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Note

A Note on Convex Approximation in L_p

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A convex function f given on $[-1, 1]$ can be approximated in L_p, $1 < p < \infty$, by convex polynomials P_n of degree at most n with the accuracy $o(n^{-2/p})$. This follows from the estimate $\|f - P_n\|_p \leqslant c \cdot n^{-2/p} \cdot \omega_2^\varphi(f, n^{-1})^{1/q}$, where $1 \leqslant p \leqslant \infty$, $p^{-1} + q^{-1} = 1$, $\varphi(x) = (1 - x^2)^{1/2}$, and $\omega_2^\varphi(f, t)$ is the Ditzian–Totik modulus of smoothness in the uniform metric.

One of the peculiarities of convex functions is that they can be approximated in $L_p[-1, 1]$ by algebraic polynomials of degree at most n as $O(n^{-2})$ when $p = 1$ (Ivanov, [2]), and as $o(n^{-2/p})$ when $1 < p < \infty$ (Stojanova, [6]). The estimates remain valid if the convex function f is approximated by convex polynomials (see Nikoltjeva-Hedberg [5] for $p = 1$, and Remarks below for $1 < p < \infty$). In the uniform metric, a convex function f can be approximated by convex algebraic polynomials of degree at most n with the accuracy $O(\omega_2^\varphi(f, n^{-1}))$, estimated in terms of the Ditzian–Totik modulus of smoothness $\omega_2^\varphi(f, t)$ (Leviatan, [3]). The estimate presented in this note naturally embraces the results indicated above.

Let

$$\omega_2^\varphi(f, t) := \sup_{\substack{0 \leqslant h \leqslant t \\ -1 \leqslant x \leqslant 1}} |\Delta_{h\varphi(x)}^2 f(x)|,$$

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where $\varphi(x) = (1 - x^2)^{1/2}$, and

$$\Delta^2_{h\varphi(x)} f(x) = f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x))$$

if $x \pm h\varphi(x) \in [-1, 1]$, and $\Delta^2_{h\varphi(x)} f(x) = 0$ elsewhere.

We prove the following theorem:

THEOREM. *For a convex function f defined on $[-1, 1]$ there exist convex polynomials P_n of degree at most n such that*

$$\|f - P_n\|_p \leqslant c \cdot n^{-2/p} \cdot \omega_2^\varphi(f, n^{-1})^{1/q}, \tag{1}$$

where $1/p + 1/q = 1$, and $1 \leqslant p \leqslant \infty$, and c is independent of n and p.

Remarks. (i) It follows from the Theorem that a convex continuous function f can be approximated in $L_p[-1, 1]$, $1 < p < \infty$, by convex polynomials of degree at most n with the accuracy $o(n^{-2/p})$.

(ii) The formula $c = c_0 \cdot \delta(f)^{1/p}$, where $\delta(f) := \max_{x \in [-1,1]} f(x) - \min_{x \in [-1,1]} f(x)$, shows how the constant in (1) depends on the function f. Being inherent in results of Ivanov's type, this dependence disappears in Leviatan's estimate when $p = \infty$.

Proof of the Theorem. It suffices to prove the theorem for a non-decreasing function f satisfying the conditions $f(-1) = 0$ and $f(1) = 1$, and polynomials P_n of degree at most $8n$ where n is large enough.

We use the method of shape-preserving approximation developed by DeVore and Yu [1], and Leviatan [3, 4]. For a convex function f this method provides convex polynomials P_n of degree at most $8n$ satisfying the condition

$$\|f - P_n\|_\infty \leqslant c \cdot \omega_2^\varphi(f, n^{-1}). \tag{2}$$

We will prove that the polynomials P_n approximate the function f in the L_1-metric so that

$$\|f - P_n\|_1 \leqslant cn^{-2}. \tag{3}$$

The estimate (1) immediately follows from (2) and (3).

We use the following properties of the partition $-1 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$ defined in [3]:

(i) $\xi_{j+1} - \xi_j \leqslant c \cdot n^{-1} \cdot (1 - \xi_j)^{1/2}$,

(ii) $\sin t_{n-j} \leqslant c \cdot (1 - \xi_j)^{1/2}$,

where $j = 0, ..., n-1$ and $t_i = i\pi/n$. These inequalities follow easily from [3, Lemma A].

Let S be the piecewise-linear function interpolating f at the points $\xi_0, \ldots, \xi_n$. Then

$$S(x) = \sum_{j=0}^{n-1} \alpha_j \varphi_j(x), \tag{4}$$

where $\alpha_0 = [\xi_0, \xi_1]f$, $\alpha_j = (\xi_{j+1} - \xi_{j-1})[\xi_{j-1}, \xi_j, \xi_{j+1}]f$ for $j = 1, \ldots, n-1$, $\varphi_j(x) = (x - \xi_j)_+$, and $[\cdots]f$ are divided differences of f. Observe that $\sum \alpha_j(1 - \xi_j) = S(1) = f(1) = 1$, and convexity of f implies that $\alpha_j \geqslant 0$.

We claim that

$$\|f - S\|_1 \leqslant cn^{-2}. \tag{5}$$

Denote by l_j the linear functions interpolating f at ξ_j and ξ_{i+1}. Then $l_0(x) = \alpha_0(x+1)$, and $l_j(x) = l_{j-1}(x) + \alpha_j(x - \xi_j)$ for $j = 1, \ldots, n-1$. Since f is convex and satisfies the conditions $f(-1) = 0$ and $f(1) = 1$, we obtain that $0 \leqslant f(x) \leqslant l_0(x)$ for $x \in [\xi_0, \xi_1]$, and $l_{j-1}(x) \leqslant f(x) \leqslant l_j(x)$ for $x \in [\xi_j, \xi_{j+1}]$. By (i), $\|f - S\|_{L_1[\xi_0, \xi_1]} \leqslant \|l_0\|_{L_1[\xi_0, \xi_1]} \leqslant c_1 n^{-2} \alpha_0 (1 - \xi_0)$ and $\|f - S\|_{L_1[\xi_j, \xi_{j+1}]} \leqslant \|l_j - l_{j-1}\|_{L_1[\xi_j, \xi_{j+1}]} \leqslant c_1 n^{-2} \alpha_j (1 - \xi_j)$. Therefore, $\|f - S\|_1 \leqslant c_1 n^{-2} \Sigma \alpha_j (1 - \xi_j) = c_1 n^{-2}$.

The polynomials P_n are defined by the formula $P_n(x) = \sum_{i=0}^{n-1} \alpha_i R_i(x)$; here $R_0(x) = 1 + x$ and for every $i = 1, \ldots, n-1$ the polynomials $R_i(x)$ of degree at most $8n$ approximate the truncated powers φ_i with the accuracy

$$|\varphi_i(x) - R_i(x)| \leqslant cn^{-1} \sin t_{n-i}\, d_{n-i}(t)^{-5}, \tag{6}$$

where $d_k(t) = 1 + n\,|t - t_k|$, $t_k = k\pi/n$, and $x = \cos t$ (see [4, Lemma 6] with $j = 2i - n$).

We claim that

$$\|S - P_n\|_1 \leqslant cn^{-2}. \tag{7}$$

Indeed, by (6)

$$\|S - P_n\|_1 \leqslant cn^{-1} \sum_{i=1}^{n-1} \alpha_i \sin t_{n-i} \cdot a_i,$$

where $a_i = \int_0^\pi d_i(t)^{-5} \sin t\, dt$. Integrating over the intervals $[t_j, t_{j+1}]$ and using the estimate $\sin t \leqslant \pi(1 + |i - j|) \sin t_{n-i}$, $t_j \leqslant t \leqslant t_{j+1}$, we obtain that $a_i \leqslant c_1 n^{-1} \sin t_{n-i}$. Therefore, by (ii),

$$\|S - P_n\|_1 \leqslant c_2 n^{-2} \sum_{i=0}^{n-1} \alpha_i (1 - \xi_i) = c_2 n^{-2}.$$

The estimates (5) and (7) yield (3). ∎

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References

1. R. De Vore and X. M. Yu, Pointwise estimates for monotone polynomial approximation, *Const. Approx.* **1** (1985), 323–331.
2. K. G. Ivanov, Approximation of convex functions by means of polynomials and polygons in L-metric, *in* "Approximation and Function Spaces (Gdansk, 1979)," pp. 287–293. North-Holland, Amsterdam/New York, 1981.
3. D. Leviatan, Pointwise estimates for convex polynomial approximation, *Proc. Amer. Math. Soc.* **98** (1986), 471–474.
4. D. Leviatan, Monotone and comonotone polynomial approxiation revisited, *J. Approx. Theory* **53** (1988), 1–16.
5. M. Nikoltjeva-Hedberg, Approximation of a convex function by convex algebraic polynomials in L_p, $1 \leqslant p < \infty$, *J. Approx. Theory* **73** (1993), 288–302.
6. M. P. Stojanova, Approximation of a convex function by algebraic polynomials in $L_p[a, b]$ $1 < p < \infty$, *Serdica* **11** (1985), 392–397.