_Q)| \leq 2$. In particular, in our situation, there exists an involution $k \in G$ exchanging $X$ and $Y$, yielding $J_Y = kJ_X k^{-1}$.

We may now reverse our point of view and assume that a candidate for $J_X$ is available, namely a permutation set on $\mathcal{O}$ with all the properties we have just described, see Section 3 for details. The previous observation may be used to test whether the given permutation set can actually arise from the tangent to an oval: if a conjugate $J'$ of $J_X$ by an involution in $G$ (not fixing $X$) is such that $|\text{Fix}(jj')| \geq 3$ holds for at least one pair of involutions $j \in J_X$ and $j' \in J'$ then the candidate for $J_X$ must be rejected.

Once our candidate for $J_X$ has positively passed this test, we have in principle reconstructed one piece of our plane, namely all points (the elements of $\mathcal{O}$ and all permutations in some conjugate of $J_X$, where the identity plays the role of the nucleus of the oval), all the tangent lines of $\mathcal{O}$ (the conjugates of $J_X$ under $G$ with their respective fixed point) and all the secant lines of $\mathcal{O}$ (to each pair of distinct points of $\mathcal{O}$ add all permutations in some conjugate of $J_X$ which exchange these points).

In order to get the whole of the plane, we must be able to define external lines. Assume that the plane $\pi$, the oval $\mathcal{O}$ and an external line $l$ are given. If $P, Q$ are distinct points on $l$ and $j_P, j_Q$ denote the corresponding involutory permutations on $\mathcal{O}$ then the product $j_P j_Q$ is fixed-point-free on $\mathcal{O}$. By Theorem 28.11 in [10] there are two orbits
$\mathcal{E}_1$, $\mathcal{E}_2$ of external lines under the action of $G$, the lines in $\mathcal{E}_1$ resp. in $\mathcal{E}_2$ admitting a dihedral group of order $2(2^{2e+1} + 2^{e+1} + 1)$ resp. $2(2^{2e+1} - 2^{e+1} + 1)$ in their stabilizers. Each external line of $\mathcal{E}_1$ resp. $\mathcal{E}_2$ contains $v_1 = 2^{2e+1} + 2^{e+1} + 1$ resp. $v_2 = 2^{2e+1} - 2^{e+1} + 1$ points which are centers of elations in $G$ and are thus obtained as conjugates in $G$ of an involution in $V_x$; all other points of the line are obtained as suitable conjugates in $G$ of an arbitrary involution $j \in J_x \setminus V_x$.

We reverse our point of view one more time and assume that we have selected $v_1$ resp. $v_2$ involutions in $G$ which should correspond to elation centers on an external line of $\mathcal{E}_1$ resp. $\mathcal{E}_2$: these involutions must be found in a dihedral subgroup $G_1$ resp. $G_2$ of $G$ of order $2v_1$ resp. $2v_2$.

We fix an involutions $j \in J_x \setminus V_x$ and test whether the products of the $v_1$ resp. $v_2$ previous involutions with a given $G$-conjugate $j'$ of $j$ are fixed-point-free on $\mathcal{E}$. If $j'$ passes the test, then further conjugates passing the test are the $G_1$-conjugates resp. $G_2$-conjugates of $j'$. The set of points of our external line which are not elation centers in $G$ must be partitioned into $G_1$-orbits resp. $G_2$-orbits of the type just described: once two such orbits are selected, we must further test whether the product of a given permutation in one orbit with an arbitrary permutation in the other orbit is fixed-point-free.

3. One-factorizations arising from the Suzuki groups

Let $e$ be a positive integer and set $r = 2e + 1$, $q = 2^e$, $t = 2^{e+1}$, $s = 2^e$, $F = GF(q)$. The mapping $F \to F$, $x \mapsto x^t$ is an automorphism of $F$ with inverse given by $x \mapsto x^s$ and we have $x^{t^2} = x^2$ for each $x \in F$. Set $F^2 = F \times F$ and $\Omega = F^2 \cup \{ \infty \}$.

For $a, b \in F$ define

$$u_{a,b} : F^2 \to F^2, \quad (x,y) \mapsto (a + x, b + y + ax^t);$$

we have

$$u_{a,b}^{-1} : F^2 \to F^2, \quad (x,y) \mapsto (a + x, b + a^{1+t} + y + ax^t).$$

Setting $U = \{ u_{a,b}; a, b \in F \}$ we see that $U$ is a sharply transitive permutation group on $F^2$. The center of $U$ is the elementary abelian subgroup $V$ of order $q$ consisting of the identity and of the involutions in $U$, i.e. $V = \{ u_{0,b}; b \in F \}$. Each non-central element of $U$ has order 4.

For $c \in F^*$ define

$$h_c : F^2 \to F^2, \quad (x,y) \mapsto (c^{-1}x, c^{-(1+t)}y);$$

we have

$$h_c^{-1} : F^2 \to F^2, \quad (x,y) \mapsto (cx, c^{1+t}y).$$
Setting \( H = \{ h_c; c \in F^* \} \) we see that \( H \) is a group of order \( q - 1 \) fixing \((0,0)\) and normalizing \( U \). The group \( HU \) is thus a semidirect product and has order \( q^2(q - 1) \).

Define

\[ w: F^2 \to F^2, \]

\[ (x, y) \mapsto ((x^1 + y)/(x^2 + xy + y^1), y/(x^2 + xy + y^1)^1). \]

If we extend the action of \( HU \) to the whole of \( \Omega \) by agreeing that \( HU \) fixes \( \infty \), then the group \( G = \langle HU, w \rangle \) is isomorphic to the Suzuki group \( Sz(q) \) and acts on \( \Omega \) in its natural 2-transitive permutation representation (see [7]). We have in particular \( G_\infty = HU \).

The following property of the Suzuki groups is probably well-known: we include a full proof since we were unable to find a convenient reference.

**Proposition 3.1.** Define \( m: F^2 \to F^2, (x, y) \mapsto (x^2, y^2) \) and set \( M = \langle m \rangle \). The normalizer of \( HU \) in \( Sym(F^2) \) is the group \( L \) of order \( rq^2(q - 1) \) generated by \( HU \) and \( M \).

**Proof.** It is easily seen that \( L \) is a group of order \( rq^2(q - 1) \) normalizing \( HU \). Let us show that if \( \lambda \in Sym(F^2) \) normalizes \( HU \) then it necessarily lies in \( L \).

Write \( \lambda \) in the form \( \lambda(x,y) = (\eta(x,y), \phi(x,y)) \) where \( \eta, \phi: F^2 \to F \) are suitable mappings. Since \( \lambda \) normalizes \( HU \) it also normalizes its characteristic subgroup \( V \). To each \( b \in F \) there corresponds \( c = \gamma(b) \in F \) with \( u_0, b \lambda = \lambda u_0, c \), yielding \( (\eta(x,y), \phi(x,y) + b) = (\eta(x,y + c), \phi(x,y + c)) \) for all \((x,y) \in F^2 \). The mapping \( b \mapsto \gamma(b) \) is a permutation of \( F \) with \( \gamma(0) = 0 \), hence \( c \) covers all elements of \( F \) as \( b \) varies in \( F \); setting \( y = 0 \) we obtain \( \eta(x,0) = \eta(x,c) \) for all \( c \in F \). We now define \( \phi(x) = \eta(x,0) \) and get \( \eta(x,y) = \phi(x) \) for all \((x,y) \in F^2 \). Since \( \lambda \) is surjective, so is \( \phi \) and therefore \( \phi \) is a permutation on \( F \). We also have \( \phi(x,0) + b = \phi(x,c) \) for \( x, b \in F \); writing \( b = \beta(c) \) with \( \beta = \gamma^{-1} \) we have in particular \( \beta(0) = 0 \) and defining \( \sigma(x) = \phi(x,0) \) we obtain \( \phi(x,y) = \sigma(x) + \beta(y) \) for all \((x,y) \in F^2 \).

The action of \( HU \) on \( F^2 \) induces an action on the first coordinate, which is given by all mappings of the form \( x \mapsto cx + d \), with \( c, d \in F, c \neq 0 \). This is precisely the action of the group \( AGL(1,F) \) on \( F \) and we know that the normalizer of \( AGL(1,F) \) in \( Sym(F) \) is precisely the group \( AGL(1,F) \), i.e. the group of all mappings of the form \( F \to F, x \mapsto cx^{2^i} + d \), with \( c, d \in F, c \neq 0, 0 \leq i \leq r - 1 \). The fact that \( \lambda \) normalizes \( HU \) implies in particular that \( \phi \) normalizes \( AGL(1,F) \), i.e. \( \phi \in AGL(1,F) \). Now every permutation in \( AGL(1,F) \) can already be found in the first coordinate of some permutation in \( L \). We may therefore assume, up to multiplication by a suitable permutation in \( L \) (which does not alter the property that \( \lambda \) normalizes \( HU \)), that the action of \( \lambda \) on the first coordinate is trivial, i.e. \( \phi(x) = x \) for \( x \in F \). We have \( \lambda(0,0) = (0,b) \) and here again we can replace \( \lambda \) by \( \lambda u_{0,b} \) and therefore assume that \( \lambda(0,0) = (0,0) \) holds, yielding \( \sigma(0) + \beta(0) = 0 \), whence \( \sigma(0) = \beta(0) = 0 \).


The characteristic subgroup $U$ is also normalized by $\lambda$; for $(a, b) \in F^2$ there exists $(c, d) \in F^2$ with $u_{a,b}\lambda = \lambda u_{c,a}$; comparing coordinates we get $a = c$ and

\[(i) \quad \sigma(x) + \beta(y) + b + ax^t = \sigma(x + a) + \beta(y + d + ax^t)\]

for $x, y, a, b \in F$, where $d = \delta(a, b)$ for a suitable function $\delta : F^2 \mapsto F$; setting $x = y = 0$, we get $b = \sigma(a) + \beta(d)$, i.e. $d = \sigma(a, b) = \beta^{-1}(b + \sigma(a))$. Substituting back into (i) we have \[\sigma(x) + \beta(y) + \sigma(a) + \beta(d) + ax^t = \sigma(x + a) + \beta(y + d + ax^t);\] setting $y = 0$:

\[(ii) \quad \sigma(x) + \sigma(a) + \beta(d) + ax^t + \sigma(a, x) + \beta(d) = \beta(d + ax^t);\]

back into (i): $\beta(y + d + ax^t) = \beta(y) + \beta(d + ax^t)$. Since $d + ax^t$ takes all values in $F$ as $a, x, b$ vary in $F$, we may set $z = d + ax^t$ and obtain $\beta(y + z) = \beta(y) + \beta(z)$ for all $y, z \in F$. We go back to (ii) one more time and get $\sigma(x) + \sigma(a) + ax^t = \sigma(x + a) + \beta(ax^t)$ for all $x, a \in F$. In particular, for $a = x$ we have $\beta(x^{t+1}) = x^{t+1}$. Since $F \mapsto F$, $x \mapsto x^{t+1}$ is a bijective mapping [10, Lemma 21.1 p. 104], we have shown that $\beta$ is the identity on $F$, i.e. $\beta(y) = y$ holds for $y \in F$ and consequently also $\sigma(x) + \sigma(a) = \sigma(x + a)$ for all $a, x \in F$.

So far we have reached for $\lambda$ the form $\lambda(x, y) = (x, \sigma(x) + y)$ where $\sigma$ is an additive function with $\sigma(0) = 0$. As $\lambda$ fixes $(0, 0)$ we see that $\lambda$ also normalizes the stabilizer of $(0, 0)$ in $HU$, namely $H$. Given $a \in F^*$ there therefore exists $b \in F^*$ with $\lambda h_a = h_b \lambda$. Comparison of coordinates yields $a = b, \sigma(a^{-1}x) + a^{-(1+t)}y = a^{-(1+t)}(\sigma(x) + y)$ for $x, y \in F$. Setting $y = 0$ we have $\sigma(a^{-1}x) = a^{-(1+t)}\sigma(x)$ and now setting $x = 1$ we obtain $\sigma(a^{-1}) = a^{-(1+t)}\sigma(1)$, whence also $\sigma(x) = x^{1+t}\sigma(1)$ for all $x \in F$. If $\sigma(1) \neq 0$ then from $\sigma(x + y) = \sigma(x) + \sigma(y)$ the relation $(x + y)^{1+t} = x^{1+t} + y^{1+t}$ would follow for all $x, y \in F$, a contradiction. Hence, $\sigma(1) = 0$ and $\sigma$ vanishes identically on $F$.

We have proved that $\lambda$ is the identity. $\Box$

With the geometric setting of Section 2 in mind, we want to study one-factorizations of the complete graph on $q^2$ vertices which have $q - 1$ prescribed one-factors and admit the one-point-stabilizer of $Sz(q)$ as an automorphism group. More precisely, in terms of permutation sets, we want to consider sharply transitive permutation sets $J$ on $F^2$ containing $V$ which are invariant under conjugation by $HU$ and consist of involutions and the identity. Furthermore, we want $V$ to be centralizer in $HU$ of one (and thus of any) involution $j \in J \setminus V$, so that $J \setminus V$ can be obtained as the set of all conjugates of $j$ under $HU$. Since $J$ is sharply transitive on $F^2$ there is precisely one involution in $J \setminus V$ mapping $(0, 0)$ to $(1, 1)$ and we may take $j$ to be this involution.

**Proposition 3.2.** Let $j \in \text{Sym}(F^2)$ be an involutory permutation mapping $(0, 0)$ to $(1, 1)$. We have that $j$ commutes with $V$ elementwise if and only if there exist an involutory permutation $g \in \text{Sym}(F)$ and a mapping $f : F \mapsto F$ with $fg = f, g(0) = 1, f(0) = f(1) = 1$ such that $j(x, y) = (g(x), f(x) + y)$ holds for all $(x, y) \in F^2$. In this case, the permutation $jv$ is fixed-point-free on $F^2$ for each $v \in V$ if and only if $g$ is fixed-point-free on $F$. 
Proof. Write \( j \) in the form \( j(x, y) = (\eta(x, y), \varphi(x, y)) \) where \( \eta, \varphi : F^2 \rightarrow F \) are suitable mappings. The condition that \( j \) commutes with \( u_{0,b} \) amounts to \( \eta(x, y + b) = \eta(x, y) \), \( \varphi(x, y + b) = \varphi(x, y) + b \) for all \( x, y, b \in F \); setting \( y = 0 \) we have in particular \( \eta(x, b) = \eta(x, 0) \) and \( \varphi(x, b) = \varphi(x, 0) + b \) for all \( x, b \in F \).

If \( j \) is given, then \( g \) and \( f \) are obtained by setting \( g(x) = \eta(x, 0) \) and \( f(x) = \varphi(x, 0) \) for all \( x \in F \), respectively; if, conversely, \( g \) and \( f \) are given then \( j \) is obtained by setting \( j(x, y) = (g(x), f(x) + y) \) for all \( (x, y) \in F^2 \).

Suppose \( j \) is such that \( j u_{0,b} \) is fixed-point-free on \( F^2 \) for each \( b \in F \) and assume \( g(x) = x \); setting \( b = f(x) \) we have then \( u_{0,b}(x, y) = (x, b + y) = (g(x), f(x) + y) = j(x, y) \), a contradiction.

If, conversely, \( g \) is fixed-point-free on \( F \), then \( (x, b + y) = (g(x), f(x) + y) \) is false for all \( (x, y) \in F^2 \), i.e. \( u_{0,b}(x, y) = j(x, y) \) is false for all \( (x, y) \in F^2 \). \( \Box \)

Suppose \( j \) satisfies all the assumptions of Proposition 3.2 and let \( g, f \) be the corresponding mappings. For \( a, b, c \in F \), \( c \neq 0 \), and \( (x, y) \in F^2 \) we have

\[
\begin{align*}
    h_c u_{a,b} j u_{a,b} h_c^{-1}(x, y) &= ((g(a + cx) + a)c^{-1} + f(a + cx) + ag(a + cx) + ac'x' + a^{1+t} + c^{1+t}y)c^{-(1+t)}.
\end{align*}
\]

Proposition 3.3. Let \( j \) be a fixed-point-free involutory permutation on \( F^2 \) mapping \((0, 0)\) to \((1, 1)\) and commuting with \( V \) elementwise; let \( g, f \) be the corresponding mappings and assume \( j u_j \) is fixed-point-free on \( F^2 \) for each \( v \in V \). If for each \( k \in H \setminus V \) the permutation \( k j k^{-1} j \) is fixed-point-free on \( F^2 \), then the mapping \( p : F \rightarrow F \) defined by \( p(x) = (f(x) + x(g(x) + x)')/(g(x) + x)^{1+t} \) is a permutation with \( p(0) = 1 \), \( p(1) = 0 \) and \( p(g(x)) = p(x) + 1 \) for each \( x \in F \).

Proof. It follows from the properties of \( g \) and \( f \) that \( p(0) = 1 \), \( p(1) = 0 \) and \( p(g(x)) = p(x) + 1 \) hold for each \( x \in F \). Assume \( x, y \) are distinct elements of \( F \) with \( p(x) = p(y) \). Set \( c = (g(y) + y)/(g(x) + x) \), \( a = y + cx \). We have \( (a, c) \neq (0, 1) \) and obtain

\[
\begin{align*}
    \{ g(y) + y = c(g(x) + x), \\
    f(y) + y(g(y) + y) = c^{1+t}(f(x) + x(g(x) + x)'); \\
    g(a + cx) + a + cx = cg(x) + cx, \\
    f(y) + y(g(y) + y)' = c^{1+t}f(x) + cx(g(y) + y)'; \\
    g(a + cx) + a = cg(x), \\
    f(y) + (y + cx)(g(y) + y)' = c^{1+t}f(x); \\
    g(a + cx) + a = cg(x), \\
    f(a + cx) + ag(a + cx)' + a^{1+t} + ac'x' + a^{1+t}y = c^{1+t}f(x) + c^{1+t}y; \\
    (g(a + cx) + a)c^{-1} = g(x), \\
    (f(a + cx) + ag(a + cx) + ac'x' + a^{1+t} + c^{1+t}y)c^{-(1+t)} = f(x) + y;
\end{align*}
\]
thus \( kjk^{-1}(x, y) = j(x, y) \) with \( k = h_{c,a,b} \) for an arbitrary \( b \in F \); since \((a, c) \neq (0, 1)\) we have \( k \in HU \setminus V \), a contradiction.

Consequently, \( p \) is injective and thus a permutation on \( F \). \( \square \)

**Proposition 3.4.** Assume \( p \in \text{Sym}(F) \) satisfies \( p(0) = 1 \), \( p(1) = 0 \). Set \( h: F \leftrightarrow F \), \( x \mapsto x + 1 \) and \( g = p^{-1}h p \). We have that \( g \) is a fixed-point-free involutory permutation on \( F \) with \( g(0) = 1 \) and the mapping \( f: F \leftrightarrow F \), \( x \mapsto x(g(x) + x)^t + p(x)(g(x) + x)^{1+t} \) satisfies \( fg = f \), \( f(0) = f(1) = 1 \). The mapping \( j: F^2 \leftrightarrow F^2 \), \((x, y) \mapsto (g(x), f(x) + y)\) is a fixed-point-free involutory permutation commuting with \( V \) elementwise with \( j(0,0) = (1,1) \) and for each \( k \in HU \setminus V \) the permutation \( kjk^{-1}j \) is fixed-point-free on \( F^2 \).

**Proof.** Since \( h \) is a fixed-point-free involutory permutation on \( F \), the permutation \( g \) is also involutory and fixed-point-free because \( g \) is conjugate to \( h \) in \( \text{Sym}(F) \). The properties \( g(0) = 1, fg = f, f(0) = f(1) = 1 \) are checked by direct calculation. We also have \( p(x) = (f(x) + x(g(x) + x)^t)/(g(x) + x)^{1+t} \) for \( x \in F \).

The mapping \( j \) is a fixed-point-free permutation on \( F^2 \) by Proposition 3.2. We only have to check that \( kjk^{-1}(x, y) = j(x, y) \) is false for all \( k \in HU \setminus V \) and all \((x, y) \in F^2 \). Assume this is not the case for some \( k = h_{c,a,b}, (a, c) \neq (0, 1) \) and \((x, y) \in F^2 \). We have thus

\[
\begin{align*}
g(a + cx) + a &= cg(x), \\
f(a + cx) + ag(a + cx)^t + ac^t x^t + a^{1+t} + c^{1+t} y &= c^{1+t}(f(x) + y); \\
g(a + cx) + a + cx &= c(g(x) + x), \\
f(a + cx) + ag(a + cx)^t + a(cx + a)^t &= c^{1+t} f(x).
\end{align*}
\]

Setting \( z = a + cx \) we get

\[
\begin{align*}
g(z) + z &= c(g(x) + x), \\
f(z) + (z + cx)(g(z) + z)^t &= c^{1+t} f(x); \\
c &= (g(z) + z)/(g(x) + x), \\
f(z) + z(g(z) + z)^t &= cx(g(z) + z)^t + c^{1+t} f(x)
\end{align*}
\]

whence \( p(z) = p(x) \), yielding \( z = x \) and consequently \( c = 1, a + x = x, a = 0 \), contradicting \((a, c) \neq (0, 1)\). \( \square \)

**Proposition 3.5.** \((q - 2)!\) is the number of fixed-point-free involutory permutations \( j \) on \( F^2 \) mapping \((0,0)\) to \((1,1)\), commuting with \( V \) elementwise and such that \( kjk^{-1}j \) is fixed-point-free for each \( k \in HU \setminus V \).

**Proof.** The number of permutations on \( F \) exchanging 0 and 1 is \((q - 2)!\). Let \( p_1 \) and \( p_2 \) be two such permutations; denote by \( g_1, f_1, j_1 \) and \( g_2, f_2, j_2 \) the mappings defined
from $p_1$ and $p_2$ respectively, as in Proposition 3.4. We have $j_1 = j_2$ if and only if $g_1 = g_2$ and $f_1 = f_2$. The relation $p_i(x) = (f_i(x) + x(g_i(x) + x))/((g_i(x) + x)^{1+})$ for $i = 1, 2$ shows then that $j_1 = j_2$ is equivalent to $p_1 = p_2$. □

Note that for a given $g \in \text{Sym}(F)$ with the required properties the number of all mappings $f$ which can be matched to $g$ is $(q - 2)!/(1 \cdot 3 \cdot 5 \cdots (q - 3)) = 2 \cdot 4 \cdot 6 \cdots (q - 2)$.

**Proposition 3.6.** Let $p$ be a permutation on $F$ exchanging 0 and 1. Let $j: F^2 \mapsto F^2$ be the corresponding mapping defined as in Proposition 3.4. The permutation set $J = V \cup \{kjk^{-1}; k \in HU\}$ is sharply transitive on $F^2$.

**Proof.** Since $kjk^{-1}j$ is fixed-point-free for each $k \in HU \setminus V$, we have $C_{HU}(j) = V$, hence $|\{kjk^{-1}; k \in HU\}| = q^2(q - 1)/q = q^2 - q$ and consequently $|J| = q^2$. All we have to show is that no two distinct permutations in $J$ have the same action on an element $\omega \in F^2$. This is clear if both are in $V$ or one is in $V$ and the other one is of the form $kjk^{-1}$. Assume $k_1j k^{-1}_1(\omega) = k_2j k^{-1}_2(\omega)$. Then $(k_2^{-1}k_1)j(k_2^{-1}k_1)^{-1}j$ fixes $\omega$, which can only occur if $k_2^{-1}k_1 \in V$, yielding $(k_2^{-1}k_1)j(k_2^{-1}k_1)^{-1} = j$, i.e. $k_1j k^{-1}_1 = k_2j k^{-1}_2$. □

The number of sharply transitive permutation sets of the kind described in Proposition 3.6 is thus $(q - 2)!$. Many of these will yield isomorphic one-factorizations: this will always be the case when the corresponding permutation sets are conjugate by a permutation in $L$; in other words the one-factorization arising from the mapping $(x,y) \mapsto (g(x), f(x) + y)$ is isomorphic to the one arising from the mapping $(x,y) \mapsto (g(x^{2^{-1}})^2, f(x^{2^{-1}})^2 + y)$.

4. One-factorizations arising from the Lüneburg planes

The Lüneburg plane of order $q^2$ can be represented as follows (see [7]). The points are the elements of $F^4$ viewed as a 4-dimensional affine space over $F$. The lines are the planes of $F^4$ whose equations in the coordinates $(x_1, x_2, y_1, y_2)$ can be written in one of the following forms when $a, b, c, d$ vary in $F$:

(i) \(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}\);  
(ii) \(\begin{pmatrix} d \\ c^2 + t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}\).

Point-line incidence is given by point-plane incidence in $F^4$. The resulting incidence structure is an affine plane $\mathcal{M}'(q)$ which is isomorphic to the one originally defined by Lüneburg in [9]. Two lines are parallel in $\mathcal{M}'(q)$ if and only if the corresponding planes are parallel in $F^4$. The projective closure of $\mathcal{M}'(q)$ is denoted by $\mathcal{M}(q)$.
Consider the following $q^2 + 1$ lines in $M(q)$:

$$l_x: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad l_{c,d}: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c^{2+t} + d' \\ c^t + d \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c^x + s + d^z \end{pmatrix},$$

$c, d \in F$.

No three of these lines are concurrent and so they form an oval $\mathcal{O}$ in the dual plane $M(q)^*$. We identify $\mathcal{O}$ with $\mathcal{O}^2$ through the mapping $l_\infty \mapsto \infty$, $l_{c,d} \mapsto (c, d)$. The plane $M(q)^*$ admits a collineation group acting on $\mathcal{O}$ as $Sz(q)$ in its natural 2-transitive permutation representation and we may assume that this action is given by the action of $G = \langle HU, w \rangle$ on $\mathcal{O}$.

Proposition 4.1. Let $P$ be the point of $M(q)^*$ obtained as the intersection of the tangent to $\mathcal{O}$ at $\infty$ with the line joining the points $(0,0), (1,1)$. Define mappings $g, f : F \rightarrow F$ by setting $g(x) = x + 1/(x^2 + 2t + x^t + 1)$, $f(x) = 1/(x^2 + t + x + 1)$ for all $x \in F$ and set $j : F^2 \rightarrow F^2$, $(x, y) \mapsto (g(x), f(x) + y)$. We have then $j = j_P$.

Proof. It is easily verified that $g$ and $f$ satisfy all the assumptions of Proposition 3.2.

In the plane $M(q)^*$ the points $(x, y)$ and $j_P(x, y)$ lie on the oval $\mathcal{O}$ and are collinear with $P$. This property determines $j_P(x, y)$ uniquely once $P$ and $(x, y)$ are given: the relation $j = j_P$ will thus be proved as soon as we show that $P, (x, y)$ and $j(x, y)$ are collinear. Going back to the plane $M(q)$ we have to show that the lines corresponding to these points are concurrent; we interpret everything in terms of planes of $F^4$ and show that the planes corresponding to these lines contain one and the same point.

Since $P$ lies on the tangent to $\mathcal{O}$ at the point $\infty$ and since the nucleus of the oval corresponds to the line at infinity of $M(q)$, we see that the plane corresponding to $P$ has equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

in fact it is parallel to the plane

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

corresponding to $\infty$ and contains the point $(1,0,0,0)$ obtained as the intersection of the planes

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

corresponding to $(0,0)$ and $(1,1)$, respectively.
The plane corresponding to the point \((x, y)\) has equation
\[
\begin{pmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ x^{2+t} + y^t \\ x^t \\ y \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x^s \\ x^{1+s} + y^s \end{pmatrix}.
\]
This plane will meet the plane
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
at the point \((1, 0, y + x^s, x^{2+t} + y^t + x^{1+s} + y^s)\); the plane corresponding to \(j(x, y) = (g(x), f(x) + y)\) will also contain this point if and only if the conditions
\[
y + x^s = f(x) + y + g(x)^s,
\]
\[
x^{2+t} + y^t + x^{1+s} + y^s = g(x)^{2+t} + (f(x) + y)^t + g(x)^{1+s} + (f(x) + y)^s
\]
are satisfied simultaneously. It is now easily checked that with the above choice for \(g\) and \(f\) this is indeed the case.

5. A characterization of the dual Lüneburg plane of order 64

Assume \(q = 8\). Let \(\mathcal{P}\) be a finite projective plane of order \(q^2\) and assume \(G \cong \text{Sz}(q)\) is a collineation group of \(\pi\) fixing an oval \(\mathcal{O}\) and acting on the points of \(\mathcal{O}\) in its natural 2-transitive permutation representation. We identify \(\mathcal{O}\) with \(\Omega\) and take \(G\) to be represented by \(\langle H, U, w \rangle\). Consider the tangent \(l_x\) to \(\Omega\) at \(\infty\) and the permutation set \(J = J_\infty\) on \(\Omega\) arising from \(l_x\). Let \(j \in J\) map \((0,0)\) to \((1,1)\) and set \(j(x,y) = (g(x), f(x) + y)\) for \((x,y) \in F^2\).

Proposition 5.1. The mappings \(g, f\) are those given by Proposition 4.1, namely
\[
g(x) = x + 1/(x^3 + x^4 + 1), \quad f(x) = 1/(x^6 + x + 1)\]
for all \(x \in F\).

Proof. We checked by computer that of all possible candidates for \(j\) given in Section 3, the one arising from the above choice of \(g\) and \(f\) is the only one with the property that \(\text{Fix}(jj') \leq 2\) holds for each permutation \(j'\) in the conjugate \(J' = wJw\).

Proposition 5.2. The dual Lüneburg plane \(M(8)^*\) is the only plane of order 64 admitting \(\text{Sz}(8)\) as a collineation group fixing an oval.

Proof. We know from Proposition 5.1 that \(J\) is uniquely determined. The discussion in Section 2 shows that our assertion will hold as soon as we can prove that external lines in the two orbits \(\mathcal{E}_1\) and \(\mathcal{E}_2\) can be reconstructed uniquely. To this purpose let \(G_1\) resp. \(G_2\) be a dihedral subgroup of order \(2v_1\) resp. \(2v_2\) of \(G\) with \(v_1 = 13, v_2 = 5\).

We checked by computer that there are precisely two \(G_1\)-orbits of conjugates of \(j\) (of size 26 each) with the property that the product of a permutation in the orbit with an
involution in $G_1$ is fixed-point-free; it follows from $65 = 2 \cdot 26 + v_1$ that an external line in $\sigma_1$ can be reconstructed uniquely.

We also verified by computer that there are precisely twelve $G_2$-orbits of conjugates of $j$ (of size 10 each) with the property that the product of a permutation in the orbit with an involution in $G_2$ is fixed-point-free; the further test to see when all products of a given permutation in one orbit with each permutation in another orbit are fixed-point-free leaves a unique set of six pairwise “compatible” orbits. Consequently an external line in $\sigma_2$ can also be reconstructed uniquely. □

Acknowledgements

The computations quoted in the previous section were performed with the software CAYLEY running on the VAX 6510 of C.I.S.I.T., the computing center of the Universitá della Basilicata.

The first author is indebted to Gary L. Ebert of the University of Delaware for introducing him to several useful features of CAYLEY.

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