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Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators

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Abstract

By making use of a general linear operator $\mathcal{I}_p^\lambda(a,c)$, the authors introduce several new subclasses of multivalent functions and investigate various inclusion relationships and argument properties associated with these subclasses. Some interesting applications involving such and other families of linear operators are also considered. The results presented here include a number of known results as their special cases.

Keywords: Analytic functions; Univalent and multivalent functions; Subordination between analytic functions; Hadamard product (or convolution); Linear operators; Carlson–Shaffer operator; Fractional derivative operator; Choi–Saigo–Srivastava operator; Strongly starlike functions; Strongly convex functions; Strongly close-to-convex functions; Inclusion relationships and argument properties; Ruscheweyh derivative operator

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1. Introduction and definitions

Let $\mathcal{A}_p$ denote the class of functions f *normalized* by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* and p-*valent* in the *open* unit disk

$$\mathbb{U} = \{z\colon\ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

If f and g are analytic in $\mathbb{U}$, we say that f is *subordinate* to g, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}_p^*(\eta)$ and $\mathcal{C}_p(\eta)$ the subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, *p-valent starlike of order* η $(0 \leqslant \eta < p)$ in $\mathbb{U}$ and *p-valent convex of order* η $(0 \leqslant \eta < p)$ in $\mathbb{U}$ (see, for details, the earlier work [27]).

Let $\mathcal{N}$ be the class of analytic functions h with $h(0) = 1$, which are *convex* and *univalent* in $\mathbb{U}$ and satisfy the following inequality:

$$\Re\{h(z)\} > 0 \quad (z \in \mathbb{U}).$$

Making use of the aforementioned principle of subordination between analytic functions, we define each of the following subclasses of $\mathcal{A}_p$:

$$\mathcal{S}_p^*(\eta; h) := \left\{ f\colon\ f \in \mathcal{A}_p \text{ and } \frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec h(z) \right\}$$
$$(0 \leqslant \eta < p;\ z \in \mathbb{U};\ h \in \mathcal{N}) \tag{1.2}$$

and

$$\mathcal{C}_p(\eta; h) := \left\{ f\colon\ f \in \mathcal{A}_p \text{ and } \frac{1}{p-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec h(z) \right\}$$
$$(0 \leqslant \eta < p;\ z \in \mathbb{U};\ h \in \mathcal{N}). \tag{1.3}$$

In particular, we set

$$\mathcal{S}_p^*\left(\eta; \left(\frac{1+z}{1-z}\right)^{\alpha}\right) =: \mathcal{S}_p^*(\eta; h_\alpha)$$
$$\left(0 \leqslant \eta < p;\ 0 < \alpha \leqslant 1;\ z \in \mathbb{U};\ h_\alpha(z) := \left(\frac{1+z}{1-z}\right)^{\alpha} \in \mathcal{N}\right) \tag{1.4}$$

and

$$\mathcal{C}_p\left(\eta; \left(\frac{1+z}{1-z}\right)^{\alpha}\right) =: \mathcal{C}_p(\eta; h_\alpha)$$
$$\left(0 \leqslant \eta < p;\ 0 < \alpha \leqslant 1;\ z \in \mathbb{U};\ h_\alpha(z) := \left(\frac{1+z}{1-z}\right)^{\alpha} \in \mathcal{N}\right). \tag{1.5}$$

We now define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p}$$
$$\left(z \in \mathbb{U};\ a \in \mathbb{R};\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right), \tag{1.6}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol (or the *shifted* factorial) defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{1.7}$$

It is easily seen from the above definitions that

$$f \in \mathcal{C}_p(\eta; h) \quad \Longleftrightarrow \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\eta; h) \tag{1.8}$$

and

$$\mathcal{S}_p^*(\eta; h_1) = \mathcal{S}_p^*(\eta) \quad \text{and} \quad \mathcal{C}_p(\eta; h_1) = \mathcal{C}_p(\eta). \tag{1.9}$$

The classes $\mathcal{S}_p^*(\eta; h)$ and $\mathcal{C}_p(\eta; h)$ were studied by Kim et al. [6] and Ma and Minda [10]. Furthermore, the *special* classes $\mathcal{S}_1^*(0; h_\alpha)$ and $\mathcal{C}_1(0; h_\alpha)$ of *strongly starlike functions of order* α in $\mathbb{U}$ and *strongly convex functions of order* α in $\mathbb{U}$, respectively, were investigated extensively by Mocanu [12] and Nunokawa [17].

Corresponding to the function $\phi_p(a, c; z)$ defined by (1.6), we introduce the following family of linear operators:

$$\mathcal{L}_p(a, c) : \mathcal{A}_p \to \mathcal{A}_p$$

by

$$\mathcal{L}_p(a, c) f(z) := \phi_p(a, c; z) * f(z) \quad (z \in \mathbb{U};\ f \in \mathcal{A}_p), \tag{1.10}$$

in terms of the Hadamard product (or convolution). Then it is easily observed from the definitions (1.6) and (1.10) that

$$\mathcal{L}_p(p+1, p) f(z) = \frac{zf'(z)}{p} \quad \text{and} \quad \mathcal{L}_p(n+p, 1) f(z) = \mathcal{D}^{n+p-1} f(z) \quad (n > -p), \tag{1.11}$$

where, in the *special* case when $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathcal{D}^n$ denotes the familiar Ruscheweyh derivative of order n ([21]; see also [5] and Eq. (1.21) below).

The operator $\mathcal{L}_p(a, c)$ was introduced and studied by Saitoh [22]. This operator is an extension of the Carlson–Shaffer operator $L_1(a, c)$ *and* the familiar fractional derivative operator D_z^λ, *each* of which has been used widely and extensively on the space of analytic and univalent functions in $\mathbb{U}$ (see, for details, [2]; see also [26]). We recall here the fact that, in their recent work, Liu and Srivastava [9] considered a *meromorphic* analogue of the linear operator $\mathcal{L}_p(a,c)$ for $p \in \mathbb{N}$.

Corresponding to the function $\phi_p(a,c;z)$ defined by (1.6), we also introduce a function $\phi_p^{\dagger}(a,c;z)$ given by

$$\phi_p(a,c;z) * \phi_p^{\dagger}(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p), \tag{1.12}$$

which leads us to the following family of linear operators $\mathcal{I}_p^{\lambda}(a,c)$ analogous to $\mathcal{L}_p(a,c)$:

$$\mathcal{I}_p^{\lambda}(a,c)f(z) := \phi_p^{\dagger}(a,c;z) * f(z)$$
$$(a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \lambda > -p;\ z \in \mathbb{U};\ f \in \mathcal{A}_p). \tag{1.13}$$

It is readily verified from the definition (1.13) that

$$\mathcal{I}_p^1(p+1,1)f(z) = f(z) \quad \text{and} \quad \mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p}, \tag{1.14}$$

$$z\big(\mathcal{I}_p^{\lambda}(a+1,c)f(z)\big)' = a\mathcal{I}_p^{\lambda}(a,c)f(z) - (a-p)\mathcal{I}_p^{\lambda}(a+1,c)f(z), \tag{1.15}$$

and

$$z\big(\mathcal{I}_p^{\lambda}(a,c)f(z)\big)' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^{\lambda}(a,c)f(z). \tag{1.16}$$

The operator $\mathcal{I}_1^{\lambda}(\mu+2,1)$ $(\lambda > -1;\ \mu > -2)$ was introduced recently by Choi et al. [3], who investigated (among other things) several inclusion relationships involving various subclasses of analytic and univalent functions, which were defined by them in terms of the operator $\mathcal{I}_1^{\lambda}(\mu+2,1)$. A *further* special case of the Choi–Saigo–Srivastava operator $\mathcal{I}_1^{\lambda}(\mu+2,1)$ was considered earlier by Noor et al. [14,16] and Liu [8].

By using the *general* linear operator $\mathcal{I}_p^{\lambda}(a,c)$, we now define a *new* subclass of $\mathcal{A}_p$ by

$$\mathcal{S}_{a,c}^{\lambda}(\eta;p;h) := \left\{ f\colon\ f \in \mathcal{A}_p \text{ and } \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^{\lambda}(a,c)f(z)\big)'}{\mathcal{I}_p^{\lambda}(a,c)f(z)} - \eta\right) \prec h(z) \right\}$$
$$(0 \leqslant \eta < p;\ h \in \mathcal{N};\ z \in \mathbb{U}). \tag{1.17}$$

We also set

$$\mathcal{S}_{a,c}^{\lambda}\left(\eta;p;\frac{1+Az}{1+Bz}\right) =: \mathcal{S}_{a,c}^{\lambda}(\eta;p;A,B) \quad (-1 \leqslant B < A \leqslant 1;\ z \in \mathbb{U}). \tag{1.18}$$

Thus, for some suitably chosen parameters a, c, λ, p, and h, the class $\mathcal{S}_{a,c}^{\lambda}(\eta;p;h)$ can be reduced to several subclasses of analytic and multivalent functions mentioned above. For example, we have

$$\mathcal{S}_{p+1,1}^{1}(\eta;p;h) = \mathcal{S}_p^*(\eta;h) \quad \text{and} \quad \mathcal{S}_{p,1}^{1}(\eta;p;h) = \mathcal{C}_p(\eta;h). \tag{1.19}$$

Finally, we put

$$\mathcal{K}_{a,c}^{\lambda}(\gamma,\delta,\eta;\,p;\,A,B) := \left\{ f\colon\ f \in \mathcal{A}_p \text{ and } \left| \arg\left(\frac{z\big(\mathcal{I}_p^{\lambda}(a,c)f(z)\big)'}{\mathcal{I}_p^{\lambda}(a,c)g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta \right\}$$
$$(0 \leqslant \eta, \gamma < p;\ 0 < \delta \leqslant 1;\ -1 \leqslant B < A \leqslant 1;\ z \in \mathbb{U};\ g \in \mathcal{S}_{a,c}^{\lambda}(\eta;\,p;\,A,B)). \tag{1.20}$$

In particular, $\mathcal{K}_{1,1}^{1}(\gamma,1,\eta;1;1,-1)$ and $\mathcal{K}_{2,1}^{1}(\gamma,1,\eta;1;1,-1)$ are the classes of *quasi-convex functions of order* γ *and type* η in $\mathbb{U}$ and *close-to-convex functions of order* γ *and type* η in $\mathbb{U}$, respectively, introduced and studied by Noor and Alkhorasani [15] and Silverman [24]. Furthermore, $\mathcal{K}_{2,1}^{1}(0,\delta,0;1;1,-1)$ is the class of *strongly close-to-convex functions of order* δ in $\mathbb{U}$ in the sense of Pommerenke [20].

In the present paper, we investigate some inclusion relationships and argument properties associated with such multivalent functions in the class $\mathcal{A}_p$ as those belonging to the subclasses $\mathcal{S}_{a,c}^{\lambda}(\eta;p;h)$ and $\mathcal{K}_{a,c}^{\lambda}(\gamma,\delta,\eta;p;A,B)$ defined by (1.17) and (1.20), respectively. The class-preserving properties involving several families of linear operators, such as the convolution operator $\mathcal{I}_p^{\lambda}(a,c)$ defined by (1.13) and the integral operator F_μ defined by (2.7) below, are also considered. Many of the earlier results given by (among others) Bernardi [1], Choi et al. [3], Libera [7], Liu [8], Noor [13], Noor and Alkhorasani [15], and Sakaguchi [23] are shown here to follow as special cases of the results presented in this paper. Thus the various inclusion relationships and argument properties associated with the function classes $\mathcal{S}_{a,c}^{\lambda}(\eta;p;h)$ and $\mathcal{K}_{a,c}^{\lambda}(\gamma,\delta,\eta;p;A,B)$ introduced here can be viewed as extensions and generalizations of numerous previously-obtained results in Geometric Function Theory. Moreover, since each of these general function classes is introduced in this paper by means of the convolution operator $\mathcal{I}_p^{\lambda}(a,c)$ which, in turn, stems eventually from such familiar operators as the Carlson–Shaffer operator $L_1(a,c)$ and the Ruscheweyh derivative operator $\mathcal{D}^{\lambda}:\mathcal{A}_1 \to \mathcal{A}_1$ defined by (cf. [21]; see also Eq. (1.11) above)

$$\mathcal{D}^{\lambda} f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) = \mathcal{L}_1(\lambda+1,1)f(z) \quad (f \in \mathcal{A}_1;\ \lambda > -1), \tag{1.21}$$

some of our results might be simplified, in these and other special cases, to results with possible geometric interpretations.

2. The main inclusion relationships

In proving our main results, we need the following lemmas.

Lemma 1 (Eenigenburg et al. [4]). *Let h be convex univalent in $\mathbb{U}$ with $h(0)=1$ and*

$$\Re\{\kappa h(z)+\nu\} > 0 \quad (\kappa,\nu \in \mathbb{C};\ z \in \mathbb{U}).$$

If q is analytic in $\mathbb{U}$ with $q(0)=1$, then the subordination

$$q(z) + \frac{zq'(z)}{\kappa q(z)+\nu} \prec h(z) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2 (Miller and Mocanu [11]). *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with*

$$\Re\{\omega(z)\} \geqslant 0 \quad (z \in \mathbb{U}).$$

If q is analytic in $\mathbb{U}$ and $q(0) = h(0)$, then the subordination

$$q(z) + \omega(z)zq'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 3 (Nunokawa et al. [18]). *Let q be analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that*

$$-\frac{\pi}{2}\alpha_1 = \arg\{q(z_1)\} < \arg\{q(z)\} < \arg\{q(z_2)\} = \frac{\pi}{2}\alpha_2 \tag{2.1}$$

for some α_1 and α_2 $(\alpha_1, \alpha_2 > 0)$ and for all z $(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \quad \textit{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)m, \tag{2.2}$$

where

$$m \geqslant \frac{1-|b|}{1+|b|} \quad \textit{and} \quad b = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \tag{2.3}$$

With the help of Lemma 1, we begin by proving an inclusion relationship for the class $\mathcal{S}^{\lambda}_{a.c}(\eta; p; h)$ given by Proposition 1 below.

Proposition 1. *Let $a \geqslant p$ and $\lambda \geqslant 0$. Then*

$$\mathcal{S}^{\lambda+1}_{a,c}(\eta; p; h) \subset \mathcal{S}^{\lambda}_{a,c}(\eta; p; h) \subset \mathcal{S}^{\lambda}_{a+1,c}(\eta; p; h) \quad (h \in \mathcal{N}).$$

Proof. First of all, we show that

$$\mathcal{S}^{\lambda+1}_{a,c}(\eta; p; h) \subset \mathcal{S}^{\lambda}_{a,c}(\eta; p; h) \quad (h \in \mathcal{N};\ \lambda \geqslant 0;\ a \geqslant p).$$

Let $f \in \mathcal{S}^{\lambda+1}_{a,c}(\eta; p; h)$ and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}^{\lambda}_{p}(a,c)f(z)\big)'}{\mathcal{I}^{\lambda}_{p}(a,c)f(z)} - \eta\right), \tag{2.4}$$

where q is analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. Applying (1.16) and (2.4), we obtain

$$(\lambda+p)\frac{\mathcal{I}^{\lambda+1}_{p}(a,c)f(z)}{\mathcal{I}^{\lambda}_{p}(a,c)f(z)} = (p-\eta)q(z) + \lambda + \eta. \tag{2.5}$$

By logarithmically differentiating both sides of (2.5) and multiplying the resulting equation by z, we have

$$\frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}^{\lambda+1}_{p}(a,c)f(z)\big)'}{\mathcal{I}^{\lambda+1}_{p}(a,c)f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z)+\lambda+\eta} \quad (z \in \mathbb{U}). \tag{2.6}$$

By applying Lemma 1 to (2.6), it follows that $q \prec h$ in $\mathbb{U}$, that is, that

$$f \in \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, h).$$

To prove the second part of Proposition 1, let $f \in \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, h)$ and put

$$s(z) = \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^{\lambda}(a+1,c)f(z)\big)'}{\mathcal{I}_p^{\lambda}(a+1,c)f(z)} - \eta\right),$$

where s is an analytic function in $\mathbb{U}$ with $s(0) = 1$ and $s(z) \neq 0$ for all $z \in \mathbb{U}$. Then, by using (1.15) *and* the arguments similar to those detailed above, it follows that $s \prec h$ in $\mathbb{U}$, which implies that $f \in \mathcal{S}_{a+1,c}^{\lambda}(\eta;\, p;\, h)$. The proof of Proposition 1 is thus completed. □

By setting

$$h(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqslant B < A \leqslant 1)$$

in Proposition 1, we have the following corollary.

Corollary 1. *Let $a \geqslant p$, $\lambda \geqslant 0$, and $-1 \leqslant B < A \leqslant 1$. Then*

$$\mathcal{S}_{a,c}^{\lambda+1}(\eta;\, p;\, A, B) \subset \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, A, B) \subset \mathcal{S}_{a+1,c}^{\lambda}(\eta;\, p;\, A, B).$$

Proposition 2. *If $f \in \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, h)$, then $F_\mu(f) \in \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, h)$, where F_μ is the integral operator defined by*

$$F_\mu(f) = F_\mu(f)(z) := \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1} f(t)\,dt \quad (\mu \geqslant 0). \tag{2.7}$$

Proof. Let $f \in \mathcal{S}_{a,c}^{\lambda+1}(\eta;\, p;\, h)$ and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z)\big)'}{\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z)} - \eta\right), \tag{2.8}$$

where q is analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. From (2.7) and (1.15), we have

$$z\big(\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z)\big)' = (\mu+p)\mathcal{I}_p^{\lambda}(a,c)f(z) - \mu\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z). \tag{2.9}$$

Then, by applying (2.9) to (2.8), we get

$$(\mu+p)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z)} = (p-\eta)q(z) + \mu + \eta. \tag{2.10}$$

Making use of the logarithmic differentiation on both sides of (2.10) and multiplying the resulting equation by z, we have

$$\frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^{\lambda}(a,c)f(z)\big)'}{\mathcal{I}_p^{\lambda}(a,c)f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z)+\mu+\eta} \quad (z \in \mathbb{U}).$$

Hence, by virtue of Lemma 1, we conclude that $q \prec h$ in $\mathbb{U}$, which implies the desired assertion that $F_\mu(f) \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; h)$. $\square$

By setting

$$h(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqslant B < A \leqslant 1)$$

in Proposition 2, we immediately get the following result.

Corollary 2. *If $f \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B)$, then $F_\mu(f) \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B)$, where F_μ is the integral operator defined by* (2.7)*.*

Remark 1. If we take $a = \mu + 1$ $(\mu > -2)$ and $c = p = 1$ in Propositions 1 and 2, we obtain the corresponding results given recently by Choi et al. [3]. Moreover, for

$$a = n + 1 \quad (n \in \mathbb{N}_0), \qquad c = \lambda = p = 1, \quad \text{and} \quad h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leqslant 1),$$

Propositions 1 and 2 would reduce to the corresponding results given earlier by Liu [8].

3. Argument properties and their consequences

Theorem 1. *Let $f \in \mathcal{A}_p$, $0 < \delta_1, \delta_2 \leqslant 1$, $0 \leqslant \gamma < p$, and $\lambda \geqslant 0$. If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\left(\mathcal{I}_p^{\lambda+1}(a,c)f(z)\right)'}{\mathcal{I}_p^{\lambda+1}(a,c)g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{a,c}^{\lambda+1}(\eta; p; A, B)$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{I}_p^{\lambda}(a,c)g(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 and α_2 $(0 < \alpha_1, \alpha_2 \leqslant 1)$ are the solutions of the following equations:

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \\ \qquad (B \neq -1), \\ \alpha_1 \quad (B = -1) \end{cases} \tag{3.1}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \\ \qquad (B \neq -1), \\ \alpha_2 \quad (B = -1), \end{cases} \tag{3.2}$$

b is given by (2.3)*, and*

$$t_1 = t_1(\lambda) := \frac{2}{\pi}\sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+\lambda)(1-B^2)}\right). \tag{3.3}$$

Proof. Let

$$q(z) = \frac{1}{p-\gamma}\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a,c)g(z)} - \gamma\right).$$

Then q is analytic in $\mathbb{U}$ with $q(0) = 1$. By using (1.16), we obtain

$$\big[(p-\gamma)q(z)+\gamma\big]\mathcal{I}_p^\lambda(a,c)g(z) = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^\lambda(a,c)f(z). \tag{3.4}$$

Differentiating both sides of (3.4) and multiplying the resulting equation by z, we find that

$$\begin{aligned}(p-\gamma)zq'(z)\mathcal{I}_p^\lambda(a,c)g(z) + \big[(p-\gamma)q(z)+\gamma\big]z\big(\mathcal{I}_p^\lambda(a,c)g(z)\big)' \\ = (\lambda+p)z\big(\mathcal{I}_p^{\lambda+1}(a,c)f(z)\big)' - \lambda z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'.\end{aligned} \tag{3.5}$$

Since $g \in \mathcal{S}_{a,c}^{\lambda+1}(\eta;\, p;\, A, B)$, by Corollary 1, it follows that $g \in \mathcal{S}_{a,c}^{\lambda}(\eta;\, p;\, A, B)$.

Next we let

$$r(z) = \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)g(z)\big)'}{\mathcal{I}_p^\lambda(a,c)g(z)} - \eta\right).$$

Then, using (1.16) once again, we have

$$(\lambda+p)\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{\mathcal{I}_p^\lambda(a,c)g(z)} = (p-\eta)r(z)+\eta+\lambda. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\frac{1}{p-\gamma}\left(\frac{z\big(\mathcal{I}_p^{\lambda+1}(a,c)f(z)\big)'}{\mathcal{I}_p^{\lambda+1}(a,c)g(z)} - \gamma\right) = q(z) + \frac{zq'(z)}{(p-\eta)r(z)+\eta+\lambda}.$$

Furthermore, by using a known result given earlier by Silverman and Silvia [25], we have

$$\left|r(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (z \in \mathbb{U};\ B \neq -1) \tag{3.7}$$

and

$$\Re\{r(z)\} > \frac{1-A}{2} \quad (z \in \mathbb{U};\ B = -1). \tag{3.8}$$

Thus, from (3.7) and (3.8), we obtain

$$(p-\eta)r(z)+\eta+\lambda = \rho\exp\left(\frac{i\pi\phi}{2}\right),$$

where, in terms of t_1 given by (3.3),

$$\frac{(p-\eta)(1-A)}{1-B}+\eta+\lambda < \rho < \frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda \quad \text{and} \quad -t_1 < \phi < t_1 \quad (B \neq -1)$$

and

$$\frac{(p-\eta)(1-A)}{2}+\eta+\lambda < \rho < \infty \quad \text{and} \quad -1 < \phi < 1 \quad (B = -1).$$

Just as we observed above, q is analytic in $\mathbb{U}$ with $q(0)=1$. We also have

$$\Re\{q(z)\}>0 \quad (z\in\mathbb{U}),$$

by applying the assertion of Lemma 2 *with*

$$\omega(z)=\frac{1}{(p-\eta)r(z)+\eta+\lambda}.$$

Hence $q(z)\neq 0$ for all $z\in\mathbb{U}$.

If there exist two points $z_1, z_2\in\mathbb{U}$ such that the condition (2.1) is satisfied, then (by Lemma 3) we obtain (2.2) under the constraint (2.3). For the first case when $B\neq -1$, we obtain

$$\begin{aligned}
&\arg\left(q(z_1)+\frac{z_1q'(z_1)}{(p-\eta)r(z_1)+\eta+a-p}\right)\\
&\quad=-\frac{\pi}{2}\alpha_1+\arg\left(1-i\left(\frac{\alpha_1+\alpha_2}{2}\right)m\left[\rho\exp\left(\frac{i\pi\phi}{2}\right)\right]^{-1}\right)\\
&\quad\leqslant-\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\left[\frac{\pi}{2}(1-\phi)\right]}{2\rho+(\alpha_1+\alpha_2)m\cos\left[\frac{\pi}{2}(1-\phi)\right]}\right)\\
&\quad\leqslant-\frac{\pi}{2}\alpha_1\\
&\qquad-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin\left(\frac{\pi}{2}t_1\right)}\right)\\
&\quad=-\frac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left(q(z_2)+\frac{z_2q'(z_2)}{(p-\eta)r(z_2)+\eta+a-p}\right)\\
&\quad\geqslant\frac{\pi}{2}\alpha_2+\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\left(\frac{\pi}{2}t_1\right)}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\lambda\right)(1+|b|)+(\alpha+\beta)(1-|b|)\sin\left(\frac{\pi}{2}t_1\right)}\right)\\
&\quad=\frac{\pi}{2}\delta_2,
\end{aligned}$$

where we have used the inequality in (2.3); δ_1, δ_2 and t_1 being given by (3.1)–(3.3), respectively. Similarly, for the second case when $B=-1$, we have

$$\arg\left(q(z_1)+\frac{z_1q'(z_1)}{(p-\eta)r(z_1)+\eta+\lambda}\right)\leqslant-\frac{\pi}{2}\alpha_1$$

and

$$\arg\left(q(z_2)+\frac{z_1q'(z_2)}{(p-\eta)r(z_2)+\eta+\lambda}\right)\geqslant\frac{\pi}{2}\alpha_2,$$

which would obviously contradict the assertion of Theorem 1. We thus complete the proof of Theorem 1. $\square$

Remark 2. The bounds asserted by Theorem 1 are not sharp *in general*. In fact, in a situation analogous to that of Theorem 1, Nunokawa et al. [19] chose to pose the corresponding *general* sharpness question as an *open* problem.

The proof of Theorem 2 below is much akin to that of Theorem 1 and so the details involved may be omitted.

Theorem 2. *Let* $f \in \mathcal{A}_p$, $0 < \delta_1, \delta_2 \leqslant 1$, $0 \leqslant \gamma < p$, *and* $a \geqslant p$. *If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a,c)g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{a,c}^\lambda(\eta; p; A, B)$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a+1,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a+1,c)g(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 *and* α_2 $(0 < \alpha_1, \alpha_2 \leqslant 1)$ *are the solutions of Eqs.* (3.1) *and* (3.2) *with* $\lambda = a - p$.

Remark 3. Just as we observed in Remark 2 above, the bounds asserted by Theorem 2 are not sharp *in general* (cf. [19]).

Remark 4. If we let $\delta_1 = \delta_2$ in Theorems 1 and 2, we get the following inclusion relationship.

Corollary 3. *Let* $a \geqslant p$, $\lambda \geqslant 0$, *and* $-1 \leqslant B < A \leqslant 1$. *Then*

$$\mathcal{K}_{a,c}^{\lambda+1}(\gamma,\delta,\eta;p;A,B) \subset \mathcal{K}_{a,c}^{\lambda}(\gamma,\delta,\eta;p;A,B) \subset \mathcal{K}_{a+1,c}(\gamma,\delta,\eta;p;A,B).$$

Remark 5. If we put

$$a = c = \lambda = p = 1, \qquad A = 1, \qquad B = -1, \quad \text{and} \quad \delta_1 = \delta_2 = 1$$

in Theorem 1, we see that every quasi-convex function of order γ and type η in $\mathbb{U}$ is a close-to-convex function of order γ and type η in $\mathbb{U}$, just as proven earlier by Noor [13] and Sakaguchi [23].

Letting $\gamma = 0$, $B \mapsto A$ $(A < 1)$, and $g(z) = z^p$ in Theorem 2, we obtain the following result.

Corollary 4. *Let* $f \in \mathcal{A}_p$ *and* $0 < \delta_1, \delta_2 \leqslant 1$. *If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{z^p}\right) < \frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a+1,c)f(z)\big)'}{z^p}\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 *and* α_2 $(0 < \alpha_1, \alpha_2 \leqslant 1)$ *are the solutions of the following equations*:

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)}{2(1+|b|)}\right)$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)}{2(1+|b|)}\right).$$

Finally, we prove an argument property asserted by Theorem 3 below.

Theorem 3. *Let* $f \in \mathcal{A}_p$, $0 < \delta_1, \delta_2 \leqslant 1$, *and* $0 \leqslant \gamma < p$. *If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a,c)g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{a,c}^\lambda(\eta;\, p;\, A, B)$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)F_\mu(f)(z)\big)'}{\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2,$$

where F_μ *is the integral operator defined by* (2.7), *and* α_1 *and* α_2 $(0 < \alpha_1, \alpha_1 \leqslant 1)$ *are the solutions of Eqs.* (3.1) *and* (3.2) *with* $\lambda = \mu$.

Proof. Let

$$q(z) = \frac{1}{p-\gamma}\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)F_\mu(f)(z)\big)'}{\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z)} - \gamma\right).$$

Since $g \in \mathcal{S}_{a,c}^\lambda(\eta;\, p;\, A, B)$, we see from Corollary 2 that $F_\mu(g) \in \mathcal{S}_{a,c}^\lambda(\eta;\, p;\, A, B)$. Using (2.9), we also have

$$\big[(p-\gamma)q(z)+\gamma\big]\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z) = (\mu+p)\mathcal{I}_p^\lambda(a,c)f(z) - \mu\mathcal{I}_p^\lambda(a,c)F_\mu(f)(z).$$

Thus, by a simple calculation, we get

$$\begin{aligned}&(\mu+p)\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z)}\\ &\quad = (p-\gamma)zq'(z) + \big[(p-\gamma)q(z)+\gamma\big]\big[(p-\eta)r(z)+\eta+\mu\big],\end{aligned}$$

where

$$r(z) = \frac{1}{p-\eta}\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z)\big)'}{\mathcal{I}_p^\lambda(a,c)F_\mu(g)(z)} - \gamma\right).$$

Hence we have

$$\frac{1}{p-\gamma}\left(\frac{z\big(\mathcal{I}_p^\lambda(a,c)f(z)\big)'}{\mathcal{I}_p^\lambda(a,c)g(z)} - \gamma\right) = q(z) + \frac{zq'(z)}{(p-\eta)r(z)+\eta+\mu}.$$

The remaining part of the proof of Theorem 3 is similar to that of Theorem 1 and so we omit the details involved. □

Taking $\delta_1 = \delta_2$ in Theorem 3, we get the following special case.

Corollary 5. *Let* $f \in \mathcal{A}_p$, $0 \leqslant \gamma < p$, *and* $0 < \delta \leqslant 1$. *If*

$$\left|\arg\left(\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{I}_p^{\lambda}(a,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \mathcal{S}_{a,c}^{\lambda}(\eta; p; A, B)$, *then*

$$\left|\arg\left(\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)F_\mu(f)(z)\right)'}{\mathcal{I}_p^{\lambda}(a,c)F_\mu(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F_μ *is the integral operator defined by* (2.7) *and* α $(0 < \alpha \leqslant 1)$ *is the solution of the following equation*:

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos(\frac{\pi}{2}t_2)}{(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu)+\alpha\sin(\frac{\pi}{2}t_2)}\right) & (B \neq -1), \\ \alpha & (B = -1) \end{cases}$$

when $t_2 = t_1(\mu)$ *given by* (3.3) *with* $\lambda = \mu$.

From Corollary 5, we easily derive the following result.

Corollary 6. *If* $f \in \mathcal{K}_{a,c}^{\lambda}(\gamma, \delta, \eta; p; A, B)$, *then* $F_\mu(f) \in \mathcal{K}_{a,c}^{\lambda}(\gamma, \delta, \eta; p; A, B)$, *where* F_μ *is the integral operator defined by* (2.7).

Remark 6. For $a = 1$ and $a = 2$, Corollary 6 *with*

$$c = \lambda = p = 1, \qquad A = 1, \qquad B = -1, \quad \text{and} \quad \delta = 1$$

yields the corresponding results obtained by Noor and Alkhorasani [15]. Furthermore, by taking

$$a = 2, \qquad c = \lambda = p = 1, \qquad \gamma = 0, \qquad A = 1, \qquad B = -1, \quad \text{and} \quad \delta = 1$$

in Corollary 6, we obtain the classical results given earlier by Bernardi [1] and Libera [7].

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