The Number of Blocks with a Given Defect Group

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Communicated by I. N. Herstein
Received August 10, 1982

An important question in modular representation theory is: When is the $p$-subgroup $D$ of the finite group $G$ a defect group for some $p$-block of $G$? Brauer, of course, showed that we need only consider the case when $D \triangleleft G$, and reduced the problem further to questions about blocks of defect 0 of $DC_D(D)/D$ and their inertia subgroups in $G$. However, when $D = 1$, for example, this is not a reduction at all.

In fact, very few conditions are known to be sufficient for $D$ to be a defect group. One condition was given by Brauer and Fowler [1], and further conditions of a similar nature were given by Tsushima [2] and Wada [3]. In this paper, we give a precise formula for the number of $p$-blocks with defect group $D$ by extending the techniques of the three above-mentioned papers, together with some new ideas. We mention some situations where group-theoretic information can be used to demonstrate the existence of $p$-blocks with a given defect group, and we obtain strong information about the way in which defect groups occur as Sylow-intersections.

Before we can state our main result, we need to fix our notation. $G$ is a finite group, $p$ is a prime, $P$ is a fixed Sylow $p$-subgroup of $G$, $D$ is a normal $p$-subgroup of $G$ (the possibility that $D = 1_G$ is not excluded). Also, $\{ y_i : 1 \leq i \leq r \}$ is a full set of representatives of those conjugacy classes of $p$-regular elements of $G$ which have defect group $D$ (where a defect group for a conjugacy class of $p$-regular elements is a Sylow $p$-subgroup of the centralizer of some element of that class) and $\{ g_j : 1 \leq j \leq n \}$ is a full set of $(P, P)$ double-coset representatives, chosen wherever possible to satisfy (simultaneously)

(i) $g_j$ is $p$-regular,
(ii) $D \subseteq \text{Syl}_p(C_G(g_j))$,
(iii) $P \cap P^g_j = D$.

If it is not possible to choose any $g_j$ which satisfies conditions (i), (ii) and

If it is not possible to choose any $g_j$ which satisfies conditions (i), (ii) and
(iii), we set \( k = 0 \). Otherwise we label so that \( g_j \) satisfies conditions (i), (ii) and (iii) for \( 1 \leq j \leq k \), but not for \( j > k \). If \( k \neq 0 \) we define an \( r \times k \) matrix \( N \) with entries in \( GF(p) \) by: for \( 1 \leq i \leq r \), \( 1 \leq j \leq k \), \( n_{ij} \) is the residue (mod \( p \)) of the number of conjugates of \( y_i \) in the coset \( g_jC_p(D) \).

We can now state our main theorem:

**Theorem A.** The number of blocks of \( G \) with defect group \( D \) is the rank of the matrix \( NN^T \) if \( k \neq 0 \), and is 0 if \( k = 0 \).

**Proof.** For \( 1 \leq i, j \leq r \), we define the set \( \Omega_{ij} \) by:

\[
\Omega_{ij} = \{(a, b): a \text{ is conjugate to } y_i, \ b \text{ is conjugate to } y_j, \ \text{and} \ a^{-1}b \in P\}.
\]

We note that if \( \Omega_{ij} \) is not empty, then \( P \) acts by conjugation on \( \Omega_{ij} \), and each orbit has length \( p^{a-d} \) (where \( |P| = p^a, |D| = p^d \)). We define the \( r \times r \) matrix \( S \) with entries in \( GF(p) \) by: \( s_{ij} \) is the residue (mod \( p \)) of \( |\Omega_{ij}|/p^{a-d} \). We will first prove that the rank of \( S \) is the number of \( p \)-blocks of \( G \) with defect group \( D \), and then we will show that \( S = NN^T \) if \( k \neq 0 \), \( S = 0 \) if \( k = 0 \).

Let \( K = \mathbb{Q}(\omega) \), where \( \omega \) is a primitive \( |G| \)th root of unity, let \( R \) be the ring of algebraic integers in \( K \), and let \( \rho \) be a prime ideal of \( R \) containing \( p \). Let \( R^* \) denote the localization of \( R \) at \( \rho \), and let \( \pi \) denote the unique maximal ideal of \( R^* \). We let \( F \) be \( R^*/\pi \), so that \( F \) is a finite field of characteristic \( p \). Also, for any group \( H \), we let \( \text{Irr}(H) \) denote the set of complex irreducible characters of \( H \). Finally, for \( a, b, c \in G \), we let \( \#(ab = c) \) denote the number of times \( c \) may be written as a product of a conjugate of \( a \) with a conjugate of \( b \).

Then the well-known formula of Burnside gives

\[
|\Omega_{ij}| = \sum_{x \in \rho} \#(y_i^{-1}y_j = x) = \sum_{x \in \rho} \frac{|G|}{|C_G(y_i)||C_G(y_j)|} \sum_{x \in \text{Irr}(G)} \frac{\chi(y_i^{-1})\chi(y_j)\chi(x^{-1})}{\chi(1)}.
\]

Changing the order of summation yields

\[
|\Omega_{ij}| = \frac{|G|}{|C_G(y_i)||C_G(y_j)|} \sum_{x \in \text{Irr}(G)} |P|\chi_p(1_p) \frac{\chi(y_i^{-1})\chi(y_j)}{\chi(1)}.
\]

Since \( D \in \text{Syl}_p(C_G(y_j)) \), we may rearrange the above equation to obtain

\[
\frac{|\Omega_{ij}|}{p^{a-d}} = \frac{1}{|C_G(y_j)|_p} \sum_{x \in \text{Irr}(G)} \frac{|G: C_G(y_j)|}{|C_G(y_j)|_p \cdot |P|\chi_p(1_p)} \frac{\chi(y_i^{-1})\chi(y_j)\chi(x^{-1})}{\chi(1)}.
\]

We note at this point that if the class sum of \( y_i \) lies in the radical of
Let $s_{ij} = 0$ for any $j$, because $[G: C_G(y_i)] \frac{x(y_i)}{x(1)} \in \pi$ for each $x \in \text{Irr}(G)$.

Let $*$ denote images in $F$. Then we have

$$
\frac{s_{ij}}{[G: C_G(y_j)]^*_{P^*}} = \frac{1}{[G: P]^*} \sum_{x \in \text{Irr}(G)} [G: C_G(y_i)] \frac{x(y_j)^*}{x(1)} \chi^{-1}(y_j^*) \chi(\eta, p, 1_p^*)
$$

Let $\{e_i: 1 \leq t \leq m\}$ be the set of primitive idempotents of $Z(FG)$ and let $\lambda_i$ be the linear character of $Z(FG)$ afforded by $e_i$ for $1 \leq t \leq m$. We define an algebra homomorphism $s: Z(FG) \to Z(FG)$ by $Xs = \sum_{t=1}^m \lambda_i(X) e_i$. We note that $s^2 = s$ and that $\ker(s) = \text{rad}(Z(FG))$.

Suppose that $F$ has $p^c$ elements. Then for each $X \in Z(FG)$ we have $(Xs)^{p^c} = \sum_{t=1}^m \lambda_i^c(X) e_i = \sum_{t=1}^m \lambda_i(X) e_t = Xs$. Now $X = Xs + (X - Xs)$, so $Xs + (X - Xs)^{p^c}$. Since $X - Xs \in \ker s$, so is nilpotent, we may repeat this argument until we eventually obtain $Xs = Xs^{p^c}$ for some $f$. In particular, $Xs$ belongs to the subalgebra of $Z(FG)$ generated by $X$.

For $1 \leq i \leq r$, let $K_i$ denote the class sum in $Z(FG)$ of the class containing $y_i$. Then $K_i \in Z(FC_G(D))$, so that $K_i s \in Z(FC_G(D))$ also. Now we label so that $\{e_i: 1 \leq t \leq q\}$ is the set of block idempotents of the blocks of $G$ which have defect group $D$. Then $\lambda_i(K_i) = 0$ if $t > q$, because $D$ is contained in the defect group of every $p$-block of $G$, and $D \in \text{Syl}_p(C_G(y_i))$.

For $1 \leq t \leq q$, $e_t$ is a linear combination of class sums of classes of $p$-regular elements whose defect groups are contained in $D$. Thus $K_i s$ is a linear combination of $K_1, K_2, ..., K_r$, since $K_i s = \sum_{t=1}^m \lambda_i(K_i) e_i$ and also $K_i s \in Z(FC_G(D))$.

We note that if $K$ is a class sum of a class of elements whose defect group does not contain $D$ then $K \in \ker s$, since $K \in \text{rad}(Z(FG))$. Also, for $1 \leq t \leq q$, $e_t = e_t s$, so it follows that $e_t$ is in fact a linear combination of $K_1, ..., K_r$.

Thus $\{K_i s: 1 \leq i \leq r\}$ and $\{e_i: 1 \leq t \leq q\}$ span the same $(q$-dimensional$)$ subspace of $Z(FG)$. We now proceed to derive an explicit expression for $K_i s$ for $1 \leq i \leq r$.

Let $B_i$ denote the $p$-block of $G$ which contains $e_t$ for $1 \leq t \leq m$. We have

$$
\frac{s_{ij}}{[G: C_G(y_j)]^*_{P^*}} = \sum_{t=1}^m \lambda_i(K_i) \frac{1}{[G: P]^*} \sum_{x \in B_i} \chi(y_j^{-1}) \chi(\eta, p, 1_p^*)
$$

Now for each $t$, we may lift $e_t$ to an idempotent of $Z(R^*G)$, and the coefficient of $y_j$ in this idempotent is $1/|G| \sum_{x \in B_i} \chi(y_j^{-1}) \chi(1)$.

Since we know that $\sum_{x \in B_i} \chi(y_j^{-1}) \chi(u) = 0$ for $u \in P^*$, the above coefficient of $y_j$ is $1/|G| \sum_{x \in B_i} \chi(y_j^{-1}) \sum_{u \in P^*} \chi(u)$ which is $1/[G: P] \sum_{x \in B_i} \chi(y_j^{-1}) \chi(\eta, p, 1_p^*)$. 

\[ \text{BLOCKS WITH A GIVEN DEFECT GROUP} \]
We see then that \( s_{ij}/[G:C_G(y_j)]_p \) is precisely the coefficient of \( K_j \) in \( \sum_{i=1}^{m} \lambda_i(K_i)e_i \). Thus for \( 1 \leq i \leq r \) we have

\[
K_j s = \sum_{j=1}^{r} \frac{s_{ij}}{[G:C_G(y_j)]_p} K_j.
\]

Since \( \{K_i: 1 \leq i \leq r\} \) is linearly independent, and \( \{K_is: 1 \leq i \leq r\} \) spans a subspace of dimension \( q \) of \( \mathbb{Z}(FG) \), it readily follows that the \( r \times r \) matrix \( S \) has rank \( q \), which is precisely the number of blocks of \( G \) with defect group \( D \).

We now show that \( S = NN^T \). For \( 1 \leq i, j \leq r \), we consider the contribution to \( n_{ij} \) from the double coset \( Pg,P \). Suppose that the coset \( g,P \) contains \( a \) conjugates of \( y_i \), and \( b \) conjugates of \( y_j \). Then so does \( xg,P \) for any \( x \in P \) (since \( xg,Px^{-1} = xg,P \) for \( x \in P \)). Then the contribution to \( |\Omega_{ij}| \) from \( Pg,mP \) is \( [P:P \cap g,mP]a \).

Now \( P \cap g,mP^{-1} \) permutes the conjugates of \( y_i \) in the coset \( g,mP \) in orbits of length \( [P \cap g,mP^{-1}: D] \), since \( D \triangleleft G \) and \( D \in \text{Syl}_p(C_G(y_i)) \). The same applies to conjugates of \( y_j \) in \( g,mP \), so that \( [P \cap g,mP^{-1}: D]^2 \) divides \( ab \).

Thus the contribution to \( |\Omega_{ij}| \) from \( Pg,mP \) is divisible by \( [P:D][P \cap g,mP^{-1}: D] \). We are only concerned, however, with the residue \( (mod \ p) \) of \( |\Omega_{ij}|/p^{a-d} \) (namely, \( s_{ij} \)). This is 0 unless

(i) \( P \cap g,mP^{-1} = D \),

(ii) the coset \( g,mP \) contains a conjugate of \( y_i \) and of \( y_j \).

By the choice of double coset representatives, \( Pg,mP \) makes no contribution to \( s_{ij} \) unless \( k \neq 0 \) and \( m \leq k \). For \( m \leq k \), the contribution to \( |\Omega_{ij}|/p^{a-d} \) from \( Pg,mP \) is \( ab \), where \( a \) is the number of conjugates of \( y_i \) in \( g,mP \) and \( b \) is the number of conjugates of \( y_j \) in \( g,mP \). Each conjugate of \( y_i \) lies in \( g,mC_D(D) \) since \( y_i \) and \( g,m \) both lie in \( C_D(D) \), and the same applies to \( y_j \). Thus \( a \equiv n_{im}(mod \ p) \) and \( b \equiv n_{jm}(mod \ p) \), so that the contribution from \( Pg,mP \) to \( s_{ij} \) is \( n_{im}n_{jm} \). Thus \( s_{ij} = \sum_{m=1}^{k} n_{im}n_{jm} \) if \( k \neq 0 \), \( s_{ij} = 0 \) if \( k = 0 \).

Hence we have proved that \( S = 0 \) if \( k = 0 \), and that \( S = NN^T \) if \( k \neq 0 \), so the proof of Theorem A is complete.

**Corollary 1.** Let \( H \) be a finite group, \( q \) be a prime, \( Q \) be a defect group for some \( q \)-block of \( H \). Then whenever \( R \) is a Sylow \( q \)-subgroup of \( H \) containing \( Q \) there is a \( q \)-regular element \( y \in H \) with \( Q \in \text{Syl}_q(C_H(y)) \) such that \( R \cap R^y = Q \).

**Proof.** Let \( N = N_p(Q) \), and let \( S \) be a Sylow \( q \)-subgroup of \( N \) containing \( (R \cap N) \). Then \( Q \) is a defect group for some \( q \)-block of \( N \), by Brauer’s First Main Theorem. By Theorem A, \( Q = S \cap S^y \) for some \( q \)-regular element \( y \) such that \( Q \in \text{Syl}_q(C_N(y)) \). Thus \( Q = (R \cap N) \cap \)
(R \cap N)^y = R \cap R^y \cap N. If R \cap R^y > Q, then R \cap R^y \cap N > Q, so we must have R \cap R^y = Q. Also if T \in \text{Syl} q(C_n(y)) with T > Q, then T \cap N > Q, whereas Q \in \text{Syl} q(C_n(y)). Hence Q \in \text{Syl} q(C_n(y)).

Remark. Corollary 1 extends the results of J. A. Green and others.

COROLLARY 2. Suppose that p = 2, and that y is an element of G of odd order with D \in \text{Syl}_2(C_n(y)) and S \cap S^y = D for each Sylow 2-subgroup, S, of G. Then D is a defect group for some 2-block, B, of G, and for any \chi \in B, we have

\[ [G: C_n(y)] \frac{\chi(y)}{\chi(1)} \not\equiv 0 \pmod{\pi}. \]

Proof. With the notation as used for the proof of Theorem A, we may assume that y = y_1. Let P_{g_m}P be a double coset which contains a conjugate of y, say, x^{-1}yx. Then P \cap g_mP^{-1} is conjugate to P \cap x^{-1}yxP^{-1}x^{-1}x, which is in turn conjugate to y^{-1}xP^{-1}y \cap xP^{-1} = (P^{x^{-1}})^y \cap P^{x^{-1}} = D, by hypothesis. Thus m \leq k. The (1, 1)-entry of NNT is \[ \sum_{m=1}^{k} n_{1m}^2 = (\sum_{m=1}^{k} n_{1m})^2. \]

However, \[ \sum_{m=1}^{k} n_{1m} \] is the residue (mod 2) of \[ [G: C_n(y)]/[P: D] \] because if \[ g_mP \] contains a conjugates of y, the total number of conjugates of y within \[ P_{g_m}P \] is \[ [P: D] a. \] Since \[ D \in \text{Syl}_2(C_n(y)), \] \[ \sum_{m=1}^{k} n_{1m} \not\equiv 0, \] so \[ NNT \not\equiv 0. \]
Furthermore, since \[ s_{11} \not\equiv 0, \] by a remark made during the proof of Theorem A, there is a character \[ \chi \in \text{Irr}(G) \] such that

\[ [G: C_n(y)] \frac{\chi(y)}{\chi(1)} \not\equiv 0 \pmod{\pi}. \]

Since \[ D \triangleleft G \] and \[ D \in \text{Syl}_2(C_n(y)), \chi \] lies in a 2-block with defect group D.

Remark. Corollary 2 has no analogue when \[ p \] is odd. For example, let \[ p = 3, G = S_3, y = (1234). \] Then \[ S \cap S^y = 1_G \] for each Sylow 3-subgroup, S, of G. Also, \[ 1_G \in \text{Syl}_3(C_n(y)). \] However, G has only one 3-block of defect 0, containing a character, \[ \chi \] of degree 6. \[ \chi \] is induced from a character of degree 3 of \[ A_5, \] so \[ \chi(y) = 0. \]

Group Theoretic Conditions Which Simplify the Calculation of the \[ \Omega_{ij} \]

Lemma 1. If \[ y_i \] and \[ y_j \] generate different normal subgroups of \[ G, \] then \[ |\Omega_{ij}| = 0. \]

Proof. Let \[ N \] be the normal subgroup of \[ G \] generated by the conjugates of \[ y_i, \] Suppose that \[ y_j \in N, \] but that \[ |\Omega_{ij}| \neq 0. \] Then, without loss of generality, \[ y_i^{-1}y_j \in P. \] Thus \[ (y_iN)^{-1}(y_jN) \in PN/N. \] Since \[ y_i \in N, \] we have
A contradiction, since \( y_jN \) is 0-regular and \( PN/N \) is a 0-group. Thus if \( |\Omega_{ij}| \neq 0 \), we have \( y_j \in N \). A similar argument, reversing the roles of \( y_i \) and \( y_j \), establishes Lemma 1.

**Lemma 2.** Suppose that \( y_i \) and \( y_j \) have coprime orders, not both 1, and that \( C_G(D) \) is 0-solvable. Then \( |\Omega_{ij}| = 0 \).

**Proof:** We prove by induction that if \( H \) is a 0-solvable group, and \( x, y \in H^* \) are 0-regular elements of coprime order, then the order of \( x^{-1}y \) is not a power of 0. The result is true if \( H \) is a 0'-group. Suppose that the result has been established for 0-solvable group of order less than \( |H| \). We may suppose that \( 0_0(H) = 1 \), for otherwise \( (x0_0(H))^{-1}(y0_0(H)) \) could not have order a power of 0 in \( H/0_0(H) \), so the order of \( x^{-1}y \) could not be a power of 0.

Thus \( 0_0(H) = 1 \). If \( x \) or \( y \) is in \( 0_0(H) \), then \( x^{-1}y \) is a 0-regular element of \( H^* \), so the order of \( x^{-1}y \) is not a power of 0. Otherwise, by induction, the order of \( (x0_0(H))^{-1}(y0_0(H)) \) is not a power of 0. In any case, the order of \( x^{-1}y \) cannot be a power of 0.

**Lemma 3.** Suppose that \( y_i \in 0_p(G) \). Then \( |\Omega_{ij}| = 0 \) if \( i \neq j \), and \( |\Omega_{ii}| = [G : C_G(y_i)] \). Thus \( s_{ij} = 0 \) for \( j \neq 1 \), and \( s_{ii} = 0 \).

**Proof:** Suppose that \( a \) is conjugate to \( y_i \), \( b \) is conjugate to \( y_j \), and that \( a^{-1}b \in P \). Since \( y_i \in 0_p(G) \), \( a \in 0_p(G) \). Hence \( a^{-1}b \) is 0-regular, as \( b \) is 0-regular. Thus \( a^{-1}b = 1 \), so \( a = b \). Hence \( |\Omega_{ij}| = 0 \) if \( i \neq j \), and \( |\Omega_{ii}| = [G : C_G(y_i)] \). Now \( s_{ii} \) is the residue (mod 0) of \( [G : C_G(y_i)] \), so \( s_{ii} \neq 0 \), as \( D \in \text{Syl}_p(C_G(y_i)) \).

**Corollary 3 (Tsushima [2]).** If \( y_1, y_2, \ldots, y_n \in 0_p(G) \), then there are at least \( m \) blocks of \( G \) with defect group \( D \).

**Proof:** With notation as in the proof of Theorem A, the matrix \( S \) has rank at least \( m \) (using Lemma 3).

**Corollary 4.** Let \( H \) be a finite group, \( T \) be a Sylow \( q \)-subgroup of \( H \) for some prime \( q \). Then there are at least as many \( q \)-blocks of \( H \) as there are conjugacy classes of \( q \)-regular elements of \( H \) meeting \( \bigcup_{\gamma \in T} 0_q(N_H(\gamma)) \).

**Proof:** We first note that if \( y \in 0_q(N_H(B)) \) for some \( q \)-subgroup \( B \) of \( H \), then \( y \in 0_q(C_H(B)) \), and that if \( B \leq K \in \text{Syl}_q(C_H(y)) \), then \( y \in 0_q(C_H(B)) \cap C_H(K) \leq 0_q(C_H(K)) = 0_q(N_H(K)) \). Now if \( L \) is another Sylow \( q \)-subgroup of \( C_H(y) \), then \( L = K^c \) for some \( c \in C_H(y) \), so that \( y = y^c \in 0_q(N_H(K^c)) = 0_q(N_H(L)) \).

Let \( \{ y_1, y_2, \ldots, y_n \} \) be a set of representatives for all the conjugacy classes of \( q \)-regular elements of \( H \) which meet \( \bigcup_{\gamma \in T} 0_q(N_H(\gamma)) \). Define an
equivalence relation \( \sim \) on \( \{y_1, \ldots, y_n\} \) by \( y_i \sim y_j \) if and only if the Sylow \( q \)-subgroups of \( C_H(y_i) \) and \( C_H(y_j) \) are conjugate in \( H \).

Relabel, if necessary, so that \( \{y_1, \ldots, y_m\} \) is one equivalence class under \( \sim \), and so that there is a subgroup, \( Q \), of \( T \) such that \( Q \in \text{Syl}_q(C_H(y_i)) \) for \( 1 \leq i \leq m \) (replacing the \( y_i \) by suitable conjugates as the need arises). Then \( y_i \in \Omega_p(N_H(Q)) \) for \( 1 \leq i \leq m \), so by Corollary 3 \( N_H(Q) \) has at least \( m \) \( q \)-blocks with defect group \( Q \), and by Brauer's First Main Theorem so has \( H \).

The same argument may be applied to each equivalence class to conclude that \( H \) has at least \( n \) \( q \)-blocks.

**Lemma 4.** Let \( y_i \) and \( y_j \) be involutions of \( G \) (with notation as in the proof of Theorem A), and suppose that \( p \) is odd. Then if \( i \neq j \), \( |\Omega_{ij}| = 0 \), and if \( i = j \), but \( y_i \) inverts no \( p \)-element of \( C_G(D)^* \), \( |\Omega_{ii}| = |G : C_G(y_i)| \).

**Proof:** If \( i \neq j \), and \( a \) is conjugate to \( y_i \), \( b \) is conjugate to \( y_j \), then \( a^{-1}b \) cannot lie in \( P \), for if it did it would have odd order and \( y_i \) would be conjugate to \( y_j \).

Thus \( |\Omega_{ij}| = 0 \) if \( i \neq j \). If \( y_i \) inverts no \( p \)-element of \( G^* \), it is easy to see that \( |\Omega_{ii}| = |G : C_G(y_i)| \), so that \( s_{ii} \neq 0 \), as \( D \in \text{Syl}_p(C_G(y_i)) \).

The following result extends that of Brauer and Fowler [1], and is implicit in the results of Wada [3].

**Corollary 5.** If \( p \) is odd, and \( y_1, y_2, \ldots, y_m \) are involutions which invert no \( p \)-element of \( C_G(D)^* \), then \( G \) has at least \( m \) \( p \)-blocks with defect group \( D \).

**Proof:** By Lemma 4, the matrix \( S \) has rank \( m \) or more, so the result follows from Theorem A.

We have seen that defect groups are Sylow intersections of a special kind. Corollary 2 says something in the opposite direction, and the next result says more in that direction.

**Corollary 6.** Let \( H \) be a finite group, and let \( Q \) be a Sylow 2-intersection of \( H \) which is maximal under inclusion (among Sylow 2-intersections). Then either \( Q \) is a defect group for some 2-block of \( H \) or (i), (ii) and (iii) are all true.

(i) \( Q \in \text{Syl}_2(\Omega_2(N_H(Q))) \).

(ii) \( N_H(Q) \) is 2-constrained, and \( C_H(Q) \) has a normal 2 complement.

(iii) \( \Omega_2(N_H(Q)) = \Omega_2(N_H(T)) \) whenever \( T \in \text{Syl}_2(H) \) with \( Q \leq T \).

**Proof:** Let \( K = N_H(Q) \). Then \( Q \) is a maximal Sylow 2-intersection within \( K \). Suppose that (i), (ii) and (iii) have been established within \( K \) (assuming that \( Q \) is not a defect group for any 2-block of \( K \)). Let \( T \) be a Sylow 2-subgroup of \( H \) with \( Q \leq T \). Then \( T \cap K > Q \). Let \( R \in \text{Syl}_2(K) \) with
(T \cap K) \leq R, and let x \in O_2(K). Then x \in O_2(N_K(R)), so x \in C_K(R), and
(T \cap K) = (T \cap K)^x. Thus T \cap T^x \cap K > Q, so T \cap T^x > Q. By the maximality of Q, x \in N_K(T).

Thus \(O_2(C_K(Q)) \leq N_K(T)\), so that \(O_2(C_K(Q)) \leq O_2(N_K(T))\) by a well-known lemma of H. Bender (since \(N_K(T)\) is certainly 2-constrained). Since \(C_K(Q)\) has a normal 2-complement, \(O_2(N_K(T)) \leq O_2(N_K(Q)).\) Thus \(O_2(N_K(Q)) = O_2(N_K(T)).\)

Suppose that Q is not a defect group for any 2-block of H. Then Q is not a defect group for any 2-block of K. From now on, then, we may work within K, and we do so. Since Q is a Sylow 2-intersection in K, we certainly have \(Q = O_2(K).\) Let \(R \in \text{Syl}_2(K).\) The fact that Q is a maximal Sylow 2-intersection in K implies that \(N_K(X) \leq N_K(R)\) whenever \(Q < X \leq R.\)

Suppose that \(Q \in \text{Syl}_2(O_2(K)).\) Then \(K = O_2(K) \cap N_K(R).\) Now \(R \subset K,\) so that \(O_2(K) \leq N_K(R).\) Hence there is an element \(x \in O_2(K)\) which is not conjugate within \(O_2(K)\) to any element of \(N_K(R).\) Since \(K = O_2(K) \cap N_K(R),\) x is not conjugate within K to any element of \(N_K(R).\)

It follows that \(Q \in \text{Syl}_2(C_K(x))\) (otherwise, without loss of generality, \(R \cap R^x > Q,\) so that \(x \in N_K(R),\) a contradiction). By Corollary 3, Q is a defect group for some 2-block of K, contrary to hypothesis. Thus \(Q \in \text{Syl}_2(O_2(K)).\)

Suppose that K is not 2-constrained. Let L be the inverse image in K of some component of \(K/O_2(K).\) Then \([L, Q] \leq O_2(K) \cap Q = 1_K,\) so \(L \leq C_K(Q).\) Also, \(Q \cap L \in \text{Syl}_2(L),\) because \(Q \cap L \leq Z(L)\) and \(L/O_2(K)\) is perfect. Thus \(Q \in \text{Syl}_2(C_K(Q)).\) Hence \(K = QC_K(Q) \cap N_K(R),\) so \(K = C_K(Q) \cap N_K(R).\) The argument used earlier for \(O_2(K)\) shows that there is an element \(x \in C_K(Q)\) which is not conjugate within K to any element of \(N_K(R).\)

Then for each \(h \in K,\) we have \(R \cap R^{hxh^{-1}} = Q,\) so that \(R^h \cap (R^h)^x = Q.\) We may write \(x = yz\) where \(y\) is a 2-element, \(z\) has odd order, \(yz = zy,\) and (replacing x by a suitable conjugate if need be) \(y \in R.\) Then \(y \in R \cap R^x,\) so that \(y \in Q.\) Thus \(R^h \cap (R^h)^x = Q\) for each \(h \in K.\) Now \(z \in C_K(Q),\) and we see easily that \(Q \in \text{Syl}_2(C_K(z)).\)

By Corollary 2, Q is a defect group for some 2-block of K, contrary to hypothesis. Thus K is 2-constrained, and \(Q \in \text{Syl}_2(QC_K(Q)).\) Also, \(Z(Q) \in \text{Syl}_2(C_K(Q)),\) so that \(C_K(Q)\) has a normal 2-complement.

It remains to prove that \(O_2(K) = 0_2(C_K(R)).\) Since \(C_K(Q)\) has a normal 2-complement, \(O_2(C_K(Q)) \leq O_2(C_K(Q)) \cap O_2(K).\) Suppose that \(O_2(K) \leq C_K(R).\) Then some \(x \in O_2(K)^k\) must be conjugate to its inverse within K (otherwise, an easy induction argument on \(|\langle y \rangle|\) shows that \([0_2(K), y] = 1_K\) for each \(y \in R).\) Thus \(C_K(x)\) does not contain a Sylow 2-subgroup of K. Hence no conjugate of x lies in \(N_K(R)\) (otherwise, without loss of generality, \(x \in N_K(R)\) and \([R, x] \leq R \cap O_2(K) = 1_K).\) Thus, for each \(h \in K\) we have \(R \cap R^{hxh^{-1}} = Q,\) so \((R^h) \cap (R^h)^x = Q.\) Hence \(Q \in \text{Syl}_2(C_K(x)),\) so by
Corollary 3, \( Q \) is a defect group for some 2-block of \( K \), contrary to hypothesis. Thus \( 0_2(K) \leq 0_2(C_K(R)) \), and the proof of Corollary 6 is complete.

**Remark.** There is no direct analogue of Corollary 6 for odd \( p \). For example, let \( H \) be the semi-direct product \( AX \) where \( X \) is a direct product of six copies of \( \mathbb{Z}_3 \) with three copies of \( \mathbb{Z}_2 \), and where \( A \) is a Frobenius group of order 21 acting on \( X \) in such a way that an element of order 7 in \( A \) acts without non-trivial fixed points on \( X \). Let \( p = 3 \), and let \( Q = O_3(H) \).

Then \( Q \) is a maximal Sylow 3-intersection of \( H \), but \( Q \) is not a defect group for any 3-block of \( H \), for there is no 3-regular element \( y \in H \) such that \( Q \in \text{Syl}_3(C_H(y)) \). Condition (iii) of Corollary 6 fails to hold within \( H \), because if \( R \) is any Sylow 3-subgroup of \( H \), \( [R, O_3(H)] \neq 1 \).

We also remark that if conditions (i), (ii) and (iii) of Corollary 6 all hold, then \( Q \) is not a defect group for any 2-block of \( H \), since each 2-block of \( N_H(Q) \) is a block of full defect.

Perhaps also worth mentioning is the following result, as it allows us to assume that we are dealing with an elementary abelian defect group when we wish to know how many \( p \)-blocks have a given defect group. (The notation is that used for Theorem A.)

**COROLLARY 7.** There is a one-to-one correspondence between \( p \)-blocks of \( G \) with defect group \( D \) and \( p \)-blocks of \( G/\Phi(D) \) with defect group \( D/\Phi(D) \).

**Proof.** Let bars denote images in \( G/\Phi(D) \). Then \( \{ \bar{y}_i : 1 \leq i \leq r \} \) is a full set of representatives of those conjugacy classes of \( p \)-regular elements with defect group \( \bar{D} \) (as any \( p \)-regular element which acts trivially on \( D/\Phi(D) \) must already centralize \( D \)). For \( 1 \leq i, j \leq r \) define the set \( \Omega^i_{ij} \) by \( \Omega^i_{ij} = \{ (\bar{a}, \bar{b}) : \bar{a} \text{ is conjugate to } y_i, \bar{b} \text{ is conjugate to } y_j, a^{-1}b \in \bar{P} \} \).

Then it is easily verified that there is a one-to-one correspondence between \( \Omega^i_{ij} \) (as defined in the proof of Theorem A) and \( \Omega^i_{ij} \) for \( 1 \leq i, j \leq r \).

Hence the matrix \( S \) (as defined for \( G \) during the proof of Theorem A) and \( S' \), for \( \bar{G} \), are identical. The rank of \( S' \) is the number of \( p \)-blocks of \( \bar{G} \) with defect group \( \bar{D} \) and the rank of \( S \) is the number of \( p \)-blocks of \( G \) with defect group \( D \), so the proof of Corollary 7 is complete.

Finally, we make some remarks about using knowledge of the \( s_{ij} \) to construct idempotents of \( Z(FG) \) (returning to the notation used in the proof of Theorem A). Once we know the \( s_{ij} \) for \( 1 \leq i, j \leq r \), we know \( K, s \) for \( 1 \leq i \leq r \). We also know that \( (Xs)^{pc} = Xs \) for each \( X \in Z(FG) \).

Let \( Y \neq 0 \) be an \( F \)-linear combination of \( K_1 s, K_2 s, \ldots, K_r s \), and let \( E_Y = \sum_{i=1}^{p-1} Y_i - Y \). Since \( Y^{pc} = Y \), it is easy to check that \( E_Y^2 = E_Y \).

In fact, \( E_Y = \sum_{i=1}^{m} - (\sum_{l=1}^{p-1} \lambda_i(Y)^l) e_i \), and we know that \( \sum_{i=1}^{p-1} \lambda_i(Y)^l = 0 \) unless \( \lambda_i(Y) = 1 \). Hence \( E_Y = \sum_{l: \lambda_i(Y) = 1} e_i \). In particular, we may
apply this argument to any non-zero $K_j s$ to produce idempotents of $Z(FG)$ (for if $K_j s \neq 0$, there is some $\lambda \in F$ such that $E_{(\lambda K_j s)} \neq 0$).

ACKNOWLEDGMENTS

I wish to thank Professor J. A. Green for some suggestions which improved the exposition of this paper, for pointing out to me the argument of Corollary 1, and for several illuminating remarks. I also wish to thank Professor M. Broué for some enlightening conversations on the subject matter of this paper, and Dr. Y. Cheng for carefully reading an earlier version of this paper, and suggesting some simplifications of my original proof.

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