A general formula for the calculation of Gaussian path-integrals in two and three euclidean dimensions

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Abstract: The method which was used in a preceding article for the analytical evaluation of Gaussian path-integrals in one euclidean dimension is generalized. As before, the final results are formulae directly applicable to all Gaussian path-integrals, this time in two and in three euclidean dimensions, respectively. The classical action is almost entirely integrated, and the proportionality factor \( F(t_b, t_a) \) in front of the exponential part is expressed in terms of a number of time-dependent functions which one encounters in the description of the motion along the classical path. The quadratic Lagrange function is kept as general as possible, e.g., involving twenty-eight terms in the three-dimensional case. The well-known time-discretization procedure for path-integrals has been avoided. Instead, one of the main steps in the theoretical development consists in applying the convolution property of quantum-mechanical Green’s functions, i.e., in three euclidean dimensions,

\[
K(r_b, t_b; r_a, t_a) = \int \int K(r_b, t_b; t, t) K(r, t; r_a, t_a) \, dr.
\]

This equality leads to a non-linear algebraic relation between \( F(t_b, t_a) \), \( F(t_b, t) \) and \( F(t, t_a) \). The solution of this equation yields the \( F \)-factor which appears in the propagator represented by the Gaussian path-integral. At two locations, the use of some remarkable determinantal identity, previously unknown to the author, has been indispensable in order to attain the desired final result. In the Appendix, the “\( n \)-dimensional” generalization of these identities is formulated.

Keywords: Gaussian path-integrals, Feynman’s formulation of quantum mechanics

1. Introduction

In January 1986, a paper was published in *Journ. Math. Phys.* by B.K. Cheng [1], in which the author calculated the propagator of the time-dependent forced harmonic oscillator with time-dependent damping by applying the well-known time-discretization method to the path-integral involved. Although the problem treated is of a reasonably high degree of complexity, it is nevertheless only one among many special cases which can be—and actually were—considered within the framework of the onedimensional Gaussian path-integrals. Recently, Grosjean and Goovaerts [3] studied the problem of finding the explicit expression of this kind of path-integrals in full generality. They considered the following Green’s function:

\[
K(x_b, t_b; x_a, t_a) := \int_{t_a}^{t_b} \frac{1}{\hbar} \int_{x_a}^{x_b} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} L\left[ x_H(t), \dot{x}_H(t), t \right] \, dt \right\} \, dx_H(t)
\]  \hspace{1cm} (1.1)

in which

\[ L(x, \dot{x}, t) = a(t)\dot{x}^2 + 2b(t)x \dot{x} + c(t)x^2 + 2d(t)x + e(t) \]  \hspace{1cm} (1.2)

and \( x = x_H(t) \), whereby \( t_a \leq t \leq t_b \), represents any continuous path (or history) connecting the initial state \((t_a, x_a)\) to the final state \((t_b, x_b)\). In (1.2), the six coefficients are arbitrary continuous functions of \( t \) except that \( a(t) \), \( b(t) \) and \( d(t) \) need also have continuous derivatives which is hardly a restriction, in practice. Their final result is

\[ K(x_b, t_b; x_a, t_a) = \exp \left[ \frac{i}{\hbar} S_{cl}(b, a) \right] \]  \hspace{1cm} (1.3)

in which \( y_1(t) \) is an arbitrarily chosen non-trivial particular solution of the homogeneous linear differential equation

\[ a(t)\dot{x} + \dot{a}(t)x + [\dot{b}(t) - c(t)]x = 0; \]  \hspace{1cm} (1.4)

\( y_2(t) \) is another particular solution of eq. (1.4) related to \( y_1(t) \) by

\[ y_2(t) = y_1(t) \int_{t_a}^{t_b} \frac{dt'}{a(t')} y_1^2(t') \]  \hspace{1cm} (1.5)

(with \( \tau \) a conveniently chosen fixed lower bound), hence of such nature that the Wronskian determinant of \( y_1(t) \) and \( y_2(t) \) is

\[ \det \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix} = \frac{1}{a(t)}; \]  \hspace{1cm} (1.6)

\[ S_{cl}(b, a) = \int_{t_a}^{t_b} L [x_{cl}(t), \dot{x}_{cl}(t), t] \, dt \]

\[ = \left[ a(t_b)\dot{x}_{cl}(t_b) + b(t_b)x_b + 2d(t_b) \right] x_b \]

\[ - \left[ a(t_a)\dot{x}_{cl}(t_a) + b(t_a)x_a + 2d(t_a) \right] x_a \]

\[ + \int_{t_a}^{t_b} \left[ \left[ e(t) - \dot{d}(t) \right] x_{cl}(t) + f(t) \right] \, dt \]  \hspace{1cm} (1.7)

where \( x = x_{cl}(t) \) is the classical path described by the point particle starting from \( x_a \) at \( t_a \) and reaching \( x_b \) at \( t_b \), in other words, \( x_{cl}(t) \) is the particular solution of the classical equation of motion

\[ a(t)\dot{x} + \dot{a}(t)x + [\dot{b}(t) - c(t)]x = e(t) - \dot{d}(t) \]  \hspace{1cm} (1.8)

uniquely determined by the boundary conditions \( x_{cl}(t_a) = x_a \), \( x_{cl}(t_b) = x_b \). The appearance of \( \dot{a}(t) \), \( \dot{b}(t) \) and \( \dot{d}(t) \) in (1.8) explains why \( a(t) \), \( b(t) \) and \( d(t) \) should have continuous derivatives.

All previously published calculations of one-dimensional path-integrals in which the action happens to be an integral whose integrand is a polynomial of at most the second degree in every dynamical variable involved, are comprised in (1.3)–(1.8). The final result of [3] was obtained without making use of the time-discretization method. Note that (1.3) is such that

\[ K(x_b, t; x_a, t) = \delta(x_b - x_a), \]

as required in quantum mechanics for the purpose of conservation of probability.
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It is the principal aim of the present article to extend the calculations and results comprised in [3], to three euclidean dimensions, maintaining the generality of the form of the Lagrange function. The calculations will largely proceed along the same lines as in [3], but at places where they become prohibitively voluminous, they will be illustrated by carrying them out in the case of two euclidean dimensions. Therefore, in the end, it will turn out that both problems have in fact been treated.

2. Theoretical development

In certain formulae, we shall make use of the following notation for the sake of brevity:

\[ x_1 = \dot{x}, \quad x_2 = \dot{y}, \quad x_3 = \dot{z}, \quad x_4 = x, \quad x_5 = y, \quad x_6 = z. \] (2.1)

In this notation, the Lagrange function under consideration reads

\[ L_G(r, \dot{r}, t) = \sum_{j=1}^{6} a_{jj}(t) x_j^2 + 2 \sum_{j=1}^{6} \sum_{k=j+1}^{6} a_{jk}(t) x_j x_k + 2 \sum_{j=1}^{6} b_j(t) x_j + c(t) \] (2.2)

which generalizes (1.2) to three euclidean dimensions. The twenty-eight coefficients in (2.2) are arbitrary continuous functions of \( t \) except that \( a_{jk}(t) \) (\( j = 1, 2, 3; \ k = 1, 2, 3, 4, 5, 6; \ j < k \)) and \( b_j(t) \) (\( j = 1, 2, 3 \)) should also have continuous derivatives. According to a theorem, first established by Feynman [2, 4], the path-integral yielding the Green’s function which connects the classical state \( a \), i.e. \( (t_a, r_a) \), and \( b \), i.e. \( (t_b, r_b) \), (provisionally with \( t_a < t_b \), solely in order to fix the ideas) is of the form:

\[ K(r_b, t_b; r_a, t_a) := \int_{t_a}^{t_b} \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} L_G[r_H(t), \dot{r}_H(t), t] \, dt \right\} \, DR_H(t) \]

\[ = F(t_b, t_a) \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} L_G[r_H(t), \dot{r}_H(t), t] \, dt \right\} \] (2.3)

in which the integral represents the classical action and \( F(t_b, t_a) \) is a factor depending on \( t_a \) and \( t_b \), but not on \( r_a \) and \( r_b \). \( F(t_b, t_a) \) should be such that

\[ K(r_b, t_b; r_a, t_a) = \delta(r_b - r_a). \] (2.4)

The problem consists in calculating \( F(t_b, t_a) \) explicitly and carrying out the integration comprised in (2.3) as far as possible.

The classical equations of motion arising from

\[ \frac{d}{dt} \frac{\partial L_G}{\partial \dot{r}} = \frac{\partial L_G}{\partial r} \] (2.5)

are

\[ a_{11} \ddot{x} + a_{12} \ddot{y} + a_{13} \ddot{z} + \dot{a}_{11} \dot{x} + (\dot{a}_{12} + a_{15} - a_{24}) \dot{y} + (\dot{a}_{13} + a_{16} - a_{34}) \dot{z} \]

\[ + (\dot{a}_{14} - a_{44}) x + (\dot{a}_{15} - a_{45}) y + (\dot{a}_{16} - a_{46}) z = b_4 - \dot{b}_1, \]

\[ a_{12} \ddot{x} + a_{22} \ddot{y} + a_{23} \ddot{z} + (\dot{a}_{12} + a_{24} - a_{15}) \dot{x} + (\dot{a}_{23} + a_{26} - a_{35}) \dot{y} + (\dot{a}_{24} - a_{45}) x + (\dot{a}_{25} - a_{55}) y + (\dot{a}_{26} - a_{56}) z = b_5 - \dot{b}_2, \]

\[ a_{13} \ddot{x} + a_{33} \ddot{y} + a_{33} \ddot{z} + (\dot{a}_{13} + a_{34} - a_{16}) \dot{x} + (\dot{a}_{13} + a_{35} - a_{26}) \dot{y} + (\dot{a}_{13} + a_{36} - a_{46}) \dot{z} + (\dot{a}_{34} - a_{46}) x + (\dot{a}_{35} - a_{56}) y + (\dot{a}_{36} - a_{66}) z = b_6 - \dot{b}_3. \] (2.6)
The general integral of this system of coupled inhomogeneous linear differential equations can be expressed as follows:

\[ x(t) = \sum_{j=1}^{6} \mathcal{C}_j f_j(t) + X(t), \quad y(t) = \sum_{j=1}^{6} \mathcal{C}_j g_j(t) + Y(t), \]

\[ z(t) = \sum_{j=1}^{6} \mathcal{C}_j h_j(t) + Z(t), \tag{2.7} \]

where \( X(t), Y(t), Z(t) \) is an arbitrarily chosen particular solution of the three coupled inhomogeneous differential equations, \( \mathcal{C}_1, \ldots, \mathcal{C}_6 \) are six real constants of integration, \( f_1(t), \ldots, f_6(t) \) are six linearly independent particular solutions of the homogeneous linear resolvent differential equation of the sixth order for \( x(t) \), resulting from the elimination of \( y, z \) and their derivatives of various orders in the system of three coupled homogeneous linear differential equations associated with (2.6), \( g_1(t), \ldots, g_6(t) \) are six linearly independent particular solutions of the sixth order resolvent differential equation for \( y(t) \) and similarly for \( h_1(t), \ldots, h_6(t) \). Once \( f_1(t), \ldots, f_6(t) \) chosen, the corresponding \( g \)- and \( h \)-functions are uniquely determined as a consequence of the coupling between the equations in the system (2.6). More details about this and the choice of \( f_1(t), \ldots, f_6(t) \) will be given further on. The classical path \( r = r_{cl}(t) \) is that solution comprised in (2.7) whereby the constants of integration take on such values that

\[ r_{cl}(t_a) = r_a, \quad r_{cl}(t_b) = r_b. \tag{2.8} \]

If

\[ \Delta(t_b, t_a) := \begin{vmatrix} f_1(t_a) & f_2(t_a) & f_3(t_a) & f_4(t_a) & f_5(t_a) & f_6(t_a) \\ f_1(t_b) & f_2(t_b) & \cdots \\ g_1(t_a) \\ g_1(t_b) \\ h_1(t_a) \\ h_1(t_b) & \cdots & h_6(t_b) \end{vmatrix}, \tag{2.9} \]

and \( \Delta_{jk}(t_b, t_a) \) represents the coefficient of the element in the \( j \)th row and \( k \)th column of \( \Delta(t_b, t_a) \), then

\[ x_{cl}(t) = \sum_{j=1}^{6} C_j(t_b, t_a) f_j(t) + X(t), \quad y_{cl}(t) = \sum_{j=1}^{6} C_j(t_b, t_a) g_j(t) + Y(t), \]

\[ z_{cl}(t) = \sum_{j=1}^{6} C_j(t_b, t_a) h_j(t) + Z(t), \tag{2.10} \]

with

\[ C_j(t_b, t_a) = \frac{1}{\Delta(t_b, t_a)} \left\{ \left[ x_a - X(t_a) \right] \Delta_{1j}(t_b, t_a) + \left[ x_b - X(t_b) \right] \Delta_{2j}(t_b, t_a) \right. \\
+ \left[ y_a - Y(t_a) \right] \Delta_{3j}(t_b, t_a) + \left[ y_b - Y(t_b) \right] \Delta_{4j}(t_b, t_a) \\
+ \left[ z_a - Z(t_a) \right] \Delta_{5j}(t_b, t_a) + \left[ z_b - Z(t_b) \right] \Delta_{6j}(t_b, t_a) \right\}. \tag{2.11} \]
should be inserted into the integrand appearing in (2.3) in order to calculate the classical action. This may seem a lengthy and complicated task, but as in the one-dimensional case, a considerable simplification arises from subjecting twenty-four terms contained in the integrand of the classical action to appropriate integration by parts and taking the equations of motion (2.6) into account. In this way, one finds

\[ \int_{t_a}^{t_b} L_{\text{cl}} [r_{\text{cl}}(t), \dot{r}_{\text{cl}}(t), t] \; dt \]

\[ = \left[ a_{11}(t_b) \dot{x}_{\text{cl}}(t_b) + a_{12}(t_b) \dot{y}_{\text{cl}}(t_b) + a_{13}(t_b) \dot{z}_{\text{cl}}(t_b) \right. \]

\[ + a_{14}(t_b) x_a + a_{15}(t_b) y_b + a_{16}(t_b) z_b + 2b_1(t_b) \] \[ x_b \]

\[ + \left[ a_{12}(t_b) \dot{x}_{\text{cl}}(t_b) + a_{22}(t_b) \dot{y}_{\text{cl}}(t_b) + a_{23}(t_b) \dot{z}_{\text{cl}}(t_b) \right. \]

\[ + a_{24}(t_b) x_a + a_{25}(t_b) y_b + a_{26}(t_b) z_b + 2b_2(t_b) \] \[ y_b \]

\[ + \left[ a_{13}(t_b) \dot{x}_{\text{cl}}(t_b) + a_{23}(t_b) \dot{y}_{\text{cl}}(t_b) + a_{33}(t_b) \dot{z}_{\text{cl}}(t_b) \right. \]

\[ + a_{34}(t_b) x_a + a_{35}(t_b) y_b + a_{36}(t_b) z_b + 2b_3(t_b) \] \[ z_b \]

\[ - \left[ a_{11}(t_a) \dot{x}_{\text{cl}}(t_a) + a_{12}(t_a) \dot{y}_{\text{cl}}(t_a) + a_{13}(t_a) \dot{z}_{\text{cl}}(t_a) \right. \]

\[ + a_{14}(t_a) x_a + a_{15}(t_a) y_a + a_{16}(t_a) z_a + 2b_1(t_a) \] \[ x_a \]

\[ + \left[ a_{12}(t_a) \dot{x}_{\text{cl}}(t_a) + a_{22}(t_a) \dot{y}_{\text{cl}}(t_a) + a_{23}(t_a) \dot{z}_{\text{cl}}(t_a) \right. \]

\[ + a_{24}(t_a) x_a + a_{25}(t_a) y_a + a_{26}(t_a) z_a + 2b_2(t_a) \] \[ y_a \]

\[ + \left[ a_{13}(t_a) \dot{x}_{\text{cl}}(t_a) + a_{23}(t_a) \dot{y}_{\text{cl}}(t_a) + a_{33}(t_a) \dot{z}_{\text{cl}}(t_a) \right. \]

\[ + a_{34}(t_a) x_a + a_{35}(t_a) y_a + a_{36}(t_a) z_a + 2b_3(t_a) \] \[ z_a \]

\[ + \int_{t_a}^{t_b} \left[ \left[ b_4(t) - \dot{b}_1(t) \right] x_{\text{cl}}(t) + \left[ b_5(t) - \dot{b}_2(t) \right] y_{\text{cl}}(t) \right. \]

\[ + \left[ b_6(t) - \dot{b}_3(t) \right] z_{\text{cl}}(t) + c(t) \] \[ dt. \]

This formula clearly generalizes (1.7).

Next, inspired by the theoretical development in Section 3 of [3], we obtain a path-integral representation of $F(t_b, t_a)$ by means of a calculation which constitutes at the same time a proof of Feynman's theorem concerning Gaussian path-integrals. We regard any history $H$ as a deviation from the classical trajectory by putting

\[ r_H(t) = r_a(t) + s(t), \quad t_a \leq t \leq t_b, \]

whereby the continuous vector function $s(t)$ satisfies

\[ s(t_a) = 0, \quad s(t_b) = 0, \]

since all histories, including the classical path, connect the classical states $a$ and $b$. Making use of the notation

\[ u_1 = \dot{u}, \quad u_2 = \dot{v}, \quad u_3 = \dot{w}, \quad u_4 = u, \quad u_5 = v, \quad u_6 = w, \]

(2.15)
whereby \( u, v, w \) are the cartesian components of \( s \), one easily finds

\[
L_G[\mathbf{r}_H(t), \dot{\mathbf{r}}_H(t), t] = L_G[\mathbf{r}_c(t), \dot{\mathbf{r}}_c(t), t] + 2 \sum_{j=1}^{6} a_{jj}(t)(x_j)_c u_j
\]

\[
+ 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t)[(x_j)_c u_k + (x_k)_c u_j] + 2 \sum_{j=1}^{6} b_j(t) u_j
\]

\[
+ \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2 + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j u_k \right],
\]

(2.16)

and consequently, for any history \( \mathbf{H} \),

\[
\int_{t_a}^{t_b} L_G[\mathbf{r}_H(t), \dot{\mathbf{r}}_H(t), t] \, dt = \int_{t_a}^{t_b} L_G[\mathbf{r}_c(t), \dot{\mathbf{r}}_c(t), t] \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{11} \dot{x}_c + a_{12} \dot{y}_c + a_{13} \dot{z}_c + a_{14} x_c + a_{15} y_c + a_{16} z_c + b_1) \dot{u} \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{12} \dot{x}_c + a_{22} \dot{y}_c + a_{23} \dot{z}_c + a_{24} x_c + a_{25} y_c + a_{26} z_c + b_2) \dot{v} \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{13} \dot{x}_c + a_{23} \dot{y}_c + a_{33} \dot{z}_c + a_{34} x_c + a_{35} y_c + a_{36} z_c + b_3) \dot{w} \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{14} \dot{x}_c + a_{24} \dot{y}_c + a_{44} x_c + a_{45} y_c + a_{46} z_c + b_4) u \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{15} \dot{x}_c + a_{25} \dot{y}_c + a_{55} x_c + a_{56} y_c + a_{56} z_c + b_5) v \, dt
\]

\[
+ 2 \int_{t_a}^{t_b} (a_{16} \dot{x}_c + a_{26} \dot{y}_c + a_{66} x_c + a_{66} y_c + a_{66} z_c + b_6) w \, dt
\]

\[
+ \int_{t_a}^{t_b} \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2(t) + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j(t) u_k(t) \right] \, dt.
\]

(2.17)

When the second integral in this right-hand side is subjected to partial integration, the integrated parts are zero on account of \( u(t_a) = 0, u(t_b) = 0 \), and the remaining integral when added to the fifth integral yields a total whose integrand vanishes in virtue of the first equation of motion comprised in (2.6). The same holds for the \( v \)- and the \( w \)-parts. Hence,

\[
\int_{t_a}^{t_b} L_G[\mathbf{r}_H(t), \dot{\mathbf{r}}_H(t), t] \, dt
\]

\[
= \int_{t_a}^{t_b} L_G[\mathbf{r}_c(t), \dot{\mathbf{r}}_c(t), t] \, dt
\]

\[
+ \int_{t_a}^{t_b} \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2(t) + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j(t) u_k(t) \right] \, dt
\]

(2.18)

in which \( s(t) (= u(t) \mathbf{1}_x + v(t) \mathbf{1}_y + w(t) \mathbf{1}_z) \) is the symbol for a continuous infinity of continuous
vector functions of \( t \) satisfying (2.14) whereas the first part of the right-hand side is fixed with regard to the path-integration process implied in \( K(r_b, t_b; r_a, t_a) \). Consequently,

\[
K(r_b, t_b; r_a, t_a) \quad = \quad \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} L_G \left[ r_{\text{cl}}(t), \dot{r}_{\text{cl}}(t), t \right] \, dt \right\}
\]

\[
\times \int_{t_a,0}^{t_b,0} \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2(t) + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j(t) u_k(t) \right] \, dt \right\} \, Ds(t)
\]

(2.19)

and comparison with (2.3) clearly shows that

\[
F(t_b, t_a) = \int_{t_a,0}^{t_b,0} \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2(t) + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j(t) u_k(t) \right] \, dt \right\} \, Ds(t).
\]

(2.20)

For the purpose of calculating the right-hand side of (2.20), a generalization is carried out, consisting in the replacement of the fixed boundary conditions (2.14) by

\[
s(t_a) = s_a, \quad s(t_b) = s_b,
\]

(2.21)

with \( s_a \) and \( s_b \) arbitrarily chosen vectors belonging to \( \mathbb{R}^3 \). In this way, we obtain

\[
F(s_b, t_b; s_a, t_a) \quad = \quad \int_{t_a, s_a}^{t_b, s_b} \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} \left[ \sum_{j=1}^{6} a_{jj}(t) u_j^2(t) + 2 \sum_{j=1}^{5} \sum_{k=j+1}^{6} a_{jk}(t) u_j(t) u_k(t) \right] \, dt \right\} \, Ds(t)
\]

(2.22)

and the connection with \( F(t_b, t_a) \) is

\[
F(t_b, t_a) = F(0, t_b; 0, t_a).
\]

(2.23)

Apart from the fact that \( s \) replaces \( r \), (2.22) is comprised in the definition of \( K(r_b, t_b; r_a, t_a) \) given in (2.3), with the sum of the first two parts of the right-hand side of (2.2) as Lagrange function. To that function, there correspond classical equations of motion given by

\[
a_{11}\ddot{u} + a_{12}\ddot{v} + a_{13}\ddot{w} + (\dot{a}_{12} + a_{15} - a_{24})\dot{v} + (\dot{a}_{13} + a_{16} - a_{34})\dot{w}
+ (\dot{a}_{14} - a_{44})u + (\dot{a}_{15} - a_{45})v + (\dot{a}_{16} - a_{46})w = 0,
\]

\[
a_{12}\ddot{u} + a_{22}\ddot{v} + a_{23}\ddot{w} + (\dot{a}_{12} + a_{24} - a_{15})\dot{u} + (\dot{a}_{22} + a_{26} - a_{35})\dot{v}
+ (\dot{a}_{24} - a_{45})u + (\dot{a}_{25} - a_{55})v + (\dot{a}_{26} - a_{56})w = 0,
\]

\[
a_{13}\ddot{u} + a_{23}\ddot{v} + a_{33}\ddot{w} + (\dot{a}_{13} + a_{34} - a_{16})\dot{u} + (\dot{a}_{23} + a_{35} - a_{26})\dot{v}
+ (\dot{a}_{34} - a_{46})u + (\dot{a}_{35} - a_{56})v + (\dot{a}_{36} - a_{66})w = 0,
\]

(2.24)

evidently being the homogeneous counterpart of (2.6).
Now, it appears that the various integrations comprised in (2.22) may be carried out explicitly. Experience has shown that it is preferable to proceed in two steps. First, a number of terms in the action integral are subjected to integration by parts:

\[
2\int_{t_a}^{t_b} a_{14}(t) \dot{u}(t) u(t) \, dt = a_{14}(t) u^2(t) \bigg|_{t=t_a}^{t=t_b} - \int_{t_a}^{t_b} \dot{a}_{14}(t) u^2(t) \, dt
\]

and similarly for the integrals with the following integrands:

\[
2a_{25} \dot{v}, \quad 2a_{36} \dot{w}, \quad a_{15} \dot{w}, \quad a_{24} \dot{u},
\]

\[
a_{16} \dot{w}, \quad a_{34} \dot{w}, \quad a_{26} \dot{v} \quad \text{and} \quad a_{35} \dot{v}.
\]

The result is

\[
F(s_b, t_b; s_a, t_a) = \exp\left(\frac{i}{\hbar} \left[\left(a_{14}(t_b) u_b^2 - a_{14}(t_a) u_a^2\right) + \left(2a_{25}(t_b) v_b^2 - 2a_{25}(t_a) v_a^2\right) + \left(2a_{36}(t_b) w_b^2 - 2a_{36}(t_a) w_a^2\right) + \left(2a_{15}(t_b) u_b v_b - 2a_{15}(t_a) u_a v_a\right) + \left(2a_{16}(t_b) u_b w_b - 2a_{16}(t_a) u_a w_a\right) + \left(2a_{26}(t_b) u_b w_b - 2a_{26}(t_a) u_a w_a\right) + \left(2a_{35}(t_b) u_b w_b - 2a_{35}(t_a) u_a w_a\right)\right] \right) \times F_1(s_b, t_b; s_a, t_a)
\]

where

\[
F_1(s_b, t_b; s_a, t_a) := \int_{t_a}^{t_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} L_1(s(t), \dot{s}(t), t) \, dt\right) \, Ds(t)
\]

with the following quadratic Lagrange function:

\[
L_1[s(t), \dot{s}(t), t] = a_{14} \dot{u}^2 + a_{22} \dot{v}^2 + a_{33} \dot{w}^2 + 2a_{12} \dot{u} \dot{v} + 2a_{13} \dot{u} \dot{w} + 2a_{23} \dot{v} \dot{w}
\]

\[
+ (a_{15} - a_{24}) \dot{u} v + (a_{16} - a_{34}) \dot{u} w + (a_{26} - a_{35}) \dot{v} w
\]

\[
+ (a_{34} - a_{16}) \dot{w} u + (a_{35} - a_{26}) \dot{w} v + (a_{44} - \dot{a}_{14}) u^2 + (a_{55} - \dot{a}_{24}) v^2
\]

\[
+ (a_{66} - \dot{a}_{36}) w^2 + (2a_{45} - \dot{a}_{15} - \dot{a}_{24}) u v + (2a_{46} - \dot{a}_{16} - \dot{a}_{34}) u w
\]

\[
+ (2a_{56} - \dot{a}_{26} - \dot{a}_{35}) v w.
\]

This first transformation was possible for any history \( s(t) \) involved in (2.22) because all these paths have the same extremities since they connect \((t_a, s_a)\) and \((t_b, s_b)\). As a consequence, the integrated parts were the same for all histories. It is easy to verify that the Eulerian equations constructed with the Lagrange function \( L_1 \) yield the same classical equations of motion as in (2.24). Note also that in virtue of (2.23) and (2.25),

\[
F(t_b, t_a) = F_1(0, t_b; 0, t_a)
\]

The second step consists in noticing that \( F_1(s_b, t_b; s_a, t_a) \) is a Gaussian path-integral to which Feynman’s theorem can be applied:

\[
F_1(s_b, t_b; s_a, t_a) = F_1(t_b, t_a) \exp\left(\frac{1}{\hbar} \int_{t_a}^{t_b} L_1[s(t), \dot{s}(t), t] \, dt\right)
\]
in which \( s = s_{cl}(t) \) denotes the classical path connecting \((t_a, s_a)\) and \((t_b, s_b)\), i.e., \( s_{cl}(t) = u_{cl}(t) + v_{cl}(t) + w_{cl}(t) \) is the solution of the system (2.24) which is uniquely determined by the boundary conditions (2.21). It is more or less remarkable that the integral in (2.29) can be worked out completely. This is shown by applying integration by parts to six of its terms:

\[
\int_{t_a}^{t_b} a_{11}(t) \dot{u}_{cl}^2(t) \, dt
\]

\[
= a_{11}(t) \dot{u}_{cl}(t) v_{cl}(t) \bigg|_{t=t_a}^{t=t_b} - \int_{t_a}^{t_b} \left[ a_{11}(t) \ddot{u}_{cl}(t) + a_{11}(t) \dot{v}_{cl}(t) \right] u_{cl}(t) \, dt,
\]

\[
\int_{t_a}^{t_b} a_{12}(t) \dot{v}_{cl}^2(t) \, dt
\]

\[
= a_{12}(t) \dot{v}_{cl}(t) w_{cl}(t) \bigg|_{t=t_a}^{t=t_b} - \int_{t_a}^{t_b} \left[ a_{12}(t) \ddot{v}_{cl}(t) + a_{12}(t) \dot{w}_{cl}(t) \right] v_{cl}(t) \, dt
\]

\[
= a_{12}(t) \dot{w}_{cl}(t) w_{cl}(t) \bigg|_{t=t_a}^{t=t_b} - \int_{t_a}^{t_b} \left[ a_{12}(t) \ddot{w}_{cl}(t) + a_{12}(t) \dot{w}_{cl}(t) \right] w_{cl}(t) \, dt,
\]

and similarly for the integrals with the integrands \( a_{22} u_{cl}^2, a_{33} w_{cl}^2, a_{13} \dot{u}_{cl} \dot{w}_{cl} \) and \( a_{23} \dot{v}_{cl} \dot{w}_{cl} \), resp. It can easily be shown that in this manner the remaining integral part stemming from the action integral in (2.29) as well as from the partial integrations has an integrand which may be arranged as a sum of three contributions, respectively proportional to \( u_{cl}(t), v_{cl}(t) \) and \( w_{cl}(t) \). The coefficient of \( u_{cl}(t) \) is precisely the left-hand side of the first differential equation comprised in (2.24) written with \( u_{cl}, v_{cl} \) and \( w_{cl} \). Since the classical path is a solution of the system (2.24), the integral in question vanishes. Similarly for the parts proportional to \( v_{cl}(t) \) and \( w_{cl}(t) \). Therefore, only the integrated terms resulting from the nine integrations by parts remain and so:

\[
F_1(s_b, t_b; s_a, t_a)
\]

\[
- F_1(t_b, t_a) \exp \left[ \frac{1}{\hbar} \left\{ \left[ a_{11}(t_b) \dot{u}_{cl}(t_b) + a_{12}(t_b) \dot{v}_{cl}(t_b) + a_{13}(t_b) \dot{w}_{cl}(t_b) \right] u_{b} + \left[ a_{12}(t_b) \dot{u}_{cl}(t_b) + a_{22}(t_b) \dot{v}_{cl}(t_b) + a_{23}(t_b) \dot{w}_{cl}(t_b) \right] v_{b} + \left[ a_{13}(t_b) \dot{u}_{cl}(t_b) + a_{23}(t_b) \dot{v}_{cl}(t_b) + a_{33}(t_b) \dot{w}_{cl}(t_b) \right] w_{b} - \left[ a_{11}(t_a) \dot{u}_{cl}(t_a) + a_{12}(t_a) \dot{v}_{cl}(t_a) + a_{13}(t_a) \dot{w}_{cl}(t_a) \right] u_{a} - \left[ a_{12}(t_a) \dot{u}_{cl}(t_a) + a_{22}(t_a) \dot{v}_{cl}(t_a) + a_{23}(t_a) \dot{w}_{cl}(t_a) \right] v_{a} - \left[ a_{13}(t_a) \dot{u}_{cl}(t_a) + a_{23}(t_a) \dot{v}_{cl}(t_a) + a_{33}(t_a) \dot{w}_{cl}(t_a) \right] w_{a} \right\} \right].
\]

(2.30)

This result can be verified to agree with (2.12) when the necessary translation is made to pass from the Lagrange function (2.2) to the Lagrange function (2.27). The absence of terms of the first and the zeroth degree in the dynamical variables in (2.27) explains why (2.30) is free of integral-signs. From (2.28) and (2.30), one deduces

\[
F(t_b, t_a) = F_1(t_b, t_a),
\]

which proves that the originally considered Gaussian path-integral \( K(r_b, t_b; r_a, t_a) \) defined in (2.3) and the Gaussian path-integral \( F(s_b, t_b; s_a, t_a) \) defined in (2.26) contain the same \( (t_b, t_a) \)-dependent proportionality factor.
2.1. Concerning the integration of the differential system (2.24)

The general integral of the system of simultaneous homogeneous linear differential equations (2.24) associated with the Lagrange function (2.27), can be expressed as follows:

\[ u(t) = \sum_{j=1}^{6} \mathcal{D}_j f_j(t), \quad v(t) = \sum_{j=1}^{6} \mathcal{D}_j g_j(t), \quad w(t) = \sum_{j=1}^{6} \mathcal{D}_j h_j(t), \]

where \( \mathcal{D}_1, \ldots, \mathcal{D}_6 \) are six real constants of integration and the \( f \)-, \( g \)- and \( h \)-functions may be regarded as being the same as those encountered in (2.7). We recall that \( f_1(t), \ldots, f_6(t) \) are six linearly independent particular solutions of the homogeneous linear resolvent differential equation of the sixth order for \( u(t) \), resulting from the elimination of \( v \), \( w \) and their derivatives of various orders in the system (2.24). The first step in the construction of the resolvent equation is the reduction of (2.24) to its normal form, in other words, solving the system with respect to \( \ddot{u}, \dot{v} \) and \( \ddot{w} \). A necessary condition in this context is that

\[ a_{12}(t) a_{13}(t) a_{21}(t) a_{23}(t) a_{31}(t) a_{32}(t) \neq 0, \quad t_a \leq t \leq t_b, \]

condition which we shall assume to hold because without it, the system (2.24) would be either incompatible or indeterminate. Once the solutions \( f_1(t), \ldots, f_6(t) \) chosen, the functions \( g_1(t), \ldots, g_6(t) \) and \( h_1(t), \ldots, h_6(t) \) are uniquely determined in virtue of the coupling between the three equations in (2.24). In the onedimensional problem treated in [3], the analogue of (2.24) is

\[ a(t) \ddot{u} + \dot{a}(t) \dot{u} + [\dot{b}(t) - c(t)] u = 0 \]

and no matter how two linearly independent particular solutions \( y_1(t) \) and \( y_2(t) \) are chosen, they satisfy

\[ y_1(t) \ddot{y}_2(t) - y_2(t) \ddot{y}_1(t) = C/a(t), \quad C \neq 0, \]

under the assumption that \( a(t) \neq 0 \). In [3], it is clearly indicated that the arbitrary proportionality factors a priori comprised in \( y_1(t) \) and \( y_2(t) \) should be taken such that \( C = 1 \) (see (1.6)), for the sake of simplicity in the final result (1.3). Here, in the three-dimensional case, the counterpart of (2.34) is that no matter how the linearly independent particular solutions \( f_1(t), \ldots, f_6(t) \) of the sixth order resolvent equation for \( u(t) \) are chosen, the following equality holds:

\[
\begin{vmatrix}
    f_1(t) & f_2(t) & f_3(t) & f_4(t) & f_5(t) & f_6(t) \\
    f_1'(t) & \ldots & f_4'(t) & f_5'(t) & f_6'(t) \\
    g_1(t) & \ldots & g_4(t) & g_5(t) & g_6(t) \\
    g_1'(t) & \ldots & g_4'(t) & g_5'(t) & g_6'(t) \\
    h_1(t) & \ldots & h_4(t) & h_5(t) & h_6(t) \\
    h_1'(t) & \ldots & h_4'(t) & h_5'(t) & h_6'(t)
\end{vmatrix} = \frac{C}{\begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\
    a_{12}(t) & a_{22}(t) & a_{23}(t) \\
    a_{13}(t) & a_{23}(t) & a_{33}(t) \end{vmatrix}},
\]

where \( C \) is an arbitrary, non-zero proportionality factor.
Near the end of the article, it will become clear that $C = 1$ is again the best choice. Each of the six $f$-solutions may be regarded as containing an arbitrary, non-zero proportionality factor, but it is sufficient in practice to adjust only one factor in order to effectuate $C = 1$. I have proved the various preceding statements, but the calculations required are prohibitively voluminous for the present article. In contrast, in the two-dimensional case, the calculations and results are not excessively long and the similarity with the three-dimensional problem is so striking that I believe it worthwhile to outline the treatment of the two-dimensional one.

The two-dimensional analogue of (2.24) is

$$
\begin{align*}
\left\{ a_{11} \ddot{u} + a_{12} \ddot{v} + \dot{a}_{11} \dot{u} + (\dot{a}_{12} + a_{15} - a_{24}) \dot{v} + (\dot{a}_{44} - a_{44}) u + (a_{15} - a_{45}) v = 0, \\
(a_{12} \ddot{u} + a_{22} \ddot{v} + (\dot{a}_{12} + a_{24} - a_{15}) \dot{u} + \dot{a}_{22} \dot{v} + (\dot{a}_{24} - a_{45}) u + (\dot{a}_{25} - a_{55}) v = 0.
\end{align*}
$$

(2.36)

Solving for $\ddot{u}$ and $\ddot{v}$, one finds:

$$
\begin{align*}
\ddot{u} &= \alpha(t) \dot{u} + \beta(t) \dot{v} + \lambda(t) u + \mu(t) v, \\
\ddot{v} &= \rho(t) \dot{u} + \sigma(t) \dot{v} + \xi(t) u + \eta(t) v,
\end{align*}
$$

(2.37)

with

$$
\begin{align*}
\alpha(t) &= A^{-1} \left[ -\dot{a}_{11} a_{22} + (\dot{a}_{12} + a_{24} - a_{15}) a_{12} \right], \\
\beta(t) &= A^{-1} \left[ - (\dot{a}_{14} - a_{44}) a_{22} + (\dot{a}_{24} - a_{45}) a_{12} \right], \\
\lambda(t) &= A^{-1} \left[ - (\dot{a}_{14} - a_{44}) a_{22} + (\dot{a}_{24} - a_{45}) a_{12} \right], \\
\mu(t) &= A^{-1} \left[ - (\dot{a}_{15} - a_{45}) a_{22} + (\dot{a}_{25} - a_{55}) a_{12} \right], \\
\rho(t) &= A^{-1} \left[ \dot{a}_{11} a_{12} - (\dot{a}_{12} + a_{24} - a_{15}) a_{11} \right], \\
\sigma(t) &= A^{-1} \left[ (\dot{a}_{12} + a_{15} - a_{24}) a_{12} - \dot{a}_{22} a_{11} \right], \\
\xi(t) &= A^{-1} \left[ (\dot{a}_{14} - a_{44}) a_{12} - (\dot{a}_{24} - a_{45}) a_{11} \right], \\
\eta(t) &= A^{-1} \left[ (\dot{a}_{15} - a_{45}) a_{12} - (\dot{a}_{25} - a_{55}) a_{11} \right],
\end{align*}
$$

(2.39)

and

$$
A = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}.
$$

(2.39′)

Next, differentiating on both sides in (2.37) and making use of (2.38) in order to eliminate $\ddot{u}$, we get

$$
\begin{align*}
u^{(3)} &= \alpha \dddot{u} + (\dot{\alpha} + \lambda + \beta \rho) \ddot{u} + \left( \dot{\lambda} + \beta \dot{\xi} \right) u + \left( \dot{\beta} + \mu + \beta \sigma \right) \dot{v} + \left( \ddot{\mu} + \beta \eta \right) v, \\
u^{(4)} &= \alpha u^{(3)} + (2\dot{\alpha} + \lambda + \beta \rho) \dddot{u} + \left( 2\dot{\lambda} + 2\dot{\beta} + 2\beta \rho + \beta \dot{\xi} + \mu \rho + \beta \rho \sigma \right) \ddot{u} \\
&\quad + \left( \dot{\lambda} + 2\dot{\beta} \dot{\xi} + \dot{\beta} \dot{\xi} + \mu \xi + \beta \sigma \xi \right) u + \left( \dot{\beta} + 2\ddot{\mu} + 2\dot{\beta} \sigma + \beta \dot{\sigma} + \beta \eta + \mu \sigma + \beta \sigma \right) \dot{v} \\
&\quad + \left( \dddot{\mu} + 2\dddot{\beta} \eta + \dddot{\beta} \dot{\eta} + \mu \eta + \beta \sigma \eta \right) v,
\end{align*}
$$

(2.40)

and the resolvent differential equation of the fourth order for $u(t)$ is given by

$$
\begin{align*}
\dddot{u} - \alpha \dddot{u} - \lambda \mu u &= 0, \\
u^{(3)} - \alpha \dddot{u} - (\dot{\alpha} + \lambda + \beta \rho) \ddot{u} &= \beta \dddot{u} + \mu + \beta \sigma \mu \dddot{u} + \beta \dddot{u} + \dot{\beta} \eta + \beta \dot{\eta} + \cdots, \\
u^{(4)} - \alpha u^{(3)} - (2\dot{\alpha} + \lambda + \beta \rho) \dddot{u} - \cdots &= \beta + 2\dddot{\mu} + 2\dot{\beta} \sigma + \cdots, \quad \dddot{\mu} + 2\dddot{\beta} \eta + \beta \dot{\eta} + \cdots.
\end{align*}
$$

(2.41)
or
\[
(\beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma) u^{(4)} - \left[ (\beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma) \alpha \right.
+ \left( \mu + 2 \beta \eta + \eta + \mu + \beta \sigma \eta \right) \beta - \left( \beta + 2 \mu + 2 \beta \sigma + \beta \delta + \beta \eta + \mu \sigma + \beta \sigma^2 \right) \mu \big] u^{(3)}
+ \left( \cdots \right) \ddot{u} + \left( \cdots \right) \dot{u} + \left( \cdots \right) u = 0.
\] (2.42')

Its general integral can be expressed by
\[
u(t) = D_1 f_1(t) + D_2 f_2(t) + D_3 f_3(t) + D_4 f_4(t)
\] (2.43)

and no matter how the four linearly independent particular solutions are chosen, we have for their Wronskian determinant
\[
\begin{vmatrix}
 f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
 f_1'(t) & f_2'(t) & f_3'(t) & f_4'(t) \\
 f_1''(t) & f_2''(t) & f_3''(t) & f_4''(t) \\
 f_1'''(t) & f_2'''(t) & f_3'''(t) & f_4'''(t)
\end{vmatrix} = C \exp \left[ \int_{t_0}^t \frac{N(t')}{D(t')} \, dt' \right],
\] (2.44)

\((t_0: \text{arbitrary lower bound})\) where \(C \neq 0\) and \(\alpha + (N/D)\) is minus the ratio of the coefficients of \(u^{(3)}\) and \(u^{(4)}\) in (2.42'). Hence,
\[
N := (\mu + 2 \beta \eta + \beta \eta + \mu + \beta \sigma \eta) \beta - (\beta + 2 \mu + 2 \beta \sigma + \beta \delta + \beta \eta + \mu \sigma + \beta \sigma^2) \mu,
\]
\[
D := \beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma
\]

and so,
\[
\frac{N}{D} = \frac{\frac{d}{dt} \left( \frac{\beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma}{\beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma} \right)}{\beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma}.
\]

Therefore, we find for the Wronskian determinant
\[
\mathcal{W}[f_1(t), f_2(t), f_3(t), f_4(t)]
\]
\[
= C \left| \begin{vmatrix}
 \beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma \\
 \beta \dot{\mu} - \beta \mu - \mu^2 + \beta^2 \eta - \beta \mu \sigma
\end{vmatrix}_{t_0}
\exp \left( \int_{t_0}^t \left( \alpha(t') + \sigma(t') \right) \, dt' \right).
\] (2.45)

From (2.37) and (2.40), we deduce
\[
v = D^{-1} \left[ \beta u^{(3)} - (\beta + \mu + \beta \alpha + \beta \sigma) \ddot{u} + (\alpha \beta - \beta \dot{\alpha} + \alpha \mu + \beta \lambda + \alpha \beta \sigma - \beta^2 \rho) \dot{u} 
- \left( \beta \lambda - \lambda \beta + \beta \mu + \beta^2 \xi - \beta \lambda \sigma \right) u \right].
\] (2.46)

\[
\ddot{v} = D^{-1} \left[ -\mu u^{(3)} + (\mu + \beta \eta + \alpha \mu) \ddot{u} - (\alpha \mu - \mu \dot{\alpha} + \alpha \beta \eta + \alpha \beta \rho) \dot{u} 
+ (\mu \lambda - \lambda \mu + \beta \mu \xi - \beta \lambda \eta) u \right].
\] (2.47)

The complete set of solutions of the system (2.36) may be written as
\[
u(t) = D_1 f_1(t) + D_2 f_2(t) + D_3 f_3(t) + D_4 f_4(t),
\]
\[
u(t) = D_1 g_1(t) + D_2 g_2(t) + D_3 g_3(t) + D_4 g_4(t),
\] (2.48)

where the particular solution \(g_1(t)\) of the resolvent differential equation of the fourth order for
\[v(t)\text{ is uniquely determined in terms of } f_1(t)\text{ by inserting this function into the righthand side of (2.46). Thus, } g_1(t)\text{ is solely a function of } f_1(t)\text{. Similarly for } g_2(t), g_3(t)\text{ and } g_4(t)\text{. In the same manner, } g_1(t), \ldots, g_4(t)\text{ are respectively determined by inserting } f_1(t), \ldots, f_4(t)\text{ into (2.47). These results enable us to calculate}
\]

\[
\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
  \dot{f}_1(t) & \dot{f}_2(t) & \dot{f}_3(t) & \dot{f}_4(t) \\
  g_1(t) & g_2(t) & g_3(t) & g_4(t) \\
  \dot{g}_1(t) & \dot{g}_2(t) & \dot{g}_3(t) & \dot{g}_4(t)
\end{vmatrix}
\]

\[= D^{-2}
\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
  \dot{f}_1(t) & \dot{f}_2(t) & \dot{f}_3(t) & \dot{f}_4(t) \\
  b_f f_1(t) - (\beta + \mu + \beta \alpha + \beta \sigma) \dot{f}_1(t) & \ldots \\
  - \mu \dot{f}_1(t) + (\mu + \beta \eta + \alpha \mu) \dot{f}_1(t) & \ldots
\end{vmatrix}
\]

\[= D^{-2} (\beta \mu - \beta \dot{\mu} + \mu^2 - \beta^2 \eta + \beta \mu \sigma) \nabla' [f_1(t), f_2(t), f_3(t), f_4(t)]
\]

\[= C' \exp\left\{ \int_{t_0}^{t'} [\alpha(t') + \sigma(t')] \, dt' \right\}, \quad C' \neq 0,
\]

(2.49)

in which \(C'\) denotes a new proportionality factor. Inserting the expressions for \(\alpha\) and \(\sigma\) comprised in (2.39), we obtain for the left-hand side of (2.49):

\[C' \exp\left\{ \int_{t_0}^{t'} \frac{-a_{11} a_{22} + a_{12} a_{12} + (a_{24} - a_{15}) a_{12} + (a_{15} - a_{24}) a_{12} - a_{22} a_{11}}{a_{11}(t') a_{22}(t') - a_{12}^2(t')} \right\}
\]

or

\[C' \frac{|a_{11}(t_0) a_{12}(t_0) - a_{12}^2(t_0)|}{|a_{11}(t) a_{22}(t) - a_{12}^2(t)|},
\]

so that

\[
\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
  \dot{f}_1(t) & \dot{f}_2(t) & \dot{f}_3(t) & \dot{f}_4(t) \\
  g_1(t) & g_2(t) & g_3(t) & g_4(t) \\
  \dot{g}_1(t) & \dot{g}_2(t) & \dot{g}_3(t) & \dot{g}_4(t)
\end{vmatrix}
\]

\[= C'' \begin{vmatrix}
  a_{11}(t) & a_{12}(t) \\
  a_{12}(t) & a_{22}(t)
\end{vmatrix},
\]

(2.50)

whereby \(C''\) is still another arbitrary non-zero proportionality factor.

In the three-dimensional problem, with (2.24) as system of basic equations, let us represent by \(\mathcal{L}(\ldots)\) some linear function of the mentioned arguments. Then, reducing (2.24) to its normal form by solving for \(\bar{u}, \bar{v}\) and \(\bar{w}\), one gets

\[
\bar{u} = \mathcal{L}_1(\dot{u}, \dot{v}, \dot{w}, u, v, w), \quad \bar{v} = \mathcal{L}_2(\dot{u}, \dot{v}, \dot{w}, u, v, w),
\]

\[
\bar{w} = \mathcal{L}_3(\dot{u}, \dot{v}, \dot{w}, u, v, w).
\]
A further differentiation in the first one of these equations leads to
\[
u^{(3)} = \mathcal{L}_2 [ \ddot{u}, \mathcal{L}_2 (\ddot{u}, \dddot{v}, \ddddot{w}, u, v, w), \mathcal{L}_3 (\ddot{u}, \dddot{v}, \ddddot{w}, u, v, w), \dot{u}, \dddot{v}, \ddddot{w}, u, v, w] \\
= \mathcal{L}_4^* (\dddot{u}, \dddot{\dddot{v}}, \dddot{\ddddot{w}}, u, v, w).
\]
Similarly,
\[
u^{(4)} = \mathcal{L}_5 [ \nu^{(3)}, \dot{u}, \mathcal{L}_3 (\ddot{u}, \dddot{v}, \ddddot{w}, u, v, w), \mathcal{L}_4 (\ddot{u}, \dddot{v}, \ddddot{w}, u, v, w), \dot{u}, \dddot{v}, \ddddot{w}, u, v, w] \\
= \mathcal{L}_5^* (\nu^{(3)}, \dot{u}, \dddot{v}, \ddddot{w}, u, v, w),
\]
\[
u^{(5)} = \mathcal{L}_6^* (\nu^{(4)}, \nu^{(3)}, \dot{u}, \dddot{v}, \ddddot{w}, u, v, w)
\]
and
\[
u^{(6)} = \mathcal{L}_7^* (\nu^{(5)}, \nu^{(4)}, \nu^{(3)}, \dot{u}, \dddot{v}, \ddddot{w}, u, v, w).
\]
The elimination of \(\dot{v}, \dddot{v}, \ddddot{v}, u, v\) and \(w\) between the equations involving \(\mathcal{L}_1, \mathcal{L}_4^*, \mathcal{L}_3^*, \mathcal{L}_6^*\) and \(\mathcal{L}_7^*\) yields the homogeneous linear resolvent differential equation of the sixth order for \(u(t)\). Its general integral can be written as
\[
u(t) = \sum_{j=1}^{6} \mathcal{D}_j f_j(t) \tag{2.51}
\]
and no matter how the particular \(f\)-solutions are chosen, their Wronskian determinant is given by
\[
\mathcal{W}[f_1(t), \ldots, f_6(t)] = C \exp\left\{-\int_{t_0}^{t} \frac{\Lambda_1(t')}{\Lambda_0(t')} \, dt'\right\}, \quad C \neq 0, \tag{2.52}
\]
if the resolvent differential equation is symbolized by
\[
\sum_{k=0}^{5} \Lambda_k(t) \frac{d^{6-k}u}{dt^{6-k}} + \Lambda_6(t)u = 0. \tag{2.53}
\]
Solving the equations involving \(\mathcal{L}_1, \mathcal{L}_4^*, \mathcal{L}_3^*, \mathcal{L}_6^*\) with respect to \(v, w, \dot{v}\) and \(\dddot{w}\), yields unambiguous formulae for these functions expressing them linearly in terms of \(u\) and its first five lower order derivatives. In this manner, the choice of \(f_j(t)\) fixes \(g_j(t), \dot{h}_j(t), \dddot{g}_j(t)\) and \(\ddot{h}_j(t)\) \((j = 1, 2, \ldots, 6)\), and in the Wronskian determinant, \(f_j(t), f_j^{(3)}(t), f_j^{(4)}(t)\) and \(f_j^{(5)}(t)\) can be expressed in terms of \(g_j(t), \dot{g}_j(t), h_j(t)\) and \(\dot{h}_j(t)\), resp. The result is:
\[
\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) & f_5(t) & f_6(t) \\
  f_1'(t) & f_2'(t) & \cdots & \cdots & \cdots & \cdots \\
  g_1(t) & \vdots & \dddot{g}_1(t) & \dot{h}_1(t) & \dddot{h}_1(t) & \dddot{h}_1(t) \\
  h_1(t) & \vdots & \dddot{h}_1(t) & \ddot{h}_1(t) & \dot{h}_1(t) & \dddot{h}_1(t) \\
\end{vmatrix} = C' \exp\left\{\int_{t_0}^{t} \mathcal{I}(t') \, dt'\right\}, \quad C' \neq 0. \tag{2.54}
\]
where the integrand \(\mathcal{I}(t')\) consists of the sum of the coefficients of \(\dot{u}\) in the first equation, \(\dddot{v}\) in the second equation and \(\ddddot{w}\) in the third equation of the differential system in normal form.
resulting from (2.24). Therefore,\[
\mathcal{F}(t') = -\frac{1}{\mathbf{a}_{11} \mathbf{a}_{12} \mathbf{a}_{13} \mathbf{a}_{12} \mathbf{a}_{22} \mathbf{a}_{23} \mathbf{a}_{13} \mathbf{a}_{23} \mathbf{a}_{33}} \int_{t'} \begin{vmatrix}
\mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\
\mathbf{a}_{12} + \mathbf{a}_{24} - \mathbf{a}_{15} & \mathbf{a}_{22} & \mathbf{a}_{23} \\
\mathbf{a}_{13} + \mathbf{a}_{34} & \mathbf{a}_{16} & \mathbf{a}_{23} & \mathbf{a}_{33}
\end{vmatrix}_{t'},
\]
\[
\mathbf{a}_{11} \mathbf{a}_{12} + \mathbf{a}_{15} - \mathbf{a}_{24} \mathbf{a}_{13} + \mathbf{a}_{12} \mathbf{a}_{22} \mathbf{a}_{23} + \mathbf{a}_{13} \mathbf{a}_{23} + \mathbf{a}_{35} - \mathbf{a}_{26} \mathbf{a}_{33}
\]
\[+ \mathbf{a}_{12} \mathbf{a}_{22} \mathbf{a}_{23} + \mathbf{a}_{13} \mathbf{a}_{23} + \mathbf{a}_{26} - \mathbf{a}_{35} \mathbf{a}_{33} \int_{t'}.
\]

With this result inserted into (2.54), there comes
\[
\mathcal{F}(t') = -\frac{d}{dt'} \begin{vmatrix}
\mathbf{a}_{11}(t') & \mathbf{a}_{12}(t') & \mathbf{a}_{13}(t') \\
\mathbf{a}_{12}(t') & \mathbf{a}_{22}(t') & \mathbf{a}_{23}(t') \\
\mathbf{a}_{13}(t') & \mathbf{a}_{23}(t') & \mathbf{a}_{33}(t')
\end{vmatrix}_{t'} \begin{vmatrix}
\mathbf{a}_{11}(t') & \mathbf{a}_{12}(t') & \mathbf{a}_{13}(t') \\
\mathbf{a}_{12}(t') & \mathbf{a}_{22}(t') & \mathbf{a}_{23}(t') \\
\mathbf{a}_{13}(t') & \mathbf{a}_{23}(t') & \mathbf{a}_{33}(t')
\end{vmatrix}_{t'}.
\]

With this result inserted into (2.54), there comes
\[
\begin{vmatrix}
\mathbf{f}_1(t) & \mathbf{f}_2(t) & \ldots & \mathbf{f}_6(t) \\
\dot{\mathbf{f}}_1(t) & \dot{\mathbf{f}}_2(t) & \ldots \\
\mathbf{g}_1(t) & \vdots \\
\dot{\mathbf{g}}_1(t) & \vdots \\
\mathbf{h}_1(t) & \vdots \\
\dot{\mathbf{h}}_1(t) & \ldots & \dot{\mathbf{h}}_6(t)
\end{vmatrix} = C'
\begin{vmatrix}
\mathbf{a}_{11}(t_0) & \mathbf{a}_{12}(t_0) & \mathbf{a}_{13}(t_0) \\
\mathbf{a}_{12}(t_0) & \mathbf{a}_{22}(t_0) & \mathbf{a}_{23}(t_0) \\
\mathbf{a}_{13}(t_0) & \mathbf{a}_{23}(t_0) & \mathbf{a}_{33}(t_0)
\end{vmatrix}
\]
\[
- \begin{vmatrix}
\mathbf{a}_{11}(t) & \mathbf{a}_{12}(t) & \mathbf{a}_{13}(t) \\
\mathbf{a}_{12}(t) & \mathbf{a}_{22}(t) & \mathbf{a}_{23}(t) \\
\mathbf{a}_{13}(t) & \mathbf{a}_{23}(t) & \mathbf{a}_{33}(t)
\end{vmatrix} C''
\]

in agreement with (2.35).

2.2. Concerning the classical path $\mathbf{s} = \mathbf{s}_c(t)$ satisfying the system (2.24) and the boundary conditions (2.21)

The $x$-component $u_{cl}(t)$ is contained in (2.51), in which the integration constants $\mathcal{D}_1, \ldots, \mathcal{D}_6$ take on the following values:
\[
\mathcal{D}_j(t_b, t_a) = \frac{1}{\Delta(t_b, t_a)} \left[ u_a \Delta_{1j}(t_b, t_a) + u_b \Delta_{2j}(t_b, t_a) + v_a \Delta_{3j}(t_b, t_a) \\
+ v_b \Delta_{4j}(t_b, t_a) + w_a \Delta_{5j}(t_b, t_a) + w_b \Delta_{6j}(t_b, t_a) \right] \quad (j = 1, 2, \ldots, 6),
\]

(2.57)
written in the notation which was already used in (2.11) (see also (2.9)). This enables us to express the classical path in terms of the boundary positions \( s, \) and \( s_b \):

\[
\begin{align*}
\dot{u}_a(t) &= \frac{1}{\Delta(t_b, t_a)} \left( \sum_{j=1}^{6} \Delta_1(t_b, t_a) f_j(t) \right) u_a + \left( \sum_{j=1}^{6} \Delta_2(t_b, t_a) f_j(t) \right) u_b \\
&+ \left( \sum_{j=1}^{6} \Delta_3(t_b, t_a) f_j(t) \right) v_a + \left( \sum_{j=1}^{6} \Delta_4(t_b, t_a) f_j(t) \right) v_b \\
&+ \left( \sum_{j=1}^{6} \Delta_5(t_b, t_a) f_j(t) \right) w_a + \left( \sum_{j=1}^{6} \Delta_6(t_b, t_a) f_j(t) \right) w_b, 
\end{align*}
\tag{2.58}
\]

and similar formulae for \( \dot{v}_a(t) \) and \( \dot{w}_a(t) \) whereby the \( f \)-functions are replaced by the \( g \)-solutions and the \( h \)-solutions, resp., according to (2.31). In the same manner, the velocity vector along the classical path is obtained by replacement of \( f_j(t) \) in (2.58) by

\[
\begin{align*}
\dot{f}_j(t) & \quad \text{for} \quad \dot{u}_a(t), \\
\dot{g}_j(t) & \quad \text{for} \quad \dot{v}_a(t), \\
\dot{h}_j(t) & \quad \text{for} \quad \dot{w}_a(t).
\end{align*}
\]

2.3. Further development

Reconsidering (2.30), the preceding results enable us to express the Gaussian path-integral \( F_1(s_b, t_b; s_a, t_a) \) also in terms of the boundary positions \( s, \) and \( s_b \):

\[
F_1(s_b, t_b; s_a, t_a) = F(t_b, t_a) \exp \left\{ \frac{1}{\hbar} \left[ A_{11}(t_b, t_a) u_b^2 + A_{22}(t_b, t_a) v_b^2 + A_{33}(t_b, t_a) w_b^2 \\
+ A_{44}(t_b, t_a) u_a^2 + A_{55}(t_b, t_a) v_a^2 + A_{66}(t_b, t_a) w_a^2 \\
+ A_{12}(t_b, t_a) u_b v_b + A_{13}(t_b, t_a) u_b w_b + A_{14}(t_b, t_a) u_a u_a \\
+ A_{15}(t_b, t_a) u_b v_a + A_{16}(t_b, t_a) u_b w_a + A_{23}(t_b, t_a) v_b v_b \\
+ A_{24}(t_b, t_a) v_b u_b + A_{25}(t_b, t_a) v_b w_b + A_{26}(t_b, t_a) v_a w_a \\
+ A_{34}(t_b, t_a) w_b u_b + A_{35}(t_b, t_a) w_b v_b + A_{36}(t_b, t_a) w_b w_a \\
+ A_{45}(t_b, t_a) u_a v_a + A_{46}(t_b, t_a) u_a w_a + A_{56}(t_b, t_a) v_a w_a \right] \right\}.
\tag{2.59}
\]

This right-hand side involves twenty-one coefficients which may be found one after the other by inserting into (2.30) the expressions of \( \dot{u}_a(t_b), \dot{v}_a(t_b), \dot{w}_a(t_b), \dot{u}_a(t_a), \dot{v}_a(t_a), \) and \( \dot{w}_a(t_a) \) in
terms of the components of \( s_b \) and \( s_a \). In this way, one finds

\[
A_{11}(t_b, t_a) = \frac{1}{\Delta(t_b, t_a)} \sum_{j=1}^{6} \Delta_{2j}(t_b, t_a) \left[ a_{11}(t_b) f_j(t_b) + a_{12}(t_b) g_j(t_b) + a_{13}(t_b) \dot{h}_j(t_b) \right],
\]

\[
A_{12}(t_b, t_a) = \frac{1}{\Delta(t_b, t_a)} \times \left\{ \sum_{j=1}^{6} \Delta_{4j}(t_b, t_a) \left[ a_{11}(t_b) f_j(t_b) + a_{12}(t_b) g_j(t_b) + a_{13}(t_b) \dot{h}_j(t_b) \right] \right. \\
\left. + \sum_{j=1}^{6} \Delta_{2j}(t_b, t_a) \left[ a_{12}(t_b) f_j(t_b) + a_{22}(t_b) g_j(t_b) + a_{23}(t_b) \dot{h}_j(t_b) \right] \right\}
\]

and similarly for the other nineteen \( A \)-coefficients.

Next, considering that \( F_i(s_b, t_b; s_a, t_a) \), being a path-integral, is actually a Green’s function belonging to some quantum-mechanical problem, the convolution theorem may be applied to it:

\[
F_i(s_b, t_b; s_a, t_a) = \int_{s_a}^{s_b} F_i(s_b, t_b; s, t) F_i(s, t; s_a, t_a) \, ds. \tag{2.60}
\]

The arbitrary positions \( s_a \) and \( s_b \) entered the theory when \( F(t_b, t_a) \) was generalized to \( F(s_b, t_b; s_a, t_a) \) (see (2.20)–(2.22)). Now, we go back to the zero-vector for \( s_a \) and \( s_b \) in (2.59) and (2.60) for the purpose of finding an equation satisfied by \( F(t, t') \). We find in virtue of (2.28) and (2.59):

\[
F(t_b, t_a) = F(t_b, t) F(t, t_a) \int_{s_a}^{s_b} \exp \left\{ \frac{i}{\hbar} \left[ A_{44}(t_b, t) u^2 + A_{55}(t_b, t) v^2 + A_{66}(t_b, t) w^2 + A_{45}(t_b, t) uv \right. \right. \\
\left. \left. + A_{46}(t_b, t) uw + A_{56}(t_b, t) vw \right] \right\} \, du \, dv \, dw. \tag{2.61}
\]

The triple integral to be evaluated is of the Gaussian type:

\[
\mathcal{J} := \int_{s_a}^{s_b} \exp \left[ \frac{i}{\hbar} \left( \mathcal{A} u^2 + 2 \mathcal{B} uv + \mathcal{C} v^2 + 2 \mathcal{D} uw + 2 \mathcal{E} vw + \mathcal{F} w^2 \right) \right] \, du \, dv \, dw,
\]

\[
\mathcal{A} = A_{44}(t_b, t) + A_{11}(t, t_a), \quad \mathcal{B} = \frac{1}{2} \left[ A_{45}(t_b, t) + A_{12}(t, t_a) \right], \\
\mathcal{C} = A_{55}(t_b, t) + A_{22}(t, t_a), \quad \mathcal{D} = \frac{1}{2} \left[ A_{46}(t_b, t) + A_{13}(t, t_a) \right], \\
\mathcal{E} = \frac{1}{2} \left[ A_{56}(t_b, t) + A_{23}(t, t_a) \right], \quad \mathcal{F} = A_{66}(t_b, t) + A_{33}(t, t_a). \tag{2.62'}
\]

The integration may be carried out, first with respect to \( w \), then with respect to \( v \) and finally with respect to \( u \). Since the exponent is purely imaginary, the integrations have to be carried out
with the required care. Convergence is in any case ensured as can be seen when one takes into account the well-known Fresnel integrals
\[ \int_{-\infty}^{+\infty} \cos(x^2) \, dx = \int_{-\infty}^{+\infty} \sin(x^2) \, dx = \left( \frac{1}{\sqrt{\pi}} \right)^{1/2}. \]

The details of one integration can be found in [3]. One finds
\[ \mathcal{J} = \left( \frac{i\pi}{\mathcal{F}} \right)^{1/2} \left( \frac{i\pi}{\mathcal{F} - \mathcal{E}^2} \right)^{1/2} \left( \frac{i\pi}{\mathcal{A} \mathcal{F} + 2 \mathcal{B} \mathcal{D} - \mathcal{A} \mathcal{E}^2 - \mathcal{C} \mathcal{B}^2 - \mathcal{F} \mathcal{B}^2} \right)^{1/2}, \quad (2.63) \]

whereby, according to the calculations in [3], each power \( \frac{1}{4} \) should denote that branch of the square root whose real part is positive. Hence, for instance,
\[ \left( \frac{i\pi}{\mathcal{F}} \right)^{1/2} = \left( \frac{-\pi}{2 |\mathcal{F}|} \right)^{1/2} [1 + (\text{sgn } \mathcal{F})i]. \quad (2.64) \]

Consequently, when one combines the three radicands in order to obtain only one square root, i.e.,
\[ \mathcal{J} = \left( \frac{(i\pi)^3}{\mathcal{A} \mathcal{A} \mathcal{F} + 2 \mathcal{B} \mathcal{D} - \mathcal{A} \mathcal{E}^2 - \mathcal{C} \mathcal{B}^2 - \mathcal{F} \mathcal{B}^2} \right)^{1/2} \]

and one would either not know the signs of \( \mathcal{F} \) and \( \mathcal{F} - \mathcal{E}^2 \) or ignore them, a minus-sign ambiguity can occur to \( \mathcal{J} \). Indeed, if the three denominators in (2.63) are positive, then \( \mathcal{J} \) is equal to its modulus times
\[ \left( \frac{1+i}{\sqrt{2}} \right)^3 \text{ or } \frac{-1+i}{\sqrt{2}}. \]

If, on the contrary, \( \mathcal{F} \) or \( \mathcal{F} - \mathcal{E}^2 \) or both these expressions are negative while the last denominator in (2.63) is positive, then \( \mathcal{J} \) is equal to its modulus times
\[ \left( \frac{1-i}{\sqrt{2}} \right)^2 \frac{1+i}{\sqrt{2}} \text{ or } \frac{1-i}{\sqrt{2}}. \]

Similarly in the case that \( \mathcal{A} \mathcal{F} + 2 \mathcal{B} \mathcal{D} - \mathcal{A} \mathcal{E}^2 - \mathcal{C} \mathcal{B}^2 - \mathcal{F} \mathcal{B}^2 \) is negative. Quantum-mechanically, however, this minus-sign ambiguity is unimportant because a Green's function connects two wave functions through a propagator formula, and each of these wave functions is only defined apart from an arbitrary phase factor. In conclusion, we may rewrite the equality (2.61) as follows:

\[ F(t_b, t_a) = \left( \frac{(i\pi)^3}{\mathcal{A}(t_b, t, t_a) \mathcal{B}(t_b, t, t_a) \mathcal{D}(t_b, t, t_a)} \right)^{1/2} F(t_b, t) F(t, t_a), \quad (2.65) \]

and choosing for the phase angle of the complex square root +45° when the denominator is positive and -45° when it is negative, can at most entail the difference of a proportionality factor \((-1)\) in front of \( F(t, t') \) and \( K(t_b, t_b; t_a, t_a) \).
Next, the determinant in (2.65) should be brought in a proper form for the purpose of solving the equation with respect to $F$. In the one-dimensional case (see [3]), the proportionality factor between $F(t_b, t_a)$ and $F(t_b, t)F(t, t_a)$ was
\[
\left( \frac{i\pi \hbar}{\mathcal{A}(t_b, t_a)} \right)^{1/2} \tag{2.66}
\]
and it was easy to show that
\[
\mathcal{A}(t_b, t, t_a) = a(t) \left[ \frac{y_2(t_b) y_1(t_a) - y_1(t_b) y_2(t_a)}{y_2(t_b) y_1(t) - y_1(t_b) y_2(t)} \right] \left[ \frac{y_2(t_b) y_1(t_a) - y_1(t_b) y_2(t_a)}{y_2(t) y_1(t_a) - y_1(t) y_2(t_a)} \right] \tag{2.66'}
\]
where the simplification was possible provided the two particular solutions $y_1(t)$ and $y_2(t)$ satisfied (1.6). Such a form of $\mathcal{A}(t_b, t, t_a)$ was, of course, ideally suited for solving an equation of the same type as (2.65). It is evident that we should try to bring the determinant in the denominator of the proportionality factor appearing in (2.65), in an analogous form. In the three-dimensional problem which is considered in this paper, however, this is a formidable task, so voluminous and complicated that I am again obliged to go over to the two-dimensional analogue at the proper moment. In order to see this, it is sufficient to look at the expressions of $\mathcal{A}, \ldots, \mathcal{F}$. Here follow two of the given as examples:
\[
\mathcal{A}(t_b, t, t_a) = A_{44}(t_b, t) + A_{11}(t, t_a)
\]
\[
= \sum_{j=1}^{6} \frac{\Delta_{2j}(t, t_a) - \Delta_{1j}(t_b, t)}{\Delta(t, t_a)} \left[ a_{11}(t) f_j(t) + a_{12}(t) g_j(t) + a_{13}(t) h_j(t) \right],
\]
\[
\mathcal{B}(t_b, t, t_a) = \frac{1}{2} [A_{45}(t_b, t) + A_{12}(t, t_a)]
\]
\[
= \frac{1}{2} \sum_{j=1}^{6} \frac{\Delta_{4j}(t, t_a) - \Delta_{3j}(t_b, t)}{\Delta(t, t_a)} \left[ a_{11}(t) f_j(t) + a_{12}(t) g_j(t) + a_{13}(t) h_j(t) \right]
\]
\[
+ \frac{1}{2} \sum_{j=1}^{6} \frac{\Delta_{2j}(t, t_a) - \Delta_{1j}(t_b, t)}{\Delta(t, t_a)} \left[ a_{12}(t) f_j(t) + a_{22}(t) g_j(t) + a_{23}(t) h_j(t) \right].
\] (2.67)

Examining the expressions of $\mathcal{A}, \ldots, \mathcal{F}$, one discovers that the determinant under consideration is of the form
\[
\begin{vmatrix}
\mathcal{A} & \mathcal{B} & \mathcal{D} \\
\mathcal{B} & \mathcal{C} & \mathcal{E} \\
\mathcal{D} & \mathcal{E} & \mathcal{F}
\end{vmatrix}
= \begin{vmatrix}
2a_{11} U_1 + 2a_{12} U_2 + 2a_{13} U_3 \\
a_{12} U_1 + a_{22} U_2 + a_{23} U_3 + a_{11} V_1 + a_{12} V_2 + a_{13} V_3 \\
a_{13} U_1 + a_{23} U_2 + a_{33} U_3 + a_{11} W_1 + a_{12} W_2 + a_{13} W_3 \\
a_{11} V_1 + a_{12} V_2 + a_{13} V_3 + a_{11} U_1 + a_{12} U_2 + a_{13} U_3 \\
a_{12} V_1 + 2a_{22} V_2 + 2a_{23} V_3 \\
2a_{13} V_1 + a_{22} V_2 + a_{23} V_3 + a_{12} W_1 + a_{22} W_2 + a_{23} W_3 \\
a_{11} W_1 + a_{12} W_2 + a_{13} W_3 + a_{11} U_1 + a_{12} U_2 + a_{13} U_3 \\
a_{12} W_1 + a_{22} W_2 + a_{23} W_3 + a_{13} V_1 + a_{23} V_2 + a_{33} V_3 \\
2a_{13} W_1 + 2a_{23} W_2 + 2a_{33} W_3
\end{vmatrix} \tag{2.68}
\]
where

\[ U_{1,2,3} = \frac{1}{2} \sum_{j=1}^{6} \left( \frac{\Delta_j(t, t_a)}{\Delta(t, t_a)} - \frac{\Delta_j(t_b, t)}{\Delta(t_b, t)} \right) \hat{f}_j(t), \; \hat{g}_j(t), \; \hat{h}_j(t), \]

\[ V_{1,2,3} = \frac{1}{2} \sum_{j=1}^{6} \left( \frac{\Delta_j(t, t_a)}{\Delta(t, t_a)} - \frac{\Delta_j(t_b, t)}{\Delta(t_b, t)} \right) \hat{f}_j(t), \; \hat{g}_j(t), \; \hat{h}_j(t), \]

\[ W_{1,2,3} = \frac{1}{2} \sum_{j=1}^{6} \left( \frac{\Delta_j(t, t_a)}{\Delta(t, t_a)} - \frac{\Delta_j(t_b, t)}{\Delta(t_b, t)} \right) \hat{f}_j(t), \; \hat{g}_j(t), \; \hat{h}_j(t). \]

The right-hand side of (2.68) may be expanded into a sum of 216 determinants of three rows and three columns. Many of them are zero on account of two or three identical rows or columns. Carrying out the required rearrangements, one finds as an intermediate result:

\[
\begin{vmatrix}
A & B & D \\
B & C & E \\
D & E & F
\end{vmatrix}
= 2 \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{vmatrix} \begin{vmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{vmatrix} + 2 \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{vmatrix} \begin{vmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{vmatrix}
\]

\[
= 2 \begin{vmatrix}
a_{11} & U_1 & a_{13} \\
a_{12} & V_1 & a_{23} \\
a_{13} & W_1 & a_{33}
\end{vmatrix} + 2 \begin{vmatrix}
a_{11} & U_1 & a_{13} \\
a_{12} & V_1 & a_{23} \\
a_{13} & W_1 & a_{33}
\end{vmatrix} + 2 \begin{vmatrix}
a_{11} & U_1 & a_{13} \\
a_{12} & V_1 & a_{23} \\
a_{13} & W_1 & a_{33}
\end{vmatrix} + 2 \begin{vmatrix}
a_{11} & U_1 & a_{13} \\
a_{12} & V_1 & a_{23} \\
a_{13} & W_1 & a_{33}
\end{vmatrix} + 2 \begin{vmatrix}
a_{11} & U_1 & a_{13} \\
a_{12} & V_1 & a_{23} \\
a_{13} & W_1 & a_{33}
\end{vmatrix}
\]

\[ (2.69) \]

The fourth, the sixth and the eight product of determinants in this right-hand side contain terms such as

\[ 2U_1 \begin{vmatrix}
a_{22} & a_{23} \\
a_{23} & a_{33}
\end{vmatrix} \begin{vmatrix}
V_2 & V_3 \\
W_2 & W_3
\end{vmatrix}, \quad 2V_1 \begin{vmatrix}
a_{12} & a_{13} \\
a_{23} & a_{33}
\end{vmatrix} \begin{vmatrix}
U_2 & U_3 \\
W_2 & W_3
\end{vmatrix}, \]
etc., which also appear in the first product of determinants. These expressions may be complemented with analogous products so as to reproduce the first product six more times. In this manner, (2.69) becomes

\[
\begin{vmatrix}
A & B & C \\
B & C & D \\
C & D & E
\end{vmatrix} = 8 \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{vmatrix} \times \begin{vmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{vmatrix} + \mathcal{P}
\] (2.70)

where \( \mathcal{P} \) denotes an algebraic sum of 360 products, mutually different apart from a few exceptions, each consisting of the fixed coefficient 2 (also noticeable in (2.69)), three \( a \)-factors and three factors among \( U_j, V_j, W_j \) (\( j = 1, 2, 3 \)). Exactly 180 of these products have the plus-sign and the other 180 products the minus-sign in front. It will be shown near the end of the article that \( \mathcal{P} = 0 \) in virtue of three equalities which follow from the system of differential equations (2.24). Hence

\[
\begin{vmatrix}
A(t_b, t, t_a) & B(t_b, t, t_a) & C(t_b, t, t_a) \\
B(t_b, t, t_a) & C(t_b, t, t_a) & D(t_b, t, t_a) \\
C(t_b, t, t_a) & D(t_b, t, t_a) & E(t_b, t, t_a)
\end{vmatrix}
= \begin{vmatrix}
a_{11}(t) & a_{12}(t) & a_{13}(t) \\
a_{17}(t) & a_{22}(t) & a_{23}(t) \\
a_{13}(t) & a_{23}(t) & a_{33}(t)
\end{vmatrix} \times \begin{vmatrix}
2U_1 & 2U_2 & 2U_3 \\
2V_1 & 2V_2 & 2V_3 \\
2W_1 & 2W_2 & 2W_3
\end{vmatrix}
\] (2.71)

in which the \((U, V, W)\)-elements of the last determinant are given in (2.68'). Making use of a certain determinantal identity, that last determinant may be transformed into

\[
\frac{\Delta(t_b, t_a)}{\Delta(t_b, t)\Delta(t, t_a)}
\begin{vmatrix}
f_1(t) & f_2(t) & \ldots & f_6(t) \\
\dot{f}_1(t) & \dot{f}_2(t) & \ldots & \dot{f}_6(t) \\
g_1(t) & \ldots & g_6(t) \\
\dot{g}_1(t) & \ldots & \dot{g}_6(t) \\
h_1(t) & \ldots & h_6(t) \\
\dot{h}_1(t) & \ldots & \dot{h}_6(t)
\end{vmatrix}
\] (2.72)

where \( \Delta(t_b, t_a) \) is given by (2.9). Recalling that \( f_1(t), \ldots, f_6(t) \) each contain a proportionality factor which can be chosen, it is always possible to take these factors in such a way that their products lead to \( C'' = 1 \) in (2.56). Actually, the adjustment of only one factor is already sufficient to effectuate \( C'' = 1 \). Under this condition, we get

\[
\begin{vmatrix}
A(t_b, t, t_a) & B(t_b, t, t_a) & C(t_b, t, t_a) \\
B(t_b, t, t_a) & C(t_b, t, t_a) & D(t_b, t, t_a) \\
C(t_b, t, t_a) & D(t_b, t, t_a) & E(t_b, t, t_a)
\end{vmatrix} = \frac{\Delta(t_b, t_a)}{\Delta(t_b, t)\Delta(t, t_a)}
\] (2.73)
and (2.65) becomes

\[ F(t_b, t_a) = \left( \frac{(i\pi\hbar)^2}{\Delta(t_b, t_a)} \right)^{1/2} F(t_b, t) F(t, t_a). \]  (2.74)

The proof of the preceding statements can be illustrated once again by considering the twodimensional case in which the calculations are sufficiently reduced to permit inclusion in this article. In two euclidean dimensions, we have as analogues of (2.65), (2.68), (2.68') and (2.9):

\[ F(t_b, t_a) = \left( \frac{(i\pi\hbar)^2}{\Delta(t_b, t_a)} \right)^{1/2} \begin{vmatrix} \mathcal{A}(t_b, t, t_a) & \mathcal{B}(t_b, t, t_a) \\ \mathcal{B}(t_b, t, t_a) & \mathcal{C}(t_b, t, t_a) \end{vmatrix} F(t_b, t) F(t, t_a) \]  (2.75)

with

\[
\begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{C} \end{vmatrix} = \begin{vmatrix} 2a_{11}U_1 + 2a_{12}U_2 & a_{11}V_1 + a_{12}V_2 + a_{12}U_1 + a_{22}U_2 \\ a_{12}U_1 + a_{22}U_2 + a_{11}V_1 + a_{12}V_2 & 2a_{12}V_1 + 2a_{22}V_2 \end{vmatrix} \]  (2.76)

where

\[
U_{1,2} = \frac{1}{2} \sum_{j=1}^{4} \left( \frac{\Delta_{2j}(t, t_a)}{\Delta(t, t_a)} - \frac{\Delta_{1j}(t_b, t)}{\Delta(t_b, t)} \right) f_j(t), \quad \dot{g}_j(t),
\]

\[
V_{1,2} = \frac{1}{2} \sum_{j=1}^{4} \left( \frac{\Delta_{4j}(t, t_a)}{\Delta(t, t_a)} - \frac{\Delta_{3j}(t_b, t)}{\Delta(t_b, t)} \right) f_j(t), \quad \ddot{g}_j(t),
\]  (2.76')

and

\[
\Delta(t_b, t_a) = \begin{vmatrix} f_1(t_a) & f_2(t_a) & f_3(t_a) & f_4(t_a) \\ f_1(t_b) & f_2(t_b) & f_3(t_b) & f_4(t_b) \\ g_1(t_a) & g_2(t_a) & g_3(t_a) & g_4(t_a) \\ g_1(t_b) & g_2(t_b) & g_3(t_b) & g_4(t_b) \end{vmatrix} \]  (2.77)

The development of (2.76) yields

\[
\begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{C} \end{vmatrix} = a_{11} a_{12} \begin{vmatrix} U_1V_2 - U_2V_1 \end{vmatrix} + a_{11} V_1 + a_{12} V_2 \]

\[ + a_{12} V_1 \begin{vmatrix} U_2 \end{vmatrix} + a_{11} V_2 \begin{vmatrix} U_1 \end{vmatrix} \]

\[ + a_{11} \begin{vmatrix} U_1 \end{vmatrix} + a_{12} \begin{vmatrix} U_2 \end{vmatrix} \]

\[ + a_{22} \begin{vmatrix} U_2 \end{vmatrix} + \left( a_{11} a_{22} - a_{12}^2 \right) \begin{vmatrix} U_1 \end{vmatrix} \begin{vmatrix} U_2 \end{vmatrix} \]

\[ \begin{vmatrix} V_1 \end{vmatrix} \begin{vmatrix} V_2 \end{vmatrix} \]
containing the analogues of (2.69) and (2.70).

Furthermore,

\[ a_{11} V_1 + a_{12} V_2 - a_{12} U_1 - a_{22} U_2 \]

\[ = \frac{1}{2 \Delta(t, t_a)} \left\{ \begin{array}{ccc}
\begin{array}{ccc}
 f_1(t_a) & \cdots & f_4(t_a) \\
g_1(t_a) & \cdots & g_4(t_a)
\end{array} & \begin{array}{ccc}
 a_{11}(t) \\
 a_{12}(t)
\end{array} \\
 f_1(t) & \cdots & f_4(t) \\
g_1(t) & \cdots & g_4(t)
\end{array} \right\} \]

\[ + \frac{1}{2 \Delta(t_b, t)} \left\{ \begin{array}{ccc}
\begin{array}{ccc}
 f_1(t_b) & \cdots & f_4(t_b) \\
g_1(t_b) & \cdots & g_4(t_b)
\end{array} & \begin{array}{ccc}
 a_{11}(t) \\
a_{12}(t)
\end{array} \\
 f_1(t) & \cdots & f_4(t) \\
g_1(t) & \cdots & g_4(t)
\end{array} \right\} \]

\[ = \frac{1}{2} \sum_{j=1}^{4} \sum_{k=1}^{4} (-1)^{j+k} \left\{ \begin{array}{ccc}
\begin{array}{ccc}
 f_j(t_a) & f_k(t_a) \\
g_j(t_a) & g_k(t_a)
\end{array} & \begin{array}{ccc}
 a_{11}(t) \\
a_{12}(t)
\end{array} \\
 f_j(t) & f_k(t) \\
g_j(t) & g_k(t)
\end{array} \right\} \]

\[ \times \left\{ \begin{array}{ccc}
\begin{array}{ccc}
 f_j(t) & f_k(t) \\
g_j(t) & g_k(t)
\end{array} & \begin{array}{ccc}
 a_{11}(t) \\
a_{12}(t)
\end{array} \\
 f_j(t) & f_k(t) \\
g_j(t) & g_k(t)
\end{array} \right\} \]

\[ + \left\{ \begin{array}{ccc}
\begin{array}{ccc}
 f_j(t) & f_k(t) \\
g_j(t) & g_k(t)
\end{array} & \begin{array}{ccc}
 a_{11}(t) \\
a_{12}(t)
\end{array} \\
 f_j(t) & f_k(t) \\
g_j(t) & g_k(t)
\end{array} \right\}, \quad (2.79) \]
in which \((j, k, m, n)\) runs over \((1, 2, 3, 4)\) and its five permutations which satisfy \(m < n\), besides \(j < k\) as implied by the summations. The last factor between braces in the preceding result may be evaluated on the basis of the fact that, in accordance with (2.48), for every \(i \in \{1, 2, 3, 4\}\),

\[
\begin{align*}
  u &= f_i(t), \\
  v &= g_i(t),
\end{align*}
\]

is a particular solution of the differential system (2.36). For two different solutions, we have

\[
\begin{align*}
  a_{11}(f_j\dot{f}_k - f_k\dot{f}_j) + a_{12}(f_j\dot{g}_k - f_k\dot{g}_j) + a_{11}(f_j\dot{f}_k - f_k\dot{f}_j) + a_{12}(f_j\dot{g}_k - f_k\dot{g}_j) \\
  + (a_{15} - a_{24})(f_j\dot{g}_k - f_k\dot{g}_j) + (a_{15} - a_{45})(f_j\dot{g}_k - f_k\dot{g}_j) &= 0
\end{align*}
\]

or

\[
\begin{align*}
  \frac{d}{dt} \left[ a_{11}(f_j\dot{f}_k - f_k\dot{f}_j) \right] + \frac{d}{dt} \left[ a_{12}(f_j\dot{g}_k - f_k\dot{g}_j) \right] - a_{12}(f_j\dot{g}_k - f_k\dot{g}_j) \\
  + (a_{15} - a_{24})(f_j\dot{g}_k - f_k\dot{g}_j) + (a_{15} - a_{45})(f_j\dot{g}_k - f_k\dot{g}_j) &= 0,
\end{align*}
\]

and, in a similar way,

\[
\begin{align*}
  \frac{d}{dt} \left[ a_{12}(g_j\dot{f}_k - g_k\dot{f}_j) \right] - a_{12}(g_j\dot{f}_k - g_k\dot{f}_j) + \frac{d}{dt} \left[ a_{22}(g_j\dot{g}_k - g_k\dot{g}_j) \right] \\
  + (a_{24} - a_{13})(g_j\dot{f}_k - g_k\dot{f}_j) + (a_{24} - a_{34})(g_j\dot{f}_k - g_k\dot{f}_j) &= 0.
\end{align*}
\]

Addition of (2.80) and (2.81) yields

\[
\begin{align*}
  \frac{d}{dt} \left[ a_{11}(f_j\dot{f}_k - f_k\dot{f}_j) + a_{12}(f_j\dot{g}_k - f_k\dot{g}_j) + a_{12}(g_j\dot{f}_k - g_k\dot{f}_j) \\
  + a_{22}(g_j\dot{g}_k - g_k\dot{g}_j) + (a_{15} - a_{24})(f_j\dot{g}_k - f_k\dot{g}_j) \right] &= 0,
\end{align*}
\]

and so, integrating with respect to \(t\), we get:

\[
\begin{align*}
  &\left| f_j(t) \quad f_k(t) \right| a_{11}(t) + \left| f_j(t) \quad f_k(t) \right| a_{12}(t) + \left| g_j(t) \quad g_k(t) \right| a_{12}(t) \\
  + \left| g_j(t) \quad g_k(t) \right| a_{22}(t) + \left| f_j(t) \quad f_k(t) \right| \left[ a_{15}(t) - a_{24}(t) \right] = \text{const.}
\end{align*}
\]

in which the constant may be put equal to either the value of the left-hand side at \(t = t_a\) or the value of the left-hand side at \(t = t_b\). Making use of (2.82) in (2.79), there comes

\[
\begin{align*}
  a_{11}V_1 + a_{12}V_2 - a_{12}U_1 - a_{22}U_2 \\
  &= \frac{1}{2\Delta(t, t_a)} \left| \begin{array}{ccc}
    f_1(t_a) & \cdots & f_4(t_a) \\
    g_1(t_a) & \cdots & g_4(t_a) \\
    f_1(t) & \cdots & f_4(t) \\
    g_1(t) & \cdots & g_4(t)
  \end{array} \right| \left[ a_{15}(t) - a_{24}(t) \right]
\end{align*}
\]
\[
\begin{align*}
&\begin{vmatrix}
  f_1(t_a) & \ldots & f_4(t_a) \\
  g_1(t_a) & \ldots & g_4(t_a) \\
  f_1(t_a) & \ldots & f_4(t_a) \\
  f_1(t_a) & \ldots & f_4(t_a)
\end{vmatrix} a_{11}(t_a) - \\
&\begin{vmatrix}
  f_1(t_a) & \ldots & f_4(t_a) \\
  g_1(t_a) & \ldots & g_4(t_a) \\
  g_1(t_a) & \ldots & g_4(t_a) \\
  f_1(t_a) & \ldots & f_4(t_a)
\end{vmatrix} a_{12}(t_a) \\
&\begin{vmatrix}
  f_1(t_a) & \ldots & f_4(t_a) \\
  g_1(t_a) & \ldots & g_4(t_a) \\
  g_1(t_a) & \ldots & g_4(t_a) \\
  f_1(t_a) & \ldots & f_4(t_a)
\end{vmatrix} a_{22}(t_a)
\end{align*}
\]

\[
= -\frac{1}{2\Delta(t_b, t)} \begin{vmatrix}
  f_1(t_b) & \ldots & f_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b)
\end{vmatrix} [a_{15}(t) - a_{24}(t)]
\]

\[
\begin{align*}
&\begin{vmatrix}
  f_1(t_b) & \ldots & f_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b)
\end{vmatrix} a_{11}(t_b) - \\
&\begin{vmatrix}
  f_1(t_b) & \ldots & f_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b)
\end{vmatrix} a_{12}(t_b) \\
&\begin{vmatrix}
  f_1(t_b) & \ldots & f_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  g_1(t_b) & \ldots & g_4(t_b) \\
  f_1(t_b) & \ldots & f_4(t_b)
\end{vmatrix} a_{22}(t_b)
\end{align*}
\]

\[
= -\frac{1}{2} [a_{15}(t) - a_{24}(t)] + \frac{1}{2} [a_{15}(t) - a_{24}(t)] = 0. \tag{2.83}
\]
In conclusion, we have:

\[
\begin{bmatrix}
A(t_b, t, t_a) & B(t_b, t, t_a) \\
B(t_b, t, t_a) & C(t_b, t, t_a)
\end{bmatrix}
= \begin{bmatrix}
a_{11}(t) & a_{12}(t) \\
a_{12}(t) & a_{22}(t)
\end{bmatrix} \times \begin{bmatrix}
2U_1 & 2U_2 \\
2V_1 & 2V_2
\end{bmatrix}
\tag{2.84}
\]

with \(U_1, U_2, V_1,\) and \(V_2\) given by (2.76').

To achieve our final goal, the last determinant in (2.84) should be cast into an appropriate form. Actually, its elements are themselves determinantal expressions which should preferably be written out in full for the sake of clarity. In order to reduce the volume of the notation, let us work with the following abbreviation:

\[
\begin{bmatrix}
\alpha_j \\
\beta_j \\
\gamma_j \\
\delta_j
\end{bmatrix} := \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\delta_1 & \delta_2 & \delta_3 & \delta_4
\end{bmatrix}
\tag{2.85}
\]

Then,

\[
\begin{bmatrix}
2U_1 & 2U_2 \\
2V_1 & 2V_2
\end{bmatrix}
= \frac{1}{\Delta^2(t, t_a)\Delta^2(t_b, t)}
\times \left\{ \begin{bmatrix}
f_j(t_a) \\
f_j(t) \\
g_j(t_a) \\
g_j(t)
\end{bmatrix} \times \begin{bmatrix}
f_j(t_a) \\
f_j(t) \\
g_j(t_a) \\
g_j(t)
\end{bmatrix} - \begin{bmatrix}
f_j(t) \\
f_j(t_a) \\
g_j(t) \\
g_j(t_a)
\end{bmatrix} \times \begin{bmatrix}
f_j(t) \\
f_j(t_a) \\
g_j(t) \\
g_j(t_a)
\end{bmatrix} \right\}
\tag{2.86}
\]

This numerator may be rewritten as

\[
\begin{bmatrix}
f_j(t) \\
g_j(t) \\
f_j(t_a) \\
g_j(t_a)
\end{bmatrix} \times \begin{bmatrix}
f_j(t) \\
g_j(t) \\
f_j(t_a) \\
g_j(t_a)
\end{bmatrix}
\times \begin{bmatrix}
f_j(t) \\
g_j(t) \\
f_j(t_a) \\
g_j(t_a)
\end{bmatrix}
\times \begin{bmatrix}
f_j(t) \\
g_j(t) \\
f_j(t_a) \\
g_j(t_a)
\end{bmatrix}
\]
This expression can be transformed into a product of four \((4, 4)\)-determinants by applying to it a determinantal identity which happens to be a generalization of

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 \\
\rho_1 & \rho_2 \\
\xi_1 & \xi_2 \\
\lambda_1 & \lambda_2 \\
\end{vmatrix} \times \begin{vmatrix}
\rho_1 & \rho_2 \\
\beta_1 & \beta_2 \\
\xi_1 & \xi_2 \\
\lambda_1 & \lambda_2 \\
\end{vmatrix} = \begin{vmatrix}
\alpha_1 & \alpha_2 \\
\rho_1 & \rho_2 \\
\xi_1 & \xi_2 \\
\lambda_1 & \lambda_2 \\
\end{vmatrix}.
\]

The identity reads:

\[
\begin{vmatrix}
\alpha_j & \rho_j \\
\sigma_j & \sigma_j \\
\lambda_j & \lambda_j \\
\mu_j & \mu_j \\
\end{vmatrix} \times \begin{vmatrix}
\rho_j & \rho_j \\
\beta_j & \beta_j \\
\xi_j & \xi_j \\
\lambda_j & \lambda_j \\
\end{vmatrix} = \begin{vmatrix}
\alpha_j & \rho_j \\
\sigma_j & \sigma_j \\
\lambda_j & \lambda_j \\
\mu_j & \mu_j \\
\end{vmatrix}.
\]

It is easy to calculate that the left-hand side of this identity involves \(2^{654208} (= 2 \times 4 \times 24^4)\) products of sixteen elements. 2322432 of these products (being \(\frac{7}{8}\) of the total) cancel one another and so, the remaining \(331776 (= 24^4)\) products, linearly combined with plus- and minus-signs, constitute the right-hand side. A verification of (2.88) can be carried out by expanding all determinants in terms of the subdeterminants of the \((2, 2)\)-minors comprised in their lowest two rows and showing that the coefficients with which the various products of such subdeterminants appear on both sides are either zero or equal. Applying the identity (2.88) to (2.86'), we obtain for (2.86):

\[
\begin{vmatrix}
2U_1 & 2U_2 \\
2V_1 & 2V_2 \\
\end{vmatrix} = \frac{1}{\Delta^2(t, t_a) \Delta^2(t_b, t)} \begin{vmatrix}
f_j(t) & f_j(t) & f_j(t_a) & f_j(t) \\
g_j(t) & g_j(t) & g_j(t_a) & g_j(t) \\
f_j(t_a) & f_j(t_b) & f_j(t_b) & f_j(t) \\
g_j(t_a) & g_j(t_b) & g_j(t_b) & g_j(t) \\
\end{vmatrix} \times \begin{vmatrix}
f_j(t) & f_j(t) & f_j(t_a) & f_j(t) \\
g_j(t) & g_j(t) & g_j(t_a) & g_j(t) \\
f_j(t_a) & f_j(t_b) & f_j(t_b) & f_j(t) \\
g_j(t_a) & g_j(t_b) & g_j(t_b) & g_j(t) \\
\end{vmatrix}.
\]

\[
\begin{vmatrix}
f_j(t) & f_j(t) & f_j(t) & f_j(t) \\
g_j(t) & g_j(t) & g_j(t_a) & g_j(t) \\
f_j(t_a) & f_j(t_b) & f_j(t_b) & f_j(t) \\
g_j(t_a) & g_j(t_b) & g_j(t_b) & g_j(t) \\
\end{vmatrix}.
\]
We have seen (cf. (2.50)) that the last determinant is always proportional to
\[
\begin{vmatrix}
  a_{11}(t) & a_{12}(t) \\
  a_{12}(t) & a_{22}(t)
\end{vmatrix}
\]^{-1}.

The proportionality factors comprised in \( f_1(t), \ldots, f_4(t) \) may be taken in such a manner that their product leads to the value \( C'' = 1 \) in (2.50). Then,
\[
\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
  f_1'(t) & f_2'(t) & f_3'(t) & f_4'(t) \\
  g_1(t) & g_2(t) & g_3(t) & g_4(t) \\
  g_1'(t) & g_2'(t) & g_3'(t) & g_4'(t)
\end{vmatrix}
= \frac{1}{\begin{vmatrix}
  a_{11}(t) & a_{12}(t) \\
  a_{12}(t) & a_{22}(t)
\end{vmatrix}},
\]
(2.90)
and therefore, in virtue of (2.75), (2.84), (2.89) and (2.90), we have
\[
F(t_b, t_a) = \left( \frac{(i\pi\hbar)^2 \Delta(t_b, t) \Delta(t, t_a)}{\Delta(t_b, t_a)} \right)^{1/2} F(t_b, t) F(t, t_a),
\]
(2.91)
which evidently is the twodimensional analogue of (2.74).

In the three-dimensional case, one deduces from the differential system (2.24), for any \((j, k) \in \{1, 2, 3, 4, 5, 6\}^2 \) whereby \( j \neq k \):
\[
\begin{vmatrix}
  f_j(t) & f_k(t) \\
  f_j'(t) & f_k'(t)
\end{vmatrix} a_{11}(t) + \begin{vmatrix}
  f_j(t) & f_k(t) \\
  g_j(t) & g_k(t)
\end{vmatrix} a_{12}(t) + \begin{vmatrix}
  f_j(t) & f_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} a_{13}(t)
+ \begin{vmatrix}
  g_j(t) & g_k(t) \\
  f_j(t) & f_k(t)
\end{vmatrix} a_{21}(t) + \begin{vmatrix}
  g_j(t) & g_k(t) \\
  g_j(t) & g_k(t)
\end{vmatrix} a_{22}(t) + \begin{vmatrix}
  g_j(t) & g_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} a_{23}(t)
+ \begin{vmatrix}
  h_j(t) & h_k(t) \\
  f_j(t) & f_k(t)
\end{vmatrix} a_{31}(t) + \begin{vmatrix}
  h_j(t) & h_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} a_{32}(t) + \begin{vmatrix}
  h_j(t) & h_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} a_{33}(t)
+ \begin{vmatrix}
  f_j(t) & f_k(t) \\
  g_j(t) & g_k(t)
\end{vmatrix} [a_{15}(t) - a_{24}(t)] + \begin{vmatrix}
  f_j(t) & f_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} [a_{16}(t) - a_{34}(t)]
+ \begin{vmatrix}
  g_j(t) & g_k(t) \\
  h_j(t) & h_k(t)
\end{vmatrix} [a_{26}(t) - a_{35}(t)] = \text{const.} = \begin{cases} \text{the l.h. side at } t = t_a, \\ \text{the l.h. side at } t = t_b, \end{cases}
\]
(2.92)
as counterpart of (2.82). By the same method as was used in (2.79) and (2.83), one deduces from this result the following equalities which are the analogues of \( a_{11}V_1 + a_{12}V_2 - a_{13}U_1 - a_{22}U_2 + a_{23}W_1 - a_{13}V_1 - a_{23}V_2 - a_{33}V_3 = 0 \):
\[
a_{12}W_1 + a_{22}W_2 + a_{23}W_3 - a_{13}V_1 - a_{23}V_2 - a_{33}V_3 = 0,
\]
\[
a_{13}U_1 + a_{23}U_2 + a_{33}U_3 - a_{11}W_1 - a_{12}W_2 - a_{13}W_3 = 0,
\]
\[
a_{11}V_1 + a_{12}V_2 + a_{13}V_3 - a_{12}U_1 - a_{22}U_2 - a_{23}U_3 = 0.
\]
(2.93)
Symbolizing the left-hand sides of these equalities respectively by $R_1$, $R_2$, and $R_3$, makes it possible to express the algebraic sum of 360 products called $F$ in (2.70) as follows:

$$F = -2\left\{ \left( a_{11}U_1 + a_{12}U_2 + a_{13}U_3 \right) R_1^2 + \left( a_{12}U_1 + a_{22}U_2 + a_{23}U_3 \right) R_1R_2 + \left( a_{13}U_1 + a_{23}U_2 + a_{33}U_3 \right) R_1R_3 \right\}$$

This right-hand side is a priori an algebraic sum of 972 products. I calculated it (by hand!) and found out that in each of the three parts between square brackets, 180 products cancel one another in pairs, thus leaving 144 products between each of the three couples of square brackets. In addition, 24 products cancel one another between the first two parts, and the same holds of course between the other two couples of parts. Hence, in the end, 360 products are left over and their algebraic sum is precisely that represented by $F$. In virtue of (2.93) (and therefore not an account of further cancellations in pairs), there comes

$$F = 0$$

which confirms (2.71).

Next, writing out in full the last determinant of the right-hand side of (2.71) making use of (2.68'), bringing all parts on the common denominator $\Delta^3(t, t_a)\Delta^3(t_b, t)$ and carrying out the necessary permutations of rows in all $(6, 6)$-determinants of the resulting numerator in order to obtain exclusively $f_j(t_a)$, $g_j(t_a)$, $h_j(t_a)$ and $f_j(t_b)$, $g_j(t_b)$, $h_j(t_b)$ in the lowest three rows, we find the counterpart of (2.86'), but this time with a minus-sign in front. To that new numerator, we now apply the following determinantal identity:
evidently making use of the same kind of abbreviated notation as defined in (2.85), with \( j \) now running from 1 to 6. The left-hand side involves \( 6687075 \, 336 \, 192 \times 10^6 \) products of thirty-six elements. Pairwise cancellations reduce the number of products to \( 1/48 \) of that quantity, i.e., \( 139314069 \, 504 \times 10^6 \) \( (= \, 720^6) \) products of thirty-six factors, the algebraic sum of which constitutes the right-hand side.

The verification of (2.95)–(2.96) could in principle be carried out in a similar manner as described for (2.88), but even for a fast computer it would be a huge task. Fortunately, there exist proofs of the “n-dimensional” generalization of the identities (2.87), (2.88) and (2.95)–(2.96) (see the Appendix).
The application of (2.95)–(2.96) within the present context yields

\[
\begin{vmatrix}
2U_1 & 2U_2 & 2U_3 \\
2V_1 & 2V_2 & 2V_3 \\
2W_1 & 2W_2 & 2W_3 \\
\end{vmatrix}
= \frac{1}{\Delta^3(t, t_a)\Delta^3(t_b, t)}
\begin{vmatrix}
f_j(t) & f_j(t) & f_j(t) \\
g_j(t) & g_j(t) & g_j(t) \\
h_j(t) & h_j(t) & h_j(t) \\
f_j(t) & f_j(t) & f_j(t) \\
g_j(t) & g_j(t) & g_j(t) \\
h_j(t) & h_j(t) & h_j(t) \\
\end{vmatrix}
\]

Choosing the proportionality factors in \(f_1(t), \ldots, f_6(t)\) in such a manner that \(C = 1\) in (2.35), we arrive at

\[
\begin{vmatrix}
a_{11}(t) & a_{12}(t) & a_{13}(t) \\
a_{12}(t) & a_{22}(t) & a_{23}(t) \\
a_{13}(t) & a_{23}(t) & a_{33}(t) \\
\end{vmatrix}
\begin{vmatrix}
2U_1 & 2U_2 & 2U_3 \\
2V_1 & 2V_2 & 2V_3 \\
2W_1 & 2W_2 & 2W_3 \\
\end{vmatrix}
= \frac{\Delta(t_b, t_a)}{\Delta(t_b, t)\Delta(t, t_a)}
\]

which proves (2.71)–(2.73) and consequently (2.74).

The equations (2.91) and (2.74) respectively corresponding to the two- and three-dimensional cases are so similar to the equation

\[
F(t_b, t_a) = \left( \frac{i\pi\hbar}{y_2(t_b) y_1(t) - y_1(t_b) y_2(t)} \right)^{1/2}
\times F(t_b, t) F(t, t_a)
\]

stemming from the one-dimensional problem treated in detail in [3], that it would be mere repetition to describe the method of solution here. I shall therefore restrict myself to simply stating the general solution of (2.91) and (2.74):

\[
F(t_b, t_a) = \exp \left[ \int_{t_a}^{t_b} \phi(t) \, dt \right] / \left[ (i\pi\hbar)^{\kappa} \Delta(t_b, t_a) \right]^{1/2}
\]

where \(\kappa\) denotes the number of euclidean dimensions and \(\phi(t)\) is an arbitrary continuous function of \(t\). Precisely which function of \(t\) is to be used in (2.100) in order to obtain a formula for propagators such as (2.3), (2.22) and (2.26), is a question which requires additional
calculations. Such calculations can also be found in [3]. Inserting (2.100) into (2.59), a comparison is made between the resulting formula for \( F_i(s_b, t_b; s_a, t_a) \) and the path-integral representation (2.26) of this Green’s function in the case of infinitesimally small \( t_b - t_a \). The former result can be expanded into suitable powers of \( t_b - t_a \) whereas to the path-integral the time-discretization method may be applied with only one time slice since \( t_b - t_a \) is assumed to be infinitesimally small. In the same way as in [3], comparing the lowest order approximations yielded the two procedures leads to \( \phi(t) = 0 \). \( \forall t \in \mathbb{R} \).

3. Conclusion

3.1. It has been shown that the quantum-mechanical Green’s function \( K(r, t; r_a, t_a) \) in three euclidean dimensions, represented in Feynman’s formalism by the Gaussian path-integral

\[
\int_{r_a}^{r_b} \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} L_G([r(t), \dot{r}(t), t]) dt \right\} Dr(t)
\]  

where \( L_G(r, \dot{r}, t) \) is given by (2.2), has the following explicit form

\[
\exp \left\{ \frac{i}{\hbar} S_{cl}(b, a) \right\} / \left\{ (i\pi \hbar)^3 \Delta(t_b, t_a) \right\}^{1/2},
\]  

in which \( S_{cl}(b, a) \), denoting the classical action between \((t_a, r_a)\) and \((t_b, r_b)\), is given by (2.12) and \( \Delta(t_b, t_a) \) is the \((6, 6)\)-determinant defined in (2.9) whereby \( f_j(t), g_j(t), h_j(t) \) \((j = 1, 2, \ldots, 6)\) are six different particular solutions of the system of coupled homogeneous linear differential equations (2.24). Each of these solutions is uniquely determined once the choice of one of the functions, for instance the \( f \)-function, is made. The six \( f \)-functions are particular solutions of the homogeneous linear resolvent differential equation of the sixth order for \( u(t) \) resulting from the elimination of \( v, w \) and their derivatives of various orders in the system (2.24). They may be arbitrarily chosen provided that the proportionality factors a priori comprised in them, are taken in such a way that

\[
\begin{vmatrix}
    f_1(t) & f_2(t) & \cdots & f_6(t) \\
    \dot{f}_1(t) & f_2(t) & \cdots & \dot{f}_6(t) \\
    g_1(t) & \cdots & g_6(t) \\
    \dot{g}_1(t) & \cdots & \dot{g}_6(t) \\
    h_1(t) & \cdots & h_6(t) \\
    \dot{h}_1(t) & \cdots & \dot{h}_6(t)
\end{vmatrix} = \frac{1}{a_{11}(t) a_{12}(t) a_{13}(t)} \begin{vmatrix}
    a_{11}(t) & a_{12}(t) & a_{13}(t) \\
    a_{12}(t) & a_{22}(t) & a_{23}(t) \\
    a_{13}(t) & a_{23}(t) & a_{33}(t)
\end{vmatrix}.
\]  

This sole condition ensures at the same time the linear independence of \( f_1(t), f_2(t), \ldots, f_6(t) \). With the power \( \frac{1}{2} \) in the denominator of (3.2), there correspond two square roots differing from one another only by the sign. For one of these signs, (3.1) and (3.2) are equal, but if in practice the correct sign is too difficult to determine, the quantum-mechanical consequence is totally unimportant because a Green’s function appearing in a propagator formula connects two quantum-mechanical state vectors each of which is only determined apart from an arbitrary phase factor.
3.2. In the case of two euclidean dimensions which is less interesting from the quantum-mechanical point of view, but can nevertheless be considered theoretically, the results may be deduced from the preceding ones in a straightforward manner. The Green’s function becomes

$$\exp\left[\frac{i}{\hbar} S_d(b, a)\right] \left[\left((i\pi\hbar)^2 \Delta(t_b, t_a)\right)^{1/2}\right]$$

(3.4)

where $S_d(b, a)$ is the expression resulting from (2.12) by deleting all terms involving the subscripts 3 and 6, and $\Delta(t_b, t_a)$ is now the $(4, 4)$-determinant (2.77) in which $f_j(t)$, $g_j(t)$ $(j = 1, 2, 3, 4)$ are four different particular solutions of the differential system (2.36). The four $f$-functions are particular solutions of the resolvent differential equation of the fourth order for $u(t)$ resulting from (2.36), chosen in such a way that

$$\begin{vmatrix}
  f_1(t) & f_2(t) & f_3(t) & f_4(t) \\
  f_1(t) & f_2(t) & \ldots & f_4(t) \\
  g_1(t) & \ldots & g_4(t) \\
  \dot{g}_1(t) & \ldots & g_4(t) 
\end{vmatrix} = \frac{1}{\left|\begin{array}{cc}
a_{11}(t) & a_{12}(t) \\
a_{12}(t) & a_{22}(t) \end{array}\right|}$$

(3.5)

3.3. A verification for $I(t_b, t_a)$. In the special case whereby $x$, $y$ and $z$ are uncoupled in the quadratic Lagrange function $L_G$, i.e.,

$$L_G(r, \dot{r}, t) = a_{11}(t)\dot{x}^2 + 2a_{14}(t)\dot{x}x + a_{44}(t)x^2 + 2b_1(t)\dot{x} + 2b_4(t)x$$

$$+ a_{22}(t)\dot{y}^2 + 2a_{25}(t)\dot{y}y + a_{55}(t)y^2 + 2b_2(t)\dot{y} + 2b_5(t)y$$

$$+ a_{33}(t)\dot{z}^2 + 2a_{36}(t)\dot{z}z + a_{66}(t)z^2 + 2h_3(t)\dot{z} + 2h_6(t)z + c(t),$$

the corresponding Green’s function $K(r_b, t_b; r_a, t_a)$ is given by the product of three one-dimensional kernel functions similar to (1.3). This can be confirmed as follows. The system (2.24) now consists of three uncoupled differential equations:

$$a_{11}(t)\ddot{u} + \ddot{a}_{11}(t)\dot{u} + \left[\ddot{a}_{14}(t) - a_{44}(t)\right]u = 0,$$

$$a_{22}(t)\ddot{v} + \ddot{a}_{22}(t)\dot{v} + \left[\ddot{a}_{25}(t) - a_{55}(t)\right]v = 0,$$

$$a_{33}(t)\ddot{w} + \ddot{a}_{33}(t)\dot{w} + \left[\ddot{a}_{36}(t) - a_{66}(t)\right]w = 0,$$

(3.6)

of course, of the same type as (1.4). Here, it no longer makes sense to consider a resolvent differential equation of the sixth order, but the general integral of (3.6) still involves six arbitrary (real) constants of integration. In the light of the theory developed in this paper (see (2.31)), that general integral may be written as follows:

$$u(t) = \mathscr{D}_1 f_1(t) + \mathscr{D}_2 f_2(t), \quad v(t) = \mathscr{D}_3 g_3(t) + \mathscr{D}_4 g_4(t),$$

$$w(t) = \mathscr{D}_5 h_3(t) + \mathscr{D}_6 h_4(t),$$

(3.7)

with $f_3(t), f_4(t), \ldots, h_4(t)$ chosen equal to zero. Each right-hand side is itself the general integral of the corresponding differential equation in (3.6). To ensure the connection with the one-dimen-
sional problem treated in [3], nothing prevents us from determining $f_2(t)$, $g_4(t)$ and $h_6(t)$ by means of

$$f_2(t) = f_1(t) \int_{\tau}^{t} \frac{dt'}{a_{11}(t')f_1^2(t')}, \quad g_4(t) = g_3(t) \int_{\tau}^{t} \frac{dt'}{a_{22}(t')g_3^2(t')},$$

$$h_6(t) = h_5(t) \int_{\tau}^{t} \frac{dt'}{a_{33}(t')h_5^2(t')}.$$  

In this way, the Wronskian determinants become

$$\mathcal{W}[f_1(t), f_2(t)] = 1/a_{11}(t), \quad \mathcal{W}[g_3(t), g_4(t)] = 1/a_{22}(t),$$  

$$\mathcal{W}[h_5(t), h_6(t)] = \frac{1}{a_{33}(t)},$$

and consequently,

$$
\begin{bmatrix}
  f_1(t) & f_2(t) & 0 & 0 & 0 & 0 \\
  f_1'(t) & f_2'(t) & 0 & 0 & 0 & 0 \\
  0 & 0 & g_3(t) & g_4(t) & 0 & 0 \\
  0 & 0 & \dot{g}_3(t) & \dot{g}_4(t) & 0 & 0 \\
  0 & 0 & 0 & 0 & h_5(t) & h_6(t) \\
  0 & 0 & 0 & 0 & \dot{h}_5(t) & \dot{h}_6(t)
\end{bmatrix} = \frac{1}{a_{11}(t)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22}(t) & 0 \\ 0 & 0 & a_{33}(t) \end{bmatrix}
$$

being the analogue of (3.3). With the particular solutions so defined, (3.2) yields

$$\exp \left[ \frac{i}{\hbar} S_{cl}(b, a) \right] / \left[ (i\pi\hbar)^3 \Delta(t_b, t_a) \right]^{1/2}$$

with

$$\Delta(t_b, t_a) = \left| \begin{array}{cccccc}
  f_1(t_a) & f_2(t_a) & 0 & 0 & 0 & 0 \\
  f_1(t_b) & f_2(t_b) & 0 & 0 & 0 & 0 \\
  0 & 0 & g_3(t_a) & g_4(t_a) & 0 & 0 \\
  0 & 0 & \dot{g}_3(t_b) & \dot{g}_4(t_b) & 0 & 0 \\
  0 & 0 & 0 & 0 & h_5(t_a) & h_6(t_a) \\
  0 & 0 & 0 & 0 & \dot{h}_5(t_b) & \dot{h}_6(t_b)
\end{array} \right|. \tag{3.9'}$$

It is clear that the denominator in (3.9) resulting from the calculations in Section 2, may be rewritten as

$$\left[ (i\pi\hbar)^3 \Delta(t_b, t_a) \right]^{1/2} = \left\{ i\pi\hbar \left[ f_2(t_b)f_1(t_a) - f_1(t_b)f_2(t_a) \right] \right\}^{1/2} \times \left\{ i\pi\hbar \left[ g_4(t_b)g_3(t_a) - g_3(t_b)g_4(t_a) \right] \right\}^{1/2} \times \left\{ i\pi\hbar \left[ h_6(t_b)h_5(t_a) - h_5(t_b)h_6(t_a) \right] \right\}^{1/2},$$

being the product of three expressions which are the counterparts of the denominator in (1.3).
Appendix

The generalization of the identities (2.87), (2.88) and (2.95)–(2.96) reads

\[
\begin{vmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{vmatrix} = \sum_{j=1}^{2n} \begin{vmatrix}
\rho_{1j}^{-1} & \rho_{2j}^{-1} & \cdots & \rho_{nj}^{-1} \\
\rho_{1j} & \rho_{2j} & \cdots & \rho_{nj} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1j} & \lambda_{2j} & \cdots & \lambda_{nj}
\end{vmatrix} \times \begin{vmatrix}
\alpha_{1j} \\
\alpha_{2j} \\
\vdots \\
\alpha_{nj}
\end{vmatrix}
\]

\[
(j = 1, 2, \ldots, 2n; \ n = 1, 2, \ldots)
\]

where

\[
T_{kl} := \begin{vmatrix}
\rho_{1j} & \rho_{2j} & \cdots & \rho_{nj} \\
\rho_{1j} & \rho_{2j} & \cdots & \rho_{nj} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1j} & \rho_{2j} & \cdots & \rho_{nj}
\end{vmatrix} \times \begin{vmatrix}
\lambda_{1j} & \lambda_{2j} & \cdots & \lambda_{nj} \\
\lambda_{1j} & \lambda_{2j} & \cdots & \lambda_{nj} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1j} & \lambda_{2j} & \cdots & \lambda_{nj}
\end{vmatrix} \times \begin{vmatrix}
\alpha_{1j} \\
\alpha_{2j} \\
\vdots \\
\alpha_{nj}
\end{vmatrix} \quad (k, l = 1, 2, \ldots, n).
\]

Recalling the abbreviation (2.85), it is clear that the above right-hand sides consist of determinants of degree \(2n\) fully specified by the written column.

I have submitted this identity to Prof. Dr. S.H. Weintraub as a problem to be published in *The Mathematical Intelligencer* (Vol. 10, No. 4, 1988) Dr. Weintraub has judged it preferable to rewrite the problem in terms of matrices. As a consequence, asking for a proof of the preceding determinantal identity may be reformulated in terms of the following problem statement:

Let \(X = (x_{ij})\), \(Y = (y_{ij})\), \(Z = (z_{ij})\) and \(W = (w_{ij})\) be \((n, 2n)\)-matrices \((n = 1, 2, \ldots)\) consisting of mutually independent elements each of which can be real or complex. Let \(Y_{kl}\) be the matrix
obtained from $Y$ by replacing the $k$th row of $Y$ by the $l$th row of $X$ ($k, l = 1, 2, \ldots, n$). Define the $(n, n)$-matrix $T = (t_{ki})$ by

$$t_{ki} = \det\begin{pmatrix} Y_{ki} \\ Z \end{pmatrix} \cdot \det\begin{pmatrix} Y \\ W \end{pmatrix} - \det\begin{pmatrix} Y_{ki} \\ W \end{pmatrix} \cdot \det\begin{pmatrix} Y \\ Z \end{pmatrix}.$$ 

Show that

$$\det(T) = \left[ \det\begin{pmatrix} Y \\ Z \end{pmatrix} \right]^{n-1} \cdot \left[ \det\begin{pmatrix} Y \\ W \end{pmatrix} \right]^{n-1} \cdot \det\begin{pmatrix} Z \\ W \end{pmatrix} \cdot \det\begin{pmatrix} X \\ Y \end{pmatrix}.$$ 

(Here, $\begin{pmatrix} X \\ Y \end{pmatrix}$ denotes the $(2n, 2n)$-matrix obtained by writing matrix $X$ on top of matrix $Y$, and similarly for the other analogous symbols.)

In the meantime, two solutions of this problem have been found and sent to me as private communications, one by K. Coolsaet, assistant at the Seminary for Algebra and Functional Analysis of the State University of Ghent, and the other by D. Constales, research fellow of the National Fund of Scientific Research, working at the same Seminary under the directorship of prof. Dr. R. Delanghe. These proofs will also be submitted to Prof. Dr. S.H. Weintraub, Editor of the column “Mathematical Entertainments” in the periodical *The Mathematical Intelligencer*.

**References**


