Flight Vehicle Attitude Determination Using the Modified Rodrigues Parameters

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Abstract

There are two attitude estimation algorithms based on the different representations of attitude errors when modified Rodrigues parameters are applied to attitude estimation. The first is multiplicative error attitude estimator (MEAE), whose attitude error is expressed by the modified Rodrigues parameters representing the rotation from the estimated to the true attitude. The second is subtractive error attitude estimator (SEAE), whose attitude error is expressed by the arithmetic difference between the true and the estimated attitudes. It is proved that the two algorithms are equivalent in the case of small attitude errors. It is possible to describe rotation without encountering singularity by switching between the modified Rodrigues parameters and their shadow parameters. The attitude parameter switching does not bring disturbance to MEAE, but it does to SEAE. This article introduces a modification to eliminate the disturbance on SEAE, and simulation results demonstrate the efficacy of the presented algorithm.

Keywords: attitude estimation; modified Rodrigues parameters; singularity; attitude error

1 Introduction

The quaternion parameterization has found wide application in flight vehicle attitude determination because it is free from singularity with a bilinear kinematic equation. However, the redundant property of the quaternion renders the singularity of the covariance matrix[1-2] and requires to normalize the estimated quaternion. This problem can be settled by way of three-dimensional parameterization. In the three-dimensional parameterization, the Rodrigues parameters and the modified Rodrigues parameters have drawn ever-increasing attention because of their simplicity and high efficiency[3-9]. Extended Kalman filter (EKF) is the main method of flight vehicle attitude estimation. When the modified Rodrigues parameters are used to represent attitude in EKF, there are two methods to describe attitude errors. In the first method, the attitude error is expressed by the modified Rodrigues parameters representing the rotation from the estimated to the true attitude (termed multiplicative error)[4]. In the second method, the attitude error is expressed by the arithmetic difference between the true and the estimated attitudes[5-6]. This study will investigate the two methods and disclose their relationship. As the modified Rodrigues parameters could not represent all rotations because of their singularity, Schaub and Junkins[7] presented an approach to avoid the singularity through switching between the modified Rodrigues parameters and their shadow parameters. Using three elements to describe the attitude, the method adopted by Schaub and Junkins eliminates the singularity and, therefore, is in widespread use[4,7-8]. However, it demands that the process of
filtering be smooth without big disturbance at
switching points. This makes it necessary to analyze
the effects of parameter switching on the estimation
process and achieve corresponding conclusions.

2 Attitude Representation of the Modified
Rodrigues Parameters

The modified Rodrigues parameters are de-

rived from the Euler’s principal rotation theorem.

Let the unit vector \( \textbf{n} \) be the principal line of rotation

and \( \theta \) the rotation angle about \( \textbf{n} \); the modi-

fied Rodrigues parameters can be defined as\[7,9\]

\[
\sigma = \tan(\theta / 4)n
\]

(1)

Denote the magnitude of \( \sigma \) by \( |\sigma| \), then \( |\sigma| \to \infty \)
as \( \theta \to 2\pi \), meaning that the modified Rodrigues

parameters are singular. The method to avoid the

singularity will be discussed in Section 5.1.

The direction cosine matrix in terms of the

modified Rodrigues parameters can be described by

\[
A(\sigma) = I - \frac{4(1 - |\sigma|^2)}{(1 + |\sigma|^2)^2} [\sigma \times] + \frac{8}{(1 + |\sigma|^2)^3} [\sigma \times]^2
\]

(2)

where

\[
[\sigma \times] = \begin{bmatrix}
0 & -\sigma_3 & \sigma_2 \\
\sigma_3 & 0 & -\sigma_1 \\
-\sigma_2 & \sigma_1 & 0
\end{bmatrix}
\]

Let the vector \( \omega \) denote the angular velocity

of the body frame with respect to the reference

frame. The kinematic equation in terms of modified

Rodrigues parameters is

\[
\dot{\sigma} = \frac{1}{4}[(1 - |\sigma|^2)I + 2[\sigma \times] + 2\sigma \sigma^T] \omega
\]

(3)

3 Attitude Estimation Algorithms Based

on the Modified Rodrigues Parameters

3.1 Sensor models

This study uses the attitude determination

mode composed of gyro and vector observations.

The gyro, whose axes are aligned with the body

axes of flight vehicles, serves to measure the angu-

lar velocity. The simple model of gyro is\[11\]

\[
\omega_m = \omega + \varepsilon + \eta_g
\]

(4)

\[
\dot{\varepsilon} = \eta_a
\]

(5)

where \( \omega_m \) is the gyro output, \( \varepsilon \) the gyro bias,

\( \eta_g \), and \( \eta_a \) are two uncorrelated zero-mean Gaussian white noises with

\[
E[\eta_g(t)\eta_g(t')] = \lambda^2(t-t')I_{3x3}
\]

\[
E[\eta_a(t)\eta_a(t')] = \lambda_a^2(t-t')I_{3x3}
\]

where \( \delta(t) \) is the Dirac \( \delta \) function.

Let \( y_k \) denote the observation vector corre-

sponding to the reference vector \( r_k \) at the time \( t_k \).

The simple model of vector observation is\[10\]

\[
y_k = A_k r_k + v_k
\]

(6)

where \( A_k \) is the direction cosine matrix given by

Eq.(2), \( v_k \) the zero-mean Gaussian white noise

with

\[
E[v_kv_k^T] = R_k \delta_k
\]

where \( \delta_k \) is the Kronecker \( \delta \) symbol.

3.2 Attitude estimation using multiplicative at-
titude error

The state vector is given by \( X = [\sigma^T \varepsilon^T]^T \).

The state equation is composed of Eq.(3) and Eq.(5).

Then the prediction equations are

\[
\dot{\sigma} = \frac{1}{4}[(1 - |\sigma|^2)I + 2[\sigma \times] + 2\dot{\sigma} \dot{\sigma}^T] \omega
\]

(7)

\[
\dot{\varepsilon} = 0
\]

(8)

where

\[
\dot{\omega} = \omega_m - \dot{\sigma}
\]

The gyro bias error \( \Delta \varepsilon \) is defined as the differ-

cence between the true value \( \varepsilon \) and the estimated

value \( \hat{\varepsilon} \), thus

\[
\Delta \varepsilon = \varepsilon - \hat{\varepsilon}
\]

(9)

From Eq.(5),

\[
\Delta \dot{\varepsilon} = \eta_a
\]

(10)

can be obtained.

The attitude error \( \Delta \sigma \) is defined as the rota-

tion from the estimated attitude \( \hat{\sigma} \) to the true atti-
dude \( \sigma \), that is,

\[
\sigma = \delta \sigma \odot \hat{\sigma} = \frac{(1 - |\sigma|^2)\delta \sigma + (1 - |\hat{\sigma}|^2)\hat{\sigma} - 2\sigma \times \hat{\sigma}}{1 + |\sigma|^2 - 28|\sigma|^2 - 28|\hat{\sigma}|^2}
\]

(11)

where “\( \odot \)” denotes the multiplication of modified

Rodrigues parameters\[9\]. By using Eq.(3) and Eq.(11)

and neglecting the higher orders, the following
equation can be derived:
The state error vector is given by
\[ \delta X = [\delta \sigma^T \Delta \varepsilon^T]^T \]  
(13)
Composed of Eq.(10) and Eq.(12), the state error equation can be written in a matrix-vector form:
\[ \delta X = F(t)\delta X + G(t)\eta \]  
(14)
where
\[ F(t) = \begin{bmatrix} \dot{\omega} & 0 \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \]  
(15)
\[ G(t) = \begin{bmatrix} \frac{1}{4} I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & \frac{1}{4} I_{3 \times 3} \end{bmatrix} \]  
(16)
\[ \eta = [\eta_v^T, \eta_w^T]^T \]  

The statistic property of the process noise \( \eta \) is given by
\[ E[\eta(t)] = 0, \ E[\eta(t)\eta^T(t')] = Q(t)\delta(t-t') \]  
where
\[ Q(t) = \begin{bmatrix} \alpha^2 I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & \alpha^2 I_{3 \times 3} \end{bmatrix} \]  

Then, the prediction equation of the covariance matrix is given by
\[ P(t) = F(t)P(t-1+F(t)^T(t)+G(t)Q(t)G^T(t) \]  
(17)
By using Eq.(6) and Eq.(11) and neglecting the higher order, the following equation can be derived:
\[ \Delta \varepsilon_k = h_k\Delta \sigma + v_k \]  
(18)
where
\[ h_k = 4\left(A(\hat{\sigma})\eta_k\right)^x \]  

Eq.(18) can be written as
\[ \Delta \varepsilon_k = H_k\delta X + v_k \]  
(19)
where
\[ H_k = [h_k, 0_{3 \times 3}] \]  
(20)
By using EKF to estimate the attitude, the program of attitude estimation at the step \( t_{k-1} - t_k \) can be summarized as follows:

(1) Propagation
Starting with the initial conditions \( \hat{X}_{k-1} = [\hat{\sigma}_{k-1}, \hat{\sigma}_{k-1}^T]^T \) and \( P_{k-1} \), Eqs.(7)-(8), and Eq.(17) can be numerically integrated at the interval \([t_{k-1}, t_k]\) to obtain the predictions \( \hat{X}_{k/k-1} \) and \( P_{k/k-1} \).

(2) Update
The gain matrix is
\[ K_k = P_{k/k-1}H_k^T[H_kP_{k/k-1}H_k^T + R_k]^{-1} \]  
(21)
The state error estimation is
\[ \delta \hat{\varepsilon}_k = K_k(y_k - \hat{y}_k) \]  
(22)
The state estimation is
\[ \hat{\sigma}_k = \hat{\sigma}_k + \Delta \hat{\varepsilon}_k \]  
(23)

The covariance matrix is
\[ P_k = (I - K_kH_k)P_{k/k-1}(I - K_kH_k)^T + K_kR_kK_k^T \]  
(25)
This algorithm is termed multiplicative error attitude estimator (MEAE).

3.3 Attitude estimation using arithmetic attitude error
The state vector is given by \( \hat{X} = [\hat{\sigma}^T, \hat{\varepsilon}^T]^T \). The overlined letters are distinguished from the non overlined ones in Section 3.2. The prediction equations are
\[ \frac{\hat{\sigma}}{\hat{\varepsilon}} = \frac{1}{4}\left(1 - \left|\hat{\sigma}\right|^2\right)I + 2\left|\hat{\sigma}\right| + 2\hat{\sigma}\hat{\sigma}^T \hat{\omega} \]  
(26)
\[ \frac{\delta \hat{\varepsilon}}{\delta \hat{\varepsilon}} = 0 \]  
(27)
where
\[ \hat{\omega} = \omega_0 - \hat{\varepsilon} \]
The gyro bias error \( \Delta \hat{\varepsilon} \) is defined as the difference between the true value \( \hat{\varepsilon} \) and the estimated value \( \hat{\varepsilon} \), that is,
\[ \Delta \hat{\varepsilon} = \hat{\varepsilon} - \hat{\varepsilon} \]  
(28)
Then
\[ \Delta \hat{\varepsilon} = \eta_0 \]  
(29)
The attitude error \( \Delta \hat{\sigma} \) is defined as the difference between the true attitude \( \hat{\sigma} \) and the estimated attitude \( \hat{\sigma} \), that is,
\[ \Delta \hat{\sigma} = \hat{\sigma} - \hat{\sigma} \]  
(30)
By using Eq.(3) and Eq.(30) and neglecting the higher order, the following equation can be derived:
\[ \Delta \hat{\sigma} = \frac{1}{4}\left(1 - \left|\hat{\sigma}\right|^2\right)I + 2\left|\hat{\sigma}\right| + 2\hat{\sigma}\hat{\sigma}^T \Delta \hat{\sigma} - 
\]  
\[ \frac{1}{4}\left(1 - \left|\hat{\sigma}\right|^2\right)I + 2\left|\hat{\sigma}\right| + 2\hat{\sigma}\hat{\sigma}^T \left(\Delta \hat{\varepsilon} + \eta_0\right) \]  
(31)
The state error vector is given by
\[ \Delta \hat{X} = [\Delta \hat{\sigma}^T, \Delta \hat{\varepsilon}^T]^T \]  
(32)
Composed of Eq.(29) and Eq.(31), the state error equation can be written in the matrix-vector form:
\[ \Delta \hat{X} = F(t)\Delta \hat{X} + \bar{G}(t)\eta \]  
(33)
where

\[
F(t) = \begin{bmatrix}
F_{11} & F_{12} \\
0_{3\times3} & 0_{3\times3}
\end{bmatrix}
\]

\[
F_{11} = \frac{1}{2}((\hat{\omega} \sigma) I + \hat{\omega} \hat{\sigma}^T - \hat{\omega} \hat{\sigma}^T - [\hat{\omega} \times])
\]

\[
F_{12} = -\frac{1}{4}((1 - |\hat{\sigma}|^2) I + 2(\hat{\sigma} \times) + 2 \hat{\sigma} \hat{\sigma}^T)
\]

\[
G(t) = \begin{bmatrix}
F_{12} & 0_{3\times3} \\
0_{3\times3} & I_{3\times3}
\end{bmatrix}
\]

Then, the prediction equation of the covariance matrix is given by

\[
\hat{P}(t) = F(t) \hat{P}(t) + \hat{P}(t) F^T(t) + G(t) Q(t) G^T(t)
\]

By using Eq.(6) and Eq.(30) and neglecting the higher order, the following equation can be derived:

\[
\Delta \hat{p}_k = \hat{H}_k \Delta \sigma + v_k
\]

where

\[
\hat{H}_k = \frac{4}{(1 + |\hat{\sigma}|^2)^2}[(A(\hat{\sigma})v_k \times)(1 - |\hat{\sigma}|^2)I + 2 \hat{\sigma} \hat{\sigma}^T - 2(\hat{\sigma} \times)]
\]

Eq.(37) can be written into

\[
\Delta \hat{p}_k = \hat{H}_k \Delta \bar{X} + v_k
\]

where

\[
\hat{H}_k = [\hat{H}_k \ 0_{3\times3}]
\]

By using EKF to estimate the attitude, the program of attitude estimation at the step \(t_{k-1} \sim t_k\) can be summarized as follows:

1. **Propagation**

   Starting with the initial conditions \(\hat{X}_{k-1} = [\hat{\theta}_{k-1} \ \hat{\omega}_{k-1}^T]^T\) and \(\hat{P}_{k-1}\), Eqs.(26)-(27), and Eq.(36) can be numerically integrated at the interval \([t_{k-1} \ \ t_k]\) to obtain the predictions \(\hat{X}_{k/k-1}\) and \(\hat{P}_{k/k-1}\).

2. **Update**

   The gain matrix is

   \[
   \hat{K}_k = \hat{H}_k \hat{P}_{k/k-1} \hat{H}_k^T (\hat{H}_k \hat{P}_{k/k-1} \hat{H}_k^T + R_k)^{-1}
   \]

   The state error estimation is

   \[
   \Delta \hat{X}_k = K_k (y_k - \hat{y}_k)
   \]

   The state estimation is

   \[
   \hat{\sigma}_k = \hat{\sigma}_{k/k-1} + \Delta \hat{\sigma}_k
   \]

   \[
   \hat{\omega}_k = \hat{\omega}_{k/k-1} + \Delta \hat{\omega}_k
   \]

   The covariance matrix is

   \[
   \hat{P}_k = (I - \hat{K}_k \hat{H}_k) \hat{P}_{k/k-1} (I - \hat{K}_k \hat{H}_k)^T + \hat{K}_k \hat{R}_k \hat{K}_k^T
   \]

   This algorithm is termed subtractive error attitude estimator (SEAE).

4 **Equivalence Proof of the Two Attitude Estimation Algorithms**

4.1 **Different errors and their relationships**

The attitude error of MEAE is expressed by multiplicative error, \(\delta \sigma\), which is defined by Eq.(11). To compare MEAE to SEAE, an arithmetic error \(\Delta \sigma\) is introduced in MEAE, which is defined as the arithmetic difference between \(\sigma\) and \(\hat{\sigma}\), that is,

\[
\Delta \sigma = \sigma - \hat{\sigma}
\]

Note that the arithmetic error, \(\Delta \sigma\), in Eq.(45) is different from the arithmetic error \(\Delta \bar{\sigma}\) in Eq.(30) in SEAE. Comparing Eq.(30) with Eq.(45), \(\hat{\sigma} - \hat{\sigma}\) if \(\Delta \sigma = \Delta \bar{\sigma}\), that is, the estimated attitudes of the two algorithms are equal, which is the conclusion required to prove. \(\Delta \sigma\) is only used in the following proving process as a middle variable. It is different from \(\delta \sigma\) in that this expresses the attitude error in MEAE. When \(\delta \sigma\) is small, from Eq.(11) and Eq.(45), it follows that

\[
\Delta \sigma = S \delta \sigma
\]

where

\[
S = (1 - |\hat{\sigma}|^2) I + 2(\hat{\sigma} \times) + 2 \hat{\sigma} \hat{\sigma}^T
\]

It can be seen from Eqs.(46)-(47) that \(\Delta \sigma\) is not equal to \(\delta \sigma\), although \(\delta \sigma\) is small.

The multiplicative state error vector, \(\delta \bar{X}\), in MEAE is defined by Eq.(13). The subtractive state error is defined by

\[
\Delta \bar{X} = [(\delta \sigma)^T \ \ \Delta \theta^T]^T
\]

From Eq.(13), Eq.(46), and Eq.(48), it follows that

\[
\Delta \bar{X} = T \delta \bar{X}
\]

where

\[
T = \begin{bmatrix}
S & 0_{3\times3} \\
0_{3\times3} & I_{3\times3}
\end{bmatrix}
\]

The multiplicative error, \(\delta \bar{\sigma}\), is introduced in SEAE quite the same way as the arithmetic error, \(\Delta \sigma\), in MEAE. The multiplicative error, \(\delta \bar{\sigma}\), in SEAE can be defined as the rotation from \(\hat{\sigma}\) to \(\bar{\sigma}\),
that is,
\[ \bar{\sigma} = \delta \sigma \otimes \hat{\sigma} \] (51)

Defining
\[ \Delta \bar{X} = [\delta \sigma^T \Delta \bar{X}]^T \] (52)
an equation similar to Eq.(49), can be obtained
\[ \Delta \bar{X} = \mathbf{T} \delta \bar{X} \] (53)
where
\[ \mathbf{T} = \begin{bmatrix} (1 - |\hat{\sigma}|)^2 & I + 2(\hat{\sigma} \times) + 2 \hat{\sigma} \hat{\sigma}^T & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \]

By replacing the above-mentioned modified Rodrigues parameters in \( \delta \sigma, \Delta \sigma, \Delta \bar{X}, \delta \bar{X}, \Delta \bar{X}, \delta X, \Delta X, \Delta \bar{X}, \) and \( \delta \bar{X} \) with their shadow parameters, \( \delta \sigma^*, \Delta \sigma^*, \Delta \bar{X}^*, \delta \bar{X}^*, \delta X^*, \Delta X^*, \Delta \bar{X}^*, \) and \( \delta \bar{X}^* \) can be obtained. By replacing the modified Rodrigues parameters in Eq.(49) and Eq.(53) with their shadow parameters, the relationships between \( \Delta X^* \) and \( \Delta \bar{X}^* \) as well as \( \Delta X^* \) and \( \delta \bar{X}^* \) can be obtained. However, this is not discussed in detail in this article.

4.2 Meaning of equivalence

The previous section has presented two algorithms, MEAE and SEAE, and, further, showed that both are equivalent when the attitude error was small. Next, the meaning of equivalence will be given.

The equivalence of two algorithms means that their estimated states and their accuracy, expressed by the covariance matrix, are equal at any time if their initial estimated states and their accuracy are equal. Because the representations of state errors in MEAE and SEAE are different, the equality of accuracy does not imply the equality of covariance matrix. The different representations of state errors in MEAE and SEAE lead to the different styles of the covariance matrices \( \mathbf{P} = \mathbf{E}[\Delta X \Delta X^T] \) and \( \bar{\mathbf{P}} = \mathbf{E}[\Delta \bar{X} \Delta \bar{X}^T] \). However, to compare these two algorithms, the styles must be made to be identical. Thus, suppose
\[ \hat{\mathbf{P}} = \mathbf{E}[\Delta X \Delta X^T] \] (54)
\( \hat{\mathbf{P}} \) and \( \bar{\mathbf{P}} \) have the same styles. From Eq.(49) and Eq.(54), it follows that
\[ \hat{\mathbf{P}} = T \mathbf{E}[\delta X \delta X^T] \mathbf{T}^T = T \mathbf{P} T^T \] (55)

Then, the equality of accuracy of MEAE and SEAE means \( \hat{\mathbf{P}} = \mathbf{P} \), that is \( T \mathbf{P} T^T = \mathbf{P} \).

4.3 Proof of equivalence

If \( \hat{\mathbf{X}}_k = \mathbf{X}_k \) and \( \hat{\mathbf{P}}_k = \mathbf{P}_k T^T \) have been proven at time \( t_k \) for MEAE and SEAE under the conditions \( \hat{\mathbf{X}}_{k-1} = \mathbf{X}_{k-1} \) and \( \hat{\mathbf{P}}_{k-1} = \mathbf{P}_{k-1} T^T \) at time \( t_{k-1} \), \( \hat{\mathbf{X}}_k = \mathbf{X}_k \) and \( \hat{\mathbf{P}}_k = \mathbf{P}_k T^T \) at any time \( t_k \) using a recursive procedure under the conditions \( \hat{\mathbf{X}}_0 = \mathbf{X}_0 \) and \( \hat{\mathbf{P}}_0 = \mathbf{P}_0 T^T \) can be obtained, that is, the two algorithms are equivalent. This will be proven at the step \( t_{k-1} \sim t_k \) according to the two stages of EKF.

(1) Propagation

As the state prediction equations of the two algorithms are identical, the predicted states \( \hat{\mathbf{X}}_{k/k-1} \) and \( \hat{\mathbf{X}}_{k/k-1} \) at time \( t_k \), obtained by integrating the equations at the interval \( [t_{k-1}, t_k] \), are equal under the condition \( \hat{\mathbf{X}}_{k-1} = \mathbf{X}_{k-1} \) at time \( t_{k-1} \). Now, efforts are made to prove \( \hat{\mathbf{P}}_{k/k-1} = \mathbf{P}_{k/k-1} T^T \), where \( \mathbf{P}_{k/k-1} \) and \( \hat{\mathbf{P}}_{k/k-1} \) are obtained by integrating Eq.(17) and Eq.(36), respectively, at the interval \( [t_{k-1}, t_k] \) under the condition \( \hat{\mathbf{P}}_{k-1} = \mathbf{P}_{k-1} T^T \). Suppose that \( \mathbf{P}(t) \) is the solution of Eq.(17), the proving can be transformed into demonstrating that \( \hat{\mathbf{P}}(t) = \mathbf{P}(t) T^T \) is the solution of Eq.(36).

By differentiating left and right sides of Eq.(55),
\[ \hat{\mathbf{P}}(t) = \mathbf{P}(t) T^T + \mathbf{P}(t) T^T + \mathbf{P}(t) T^T \] (56)
can be obtained. Substituting Eq.(17) into Eq.(56) yields
\[ \hat{\mathbf{P}}(t) = \mathbf{P}(t) T^T + \mathbf{P}(t) T^T + \mathbf{P}(t) T^T \] (57)
From Eq.(15), Eq.(34), and Eq.(50), it follows that
\[ \hat{\mathbf{F}}(t) = \mathbf{F}(t) T^T \] (58)
From Eq.(16), Eq.(35), and Eq.(50), it follows that
\[ \hat{\mathbf{G}}(t) = \mathbf{G}(t) \] (59)
By substituting Eqs.(58)-(59) into Eq.(57),
\[ \hat{\mathbf{P}}(t) = \mathbf{F}(t) \hat{\mathbf{P}}(t) + \mathbf{F}(t) \mathbf{F}^T(t) + \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \] (60)
is obtained. Thus \( \hat{\mathbf{P}}(t) \) is the solution of Eq.(36). Therefore, \( \hat{\mathbf{P}}(t) = \hat{\mathbf{P}}(t) \), that is \( \hat{\mathbf{P}}(t) = \mathbf{P}(t) T^T \). Then \( \hat{\mathbf{P}}_{k/k-1} = \mathbf{P}_{k/k-1} T^T \).
(2) Update

From Eq.(20), Eq.(39), and Eq.(50),
\[ \dot{H}_k = H_k T^{-1} \quad (60) \]
is derived. Substituting \( \dot{P}_{k/k-1} = TP_{k/k-1} T^T \) and
Eq.(60) into Eq.(40) leads to
\[ \dot{K}_k = T P_{k/k-1} H_k^T \{ H_k^{-1} P_{k/k-1} H_k^T + R \}^{-1} \quad (61) \]
From Eq.(21) and Eq.(61), it follows that
\[ \dot{K}_k = TK_k \quad (62) \]
From Eq.(22), Eq.(41), and Eq.(62), it follows that
\[ \dot{G} = G_{\dot{X}_k} T \dot{X}_k \quad (63) \]
or
\[ \dot{G} = S \delta \dot{\sigma} \quad (64) \]
Considering \( \ddot{X}_k = \delta \dot{X}_k \) and \( \ddot{\sigma}_k = \delta \dot{\sigma}_k \),
\[ \dot{X}_k = \delta \dot{X}_k + \Delta \dot{X}_k = \delta \dot{X}_k + \Delta \dot{\sigma} = \dot{\sigma}_k \quad (65) \]
\[ \dot{\sigma}_k = \delta \dot{\sigma}_k + \Delta \dot{\sigma}_k = \delta \dot{\sigma}_k + S \delta \dot{\sigma}_k \quad (66) \]
can be obtained. From Eq.(11) and Eq.(47), it follows that
\[ \dot{\sigma}_k = \dot{\sigma}_k \quad (67) \]
From Eq.(23) and Eqs.(64)-(65),
\[ \dot{\sigma}_k = \dot{\sigma}_k \quad (68) \]
can be obtained. Eq.(63) and Eq.(66) indicate that the estimated states of the two algorithms are equal, that is,
\[ \dot{X}_k = \dot{X}_k \]
Substituting \( \ddot{X}_k = T \dot{X}_k \), Eq.(60) and Eq.
(62) into Eq.(44) leads to
\[ \ddot{P}_k = T (I - K_k H_k) P_{k/k-1} (I - K_k H_k)^T + K_k R_k K_k^T T^T \quad (67) \]
From Eq.(25) and Eq.(67), it follows that
\[ \ddot{P}_k = TP_k T^T \]
This ends the proving process.

5 All-attitude Estimation Algorithms

5.1 Switching method to avoid singularity of modified Rodrigues parameters

The modified Rodrigues parameters could not represent all the attitudes because of their singularity. To avoid it, it is reasonable to switch the modified Rodrigues parameters to the shadow parameters. The shadow parameters are\(^{(7,9)}\)
\[ \sigma^s = -\cot(\theta / 4) n \quad (68) \]
The equation to transform the shadow parameters of modified Rodrigues parameters \( \sigma^s \) to the direction cosine matrix and the kinematic equation in terms of \( \sigma^s \) are exactly the same as Eqs.(2)-(3). Therefore, \( \sigma \) in MEAE and SEAE can be replaced with \( \sigma^s \).

From Eq.(1) and Eq.(68), it follows that
\[ |\sigma| = |\sigma^s| = 1 \]
when \( |\sigma| > 1 \) and \( |\sigma| < 1 \), when \( |\sigma^s| > 1 \). Thus, the singularities can be avoided by switching between \( \sigma \) and \( \sigma^s \). When using \( \sigma \) to represent the attitude, switch from \( \sigma \) to \( \sigma^s \) if \( |\sigma| > 1 \), then
\[ \sigma^s = -|\sigma| |\sigma^s|^2 \quad (69) \]
When using \( \sigma^s \) to represent the attitude, switch from \( \sigma^s \) to \( \sigma \) if \( |\sigma^s| > 1 \), then
\[ \sigma = -|\sigma^s| |\sigma^s|^2 \quad (70) \]
Thus, ensure that the magnitude of \( \sigma \) or \( \sigma^s \) will never exceed 1, which results in avoiding the singularity.

5.2 Effects of attitude parameter switching on attitude estimation

This section analyzes the effects of attitude parameter switching on the attitude estimation. This means to clarify whether the filters are disturbed at the switching points and give methods to deal with the disturbances if any.

Let the attitude error be represented by multiplicative error of MEAE, and from Eq.(11) and Eq.(47), it follows that
\[ \dot{\sigma}_k = \dot{\sigma}_k \quad (71) \]
Eq.(71) indicates that there is no jump of attitude error vector at the switching point. Then, there is no disturbance in MEAE because the EKF is actually the Kalman filtering of state error vector.

The attitude error of SEAE is represented by arithmetic error. According to the definition of attitude error and by neglecting the higher order,
\[ \Delta \dot{\sigma} = \frac{2 \dot{\sigma} \dot{\sigma}^T}{|\dot{\sigma}|^2} I \Delta \sigma \quad (72) \]
can be obtained. It can be seen from Eq.(72) that
$\Delta \sigma = \Delta \sigma$ is impossible. Therefore, there will be disturbances in SEAE if no modification is performed.

Similar to Eq.(71),

$$\delta \sigma = \delta \sigma$$

Thus

$$\delta \mathbf{X} = \delta \mathbf{X}$$

From Eq.(53), it follows that

$$\bar{P} = E[\Delta \mathbf{X} \Delta \mathbf{X}^T] = \bar{E}E[\delta \mathbf{X} \delta \mathbf{X}^T] \bar{T}^T$$

Similar to Eq.(74),

$$\bar{P}^s = E[\Delta \mathbf{X}^s (\Delta \mathbf{X}^s)^T] = \bar{P}^s E[\delta \mathbf{X}^s (\delta \mathbf{X}^s)^T] (\bar{T}^s)^T$$

can be obtained, where

$$\bar{T}^s = \left[ (1-|\tilde{\sigma}^s|^2) I + 2(\tilde{\sigma}^s \times) + 2 \tilde{\sigma}^s (\tilde{\sigma}^s)^T \right] \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}$$

From Eqs.(73)-(75), it follows that

$$\bar{P}^s = (\bar{T} \bar{T}^{-1}) \bar{P} (\bar{T} \bar{T}^{-1})^T$$

It is clear that there will be a jump of covariance matrix at the switching point in SEAE. Therefore, the covariance matrix should be modified at the switching point by using Eq.(76).

6 Simulation and Discussions

To evaluate the efficacy of the presented algorithms, numerical simulation is performed on a small satellite in the 600 km orbit. The sensors include a gyro and three-axis magnetometers (TAM). The parameters for the simulation are given as follows: the gyro angle random walk is $\lambda_\psi = 1 (\cdot) \cdot h^{3/2}$, the gyro rate random walk is $\lambda_\nu = 4 (\cdot) \cdot h^{3/2}$, the time interval between time updates of gyro is 0.1 s, the time interval between time updates of TAM is 1 s, the magnetic field reference is modeled with a tenth-order International Geomagnetic Reference Field model, TAM measurement noise is modeled with a zero-mean Gaussian white-noise process with a standard deviation of 300 nT, the initial attitude is set to be $\mathbf{\sigma}(t_0) = [\tan 2.5^\circ \ \tan 2.5^\circ \ \tan 2.5^\circ]^T$, the angular velocity is $\mathbf{\omega} = [5 \ 5 \ 5]^T (\cdot) \cdot s^{-1}$, the initial bias of gyro is $\mathbf{g}(t_0) = [2 \ 2 \ 2]^T (\cdot) \cdot h^{-1}$, the initial estimation of attitude and gyro bias in the filter is set to be zero vectors.

Fig.1 shows the attitude estimation results of MEAE, whereas Fig.2 shows the attitude estimation results of SEAE, which is modified with Eq.(76). $\delta \phi$, $\delta \psi$, $\delta \theta$ in the figures are the errors of the roll, the yaw, and the pitch angle, respectively. It can be seen from the figures that, consistent with the previous theoretical analysis, the performance of the two algorithms is about on par when the filtering has converged. To analyze the effects of attitude parameter switching on attitude estimation, Fig.3 shows the attitude estimation results of SEAE without modification by Eq.(76). Comparing Fig.2 with Fig.3, it is clear that, without modification, distur-
bances take place, of which the effects could never be neglected. As a result, the modification by Eq.(76) is needed if all-attitude estimation is performed with SEAE.

7 Conclusions

According to the different representations of attitude errors, two attitude estimation algorithms, MEAE and SEAE, are presented in this article. It turns out to be that the two algorithms are equivalent in the case of small attitude errors. The attitude parameter switching does not cause disturbances in MEAE, but it does in SEAE, for which, therefore, modification is introduced to eliminate it. Theoretical analysis and simulation results demonstrate that, to express the attitude error, the selection of either arithmetic error or multiplicative error between the true and the estimated attitudes could acquire good attitude estimation. However, the multiplicative error is preferred to express the attitude error because it obviates the need for the modification to the covariance matrix at the switching point of attitude parameters.

References


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