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A Morse complex for infinite dimensional manifolds—part I

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Abstract

In this paper and in the forthcoming Part II, we introduce a Morse complex for a class of functions f defined on an infinite dimensional Hilbert manifold M, possibly having critical points of infinite Morse index and co-index. The idea is to consider an infinite dimensional subbundle—or more generally an essential subbundle—of the tangent bundle of M, suitably related with the gradient flow of f. This Part I deals with the following questions about the intersection W of the unstable manifold of a critical point x and the stable manifold of another critical point y: finite dimensionality of W, possibility that different components of W have different dimension, orientability of W and coherence in the choice of an orientation, compactness of the closure of W, classification, up to topological conjugacy, of the gradient flow on the closure of W, in the case $\dim W = 2$.

Keywords: Gradient flow; Strongly indefinite functionals; Hilbert Grassmannian

Contents

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0. Introduction

Morse theory [Mor25] relates the topology of a compact differentiable manifold M to the combinatorics of the critical points of a smooth Morse function $f : M \to \mathbb{R}$: if $\beta_q(M) = \operatorname{rank} H_q(M)$ denotes the qth Betti number of M, and $c_q(f)$ is the number of critical points x of f with Morse index $m(x) = q$, then the identity

$$\sum_{q=0}^{\dim M} c_q(f) t^q = \sum_{q=0}^{\dim M} \beta_q(M) t^q + (1+t) Q(t) \tag{0.1}$$

holds, with Q a polynomial with positive integer coefficients. Denoting by $C_q(f)$ the free Abelian group generated by the critical points of f of index q, $q = 0, 1, \ldots, \dim M$, it is readily seen that (0.1) is implied[1] by the existence of homomorphisms $\partial_q : C_q(f) \to C_{q-1}(f)$ making $\{C_*(f), \partial_*\}$ a chain complex, whose homology groups are isomorphic to the singular $\mathbb{Z}$-homology groups of M:

$$H_q(\{C_*(f), \partial_*\}) = \frac{\ker \partial_q}{\operatorname{ran} \partial_{q+1}} \cong H_q(M). \tag{0.2}$$

A chain complex with the above properties is indeed provided by a suitable cellular filtration of M. More precisely, if we fix a Riemannian structure on M such that the corresponding gradient flow of f, i.e. the integral flow $\phi : \mathbb{R} \times M \to M$ of the vector field $-\operatorname{grad} f$, is Morse–Smale,[2] then the open subsets

$$M^q := \bigcup_{\substack{x \in \operatorname{crit}(f) \\ m(x) \leqslant q}} \phi([0, +\infty[\times U_x), \quad q = 0, 1, \ldots, \dim M$$

for U_x a suitable small neighborhood of x, constitute a cellular filtration of M, such that

$$H_q(M^q, M^{q-1}) \cong C_q(f).$$

So we get the boundary homomorphism

$$\partial_q : C_q(f) \cong H_q(M^q, M^{q-1}) \to H_{q-1}(M^{q-1}, M^{q-2}) \cong C_{q-1}(f) \tag{0.3}$$

and the classical isomorphism between the homology of the cellular chain complex (0.3) and the singular homology of M (see [Dol80, Section V.1]) implies (0.2).

[1] The two facts would actually be equivalent if we were using coefficients in a field, instead of the ring $\mathbb{Z}$.

[2] Here one needs just that the unstable manifold $W^u(x)$ and the stable manifold $W^s(y)$ have empty intersection, for every pair of distinct critical points x, y with $m(x) \leqslant m(y)$.

The boundary homomorphism ∂_q constructed above has also the following combinatorial description, in terms of the intersections between the unstable manifolds $W^u(x)$ and the stable manifolds $W^s(y)$ of pairs of critical points.[3] Since $\dim W^u(x) = m(x)$ and $\dim W^s(y) = \dim M - m(y)$, the intersection $W^u(x) \cap W^s(y)$ is a submanifold of dimension $m(x)-m(y)$. An arbitrary choice of an orientation for each unstable manifold $W^u(x)$ determines a co-orientation (i.e. an orientation of the normal bundle) for each stable manifold $W^s(x)$, and thus an orientation for each intersection[4] $W^u(x) \cap W^s(y)$. When $m(x) = q$ and $m(y) = q-1$, $W^u(x) \cap W^s(y)$ consists of finitely many gradient flow lines, each of which can be counted as $+1$ or as -1, depending on whether its orientation agrees with the direction of the gradient flow or not. The algebraic sum of these numbers gives an integer $n(x, y)$, and ∂_q can be expressed in terms of the generators of $C_q(f)$ and $C_{q-1}(f)$ as

$$\partial_q x = \sum_{\substack{y \in \mathrm{crit}(f) \\ m(y)=q-1}} n(x, y)\, y \quad \text{for } x \in \mathrm{crit}(f), \quad m(x) = q. \tag{0.4}$$

The Morse complex $\{C_*(f), \partial_*\}$ depends on the choice of the Riemannian structure on M (a different Riemannian structure would produce a different gradient flow) and on the choice of the orientations of the unstable manifolds, but the isomorphism class of such a chain complex depends just on the function f.

The approach described above was essentially clear to the pioneers of Morse theory, such as Thom [Tho49] and Milnor [Mil63,Mil65], and to people in dynamical systems, such as Smale [Sma60,Sma61,Sma67] and Franks [Fra79,Fra80], but it has received increasing attention after the works of Witten [Wit82] and Floer [Flo89]. See the systematic study by Schwarz [Sch93], and Weber's thesis [Web93]. The observation on the invariance of the isomorphism class of the Morse complex is due to Cornea and Ranicki [CR03], together with more striking rigidity results.

Already in the sixties, Morse theory had been generalized to infinite dimensional Hilbert manifolds (manifolds modeled on a Hilbert space) by Palais [Pal63], and Smale [Sma64a,Sma64b], and had been successfully applied to many variational problems (see the expository papers of Bott [Bot82,Bot88], the books of Klingenberg [Kli78,Kli82], of Mawhin and Willem [MW89], of Chang [Cha93], and references therein). Indeed, the compactness of M can be replaced by a compactness assumption on f, the well known Palais–Smale condition ((PS) for short): any sequence $(p_n) \subset M$ such that $f(p_n)$ is bounded and $\|Df(p_n)\|$ is infinitesimal must be compact. If M is a Hilbert manifold endowed with a complete Riemannian structure, and $f \in C^2(M, \mathbb{R})$ is a Morse function, bounded below and satisfying (PS), then the Morse relations (0.1) still hold, the difference being that now (0.1) is an equality between formal power series, with coefficients in $\mathbb{N} \cup \{\infty\}$.

[3] Here one needs that $W^u(x)$ and $W^s(y)$ meet transversally just when $m(x) - m(y) \leqslant 1$.

[4] Indeed by transversality, a normal bundle of $W^u(x) \cap W^s(y)$ in $W^u(x)$ is also the restriction of a normal bundle of $W^s(y)$ in M, so it is oriented, and together with the orientation of $W^u(x)$, it determines an orientation of $W^u(x) \cap W^s(y)$. Notice that the manifold M needs not be orientable.

However, (0.1) takes into account only critical points with finite Morse index, the ultimate reason being that the closed ball of an infinite dimensional Hilbert space is retractable onto its boundary, so that critical points with infinite Morse index are invisible to homotopy theory. It was Floer [Flo88a,Flo88b,Flo88c,Flo89] who observed that the Morse complex approach is suitable to deal with critical points of infinite Morse index and co-index: even if the unstable and stable manifolds are infinite dimensional, one may still hope the dimension of their intersection to be finite. In this case, one could try to see (0.4) not as a description, but rather as the definition of a chain complex. In this way, Floer was able to develop the analogue of Morse theory in a case where the gradient flow ODE is replaced by a Cauchy–Riemann type PDE, which does not even determine a local flow (so that there are no stable and unstable manifolds). The resulting theory, known as Floer homology, plays now a central role in symplectic geometry (see [HZ94,Sal99] and references therein).

In the present paper, and in the forthcoming Part II, we introduce and study the Morse complex for gradient-like flows on infinite dimensional Hilbert manifolds. The results we present are a far reaching generalization of a previous work on a special class of functionals on Hilbert spaces [AM01]. See also [AvdV99] for a construction of the Morse complex for the energy functional of an elliptic system, and Chapter 6 in Jost's book [Jos02] for a general approach to the Morse complex. More precisely, we give an answer to the following questions.

(i) When is $W^u(x) \cap W^s(y)$ a finite dimensional manifold?
(ii) How can we give coherent orientations to the manifolds $W^u(x) \cap W^s(y)$?
(iii) When is the closure of $W^u(x) \cap W^s(y)$ compact?
(iv) Having defined ∂_q by (0.4), how do we prove that $\partial_{q-1} \circ \partial_q = 0$?
(v) Which form of transversality is generic?
(vi) How do we recover the classical infinite dimensional Morse theory?
(vii) How can we compute the homology of the Morse complex?

In the present paper, we address questions (i)–(iv), leaving questions (v)–(vii) to Part II. We wish to emphasize the fact that these questions are only formally analogue to corresponding issues in Floer homology. Indeed, since in our case the gradient-like vector field determines a C^1 local flow, some of the problems above can be dealt by dynamical systems techniques. On the other hand, finite dimensionality and compactness results do not come from elliptic estimates, but involve different ideas. In particular, the study of some infinite dimensional Grassmannians, of ordinary differential operators on Hilbert spaces, and the use of Hausdorff measures of non-compactness turn out to be important tools.

We conclude this introduction by giving an informal description of our results.

0.1. Finite dimensional intersections

Let f be a C^2 Morse function on a paracompact Hilbert manifold M. Let F be a C^1 Morse vector field on M, having f as a non-degenerate Lyapunov function: this means that $Df(p)[F(p)] < 0$ for every $p \in M$ which is not a rest point of F, that the Jacobian of F at every rest point x—denoted by $\nabla F(x)$—is a hyperbolic operator, and that the quadratic form $D^2 f(x)$ is coercive on $V^-(\nabla F(x))$, the negative eigenspace of $\nabla F(x)$, while $-D^2 f(x)$ is coercive on the positive eigenspace $V^+(\nabla F(x))$. Under these assumptions, x is a rest point of F if and only if it is a critical point of f. Typically, $F = -\operatorname{grad} f$, the negative gradient of f with respect to some Riemannian metric on M, or $F = -h \operatorname{grad} f$, for some positive function h.

The unstable and stable manifolds of a critical point x are C^1 submanifolds of dimension the Morse index and co-index of x. When the critical points x and y have infinite index and co-index, respectively, the intersection $W^u(x) \cap W^s(y)$ can be infinite dimensional: consider for example the restriction of a continuous linear form f on a Hilbert space H to the unit sphere S of H. Its critical points are a maximum point x and a minimum point $-x$, and $W^u(x) \cap W^s(-x) = S \setminus \{x, -x\}$.

What is more striking, if x and y are critical points of f with infinite Morse index and co-index, the dimension of the intersection between their unstable and stable manifolds (with respect to the negative gradient flow of f) depends on the metric on M: indeed, if all the critical points of a Morse function f have infinite Morse index and co-index, and $a : \mathrm{crit}(f) \to \mathbb{Z}$ is any function, then M supports a metric g—uniformly equivalent to a given one—such that the corresponding negative gradient flow of f has the property that for every pair of critical points x, y the intersection $W^u(x) \cap W^s(y)$ is transverse and has dimension $a(x) - a(y)$ (see [AM04b]).

Therefore, some extra structure on the manifold M is needed: we will assume the existence of a subbundle $\mathcal{V}$ of TM, which can be used to make comparisons. More precisely, the object of our study will be a quartet $(M, F, f, \mathcal{V})$, where f is a non-degenerate Lyapunov function for the Morse vector field F, and the subbundle $\mathcal{V}$ of TM is compatible to F, meaning that:

(C1) for every rest point x, $V^+(\nabla F(x))$, the positive eigenspace of the Jacobian of F at x, is a compact perturbation of $\mathcal{V}(x)$ (this means that the corresponding orthogonal projectors have compact difference);

(C2) denoting by $\mathcal{P}$ a projector onto $\mathcal{V}$, $(L_F \mathcal{P})(p)\mathcal{P}(p)$ is a compact linear operator on T_pM, for every $p \in M$ (here $L_F \mathcal{P}$ denotes the Lie derivative of the tensor $\mathcal{P}$ along the vector field F).

Assumption (C1) allows us to define the *relative Morse index of a rest point x with respect to* $\mathcal{V}$ to be the integer

$$\begin{aligned} m(x, \mathcal{V}) &:= \dim(V^+(\nabla F(x)), \mathcal{V}(x)) \\ &= \dim V^+(\nabla F(x)) \cap \mathcal{V}(x)^\perp - \dim V^+(\nabla F(x))^\perp \cap \mathcal{V}(x). \end{aligned}$$

Notice that $m(x, \mathcal{V})$ can be negative. A subbundle $\mathcal{V} = \mathcal{P}(TM)$ is invariant for the differential of the integral flow of a vector field X if and only if $(L_X \mathcal{P})\mathcal{P} = 0$. Assumption (C2) says that $\mathcal{V}$ is *essentially invariant* for the linearized flow of F, meaning that the differential of the flow of F maps $\mathcal{V}$ into a compact perturbation of $\mathcal{V}$. Assumptions (C1) and (C2) are automatically fulfilled when all the critical points have finite index, by choosing $\mathcal{V} = (0)$: in this case $m(x, (0))$ is the usual Morse index.

Our first result will be that if (C1) and (C2) hold, and $W^u(x)$, $W^s(y)$ meet transversally, then their intersection is finite dimensional, and

$$\dim W^u(x) \cap W^s(y) = m(x, \mathcal{V}) - m(y, \mathcal{V}), \tag{0.5}$$

which is the first step for the construction of the Morse complex. A useful tool, in the proof of this result and in transversality questions, will be the study, presented in [AM03b], of the Fredholm properties of the differential operator

$$\frac{d}{dt} - A(t) : C_0^1(\mathbb{R}, H) \to C_0^0(\mathbb{R}, H),$$

where the subscript 0 means vanishing at infinity, and A is a continuous path of bounded operators on the Hilbert space H, converging to hyperbolic operators for $t \to \pm\infty$.

As we shall see, the usefulness of conditions (C1) and (C2) lies in the fact that they are both *stable* and *convex*.

In many cases, the choice of the subbundle $\mathcal{V}$ for which (C1) and (C2) hold, is suggested by the problem itself: for example, this is the case of semi-linear equations, where f is a lower order perturbation of a non-degenerate quadratic form on a Hilbert space, and of many functionals coming from geometric problems, such as the energy of curves on a semi-Riemannian manifold. In other cases, (C1) and (C2) just hold locally: one finds an open covering $\{U_j \mid j \in J\}$ of M and subbundles $\mathcal{V}_j$ of TU_j, which satisfy (C1), (C2), and are such that $\mathcal{V}_i|_{U_i \cap U_j}$ is a compact perturbation of $\mathcal{V}_j|_{U_i \cap U_j}$, for any $i, j \in J$. That is, (C1) and (C2) hold with respect to an *essential subbundle*. In such a situation the intersection of the unstable and stable manifolds are finite dimensional, but no formula like (0.5) can possibly hold. Indeed, we will show an example of a Morse function on $S^1 \times H$, H an infinite dimensional Hilbert space, with two rest points x, y, such that different components of the transverse intersection $W^u(x) \cap W^s(y)$ have different dimension. This is a purely infinite dimensional phenomenon, related to the fact that the general linear group of an infinite dimensional Hilbert space is contractible (see [Kui65]). Formula (0.5) will hold in the intermediate situation in which $\dim(\mathcal{V}_i, \mathcal{V}_j) = 0$ for every $i, j \in J$: in this case we will say that (C1) and (C2) hold with respect to a (0)-*essential subbundle*.

These facts are closely related to Cohen, Jones, and Segal's use of polarizations to understand the homotopy theory which lies behind Floer homology [CJS95].

0.2. Coherent orientations

As we have seen, when M is finite dimensional—or more generally when the rest points have finite Morse index—$W^u(x) \cap W^s(y)$ is orientable. In the case of infinite Morse indices and co-indices, however, $W^u(x) \cap W^s(y)$ needs not be orientable: indeed we will provide an example showing that such a transverse intersection can be diffeomorphic to $Z \times \mathbb{R}$, where Z is any manifold.

The existence of a subbundle $\mathcal{V}$ satisfying (C1) and (C2) will imply that all the intersections $W^u(x) \cap W^s(y)$ are orientable, and it will allow us to define their orientations in a coherent way. The starting point is the fact that Fredholm pairs (i.e. pairs (V, W) of closed linear subspaces of a Hilbert space H with $\dim V \cap W < \infty$, $\operatorname{codim}(V+W) < \infty$) can be oriented: an orientation of (V, W) is by definition an orientation of the finite dimensional space $(V \cap W) \times (H/(V + W))^*$. Actually, a determinant bundle can be defined on the space of Fredholm pairs, extending the determinant bundle on the space of Fredholm operators, defined by Quillen [Qui85]. Together with the fact that the fundamental group of the space of Fredholm pairs (V, W) with $\dim V = \dim W = \infty$, is $\mathbb{Z}_2$, this implies that the notion of orientation of a Fredholm pair shares all the good properties of orientations of finite dimensional spaces.

For every rest point x, one fixes an orientation of the Fredholm pair $(T_x W^s(x), \mathcal{V}(x))$. Assumptions (C1) and (C2) guarantee that $(T_p W^s(x), \mathcal{V}(p))$ is a Fredholm pair, for every $p \in W^s(x)$. Hence, the orientation chosen at x propagates to all the stable manifold of x. The way of orienting $W^u(x) \cap W^s(y)$ is then similar to what we have described in the case of a finite dimensional M.

If conditions (C1) and (C2) hold with respect to a (0)-essential subbundle, coherent orientations cannot be defined, and one obtains just a Morse complex with $\mathbb{Z}_2$ coefficients. Bott periodicity theorem [Bot59] can be used to find the obstructions to have a Morse complex with integer coefficients: they are given by the homotopy groups $\pi_i(M)$, with $i \equiv 1, 2, 3, 5 \bmod 8$.

0.3. Relative compactness of the intersections

When the rest point x has a finite Morse index, the (PS) condition [5] and the completeness of the flow imply that the intersection $W^u(x) \cap W^s(y)$ has compact closure in M. When the indices are infinite, even if (C1-2) guarantee that $W^u(x) \cap W^s(y)$ is finite dimensional, we cannot conclude that its closure is compact: for instance, it may consist of infinitely many isolated curves, with no cluster points besides x and y.

The reason is that (C1-2) are local assumptions, while compactness involves a global condition: we shall need a global version of condition (C2). Let us assume for simplicity that the subbundle $\mathcal{V}$ of TM has a *global presentation*, that is a submersion $\mathcal{Q} : M \to N$ into a complete Riemannian Hilbert manifold N such that $\mathcal{V}(p) = \ker D\mathcal{Q}(p)$. We will denote by $\beta_X(A)$ the Hausdorff measure of non-compactness of the subset A of a metric space X, that is the infimum of all positive numbers r such that A can be covered by finitely many balls of radius r. The new assumption is:

(C3) (i) $D\mathcal{Q} \circ F$ is bounded;
(ii) for every $q \in N$ there exist $\delta > 0$ and $c \geqslant 0$ such that $\beta_{TN}(D\mathcal{Q}(F(A))) \leqslant c\, \beta_N(\mathcal{Q}(A))$, for any A in a $\mathcal{Q}^{-1}(B_\delta(q))$.

This condition implies (C2) by differentiation. Condition (C3) is also stable and convex, in a sense to be specified.

[5] In this contest, a (PS) sequence is a sequence $(p_n) \subset M$ such that $(f(p_n))$ is bounded and $(Df(p_n)[F(p_n)])$ is infinitesimal.

We shall prove that conditions (C1) and (C3), together with (PS) and the completeness of the flow, imply that $W^u(x) \cap W^s(y)$ has compact closure in M, for every pair of critical points x, y.

This compactness result will be proved in the more general setting of a flow which preserves an *essentially vertical family* $\mathcal{F}$ of subsets of M, with respect to a *strong integrable structure* for an essential subbundle $\mathcal{E}$ of TM. When $\mathcal{E}$ is the essential class of a subbundle $\mathcal{V}$ with a global presentation $\mathcal{Q}$, one chooses $\mathcal{F}$ to be the family of subsets $A \subset M$ such that $\mathcal{Q}(A)$ is pre-compact. More-generally, one can deal with a suitable presentation of $\mathcal{E}$ consisting of an open covering $\{M_i\}_{i\in I}$ of M and of semi-Fredholm maps with non-negative index $\mathcal{Q}_i : M_i \to N_i$, such that $\mathcal{E}(p) = [\ker D\mathcal{Q}(p)]$ for every $p \in M_i$.

0.4. The boundary homomorphism

Assume that $(M, F, f, \mathcal{V})$ satisfies (C1-3) and (PS), and that the stable and unstable manifolds of rest points meet transversally. For $q \in \mathbb{Z}$, we can define $C_q(F)$ to be the free Abelian group generated by the rest points x with $m(x, \mathcal{V}) = q$. In order to define the homomorphism $\partial_q : C_q(F) \to C_{q-1}(F)$, we just need the last condition

(C4) for any $q \in \mathbb{Z}$, f is bounded below on the set of critical points x of relative Morse index $m(x, \mathcal{V}) = q$,

which guarantees that the sum appearing in (0.4) is finite.

The boundary property $\partial_{q-1} \circ \partial_q = 0$ comes from the possibility of describing exactly the flow on the closure of each component of $W^u(x) \cap W^s(y)$, when $m(x, \mathcal{V}) - m(y, \mathcal{V}) = 2$: such a flow is either topologically conjugated to the exponential flow $(t, z) \mapsto e^t z$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, or it is topologically conjugated to the shift flow $(t, (u, v)) \mapsto (u + t, v + t)$ on $[-\infty, +\infty] \times [-\infty, +\infty]$. In the latter case, the orientation of this component is the product orientation of its sides. These results will be proved by hyperbolic dynamical systems techniques, which in this case seem more natural than the gluing method used in Floer homology.

The resulting complex $\{C_*(F), \partial_*\}$ is said the *Morse complex* of F. If F_1 and F_2 are two Morse vector fields having the same non-degenerate Lyapunov function f, the Morse complexes of F_1 and of F_2 are isomorphic. In particular, their homology depends only on the Lyapunov function f, and it will be said the *Morse homology* of f and denoted by $H_*(f)$.

0.5. Transversality

The transversality of the intersection of stable and unstable manifolds will be achieved by perturbing the vector field F. Small perturbations in a suitable class of vector fields keep the conditions (C1-4) and (PS) valid: in this sense, these conditions were said to be stable. When one restricts the attention to the class of gradient vector fields, transversality can be achieved by using rank 2 perturbations of the given Riemannian metric. A difference with respect to the finite dimensional case is the regularity requirement. Indeed, high regularity of F is needed to apply Sard–Smale theorem, and such a regularity cannot be obtained by smoothing the vector field F, because C^{k+1} functions on an infinite dimensional Hilbert space are not C^k dense (see [NS73,LL86]). As a consequence, we shall assume $F \in C^2(M)$, and we will achieve transverse intersections of $W^u(x)$ and $W^s(y)$ whenever $m(x,\mathcal{V}) - m(y,\mathcal{V}) \leqslant 2$, which is what we need for the construction of the Morse complex.

0.6. Relationship with classical infinite dimensional Morse theory

In the case of f bounded below, satisfying (PS), and with critical points of finite Morse index, we shall prove that the Morse homology of f is isomorphic to the singular homology of M, a result which agrees with the Morse relations proved by Palais. This will be a simple generalization of the cellular filtration argument described for the compact case.

From this fact, it is easy to determine the Morse complex of some classes of vector fields having rest points of infinite Morse index and co-index. For instance, if $M = M^- \times M^+$ is the product of two infinite dimensional Hilbert manifolds, endowed with a complete product Riemannian structure, and the Morse function $f : M \to \mathbb{R}$ has the special form

$$f(p^-, p^+) = f^+(p^+) - f^-(p^-), \tag{0.6}$$

where $f^+ : M^+ \to \mathbb{R}$, $f^- : M^- \to \mathbb{R}$ are bounded below and satisfy (PS), then

$$F = -\left(\frac{\operatorname{grad} f^-}{1 + \|\operatorname{grad} f^-\|^2}, \frac{\operatorname{grad} f^+}{1 + \|\operatorname{grad} f^+\|^2} \right)$$

satisfies (C1-3) with respect to the subbundle $\mathcal{V} = TM^- \times (0)$, with global presentation the submersion $\mathcal{Q} : M \to M^+$, $(p^-, p^+) \mapsto p^+$. Notice that F has the form $F(p^-, p^+) = (F^-(p^-), F^+(p^+))$. It is easy to see that the Morse complex of F is

$$C_q(F) = (C_*(F^+) \otimes C_{-*}(F^-))_q = \bigoplus_{\substack{(q^-,q^+)\in\mathbb{N}^2 \\ q^+ - q^- = q}} C_{q^+}(F^+) \otimes C_{q^-}(F^-) \quad \forall q \in \mathbb{Z}$$

and the Morse homology of f is

$$H_q(f) \cong (H_*(M^+) \otimes H_{-*}(M^-))_q = \bigoplus_{\substack{(q^-,q^+)\in\mathbb{N}^2 \\ q^+ - q^- = q}} H_{q^+}(M^+) \otimes H_{q^-}(M^-) \quad \forall q \in \mathbb{Z}. \tag{0.7}$$

0.7. Computation of the homology and functoriality

In the case of infinite Morse indices and co-indices, the topology of M is not immediately related to the Morse homology of f. However, the homology groups $H_q(f)$ are still considerably stable.

The key ingredient to compute the Morse homology groups will be the fact that Morse homology is a functor from the class of Morse functions with a gradient-like vector field satisfying (C1-4) and (PS), seen as a small category with the usual order relation, to the category of Abelian groups: to every inequality $f_0 \geqslant f_1$ is associated a homomorphism

$$\phi_{f_0 f_1} : H_*(f_0) \to H_*(f_1),$$

in such a way that $\phi_{f_1 f_2} \circ \phi_{f_0 f_1} = \phi_{f_0 f_2}$, and $\phi_{ff} = \mathrm{id}$ (actually, $\phi_{\theta \circ f f} = \mathrm{id}$, for $\theta(s) \geqslant s$ a strictly increasing smooth function). The idea for the definition of $\phi_{f_0 f_1}$ comes from the following observation: every chain homomorphism $\psi : \{C^0_*, \partial^0_*\} \to \{C^1_*, \partial^1_*\}$ comes from a boundary operator $\partial_q : C^0_q \oplus C^1_{q+1} \to C^0_{q-1} \oplus C^1_q$, the cone of ψ, namely

$$\partial_q = \begin{pmatrix} \partial^0_q & 0 \\ \psi & -\partial^1_{q+1} \end{pmatrix}. \tag{0.8}$$

With this in mind, we will construct a Morse function $f : \mathbb{R} \times M \to \mathbb{R}$, of the form

$$f(s, p) = \chi(s) f_0(p) + (1 - \chi(s)) f_1(p) + \varphi(s),$$

with χ a monotone smooth function such that $\chi(s) = 1$ for $s \leqslant 0$, and $\chi(s) = 0$ for $s \geqslant 1$, while $\varphi(s) = 2s^3 - 3s^2 + 1$ has a non-degenerate maximum at 0 and a non-degenerate minimum at 1. The function f is a non-degenerate Lyapunov function for a Morse vector field on $\mathbb{R} \times M$ satisfying (C1-4) and (PS), and the boundary operator in the associated Morse complex has the form (0.8). This allows us to define $\phi_{f_0 f_1}$ as the homomorphism induced by the chain homomorphism ψ.

We wish to emphasize that this functorial approach is possible thanks to the fact that the conditions (C1-4), and (PS) naturally pass from the functions f_0, f_1 to the *cone function* f: in this sense, these conditions were said to be convex.

In particular, two functions f_0 and f_1 such that $c := \|f_1 - f_0\|_\infty$ is finite, have always isomorphic Morse homologies, as implied by the functoriality applied to the inequalities

$$f_0 - c \leqslant f_1 \leqslant f_0 + c, \quad f_1 - c \leqslant f_0 \leqslant f_1 + c.$$

For example, let $f : M^- \times M^+ \to \mathbb{R}$ be a Morse function satisfying (PS) and such that $F = -\mathrm{grad}\, f / (1 + \|\mathrm{grad}\, f\|^2)$ satisfies (C1-4) with respect to the subbundle $\mathcal{V} =$

$TM^- \times (0)$. If f has bounded distance from a function of the form (0.6), still satisfying the same assumptions, the Morse homology of f is given by (0.7). More generally, if there exists $c > 0$ such that

$$\frac{1}{c} f^+(p^+) - cf^-(p^-) - c \leqslant f(p^-, p^+) \leqslant cf^+(p^+) - \frac{1}{c} f(p^-) + c,$$

we deduce the existence of a surjective homomorphism

$$H_q(f) \to \bigoplus_{\substack{(q^-,q^+)\in\mathbb{N} \\ q^+-q^-=q}} H_{q^+}(M^+) \otimes H_{q^-}(M^-),$$

which implies lower estimates on the number of critical points of f of a given relative Morse index.

1. Essential subbundles of a Hilbert bundle

In this section, we will fix some basic facts about the Grassmannian of a Hilbert space and some related constructions. We refer to Appendix A for more details.

1.1. Hilbert Grassmannians

If E and F are Banach spaces, $\mathcal{L}(E, F)$ will denote the space of bounded linear operators from E to F, while $\mathcal{L}_c(E, F)$ will denote the subspace consisting of compact operators. In the case $F = E$, we will simply write $\mathcal{L}(E)$ and $\mathcal{L}_c(E)$.

Let H be an infinite dimensional separable real Hilbert space. The orthogonal projection onto a closed linear subspace $V \subset H$ will be denoted by P_V, while the orthogonal complement of V will be indicated by $V^\perp$. We will denote by $\mathrm{Gr}(H)$ the *Grassmannian* of H, that is the space of all closed linear subspaces of H, endowed with the operator norm topology. By $\mathrm{Gr}_{\infty,\infty}(H)$ we will denote the connected component of $\mathrm{Gr}(H)$ consisting of subspaces of infinite dimension and infinite codimension. The other connected components of $\mathrm{Gr}(H)$ are the subsets $\mathrm{Gr}_{n,\infty}(H)$, the set of linear subspaces of H of dimension n, and $\mathrm{Gr}_{\infty,n}(H)$, the set of linear subspaces of H of codimension n.

1.2. Compact perturbations and essential Grassmannians

Given $V, W \in \mathrm{Gr}(H)$, we will say that V is a *compact perturbation* of W if $P_V - P_W$ is a compact operator. In this case, the *relative dimension* of V with respect to W is the integer

$$\dim(V, W) = \dim V \cap W^\perp - \dim V^\perp \cap W.$$

Given $m \in \mathbb{N}$, the (m)-*essential Grassmannian* $\mathrm{Gr}_{(m)}(H)$ is the quotient space of $\mathrm{Gr}(H)$ by the equivalence relation

$$\{(V,W) \in \mathrm{Gr}(H)^2 | V \text{ is a compact perturbation of } W, \\ \text{and } \dim(V,W) \in m\mathbb{Z}\}.$$

By $\mathrm{Gr}^*_{(m)}(H)$ we will denote the quotient of $\mathrm{Gr}_{\infty,\infty}(H)$ by the same equivalence relation. The space $\mathrm{Gr}_{(1)}(H)$ is called just the *essential Grassmannian* of H. If $[W] \in \mathrm{Gr}_{(m)}(H)$ and $V \in \mathrm{Gr}(H)$ is a compact perturbation of an element (hence every element) of the class $[W]$, then $\dim(V,[W]) := \dim(V,W)$ is well defined as an integer modulo m.

1.3. Essential subbundles

Fix some $k \in \mathbb{N} \cup \{\infty\}$. Let B be a topological space if $k = 0$, or a C^k Banach manifold if $k \geqslant 1$, and let $\mathcal{H} \to B$ be an H-bundle on B, that is a C^k fiber bundle with base space B, total space $\mathcal{H}$, typical fiber the Hilbert space H, and structure group $\mathrm{GL}(H)$. Since the Hilbert space H is infinite dimensional, the group $\mathrm{GL}(H)$ is contractible (see [Kui65]), so the above bundle is always trivial.

We can associate to the C^k Hilbert bundle $\mathcal{H} \to B$ the C^k fiber bundles

$$\mathrm{Gr}(\mathcal{H}) = \bigcup_{b \in B} \mathrm{Gr}(\mathcal{H}_b) \to B, \quad \mathrm{Gr}_{(m)}(\mathcal{H}) = \bigcup_{b \in B} \mathrm{Gr}_{(m)}(\mathcal{H}_b) \to B, \quad m \in \mathbb{N}.$$

The spaces $\mathrm{Gr}(H)$ and $\mathrm{Gr}_{(m)}(H)$ admit natural analytic structures, so the above bundles have C^k structures. A C^k section of $\mathrm{Gr}(\mathcal{H}) \to B$ is just a C^k subbundle of $\mathcal{H} \to B$. Similarly, a C^k section of $\mathrm{Gr}_{(m)}(\mathcal{H}) \to B$ will be called a C^k (m)-*essential subbundle of* $\mathcal{H} \to B$, or just a C^k *essential subbundle* in the case $m = 1$.

1.4. Lifting properties

The following questions arise naturally: when is an (m)-essential subbundle, $m \in \mathbb{N}$, liftable to a true subbundle? when is an (m)-essential subbundle, $m \geqslant 1$, liftable to an (hm)-essential subbundle, for $h \in \mathbb{N}$? We shall discuss these questions in the nontrivial case of subbundles with infinite dimension and codimension.

Since the Hilbert bundle $\mathcal{H} \to B$ is trivial, the (m)-essential subbundle we wish to lift can be identified with a map

$$\mathcal{E} : B \to \mathrm{Gr}^*_{(m)}(H)$$

and we are looking at the lifting problems

$$\begin{tikzcd} & \mathrm{Gr}_{\infty,\infty}(H) \arrow[d] \\ B \arrow[ur, dashed] \arrow[r, "\mathcal{E}"] & \mathrm{Gr}^*_{(m)}(H) \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Gr}^*_{(hm)}(H) \arrow[d] \\ B \arrow[ur, dashed] \arrow[r, "\mathcal{E}"] & \mathrm{Gr}^*_{(m)}(H) \end{tikzcd}$$

In the first diagram, the vertical map is a fibration from a contractible space, so the (m)-essential subbundle $\mathcal{E}$ is liftable to a true subbundle if and only if the map $\mathcal{E}$ is null-homotopic. It can be proved that $\mathrm{Gr}^*_{(0)}(H)$ is simply connected, while the fundamental group of $\mathrm{Gr}_{(m)}(H)$ for $m \geqslant 1$ is infinite cyclic. Furthermore, $\pi_i(\mathrm{Gr}_{(m)}(H)) \cong \pi_{i-1}$ $(\mathrm{BO}(\infty))$ for $i \geqslant 2$, where $\mathrm{BO}(\infty)$ denotes the classifying space of the infinite real orthogonal group. Hence, using Bott periodicity theorem, we deduce that $\mathcal{E}$ is null homotopic if and only the homomorphism

$$\mathcal{E}_* : \pi_i(B) \to \pi_i(\mathrm{Gr}^*_{(m)}(H))$$

vanishes for every $i \equiv 1, 2, 3, 5 \bmod 8$. In particular, every (m)-essential subbundle is liftable to a true subbundle when $\pi_i(B) = 0$ for every $i \equiv 1, 2, 3, 5 \bmod 8$.

In the second diagram, the vertical arrow is a covering map, and the image of the induced homomorphism

$$\pi_1(\mathrm{Gr}^*_{(hm)}(H)) \to \pi_1(\mathrm{Gr}^*_{(m)}(H)) = \mathbb{Z}$$

is the subgroup $h\mathbb{Z}$, so the (m)-essential subbundle $\mathcal{E}$ is liftable to a (hm)-essential subbundle if and only if $\mathcal{E}_*(\pi_1(B)) \subset h\mathbb{Z}$. In particular, every (m)-essential subbundle is liftable to a (0)-essential subbundle when B is simply connected.

In this paper, we will be mainly interested in subbundles and essential subbundles of the tangent bundle TM of a Hilbert manifold M (that is a paracompact manifold modeled on the Hilbert space H). Notice that, since M is locally contractible, any (m)-essential subbundle $\mathcal{E}$ is locally liftable to a true subbundle, which will be called a *local representative* of $\mathcal{E}$.

1.5. Integrable essential subbundles of TM

An essential subbundle $\mathcal{E}$ of TM is called *integrable* if M admits an atlas whose charts $\varphi : \mathrm{dom}(\varphi) \subset M \to H$ map $\mathcal{E}$ into the essential subbundle represented by a constant closed linear subspace $V \subset H$:

$$\forall p \in \mathrm{dom}(\varphi) \quad D\varphi(p)\mathcal{E}(p) = [V]. \tag{1.1}$$

If φ and ψ are two such charts, the transition map $\tau = \varphi \circ \psi^{-1} : \mathrm{dom}\,(\tau) \subset H \to H$ satisfies

$$D\tau(\xi)V \text{ is a compact perturbation of } V \quad \forall \xi \in \mathrm{dom}\,(\tau).$$

By Proposition A.4, the above fact is equivalent to

$$QD\tau(\xi)(I - Q) \text{ is a compact operator} \quad \forall \xi \in \mathrm{dom}\,(\tau), \tag{1.2}$$

Q being a projector with kernel V. An atlas $\mathcal{A}$ of M satisfying (1.1) and (1.2) form an *integrable structure modeled on* (H, V) for the essential subbundle $\mathcal{E}$.

Conversely, an atlas of M whose transition maps satisfy (1.2) defines an integrable essential subbundle of *TM*. Such an essential subbundle is liftable to an (m)-essential subbundle $m \in \mathbb{N}$, if and only if

$$\dim(D\tau(\xi)V, V) \equiv 0 \mod m$$

for every transition map τ and every $\xi \in \mathrm{dom}\,(\tau)$.

Considering integrable essential subbundles will be important starting from Section 6. Actually, we will be interested in the following stronger version of integrability.

Definition 1.1. Let V be a closed linear subspace of the Hilbert space H, and let $Q \in \mathcal{L}(H)$ be a projector with kernel V. A strong integrable structure modeled on (H, V) for the essential subbundle $\mathcal{E}$ of *TM* is atlas $\mathcal{A}$ of M such that:

(i) for every $\varphi \in \mathcal{A}$ and every $p \in \mathrm{dom}\,(\varphi)$, $D\varphi(p)\mathcal{E}(p) = [V]$;
(ii) for every $\varphi, \psi \in \mathcal{A}$ the transition map $\tau = \varphi \circ \psi^{-1} : \mathrm{dom}\,(\tau) \subset H \to H$ satisfies

$$QA \text{ is pre-compact if and only if } Q\tau(A) \text{ is pre-compact},$$

for every bounded $A \subset \mathrm{dom}\,(\tau)$.

Since the set *QA* is pre-compact if and only if the projection of *A* into the quotient space H/V is pre-compact, the above definition does not depend on the choice of the projector *Q*, but only on the subspace *V*.

Let τ be a transition map satisfying condition (ii) of the above definition, and let $\xi \in \mathrm{dom}\,(\tau)$. Then the restriction of the map $Q\tau$ to the set $\mathrm{dom}\,(\tau) \cap (\xi + V)$ is a compact map (i.e. it maps bounded sets into pre-compact sets). Therefore its differential at ξ, namely the linear operator $QD\tau(\xi)|_V$ is compact, implying (1.2). Hence a strong integrable structure is also an integrable structure. The notion of a strong integrable structure is strictly more restrictive, because a nonlinear map whose differential at every point is compact need not be compact.

Remark 1.2. Assume that W is a compact perturbation of the closed linear subspace V. Notice that if $A \subset H$

$$P_{W^\perp} A \subset P_{V^\perp} A + (P_V - P_W) A.$$

Then if A is bounded $P_{V^\perp} A$ is pre-compact if and only if $P_{W^\perp} A$ is pre-compact. Therefore, a strong integrable structure modeled on (H, V) is also a strong integrable structure modeled on (H, W).

1.6. Presentations of an essential subbundle

A natural way to construct an integrable subbundle of TM is to consider the kernel of a submersion, or of a family of submersions with matching kernels. We wish to describe the essential version of this construction.

The following lemma can be considered the essential version of the fact that in suitable charts a submersion is a linear projection.

Lemma 1.3. *Let M and N be manifolds modeled on the Hilbert spaces H and E, respectively. Let $\mathcal{Q} : M \to N$ be a C^k, $k \geqslant 1$, semi-Fredholm map with constant non-negative index. Then there exists a projector $Q \in \mathcal{L}(H)$ such that, denoting by V its kernel, there holds: for every $p \in M$ there exists a C^k chart $\varphi : U \to H$, $p \in U \subset M$, such that*

(i) *$\varphi(U)$ is bounded*;
(ii) *for every $\xi \in \varphi(U)$*, $\ker D(\mathcal{Q} \circ \varphi^{-1})(\xi)$ *is a compact perturbation of V, with*

$$\dim(\ker D(\mathcal{Q} \circ \varphi^{-1})(\xi), V) = \dim \operatorname{coker} D\mathcal{Q}(\varphi^{-1}(\xi));$$

(iii) *for every $A \subset \varphi(U)$, $\mathcal{Q}(\varphi^{-1}(A))$ is pre-compact if and only if QA is compact.*

In most applications, the index of $\mathcal{Q}$ will be $+\infty$.

Proof. The matter being local, we may assume that M is an open subset of the Hilbert space H, that $p = 0$, that N is an open subset of the Hilbert space E, and that $\mathcal{Q}(0) = 0$. Since $\operatorname{ind} D\mathcal{Q}(0) \geqslant 0$, there is $T \in \mathcal{L}(H, E)$ with finite rank such that $D\mathcal{Q}(0) + T$ is surjective. By the open mapping theorem, $D\mathcal{Q}(0)+T$ has a linear bounded right inverse $R \in \mathcal{L}(E, H)$. Let $Q := R(D\mathcal{Q}(0) + T) \in \mathcal{L}(H)$ be the associated linear projection, and set $V := \ker Q$. Since the index of $\mathcal{Q}$ is constant (i.e. it does not take different values on different connected components of M), by applying a linear conjugacy the same Q and V can be used for every point $p \in M$.

The map $R(\mathcal{Q}+T) : M \to \operatorname{ran} Q$ is a local submersion at 0, with differential at 0 equal to Q. Therefore, there exists a neighborhood $U \subset M$ of 0 and a C^k diffeomorphism $\varphi : U \to H$ such that $\varphi(0) = 0$, $D\varphi(0) = I$, $\varphi(U)$ and $\mathcal{Q}(U)$ bounded (so that (i) holds), and

$$R(\mathcal{Q} + T) \circ \varphi^{-1}(\xi) = Q\xi \quad \forall \xi \in \varphi(U). \tag{1.3}$$

Differentiating we get $RD(Q \circ \varphi^{-1})(\xi) + RT D\varphi^{-1}(\xi) = Q$. Since R is injective and since T has finite rank, Proposition A.3 implies that

$$\ker D(\mathcal{Q} \circ \varphi^{-1})(\xi) = \ker RD(\mathcal{Q} \circ \varphi^{-1})(\xi)$$

is a compact perturbation of $\ker Q = V$, and

$$\begin{aligned} \dim(\ker D(\mathcal{Q} \circ \varphi^{-1})(\xi), V) &= -\dim(\operatorname{ran} RD(\mathcal{Q} \circ \varphi^{-1})(\xi), \operatorname{ran} Q) \\ &= -\dim(\operatorname{ran} D(\mathcal{Q} \circ \varphi^{-1})(\xi), E) \\ &= \dim \operatorname{coker} D\mathcal{Q}(\varphi^{-1}(\xi)) D\varphi^{-1}(\xi) \\ &= \dim \operatorname{coker} D\mathcal{Q}(\varphi^{-1}(\xi)), \end{aligned}$$

proving (ii). By (1.3), for every $A \subset \varphi(U)$,

$$QA \subset R\mathcal{Q}(\varphi^{-1}(A)) + \operatorname{ran} RT, \quad \mathcal{Q}(\varphi^{-1}(A)) \subset (D\mathcal{Q}(0) + T)QA + \operatorname{ran} T,$$

so claim (iii) follows from the fact that $\varphi(U)$ and $\mathcal{Q}(U)$ are bounded, and from the fact that T has finite rank. □

Proposition 1.4. *Consider an open covering $\{M_i\}_{i \in I}$ of M, a family of infinite dimensional Hilbert manifolds $\{N_i\}_{i \in I}$ modeled on E, and a family of semi-Fredholm C^k, $k \geqslant 1$, maps $\mathcal{Q}_i : M_i \to N_i$ with the same constant non-negative index, such that for any $i, j \in I$ and for any $A \subset M_i \cap M_j$,*

$$\mathcal{Q}_i(A) \textit{ is pre-compact if and only if } \mathcal{Q}_j(A) \textit{ is pre-compact}. \tag{1.4}$$

Then the family $\{\ker D\mathcal{Q}_i(p) \mid p \in M_i\}$, $i \in I$, defines a C^{k-1} essential subbundle $\mathcal{E}$ of TM. The atlas $\mathcal{A}$ consisting of all the charts φ of M satisfying properties (i)–(iii) *of Lemma 1.3 applied to all the maps $\mathcal{Q}_i$ is a strong integrable structure modeled on (H, V) for $\mathcal{E}$. This atlas is such that for every $\varphi \in \mathcal{A}$ and every $A \subset \operatorname{dom}(\varphi) \subset M_i$,*

$$Q\varphi(A) \textit{ is pre-compact if and only if } \mathcal{Q}_i(A) \textit{ is pre-compact}. \tag{1.5}$$

Moreover, for every $p \in M_i \cap M_j$ the operator $D\mathcal{Q}_i(p)\, D\mathcal{Q}_j(p)^* \in \mathcal{L}(T_{\mathcal{Q}_j(p)}N_j, T_{\mathcal{Q}_i(p)}N_i)$ is Fredholm, and $\mathcal{E}$ is liftable to an (m)-essential subbundle, $m \in \mathbb{N}$, if and only if

$$\operatorname{ind}(D\mathcal{Q}_i(p)\, D\mathcal{Q}_j(p)^*) \equiv 0 \mod m \quad \forall i, j \in I,\ \forall p \in M_i \cap M_j.$$

Proof. Let us prove that for any $p \in M_i \cap M_j$ the subspace $\ker D\mathcal{Q}_i(p)$ is a compact perturbation of $\ker D\mathcal{Q}_j(p)$. Since $D\mathcal{Q}_i(p)$ has finite corank, we can find a C^1

embedded finite dimensional open disk $D \subset N_i$ with $\overline{D}$ compact, such that $\mathcal{Q}_i(p) \in D$ and the map $\mathcal{Q}_i$ is transverse to D. Then $\mathcal{Q}_i^{-1}(D)$ is a C^1 submanifold of M, and by our assumption the map $\mathcal{Q}_j|_{\mathcal{Q}_i^{-1}(D)\cap M_j}$ is compact. Therefore its differential, namely the restriction of $D\mathcal{Q}_j$ to the subspace $T_p\mathcal{Q}_i^{-1}(D) \supset \ker D\mathcal{Q}_i(p)$ is compact. In particular, the restriction of $D\mathcal{Q}_j(p)$ to $\ker D\mathcal{Q}_i(p)$ is compact, and similarly the restriction of $D\mathcal{Q}_i(p)$ to $\ker D\mathcal{Q}_j(p)$ is compact. Hence Proposition A.5 implies that $\ker D\mathcal{Q}_i(p)$ is a compact perturbation of $\ker D\mathcal{Q}_j(p)$, as we wished to prove, and that

$$\begin{aligned}\operatorname{ind}(D\mathcal{Q}_i(p)\, D\mathcal{Q}_j(p)^*) = \dim\operatorname{coker} D\mathcal{Q}_j(p) - \dim\operatorname{coker} D\mathcal{Q}_i(p)\\ + \dim(\ker D\mathcal{Q}_i(p), \ker D\mathcal{Q}_j(p)).\end{aligned} \tag{1.6}$$

Now let φ and ψ, $\operatorname{dom}(\varphi) \subset M_i$, $\operatorname{dom}(\psi) \subset M_j$, be two charts satisfying conditions (i)–(iii) of Lemma 1.3 applied to $\mathcal{Q}_i$ and $\mathcal{Q}_j$, respectively (possibly $i = j$). Let $\tau = \varphi \circ \psi^{-1}$ be the transition map. If $A \subset \operatorname{dom}(\tau) = \psi(\operatorname{dom}(\varphi) \cap \operatorname{dom}(\psi))$, by Lemma 1.3(iii) QA is pre-compact if and only if $\mathcal{Q}_j(\psi^{-1}(A))$ is pre-compact, by (1.4) if and only if $\mathcal{Q}_i(\psi^{-1}(A)) = \mathcal{Q}_i(\varphi^{-1}(\tau(A)))$ is pre-compact, and again by Lemma 1.3(iii) if and only if $Q\tau(A)$ is pre-compact. This proves condition (ii) of Definition 1.1, and proves that the atlas $\mathcal{A}$ satisfies (1.5).

Finally, let $p \in M_i$. By Lemma 1.3(ii), there is a neighborhood U_p of p and a C^k submersion $\widetilde{\mathcal{Q}}_p := Q\varphi : U_p \to \operatorname{ran} Q$ into a Hilbert space such that for any $q \in U_p$, $\ker D\mathcal{Q}_i(q)$ is a compact perturbation of $\ker D\widetilde{\mathcal{Q}}_p(q)$, and

$$\dim(\ker D\mathcal{Q}_i(q), \ker D\widetilde{\mathcal{Q}}_p(q)) = \dim\operatorname{coker} D\mathcal{Q}_i(q). \tag{1.7}$$

Then the family $\{\ker D\mathcal{Q}_i \mid i \in I\}$ defines the same C^{k-1} essential subbundle of TM as the one defined by the family

$$\left\{\ker D\widetilde{\mathcal{Q}}_p \mid p \in M\right\}. \tag{1.8}$$

If $q \in U_p \cap U_{p'} \cap M_i \cap M_j$, by (1.6) and (1.7) we obtain (see formula (A.2))

$$\begin{aligned}\dim(\ker D\widetilde{\mathcal{Q}}_p(q), \ker \widetilde{\mathcal{Q}}_{p'}(q)) &= \dim(\ker D\widetilde{\mathcal{Q}}_p(q), \ker D\mathcal{Q}_i(q))\\ &\quad + \dim(\ker D\mathcal{Q}_i(q), \ker D\mathcal{Q}_j(q)) + \dim(\ker D\mathcal{Q}_j(q), \ker D\widetilde{\mathcal{Q}}_{p'}(q))\\ &= \operatorname{ind}(D\mathcal{Q}_i(q)\, D\mathcal{Q}_j(q)^*),\end{aligned}$$

so (1.8) defines an (m)-essential subbundle of TM if and only if

$$\operatorname{ind}(D\mathcal{Q}_i(q)\,D\mathcal{Q}_j(q)^*) \equiv 0 \mod m \quad \forall i, j \in I, \ \forall q \in M_i \cap M_j. \qquad \square$$

The above proposition suggests the following:

Definition 1.5. A strong presentation of the essential subbundle $\mathcal{E}$ of TM consists of an open covering $\{M_i\}_{i \in I}$ of M, a family of manifolds N_i, $i \in I$, modeled on the Hilbert space E, a family of semi-Fredholm C^1 maps $\mathcal{Q}_i : M_i \to N_i$ with the same constant non-negative index such that:

(i) for every $i \in I$ and every $p \in M_i$, the kernel of $D\mathcal{Q}_i(p)$ belongs to the essential class $\mathcal{E}(p)$;
(ii) for every $i, j \in I$ and every $A \subset M_i \cap M_j$, $\mathcal{Q}_i(A)$ is pre-compact if and only if $\mathcal{Q}_j(A)$ is pre-compact.

Proposition 1.4 states among other facts that a strong presentation of $\mathcal{E}$ determines a strong integrable structure for $\mathcal{E}$.

2. Morse vector fields and subbundles

2.1. Definitions and basic facts

Let M be a paracompact manifold of class C^2, modeled on the infinite dimensional separable real Hilbert space H. Let F be a tangent vector field of class C^1 on M. This field determines a local flow on M,

$$\phi \in C^1(\Omega(F), M), \quad \partial_t \phi(t, p) = F(\phi(t, p)), \quad \phi(0, p) = p,$$

where $\Omega(F) \subset \mathbb{R} \times M$ is the maximal set of existence for the solutions of this ordinary differential equation. We will also use the notation $\phi_t(p) = \phi(t, p)$.

A *rest point* of F is a point $x \in M$ such that $F(x) = 0$. The set of rest points of F is denoted by $\operatorname{rest}(F)$. If $x \in \operatorname{rest}(F)$, the Jacobian of F at x, $\nabla F(x)$, is the bounded linear operator on $T_x M$ defined as

$$\nabla F(x)\xi = L_X F(x) \quad \text{for } X \text{ a tangent vector field such that } X(x) = \xi \in T_x M,$$

where $L_X F$ denotes the Lie derivative of F along X. Indeed, the fact that $F(x) = 0$ implies that $L_X F(x)$ depends only on the value of X at x.

We recall that an operator $L \in \mathcal{L}(H)$ is said *hyperbolic* if $\sigma(L) \cap i\mathbb{R} = \emptyset$. In this case, the decomposition of the spectrum of L into the subset with positive real part and the one with negative real part determines an L-invariant splitting $H = V^+(L) \oplus V^-(L)$.

A point $x \in \mathrm{rest}\,(F)$ is said *hyperbolic* if the operator $\nabla F(x)$ is hyperbolic. In this case, the *linear unstable space* H_x^u and the *linear stable space* H_x^s, are defined as

$$H_x^u := V^+(\nabla F(x)), \quad H_x^s := V^-(\nabla F(x)).$$

A vector field F all of whose rest points are hyperbolic is said a *Morse vector field*.

A *Lyapunov function* for the vector field F is a function $f \in C^1(M)$ such that

$$Df(p)[F(p)] < 0 \quad \forall p \in M \setminus \mathrm{rest}\,(F). \tag{2.1}$$

In particular, $t \mapsto f(\phi(t,p))$ is strictly decreasing if $p \notin \mathrm{rest}\,(F)$. Note that every critical point of f must be a rest point of F. If x is a hyperbolic rest point for F, then it is a critical point of f, as it easily follows from a first-order expansion of F at x.

If the vector field F is Morse, we shall ask the Lyapunov function to be *non-degenerate*: f is twice differentiable at every rest point x and, denoting by $D^2f(x)$ the second differential of f at x, seen as a symmetric bounded bilinear form, we have that $\xi \mapsto D^2f(x)[\xi,\xi]$ is coercive on H_x^s, while $\xi \mapsto -D^2f(x)[\xi,\xi]$ is coercive on H_x^u. The Morse vector field F is said *gradient-like* if it has a non-degenerate Lyapunov function.

2.2. The relative Morse index

For $\mathcal{V}$ a subbundle of TM of class C^1, consider the following compatibility condition between F and $\mathcal{V}$:

(C1) for every x rest point of F, the linear unstable space H_x^u is a compact perturbation of $\mathcal{V}(x)$.

If (C1) holds, the *relative Morse index* of $x \in \mathrm{rest}\,(F)$ is the integer

$$m(x,\mathcal{V}) := \dim(H_x^u, \mathcal{V}(x))$$

and the sets

$$\mathrm{rest}_q(F) := \{x \in \mathrm{rest}\,(F) \mid m(x,\mathcal{V}) = q\}, \quad q \in \mathbb{Z},$$

constitute a partition of $\mathrm{rest}\,(F)$.

Condition (C1) clearly depends only on the essential class of $\mathcal{V}$. Therefore, it makes sense to talk about vector fields which satisfy (C1) with respect to an essential subbundle. More precisely, the C^1 Morse vector field F satisfies (C1) with respect to the essential subbundle $\mathcal{E}$ if for every rest point x of F the unstable space H_x^u belongs to the essential class $\mathcal{E}(x)$. In this more general situation, there is no relative Morse index. However, if the essential subbundle $\mathcal{E}$ comes from an (m)—essential subbundle—still denoted by $\mathcal{E}$—then the relative Morse index of $x \in \mathrm{rest}\,(F)$ is an integer modulo

m—denoted by $m(x,\mathcal{E})$. In particular, if $\mathcal{E}$ is a (0)-essential subbundle, the relative Morse index is still integer valued.

2.3. Essentially invariant subbundles

We shall say that the C^1 subbundle $\mathcal{V}$ is *invariant with respect to F at $p \in M$* if

$$(L_F\mathcal{P})(p)\mathcal{P}(p) = 0, \tag{2.2}$$

where $\mathcal{P}$ is a projector onto $\mathcal{V}$ of *TM*: $\mathcal{P}$ is a C^1 section of the Banach bundle of linear endomorphisms of *TM* such that for every $p \in M$, $\mathcal{P}(p) \in \mathcal{L}(T_pM)$ is a projector onto $\mathcal{V}(p)$. This notion does not depend on the choice of the projector $\mathcal{P}$, but only on the subbundle $\mathcal{V}$. Indeed, if $\mathcal{P}$ and $\mathcal{Q}$ are two projectors onto $\mathcal{V}$, we have the identity

$$(L_F\mathcal{Q})\mathcal{Q} = (I-\mathcal{Q})(L_F\mathcal{P})\mathcal{P}\mathcal{Q}, \tag{2.3}$$

which can be verified by taking the Lie derivative of the identities $\mathcal{P}\mathcal{Q} = \mathcal{Q} = \mathcal{Q}^2$. This definition is motivated by the well-known fact that (2.2) holds for any $p \in M$ if and only if the subbundle $\mathcal{V}$ is invariant under the action of the local flow ϕ, that is $D\phi_t(p)\mathcal{V}(p) = \mathcal{V}(\phi_t(p))$ for every $(t,p) \in \Omega(F)$.

Similarly, we shall say that $\mathcal{V}$ is *essentially invariant with respect to F at p* if $(L_F\mathcal{P})(p)\mathcal{P}(p)$ is a compact endomorphism of T_pM. Again, (2.3) shows that this notion depends only on $\mathcal{V}$. By Proposition A.4, $\mathcal{V}$ is essentially invariant with respect to F at every $p \in M$ if and only if $D\phi_t(p)\mathcal{V}(p)$ is a compact perturbation of $\mathcal{V}(\phi_t(p))$, for every $(t,p) \in \Omega(F)$. The second compatibility condition between F and $\mathcal{V}$ is:

(C2) $\mathcal{V}$ is essentially invariant with respect to F at any point $p \in M$.

Also this condition can be stated for an essential subbundle. Indeed, an essential subbundle $\mathcal{E}$ of *TM* will be said *invariant with respect to F at p* if a local representative of $\mathcal{E}$ at p is essentially invariant with respect to F at p. This notion does not depend on the choice of the local representative of $\mathcal{E}$ at p: if $\mathcal{V}$ and $\mathcal{W}$ are two such local representatives on some neighborhood U of p, and $\mathcal{P}$, $\mathcal{Q}$ are the orthogonal projectors onto $\mathcal{V}$, $\mathcal{W}$, with respect to some Riemannian structure on M, we have that $\mathcal{P}(q) - \mathcal{Q}(q) \in \mathcal{L}_c(T_qM)$ for any $q \in U$, so $(L_F(\mathcal{P}-\mathcal{Q}))(p) \in \mathcal{L}_c(T_pM)$, and the identity

$$(L_F\mathcal{P})\mathcal{P} - (L_F\mathcal{Q})\mathcal{Q} = (L_F\mathcal{P})(\mathcal{P}-\mathcal{Q}) + (L_F(\mathcal{P}-\mathcal{Q}))\mathcal{Q},$$

shows that $(L_F\mathcal{P})\mathcal{P}$ is compact if and only if $(L_F\mathcal{Q})\mathcal{Q}$ is compact. Hence, we shall say that the C^1 vector field F satisfies (C2) with respect to the C^1 essential subbundle $\mathcal{E}$ of *TM* if $\mathcal{E}$ is invariant with respect to F at every $p \in M$.

Proposition 2.1. *Let $\mathcal{E}$ be a C^1 essential subbundle of TM. Then the set of C^1 vector fields on M which satisfy* (C2) *with respect to $\mathcal{E}$ is a $C^1(M)$-module.*

Proof. Everything follows from the formulas

$$(L_{X+Y}\mathcal{P})\mathcal{P} = (L_X\mathcal{P})\mathcal{P} + (L_Y\mathcal{P})\mathcal{P},$$
$$(L_{hX}\mathcal{P})\mathcal{P}\xi = h(L_X\mathcal{P})\mathcal{P}\xi + Dh[\mathcal{P}\xi](\mathcal{P} - I)X \quad \forall \xi \in TM,$$

where $h \in C^1(M)$. □

Examples: We conclude this section with some simple examples.

Example 2.2 (Vector fields whose rest points have finite Morse index or finite Morse co-index) Consider the classical situation of a Morse vector field F all of whose rest points have finite Morse index. Then (C1) and (C2) hold with respect to the trivial subbundle $\mathcal{V} = (0)$. With such a $\mathcal{V}$ indeed, (C2) is fulfilled by any vector field, while (C1) is equivalent to asking the unstable space of every rest point to be finite dimensional. In this case, $m(x, (0))$ is the usual Morse index of the rest point x.

Similarly, a Morse vector field all of whose rest points have finite Morse co-index satisfies (C1) and (C2) with respect to the trivial subbundle $\mathcal{V} = TM$, and $-m(x, TM)$ is the co-index of the rest point x.

Example 2.3 (*Perturbations of a non-degenerate quadratic form*). Assume that $M = H$ is a Hilbert space, and consider a function of the form

$$f(\xi) = \frac{1}{2}\langle L\xi, \xi\rangle + b(\xi),$$

where $L \in \mathcal{L}(H)$ is self-adjoint invertible, and $b \in C^2(H)$. Let F be the (negative) gradient vector field of f,

$$F(\xi) = -\operatorname{grad} f(\xi) = -L\xi - \operatorname{grad} b(\xi)$$

and consider the constant subbundle $V = V^-(L)$. In this case, condition (C2) means asking that

$$(L_{\operatorname{grad} f} P_V)(\xi) P_V = [P_V, \operatorname{Hess} f(\xi)] P_V = [P_V, \operatorname{Hess} b(\xi)] P_V$$

should be compact for every $\xi \in H$. In particular, if we assume that the Hessian of b at every point is compact, condition (C2) holds. Since the negative eigenspace of a compact perturbation of L is a compact perturbation of V (Proposition B.1), also condition (C1) holds.

Example 2.4 (*Product manifolds*). Assume that $M = M^- \times M^+$ is the product of two Hilbert manifolds, and consider the subbundle $\mathcal{V} = TM^- \times (0)$ of *TM*. Fix some Riemannian structure on M^- and on M^+, and consider the product Riemannian structure on M. Let $F = -\text{grad}\, f$ be the negative gradient of a Morse function on M. Then F satisfies (C1) with respect to $\mathcal{V}$ if and only if for every critical point x the Hessian of f at x decomposes as $\text{Hess}\, f(x) = L_x + K_x$, where L_x is self-adjoint, invertible, and $V^-(L_x) = \mathcal{V}(x)$, while K_x is a compact endomorphism of $T_x M$. Moreover, F satisfies (C2) with respect to $\mathcal{V}$ if and only if for every $p \in M$ the operator

$$(L_F \mathcal{P})(p)\mathcal{P}(p) = (\nabla_{\text{grad}\, f} \mathcal{P}(p) + [\mathcal{P}(p), \text{Hess}\, f(p)])\mathcal{P}(p)$$

is compact, where $\mathcal{P}$ denotes the orthogonal projection onto $\mathcal{V}$.

Example 2.5 (*Semi-Riemannian geodesics [AM04a]*). Let Q be an n-dimensional manifold, endowed with a semi-Riemannian structure h, that is a symmetric non-degenerate bilinear form on TQ. Denote by (n^+, n^-) the signature of h, $n^+ + n^- = n$. The semi-Riemannian structure h induces a Levi-Civita covariant derivation ∇, and the geodesics, i.e. the solutions q of the second order ODE $\nabla_{\dot{q}}\dot{q} = 0$, joining two fixed points $q_0, q_1 \in Q$ are the critical points of the energy functional

$$f(q) = \frac{1}{2} \int_0^1 h(\dot{q}(t), \dot{q}(t))\, dt,$$

on the Hilbert manifold $M := \left\{ q \in W^{1,2}([0,1], Q) \mid q(0) = q_0, q(1) = q_1 \right\}$ consisting of paths in Q of Sobolev class $W^{1,2}$ joining q_0 and q_1. When $n^+ \neq 0$ and $n^- \neq 0$, all the critical points of f have infinite Morse index and co-index. Assume that TQ has an *integrable* subbundle V of dimension n^- such that h is strictly negative on V, and set

$$\mathcal{V}(q) = \left\{ \zeta \in T_q M = q^*(TQ) \mid \zeta(t) \in V(q(t))\ \forall t \in [0,1] \right\} \quad \forall q \in M.$$

The integrability of V is reflected into the integrability of $\mathcal{V}$, and this fact can be used to build a class of Riemannian structures on M—equivalent to the standard $W^{1,2}$ metric—such that $\text{grad}\, f$ satisfies (C1) and (C2) with respect to $\mathcal{V}$. In this situation, it can also be proved that the relative Morse index $m(q, \mathcal{V})$ of the geodesic q coincides with the Maslov index of a suitable path of Lagrangian subspaces, obtained by looking at the Hamiltonian system on the cotangent bundle of Q generated by the Legendre transform $H : T^*Q \to \mathbb{R}$ of the Lagrangian $L : TQ \to \mathbb{R}$, $L(\zeta) = 1/2\, h(\zeta, \zeta)$.

3. Finite dimension of $W^u(x) \cap W^s(y)$

3.1. Stable and unstable manifolds

The unstable and stable manifolds of a hyperbolic rest point x are the sets

$$W^u(x) := \{ p \in M \mid]-\infty, 0] \times \{p\} \subset \Omega(F) \text{ and } \phi(t,p) \to x \text{ for } t \to -\infty \},$$
$$W^s(x) := \{ p \in M \mid [0, +\infty[\times \{p\} \subset \Omega(F) \text{ and } \phi(t,p) \to x \text{ for } t \to +\infty \}$$

and classical results in the theory of dynamical systems imply that $W^u(x)$ and $W^s(x)$ are the images of injective C^1 immersions of H^u_x and H^s_x, respectively, and that

$$T_x W^u(x) = H^u_x, \qquad T_x W^s(x) = H^s_x.$$

In general, they need not be embedded submanifolds. Starting from Section 6 however, we will restrict our attention to gradient-like vector fields, for which $W^u(x)$ and $W^s(x)$ are embedded submanifolds (see also Appendix C).

Proposition 3.1. *Let $\mathcal{E}$ be an essential subbundle of TM, and let x be a hyperbolic rest point of the C^1 vector field F on M. Then the following facts are equivalent*:

(i) *H^u_x belongs to the essential class $\mathcal{E}(x)$, and $\mathcal{E}$ is invariant with respect to F at every $p \in W^u(x)$;*
(ii) *the tangent space $T_p W^u(x)$ belongs to the essential class $\mathcal{E}(p)$ for every $p \in W^u(x)$.*

If either (i) *or* (ii) *holds, and if $\mathcal{E}$ is liftable to an (m)-essential subbundle—still denoted by $\mathcal{E}$—then we have the identity between integers modulo m*

$$\dim(T_p W^u(x), \mathcal{E}(p)) = m(x, \mathcal{E}) \quad \forall p \in W^u(x).$$

Proof. Let $p \in W^u(x)$ and define $u : [-\infty, 0] \to M$ by $u(t) := \phi_t(p)$ for $t > -\infty$, and $u(-\infty) = x$. If $\psi : U \to H$, $x \in U$, is a local chart mapping the open set U diffeomorphically into the Hilbert space H, then for T large $\psi \circ \phi_{-T} : \phi^{-1}_{-T}(U) \to H$ is a local chart whose domain contains $u([-\infty, 0])$. Therefore, since both the assertions of the theorem are invariant with respect to differentiable conjugacy, we may assume that M is an open subset of H. The set $\phi([-\infty, 0] \times \{p\})$ has a contractible neighborhood U, and we can find a C^1 map $\mathcal{P} : U \to \mathcal{L}(H)$ such that $\mathcal{P}(\xi)$ is a projector onto a subspace in the essential class $\mathcal{E}(\xi)$, for every $\xi \in U$.

Set $P := \mathcal{P}(x)$, and let $R : [-\infty, 0] \to \mathrm{GL}(H)$ be such that $R(t)P = \mathcal{P}(u(t))R(t)$, $R(-\infty) = I$, and $R'(t) \to 0$ for $t \to -\infty$. Set

$$X(t) := R(t)^{-1} D\phi_t(p) R(0).$$

Then X solves $X' = AX$, $X(0) = I$, where

$$A(t) = R(t)^{-1} R'(t) + R(t)^{-1} DF(u(t)) R(t) \in \mathcal{L}(H)$$

converges to the hyperbolic operator $A(-\infty) = DF(x)$ for $t \to -\infty$. Let

$$W^u_A := \left\{ \xi \in H \mid \lim_{t \to -\infty} X(t)\xi = 0 \right\},$$

be the linear unstable space of the path of operators A (see Appendix B). Then

$$T_{u(t)}W^u(x) = R(t)X(t)W^u_A, \quad T_x W^u(x) = V^+(A(-\infty)). \tag{3.1}$$

Differentiating $R(t)P = \mathcal{P}(u(t))R(t)$ we obtain the identity

$$R'(t)P = D\mathcal{P}(u(t))[F(u(t))]R(t) + \mathcal{P}(u(t))R'(t),$$

from which an easy computation gives

$$[A(t), P]P = R(t)^{-1}\left(D\mathcal{P}(u(t))[F(u(t))] + [\mathcal{P}(u(t)), DF(u(t))]\right)\mathcal{P}(u(t))R(t).$$

So by the usual expression for the Lie derivative,

$$[A(t), P]P = R(t)^{-1}(L_F\mathcal{P})(u(t))\mathcal{P}(u(t))R(t) \tag{3.2}$$

and the equivalence of (i) and (ii) follows form (3.1), (3.2), and Proposition B.3.

Assume now that $\mathcal{E}$ comes from an (m)-essential subbundle. Since $W^u(x)$ is connected and the relative dimension is a continuous function, for every $p \in W^u(x)$ we have the following identity between integers modulo m

$$\dim(T_p W^u(x), \mathcal{E}(p)) = \dim(T_x W^u(x), \mathcal{E}(x)) = \dim(H^u_x, \mathcal{E}(x)) = m(x, \mathcal{E}). \qquad \square$$

Recall that a pair of closed subspaces (V, W) of the Hilbert space H is said a *Fredholm pair* if $V \cap W$ has finite dimension and $V + W$ has finite codimension, in which case we define the index of (V, W) to be

$$\mathrm{ind}\,(V, W) = \dim V \cap W - \mathrm{codim}(V + W).$$

The space of Fredholm pairs of H, denoted by $\mathrm{Fp}(H)$, is an open subspace of $\mathrm{Gr}(H) \times \mathrm{Gr}(H)$, and the index is a continuous function. If $\mathcal{H} \to B$ is a C^k Hilbert bundle, there is an associated C^k bundle

$$\mathrm{Fp}(\mathcal{H}) = \bigcup_{b \in B} \mathrm{Fp}(\mathcal{H}_b) \to B.$$

The above proposition has the following corollary.

Corollary 3.2. *Assume that the Morse vector field F satisfies* (C1-2) *with respect to a subbundle $\mathcal{V}$ of TM. Then for every rest point x*:

(i) *for any $p \in W^u(x)$, $T_p W^u(x)$ is a compact perturbation of $\mathcal{V}(p)$, and* $\dim(T_p W^u(x), \mathcal{V}(p)) = m(x, \mathcal{V})$;
(ii) *for any $p \in W^s(x)$, $(T_p W^s(x), \mathcal{V}(p))$ is a Fredholm pair of index $-m(x, \mathcal{V})$.*

Proof. Assertion (i) follows immediately from Proposition 3.1 and from the continuity of the relative dimension. By (C1) and Proposition A.2, $(T_x W^s(x), \mathcal{V}(x)) = (H_x^s, \mathcal{V}(x))$ is a Fredholm pair of index

$$\mathrm{ind}\,(T_x W^s(x), \mathcal{V}(x)) = \mathrm{ind}\,(H_x^s, H_x^u) + \dim(\mathcal{V}(x), H_x^u) = -m(x, \mathcal{V}).$$

Therefore, $(T_p W^s(x), \mathcal{V}(p))$ is a Fredholm pair of the same index for any p in a neighborhood U of x in the intrinsic topology of the immersed submanifold $W^s(x)$. The backward evolution of U by ϕ is the whole $W^s(x)$, so assertion (ii) follows from the fact that the tangent bundle of $W^s(x)$ is invariant, and $\mathcal{V}$ is essentially invariant under the action of ϕ. $\square$

3.2. Intersections

Recall that two immersed submanifolds $N, O \subset M$ have a *transverse intersection* if for every $p \in N \cap O$ there holds $T_p N + T_p O = T_p M$. In this case, $N \cap O$ is an immersed submanifold of M, and $T_p(N \cap O) = T_p N \cap T_p O$. Similarly, $N, O \subset M$ have a *Fredholm intersection* if for every $p \in N \cap O$, $(T_p N, T_p O)$ is a Fredholm pair of subspaces of $T_p M$. We are now ready to state the result about the dimension of the intersection of the unstable and the stable manifolds.

Theorem 3.3. *Let $k \in \mathbb{N}$, let $\mathcal{E}$ be a (k)-essential subbundle of TM, and assume that the Morse vector field F satisfies* (C1-2) *with respect to $\mathcal{E}$. Let x, y be two rest points of F. Then $W^u(x)$ and $W^s(y)$ have Fredholm intersection, with the number*

$$\mathrm{ind}\,(T_p W^u(x), T_p W^s(y)), \qquad p \in W^u(x) \cap W^s(y),$$

depending only on the homotopy class of the curve $t \mapsto \phi(t, p)$ in the space of continuous paths $u : \overline{\mathbb{R}} \to M$ such that $u(-\infty) = x$, $u(+\infty) = y$. Furthermore

$$\mathrm{ind}\,(T_p W^u(x), T_p W^s(y)) \equiv m(x, \mathcal{E}) - m(y, \mathcal{E}) \mod k \tag{3.3}$$

for every $p \in W^u(x) \cap W^s(y)$.

In particular, if $W^u(x)$ and $W^s(y)$ have non-empty transverse intersection, then $W^u(x) \cap W^s(y)$ is an immersed finite dimensional submanifold of M, the dimension of the component W_p of $W^u(x) \cap W^s(y)$ containing p depends only on the homotopy class of the curve $t \mapsto \phi(t, p)$, and

$$\dim W_p \equiv m(x, \mathcal{E}) - m(y, \mathcal{E}) \mod k.$$

In particular, when F satisfies (C1-2) with respect to a (0)-essential subbundle $\mathcal{E}$, then all the components of the transverse intersection $W^u(x) \cap W^s(y)$ have the same dimension $m(x, \mathcal{E}) - m(y, \mathcal{E})$.

Proof of Theorem 3.3. Let us fix two points $p_0, p_1 \in W^u(x) \cap W^s(y)$ such that their orbits are homotopic in the space of paths $u : \overline{\mathbb{R}} \to M$ with $u(-\infty) = x$, $u(+\infty) = y$. In other words, there exists a continuous map

$$h : \overline{\mathbb{R}} \times [0, 1] \to M,$$

such that $h(-\infty, s) = x$, $h(+\infty, s) = y$, $h(t, i) = \phi(t, p_i)$, for $i = 0, 1$.

The map $\overline{\mathbb{R}} \times [0, 1] \to \mathrm{Gr}_{(k)}(TM)$, $(t, s) \mapsto \mathcal{E}(h(t, x))$, is liftable to a map $\mathcal{W} : \overline{\mathbb{R}} \times [0, 1] \to \mathrm{Gr}(TM)$ such that $\mathcal{W}(-\infty, \cdot)$ and $\mathcal{W}(+\infty, \cdot)$ are constant maps. By Proposition 3.1, $T_{h(t,s)} W^u(x)$ is a compact perturbation of $\mathcal{W}(t, s)$ and $\dim(T_{h(t,s)} W^u(x), \mathcal{W}(t, s)) = \dim(H^u_x, \mathcal{W}(-\infty, \cdot))$, for any $t < +\infty$. Using an argument analogous to the proof of Corollary 3.2(ii), it is easy to see that $(T_{h(t,s)} W^s(y), \mathcal{W}(t, s))$ is a Fredholm pair of index $-\dim(H^u_y, \mathcal{W}(+\infty, \cdot))$, for any $t > -\infty$. Then by Proposition A.2, $(T_{h(t,s)} W^s(y), T_{h(t,s)} W^u(x))$ is a Fredholm pair of index $\dim(H^u_x, \mathcal{W}(-\infty, \cdot)) - \dim (H^u_y, \mathcal{W}(+\infty, \cdot))$. In particular, $(T_{p_0} W^s(y), T_{p_0} W^u(x))$ and $(T_{p_1} W^s(y), T_{p_1} W^u(x))$ are Fredholm pairs of the same index

$$\begin{aligned} \operatorname{ind}(T_{p_0} W^s(y), T_{p_0} W^u(x)) &= \operatorname{ind}(T_{p_1} W^s(y), T_{p_1} W^u(x)) \\ &= \dim(H^u_x, \mathcal{W}(-\infty, \cdot)) - \dim(H^u_y, \mathcal{W}(+\infty, \cdot)) \end{aligned}$$

and the above formula implies (3.3). Finally, the statements which assume transversality follow from the fact that, under such an assumption,

$$\operatorname{ind}(T_p W^u(x), T_p W^s(y)) = \dim T_p W^u(x) \cap T_p W^s(y). \qquad \square$$

Remark 3.4. We wish to remark that (C1-2) are asymmetric conditions: if F satisfies (C1-2) with respect to a subbundle $\mathcal{V}$, there need not exist a subbundle $\mathcal{W}$ such that $-F$ satisfies (C1-2) with respect to $\mathcal{W}$. Such an asymmetry is reflected into Corollary 3.2, which states that $TW^u(x)$ is a compact perturbation of $\mathcal{V}$—a closed condition—while $TW^s(x)$ is in Fredholm pair with $\mathcal{V}$—an open condition. If we symmetrize (C1-2) we obtain the following stronger assumptions: if $\mathcal{P}$ is a projector on *TM*, with image $\mathcal{V}$ and kernel $\mathcal{W}$, there holds (C1'): H^u_x is a compact perturbation of $\mathcal{V}(x)$, H^s_x is a compact perturbation of $\mathcal{W}(x)$ for every $x \in \operatorname{rest}(F)$, and (C2'): $(L_F \mathcal{P})(p)$ is compact for any $p \in M$. This setting—actually its essential version—is closer to the setting of a polarized manifold (see [CJS95]).

4. Which manifolds can be obtained as $W^u(x) \cap W^s(y)$

4.1. Arbitrary gradient-like vector fields

Let F be a gradient-like Morse–Smale vector field on the Hilbert manifold M, with Lyapunov function f. If $x, y \in \mathrm{rest}\,(F)$ and $f(y) < c < f(x)$, the set $Z = W^u(x) \cap W^s(y) \cap f^{-1}(\{c\})$ is a submanifold (non-necessarily closed), being the transverse intersection in $f^{-1}(\{c\})$ of the submanifolds $W^u(x) \cap f^{-1}(\{c\})$ and $W^s(y) \cap f^{-1}(\{c\})$, and ϕ subordinates a diffeomorphism from $\mathbb{R} \times Z$ onto $W^u(x) \cap W^s(y)$.

When M is finite dimensional, there are limitations on the topological type of Z, e.g. if M is compact and there are no rest points z with $f(y) < f(z) < f(x)$, then Z is the transverse intersection in $f^{-1}(\{c\})$ of two spheres. When M is infinite dimensional, and the rest points x, y have infinite Morse index and co-index, there are no limitations at all on the topological type of Z, the reason being that any manifold can be obtained as the transverse intersection of two infinite dimensional spheres.

More precisely, for any Hilbert manifold Z (finite dimensional or not) there is a gradient like Morse vector field F on the Hilbert space H, with a non-degenerate Lyapunov function f, having exactly two rest points x, y with $f(y) < 0 < f(x)$, such that $W^u(x) \cap f^{-1}(\{0\})$ and $W^s(y) \cap f^{-1}(\{0\})$ are infinite dimensional spheres intersecting transversally in $f^{-1}(\{0\})$ at a closed submanifold diffeomorphic to Z. Notice that in this case, the closure of $W^u(x) \cap W^s(y)$ is $(W^u(x) \cap W^s(y)) \cup \{x, y\}$, which is homeomorphic to the suspension of Z.

The construction relies on the following lemma.

Lemma 4.1. *Let Z be a closed infinite codimensional submanifold of a Hilbert manifold M. Then there exists a smooth map $\varphi : M \to H$ such that 0 is a regular value and $Z = \varphi^{-1}(\{0\})$.*

Proof. A suitable tubular neighborhood U of Z is diffeomorphic to the normal bundle of Z. Since Z has infinite codimension, such a bundle is trivial. Therefore, there exists a submersion $\psi : U \to H$ such that $\psi^{-1}(\{0\}) = Z$. Since $H \setminus \overline{B}$, B denoting the open unit ball of H, is diffeomorphic to H (see [Bes66]), it is easy to define a smooth map $\varphi : M \to H$ which agrees with ψ on a neighborhood $U' \subset U$ of Z and such that $\varphi(M \setminus U') \subset H \setminus \overline{B}$, so that $\varphi^{-1}(\{0\}) = Z$. $\square$

Let F_0 be the vector field on $H \times H$

$$F_0(\xi, \eta) = (\xi, -\chi(\|\xi\|)\eta),$$

where $\chi \in C^\infty(\mathbb{R})$ is decreasing, $\chi(s) = 1$ for $s \leqslant \frac{1}{3}$ and $\chi(s) = 0$ for $s \geqslant \frac{2}{3}$. The vector field F_0 has a unique rest point $o = (0, 0)$, with $W^u(o) = H \times \{0\}$, and has a non-degenerate Lyapunov function

$$f_0(\xi, \eta) = 1 - \|\xi\|^2 + \chi(\|\xi\|)\|\eta\|^2.$$

Let B be the open unit ball of H and let S be its boundary. We can embed Z as a closed infinite codimensional submanifold of S. By the above lemma, there exists a smooth map $\varphi : S \to H$ such that 0 is a regular value and $Z = \varphi^{-1}(\{0\})$. Let C_1 and C_2 be two copies of the Hilbert manifold with boundary $\overline{B} \times H$, and consider the Hilbert manifold $M := C_1 \cup_\Phi C_2$, where the gluing map Φ is

$$\Phi : \partial C_1 = S \times H \to S \times H = \partial C_2 \quad (\xi, \eta) \mapsto (\xi, \eta + \varphi(\xi)).$$

Let F be the vector field on M coinciding with F_0 on C_1 and with $-F_0$ on C_2, and let $f : M \to \mathbb{R}$ be the function coinciding with f_0 on C_1 and with $-f_0$ on C_2. It is readily seen that F and f are well defined and smooth, and that f is a non-degenerate Lyapunov function for F. By construction, C_1 is negatively invariant for the flow of F, C_2 is positively invariant, and there are exactly two rest points $x = (0,0) \in C_1$ and $y = (0,0) \in C_2$. Moreover, $f^{-1}(\{0\}) = \partial C_1 = \partial C_2$, and

$$\begin{aligned} W^s(y) \cap \partial C_2 &= S \times \{0\}, \\ W^u(x) \cap \partial C_2 = \Phi(W^u(x) \cap \partial C_1) = \Phi(S \times \{0\}) &= \operatorname{graph} \varphi. \end{aligned}$$

Hence

$$W^s(y) \cap W^u(x) \cap \partial C_2 = (S \times \{0\}) \cap \operatorname{graph} \varphi = \varphi^{-1}(\{0\}) \times \{0\} = Z \times \{0\},$$

the intersection being transversal, as required. Finally, since the gluing map Φ is isotopic to the identity map on $S \times H$, M is diffeomorphic to $(\overline{B} \times H) \cup_{\mathrm{id}} (\overline{B} \times H) = (\overline{B} \cup_{\mathrm{id}} \overline{B}) \times H$. Being an infinite dimensional sphere, $\overline{B} \cup_{\mathrm{id}} \overline{B}$ is diffeomorphic to H (again by [Bes66]), hence M is diffeomorphic to H.

4.2. Gradient-like vector fields satisfying (C1-2)

In particular, if Z has components of different dimension, the above example shows that in the case of infinite Morse indices and co-indices, the components of $W^u(x) \cap W^s(y)$ may have different dimension. Actually, the discussion of Section 3 suggests that this phenomenon may happen also when F satisfies (C1-2) with respect to an essential subbundle, provided M is not simply connected. Indeed this is the case, as it is shown by the following example, where the vector field is actually the gradient of a smooth function.

We recall some pieces of notation from Appendix B. If $A : [-\infty, +\infty] \to \mathcal{L}(H)$ is a path of bounded linear operators with $A(-\infty)$ and $A(+\infty)$ hyperbolic, we denote by $X_A : \mathbb{R} \to \mathrm{GL}(H)$ the solution of the linear Cauchy problem $X_A'(t) = A(t) X_A(t)$, $X_A(0) = I$, and we consider the closed linear subspaces

$$W_A^s = \left\{ \xi \in H \mid \lim_{t \to +\infty} X_A(t)\xi = 0 \right\}, \quad W_A^u = \left\{ \xi \in H \mid \lim_{t \to -\infty} X_A(t)\xi = 0 \right\}.$$

Let $M = H \times \mathbb{T}^1$, with $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $H = H^- \oplus H^+$ be an orthogonal splitting such that $H^-, H^+ \in \mathrm{Gr}_{\infty,\infty}(H)$, with associated projectors P^-, P^+. Let $k \geqslant 1$ be an integer. By Proposition B.5 there exists $A \in C^\infty(\mathbb{R}, \mathrm{GL}(H) \cap \mathrm{Sym}(H))$ with $A(t) = P^+ - P^-$ for $t \notin]0,1[$, such that $W_A^s + W_A^u = H$ and $\dim W_A^s \cap W_A^u = k$. Consider the smooth tangent vector field on M

$$F(\xi, s) = \begin{cases} ((P^+ - P^-)\xi, \sin s) & \text{for } s \notin [\pi/2, \sigma(1)], \\ (A(\tau(s))\xi, \sin s) & \text{for } s \in]0, \pi[, \end{cases} \qquad (\xi, s) \in H \times \mathbb{T}^1,$$

where $\tau(s) = \log \tan(s/2)$ for $0 < s < \pi$, and $\sigma(t) = \tau^{-1}(t) = 2 \arctan e^t$, $t \in \mathbb{R}$. So $\sigma' = \sin \sigma$ and $\tau' = \cosh \tau$.

The rest points of F, $x = (0,0)$ and $y = (0, \pi/2)$, are hyperbolic, with stable and unstable linear spaces

$$H_x^s = H^- \times (0), \quad H_x^u = H^+ \times \mathbb{R}, \quad H_y^s = H^- \times \mathbb{R}, \quad H_y^u = H^+ \times (0). \tag{4.1}$$

The flow of F, $\phi : \mathbb{R} \times M \to M$, is readily seen to be

$$\phi_t(\xi, s) = \begin{cases} (e^{t(P^+ - P^-)}\xi, -\sigma(t + \tau(s))) & \text{for } -\pi < s < 0, \\ (e^{t(P^+ - P^-)}\xi, s) & \text{for } s = 0 \text{ or } s = \pi, \\ (X_A(t + \tau(s)) X_A(\tau(s))^{-1}\xi, \sigma(t + \tau(s))) & \text{for } 0 < s < \pi. \end{cases} \tag{4.2}$$

As a consequence, the unstable manifold of x and the stable manifold of y are the sets

$$W^u(x) = (H^+ \times] - \pi, 0]) \cup \bigcup_{0<s<\pi} X_A(\tau(s)) W_A^u \times \{s\},$$

$$W^s(x) = (H^- \times [-\pi, 0[) \cup \bigcup_{0<s<\pi} X_A(\tau(s)) W_A^s \times \{s\},$$

with tangent spaces

$$T_{(\xi,s)} W^u(x) = \begin{cases} H^+ \times \mathbb{R} & \text{for } (\xi, s) \in H^+ \times] - \pi, 0], \\ X_A(\tau(s)) W_A^u \oplus \mathbb{R} F(\xi, s) & \text{otherwise,} \end{cases}$$

$$T_{(\xi,s)} W^s(y) = \begin{cases} H^- \times \mathbb{R} & \text{for } (\xi, s) \in H^- \times [-\pi, 0[, \\ X_A(\tau(s)) W_A^s \oplus \mathbb{R} F(\xi, s) & \text{otherwise.} \end{cases}$$

Therefore, the unstable manifold of x and the stable manifold of y meet transversally, and their intersection

$$W^u(x) \cap W^s(y) = (\{0\} \times] - \pi, 0[) \cup \bigcup_{0<s<\pi} X_A(\tau(s))(W_A^u \cap W_A^s) \times \{s\},$$

consists of two connected components, one of which one-dimensional, the other one of dimension $k+1$.

We are going to show that the vector field F satisfies conditions (C1) and (C2) with respect to a (non-liftable) essential subbundle of TM. Consider the two subbundles of $T(H\times[-\pi,0])$ and $T(H\times]0,\pi[)$,

$$V_1(\xi,s)=H^+\times(0)\quad\text{for }(\xi,s)\in H\times[-\pi,0],$$
$$V_2(\xi,s)=X_A(\tau(s))Y\times(0)\quad\text{for }(\xi,s)\in H\times]0,\pi[,$$

where Y is a closed supplement of $W_A^s\cap W_A^u$ in $W_A^u=H^+$. Since $A(\tau(s))=P^+-P^-$ for $0<s\leqslant\pi/2$, $V_2(\xi,s)=Y\times(0)$ for any $(\xi,s)\in H\times]0,\pi/2]$. Moreover, since $H=W_A^s\oplus Y$, by Theorem B.2(iii),

$$\operatorname{dist}(V_2(\xi,s),H^+\times(0))=\operatorname{dist}(X_A(\tau(s))Y,V^+(A(+\infty)))\to 0\quad\text{for }s\to\pi-.$$

Therefore, V_1 and V_2 define a C^0 essential subbundle $\mathcal{E}$ of TM. In order to show that $\mathcal{E}$ is of class C^1, we have to verify that

$$\frac{d}{ds}P_{X_A(\tau(s))Y}=\tau'(s)Q'(\tau(s))=\cosh\tau(s)Q'(\tau(s))\to 0\quad\text{for }s\to\pi-,\tag{4.3}$$

where $Q(t)=P_{X_A(t)Y}$. For $t_0\geqslant 1$ large, $X_A(t_0)Y\subset H^+\times H^-$ is the graph of some operator $L\in\mathcal{L}(H^+,H^-)$, so

$$X_A(t)Y=X_{P^+-P^-}(t-t_0)X_A(t_0)Y=\left\{\begin{pmatrix}e^{t-t_0}&0\\0&e^{t_0-t}\end{pmatrix}\begin{pmatrix}\xi\\L\xi\end{pmatrix}\,\middle|\,\xi\in H^+\right\}$$

from which we deduce that $Q(t)-P^+=O(e^{-2t})$ for $t\to+\infty$. By identity (B.1), Q solves the Riccati equation

$$Q'=(I-Q)AQ+QA(I-Q)$$

and since $A(t)=P^+-P^-$ for $t\geqslant 1$, we obtain

$$Q'(t)=2(Q(t)-P^+)P^-(Q(t)-I)+2(Q(t)-I)P^-(Q(t)-P^+)=O(e^{-2t})$$

for $t\to+\infty$, which proves (4.3).

By (4.1) the vector field F satisfies (C1) with respect to the essential subbundle $\mathcal{E}$. By (4.2),

$$D\phi_t(\xi, s)[(\eta, 0)] = \begin{cases} (e^{t(P^+ - P^-)}\eta, 0) & \text{for } \pi \leqslant s \leqslant 0, \\ (X_A(t+\tau(s))X_A(\tau(s))^{-1}\eta, 0) & \text{for } 0 < s < \pi \end{cases}$$

for every $t \in \mathbb{R}$, $(\xi, s) \in M$, $\eta \in H$. Therefore, the subbundle V_1 is invariant with respect to F. Since $X_A(t+\tau)X_A(\tau)^{-1} = X_{A(\cdot+\tau)}(t)$, also the subbundle V_2 is invariant with respect to F. Hence $(L_F P_{V_i})P_{V_i} = 0$, for $i = 1, 2$, and F satisfies (C2) with respect to the essential subbundle $\mathcal{E}$.

The smooth function

$$f(\xi, s) = \begin{cases} -\frac{1}{2}\langle (P^+ - P^-)\xi, \xi\rangle + \cos s & \text{for } s \notin [\pi/2, \sigma(1)], \\ -\frac{1}{2}\langle A(\tau(s))\xi, \xi\rangle + \cos s & \text{for } s \in]0, \pi/2[, \end{cases}$$

satisfies

$$Df(\xi, s)[F(\xi, s)] = \begin{cases} -\|\xi\|^2 - \sin^2 s & \text{for } s \notin [\pi/2, \sigma(1)], \\ -\|A(\tau(s))\xi\|^2 - \sin^2 s & \\ \quad -\frac{1}{2}\cosh\tau(s)\langle A'(\tau(s))\xi, \xi\rangle & \text{for } s \in]0, \pi/2[. \end{cases}$$

Since $A(t)$ is invertible for any t and $A(t)$ is constant for $t \notin (0, 1)$, we find

$$Df(\xi, s)[F(\xi, s)] \leqslant -\delta\|\xi\|^2 - \sin^2 s \quad \text{for } \|\xi\| < r \tag{4.4}$$

for suitable $\delta > 0$, $r > 0$, so f is a non-degenerate Lyapunov function for F on the open set $B_r(0) \times \mathbb{T}^1$. Actually, on $B_r(0) \times \mathbb{T}^1$ the vector field F is the gradient of $-f$ with respect to a smooth metric of the form $\alpha_p(\zeta_1, \zeta_2) = \langle T(p)\zeta_1, \zeta_2\rangle$, $p \in M$, $\zeta_1, \zeta_2 \in T_pM = H \oplus \mathbb{R}$. Here $T \in C^\infty(B_r(0) \times \mathbb{T}^1, \mathrm{Sym} \cap \mathrm{GL}(H \oplus \mathbb{R}))$ is positive, and verifies

$$T(\xi, s) = I \quad \text{for } s \notin [\pi/2, \sigma(1)], \quad T(p)F(p) = -\mathrm{grad}\, f(p) \quad \text{for } p \in B_r(0) \times \mathbb{T}^1,$$

where $\mathrm{grad}\, f$ denotes the gradient of f with respect to the flat metric on $H \times \mathbb{T}^1$. Such a map T can be easily found because by (4.4), $\langle F(p), -\mathrm{grad}\, f(p)\rangle > 0$ for every $p \in B_r(0) \times [\pi/2, \sigma(1)]$.

5. Orientation of $W^u(x) \cap W^s(y)$

The first example of the previous section shows that the transverse intersection of an unstable and a stable manifold of two rest points with infinite Morse index and co-index, even if finite dimensional, needs not be orientable. This intersection will be orientable when the vector field satisfies (C1-2) with respect to a subbundle of *TM*.

5.1. Orientation of Fredholm pairs

We need to recall some facts about the orientation bundle on the space of Fredholm pairs. See Appendix A for more details. For H a real Hilbert space and $n \in \mathbb{N}$, we denote by $\mathrm{Or}(\mathrm{Gr}_{n,\infty}(H)) \to \mathrm{Gr}_{n,\infty}(H)$ the orientation bundle of the Grassmannian of n-dimensional subspaces of H: the fiber of $X \in \mathrm{Gr}_{n,\infty}(H)$ consists of the two orientations of X. Similarly, $\mathrm{Or}(\mathrm{Fp}(H)) \to \mathrm{Fp}(H)$ denotes the orientation bundle of the space of Fredholm pairs: the fiber of $(V, W) \in \mathrm{Fp}(H)$ consists of the two orientations of the finite dimensional vector space $(V \cap W) \times (H/(V+W))^*$. This bundle is actually a double covering of $\mathrm{Fp}(H)$. If $\mathcal{H} \to B$ is a Hilbert bundle, we obtain the bundles $\mathrm{Or}(\mathrm{Gr}_{n,\infty}(\mathcal{H})) \to B$, $\mathrm{Or}(\mathrm{Fp}(\mathcal{H})) \to B$, where

$$\mathrm{Or}(\mathrm{Gr}_{n,\infty}(\mathcal{H}) = \bigcup_{b\in B} \mathrm{Or}(\mathrm{Gr}_{n,\infty}(\mathcal{H}_b)), \quad \mathrm{Or}(\mathrm{Fp}(\mathcal{H}) = \bigcup_{b\in B} \mathrm{Or}(\mathrm{Fp}(\mathcal{H}_b))$$

and the maps

$$\mathrm{Or}(\mathrm{Gr}_{n,\infty}(\mathcal{H})) \to \mathrm{Gr}_{n,\infty}(\mathcal{H}), \quad \mathrm{Or}(\mathrm{Fp}(\mathcal{H})) \to \mathrm{Fp}(\mathcal{H})$$

are double coverings.

Let $n \in \mathbb{N}$. If $\mathcal{S}(n, \mathrm{Fp})$ denotes the open set consisting of all $(X, (V, W))$ in $\mathrm{Gr}_{n,\infty}(H) \times \mathrm{Fp}(H)$ such that $X \cap V = (0)$, the map $\mathcal{S}(n, \mathrm{Fp}) \to \mathrm{Fp}(H)$, $(X, (V, W)) \mapsto (X \oplus V, W)$, is continuous, and it lifts to a continuous map—the product of orientations;

$$(o_X, o_{(V,W)}) \mapsto o_X \bigwedge o_{(V,W)},$$

from the corresponding open subset of $\mathrm{Or}(\mathrm{Gr}_{n,\infty}(H)) \times \mathrm{Or}(\mathrm{Fp}(H))$ to $\mathrm{Or}(\mathrm{Fp}(H))$. If $\alpha : B \to \mathrm{Gr}_{n,\infty}(\mathcal{H})$, $\beta, \gamma : B \to \mathrm{Fp}(\mathcal{H})$ are continuous sections such that

$$\alpha(b) \cap \beta_1(b) = (0) \quad \text{and} \quad (\alpha(b) \oplus \beta_1(b), \beta_2(b)) = \gamma(b) \quad \forall b \in B$$

for any choice of liftings of two of α, β, γ to the orientation bundles, there is a unique lifting of the third one such that $\hat{\alpha}(b) \bigwedge \hat{\beta}(b) = \hat{\gamma}(b)$ for every $b \in B$.

5.2. Orientation of $W^u(x) \cap W^s(y)$

Let $\mathcal{V}$ be a C^1 subbundle of TM, and let us assume that the Morse vector field F satisfies (C1-2) with respect to $\mathcal{V}$. By (C1) for every rest point x the pair $(H_x^s, \mathcal{V}(x))$ is a Fredholm pair. Let us fix arbitrarily an orientation o_x of $(H_x^s, \mathcal{V}(x))$. Let $x, y \in \mathrm{rest}\,(F)$ be such that $W^u(x)$ and $W^s(y)$ have a non-empty and transverse intersection. By Theorem 3.3, $W^u(x) \cap W^s(y)$ is an immersed submanifold of dimension $n = m(x, \mathcal{V}) - m(y, \mathcal{V})$. In this section, we will prove that $W^u(x) \cap W^s(y)$ is orientable, and we will show how an orientation of such a manifold is determined by the orientations o_x and o_y.

Let $h^u_x : H^u_x \to M$ and $h^s_y : H^s_y \to M$ be injective C^1 immersions onto $W^u(x)$ and $W^s(y)$ such that $h^u_x(0) = x$ and $h^s_y(0) = y$. Then $W^u(x) \cap W^s(y)$ is the image of the injective immersion $h = h^u_x \circ p^u = h^s_y \circ p^s : W \to M$ coming from the fiber product square of the transverse maps h^u_x and h^s_y:

$$\begin{tikzcd} W \arrow[r, "p^s"] \arrow[d, "p^u"'] \arrow[dr, "h"] & H^s_y \arrow[d, "h^s_y"] \\ H^u_x \arrow[r, "h^u_x"] & M, \end{tikzcd} \qquad W = \left\{ (\xi, \eta) \in H^u_x \times H^s_y \mid h^u_x(\xi) = h^s_y(\eta) \right\}.$$

Giving an orientation to $W^u(x) \cap W^s(y)$ is equivalent to lifting the section

$$\tau : W \to \mathrm{Gr}_{n,\infty}(h^*(TM)), \quad w \mapsto T_{h(w)}(W^u(x) \cap W^s(y)),$$

to a section $\hat{\tau} : W \to \mathrm{Or}(\mathrm{Gr}_{n,\infty}(h^*(TM)))$.

Since H^s_y is contractible, the section

$$H^s_y \to \mathrm{Fp}(h^{s\,*}_y(TM)), \quad \eta \mapsto (T_{h^s_y(\eta)} W^s(y), \mathcal{V}(h^s_y(\eta))),$$

has a unique lifting $H^s_y \to \mathrm{Or}(\mathrm{Fp}(h^{s\,*}_y(TM)))$ whose value at 0 is o_y. By composition with the projection $p^s : W \to H^s_y$, we obtain the section

$$\omega : W \to \mathrm{Fp}(h^*(TM)), \quad w \mapsto (T_{h(w)} W^s(y), \mathcal{V}(h(w)))$$

and a lifting of ω, $\hat{\omega} : W \to \mathrm{Or}(\mathrm{Fp}(h^*(TM)))$.

Let $Y : W \to \mathrm{Gr}(h^*(TM))$ be a continuous section of linear supplements of $T(W^u(x) \cap W^s(y))$ in $TW^s(y)$, that is

$$T_{h(w)} W^s(y) = T_{h(w)}(W^u(x) \cap W^s(y)) \oplus Y(w) \quad \forall w \in W.$$

By the transversality of the intersection of $W^u(x)$ and $W^s(y)$, $Y(w)$ is a linear supplement of $T_{h(w)} W^u(x)$ in $T_{h(w)} M$, so by Theorem B.2(iii),

$$\lim_{t \to -\infty} D\phi_t(h(w)) Y(w) = H^s_x$$

and the limit is locally uniform in W. Therefore the map $A : [-\infty, 0] \times W \to \mathrm{Fp}(TM)$ defined by

$$A(t, w) = \begin{cases} (D\phi_t(h(w))Y(w), \mathcal{V}(\phi_t(h(w)))) & \text{for } t > -\infty, \\ (H^s_x, \mathcal{V}(x)) & \text{for } t = -\infty, \end{cases}$$

is continuous. Let $\hat{A}: [-\infty, 0] \times W \to \mathrm{Or}(\mathrm{Fp}(TM))$ be the unique lifting of A such that $\hat{A}(-\infty, w) = o_x$ for any $w \in W$. By restriction to $\{0\} \times W$, we obtain the section

$$\alpha : W \to \mathrm{Fp}(h^*(TM)), \quad w \mapsto (Y(w), \mathcal{V}(h(w)))$$

and a lifting of α, $\hat{\alpha} : W \to \mathrm{Or}(\mathrm{Fp}(h^*(TM)))$.

By construction,

$$\tau(w) \cap \alpha_1(w) = (0) \text{ and } (\tau(w) \oplus \alpha_1(w), \alpha_2(w)) = \omega(w) \quad \forall w \in W,$$

so τ has a unique lifting $\hat{\tau} : W \to \mathrm{Or}(\mathrm{Gr}_{n,\infty}(h^*(TM)))$ such that

$$\hat{\tau}(w) \bigwedge \hat{\alpha}(w) = \hat{\omega}(w) \quad \forall w \in W,$$

which provides us with an orientation of $W^u(x) \cap W^s(y)$.

Since the set of linear supplements of $T_p(W^u(x) \cap W^s(y))$ in $T_p W^u(x)$ is connected, the orientation we have defined does not depend on the choice of Y. The construction shows that it does not depend on the choice of the immersions h_x^u and h_y^s.

6. Compactness of $W^u(x) \cap W^s(y)$

6.1. The Palais–Smale condition

Let F be a gradient-like Morse vector field on M. Then the stable and unstable manifolds are (embedded) submanifolds, and so are their transverse intersections. We would like to state a set of assumptions which imply that $W^u(x) \cap W^s(y)$ is pre-compact, i.e. it has compact closure in M. The first assumption is a version of the well known Palais–Smale condition:

Definition 6.1. Let F be a C^1 vector field on M, and $f \in C^1(M)$ be a Lyapunov function for F. A (*PS*) sequence for (F, f) is a sequence $(p_n) \subset M$ with $f(p_n)$ bounded and $Df(p_n)[F(p_n)] \to 0$. The pair (F, f) satisfies the (*PS*) condition if every (*PS*) sequence is compact. We shall say that F satisfies (*PS*) if (F, f) satisfies (*PS*) for some Lyapunov function f.

When F is the negative gradient of a function f with respect to some Riemannian metric on M, one finds the usual notion of a Palais–Smale sequence: $f(p_n)$ bounded and $\|Df(p_n)\|$ infinitesimal.

Since $Df(p)[F(p)] < 0$ for $p \notin \mathrm{rest}\,(F)$, the limit points of the (PS) sequences are rest points of F. If (F, f) satisfies (PS), then the set $\mathrm{rest}\,(F) \cap \{a \leqslant f \leqslant b\}$ is compact for every $a, b \in \mathbb{R}$. If moreover F is a Morse vector field, this set is also discrete, hence finite.

Remark 6.2 (*Genesis of (PS) sequences*). Let $(t_n, p_n) \in \Omega(F)$ be such that $t_n \to +\infty$, and $f(p_n)$, $f(\phi(t_n, p_n))$ are bounded. Then by the mean value theorem there exists $s_n \in [0, t_n]$ such that $(\phi(s_n, p_n))$ is a (PS) sequence for (F, f).

The second assumption is the completeness of F, that is the fact that the local flow $\phi(t, \cdot)$ of F is defined for every t, i.e. $\Omega(F) = \mathbb{R} \times M$.

Remark 6.3. Notice that multiplying F by a positive function does not change $W^u(x) \cap W^s(y)$, whereas it may have an effect on the validity of the (PS) and the completeness assumption. For instance, multiplication by a suitably small function makes the vector field complete, while multiplication by a suitable large function makes (PS) true, when $\mathrm{rest}\,(F) \cap f^{-1}([a, b])$ is compact for every $a, b \in \mathbb{R}$. The two assumptions are meaningful here only when considered together.

6.2. Essentially vertical families

As we shall see, (PS) condition and the completeness imply that $W^u(x) \cap W^s(y)$ has compact closure, when either all the rest points of F have finite Morse index, or they have finite Morse co-index. However, they are not sufficient in the general case.

The first assumption we need in the general case is that the essential subbundle $\mathcal{E}$ of TM should have a strong integrable structure $\mathcal{A}$ modeled on (H, V) (see Definition 1.1). In this case, denoting by Q a linear projector with kernel V, we can introduce the following:

Definition 6.4. A family $\mathcal{F}$ of subsets of M is called an essentially vertical family for the strong integrable structure $\mathcal{A}$ of $\mathcal{E}$ if it satisfies:

(i) it is an ideal of $\mathcal{P}(M)$, meaning that it is closed for finite unions and if $A \in \mathcal{F}$ then any subset of A is also in $\mathcal{F}$;
(ii) every point p has a neighborhood U which is the domain of a chart $\varphi \in \mathcal{A}$ such that every set $A \subset U$ with $\varphi(A)$ bounded belongs to $\mathcal{F}$ if and only if $Q\varphi(A)$ is pre-compact.

Once an essentially vertical family $\mathcal{F}$ has been fixed, its elements will be called essentially vertical sets. Clearly, there can be many different essentially vertical families associated to the same strong integrable structure of $\mathcal{E}$, because only the “small” essentially vertical subsets are determined.

The family $\mathcal{F}$ will be called *positively invariant* if it is closed under the positive action of the flow ϕ: for every $A \in \mathcal{F}$ and every $t \geqslant 0$, the set $\phi([0, t] \times A)$ is in $\mathcal{F}$.

The main result of this section is the following compactness theorem.

Theorem 6.5. *Assume that the Morse gradient-like vector field F is complete, satisfies* (C1) *with respect to an essential subbundle $\mathcal{E}$ of TM, and satisfies* (*PS*). *Assume also that $\mathcal{E}$ has a strong integrable structure $\mathcal{A}$ modeled on (H, V) and an essentially vertical family $\mathcal{F}$, which is positively invariant for the flow of F.*

Let $(p_n) \subset M$, $(s_n) \subset (-\infty, 0]$, $(t_n) \subset [0, +\infty)$, be such that $(\phi(s_n, p_n))$ converges to a rest point x, while $(\phi(t_n, p_n))$ converges to a rest point y. Then the sequence (p_n) is compact.

An immediate consequence is the following corollary.

Corollary 6.6. *Under the assumptions of Theorem 6.5, for every $x, y \in \mathrm{rest}\,(F)$ the intersection $W^u(x) \cap W^s(y)$ is pre-compact.*

In order to prove the above theorem, we need to establish the following lemma.

Lemma 6.7. *Let x be a rest point of F. Then x has a fundamental system of neighborhoods U such that*:

(i) *the set $W^u(x) \cap U$ is essentially vertical*;
(ii) *if $A \subset U$ is essentially vertical, then $A \cap W^s(x)$ is pre-compact.*

Furthermore, if f is a non-degenerate Lyapunov function for F, for any sequence $(p_n) \subset U$ converging to x there holds:

(iii) *if $t_n \geqslant 0$ is such that $\phi(t_n, p_n) \in \partial U$ then the set $\{\phi(t_n, p_n) \mid n \in \mathbb{N}\}$ is essentially vertical, and*

$$\limsup_{n\to\infty} f(\phi(t_n, p_n)) < f(x).$$

(iv) *if $t_n \leqslant 0$ is such that $\phi(t_n, p_n) \in \partial U$ then the set $\{\phi(t_n, p_n) \mid n \in \mathbb{N}\}$ has a pre-compact intersection with any essentially vertical subset of M, and*

$$\liminf_{n\to\infty} f(\phi(t_n, p_n)) > f(x).$$

Proof. By choosing a chart $\varphi \in \mathcal{A}$ satisfying property (ii) in Definition 6.4, we can identify a neighborhood U of x in M with a bounded neighborhood—still denoted by U—of 0 in H, in such a way that x corresponds to 0, the essential subbundle $\mathcal{E}$ is represented by the constant subbundle V, the kernel of a projector Q, and the essentially vertical subsets $A \subset U$ are those for which QA is pre-compact.

Let $H = H^u \oplus H^s$, with projections P^u, P^s, be the splitting into the linear unstable and the stable spaces of the hyperbolic rest point 0. Endow H with an adapted norm $\|\cdot\|$ (see Appendix C), and denote by $H^u(r)$, $H^s(r)$ the closed r-balls of H^u and H^s, respectively. Up to reducing the neighborhood U, we may assume that $U = H^u(r) \times H^s(r)$, where $r > 0$ is so small that all the results of Appendix C hold.

By (C1), H^u is a compact perturbation of V. Therefore, we may assume that Q is a compact perturbation of P^s. By Remark 1.2 and by the boundedness of U, a subset $A \subset U$ is essentially vertical if and only if $P^s A$ is pre-compact. In particular, the graph of a map $\sigma : H^u(r) \to H^s(r)$ is essentially vertical if and only if the map σ is compact.

Let $\sigma_0 : H^u(r) \times H^s(r)$ be a 1-Lipschitz map. By the graph transform method (Proposition C.5), for every $t \geqslant 0$ the set

$$\left\{\phi(t,\xi) \mid \xi \in \operatorname{graph}\sigma_0 \text{ and } \phi([0,t]\times\{\xi\}) \subset H^u(r)\times H^s(r)\right\}$$

is the graph of a 1-Lipschitz map $\sigma_t : H^u(r) \to H^s(r)$, and σ_t converges uniformly to a map σ^u for $t \to +\infty$, with $\operatorname{graph}\sigma^u = W^u_{\mathrm{loc},r}(0)$, the local unstable manifold of 0. If σ_0 is a compact map—for example $\sigma_0 = 0$—the fact that the family $\mathcal{F}$ is positively invariant implies that σ_t is a compact map for every $t \geqslant 0$. By the uniform convergence, σ^u is also compact, so the local unstable manifold $W^u_{\mathrm{loc},r}(0)$ is an essentially vertical set. By Theorem C.7, $W^u_{\mathrm{loc},r}(0) = W^u(x) \cap U$, proving (i).

The local stable manifold $W^s_{\mathrm{loc},r}(0)$ is the graph of a 1-Lipschitz map $\sigma^s : H^s(r) \to H^u(r)$. Let $A \subset U$ be an essentially vertical subset, that is $P^s A$ is pre-compact. Then

$$A \cap W^s_{\mathrm{loc},r}(0) = \operatorname{graph}\sigma^s|_{P^s A}$$

is also pre-compact. By Theorem C.7, $W^s_{\mathrm{loc},r}(0) = W^s(x) \cap U$, proving (ii).

Let $(p_n) \subset U$ be a sequence converging to 0, and $t_n \geqslant 0$ such that $\phi(t_n, p_n) \in \partial U$. By Lemma C.4,

$$\limsup_{n\to\infty} f(\phi(t_n,p_n)) < f(0),$$

$$\lim_{n\to\infty} \operatorname{dist}(\phi(t_n,p_n), W^u_{\mathrm{loc},r}(0) \cap \partial U) = 0.$$

The first limit proves part of assertion (iii). By the second limit, we can find a sequence $(q_n) \subset W^u_{\mathrm{loc},r}(0)$ such that $\|\phi(t_n,p_n) - q_n\|$ is infinitesimal. In particular $\|P^s\phi(t_n,p_n) - P^s q_n\|$ is infinitesimal. By (i), the sequence $(P^s q_n)$ is compact. So also the sequence $(P^s\phi(t_n,p_n))$ is compact. This proves that the set $\{\phi(t_n,p_n) \mid n \in \mathbb{N}\}$ is essentially vertical, concluding the proof of (iii).

The fact that σ^s is 1-Lipschitz implies that

$$\|P^u\xi - \sigma^s(P^s\xi)\| \leqslant \sqrt{2}\operatorname{dist}(\xi, \operatorname{graph}\sigma^s) \quad \forall \xi \in U. \tag{6.1}$$

Indeed, if $\xi \in U$ and $c > 1$ we can find $\eta \in \operatorname{graph}\sigma^s$ such that

$$\|\xi - \eta\| \leqslant c \operatorname{dist}(\xi, \operatorname{graph}\sigma^s).$$

Since $P^u\eta = \sigma^s(P^s\eta)$ and since σ^s is 1-Lipschitz,

$$\begin{aligned}\|P^u\xi - \sigma^s(P^s\xi)\| &\leqslant \|P^u\xi - P^u\eta\| + \|\sigma^s(P^s\eta) - \sigma^s(P^s\xi)\| \\ &\leqslant \|P^u\xi - P^u\eta\| + \|P^s\eta - P^s\xi\| \leqslant \sqrt{2}\|\xi - \eta\| \\ &\leqslant c\sqrt{2}\operatorname{dist}(\xi, \operatorname{graph}\sigma^s)\end{aligned}$$

and since $c > 1$ is arbitrary, (6.1) follows.

Now assume that $t_n \leqslant 0$ are such that $\phi(t_n, p_n) \in \partial U$. By Lemma C.4 applied to $-F$,

$$\liminf_{n\to\infty} f(\phi(t_n, p_n)) > f(0),$$

$$\lim_{n\to\infty} \operatorname{dist}(\phi(t_n, p_n), W^s_{\mathrm{loc},r}(0) \cap \partial U) = 0.$$

The first limit proves part of assertion (iv). Let $A \subset U$ be an essentially vertical set, that is $P^s A$ is pre-compact. If the set $A \cap \{\phi(t_n, p_n) \mid n \in \mathbb{N}\}$ is infinite (otherwise there is nothing to prove), its elements form a subsequence $(\phi(t_{n_k}, p_{n_k}))$ such that the sequence $(P^s \phi(t_{n_k}, p_{n_k}))$ is compact. By the continuity of σ^s, also the sequence $(\sigma^s(P^s \phi(t_{n_k}, p_{n_k})))$ is compact. By (6.1),

$$\begin{aligned}\|P^u \phi(t_{n_k}, p_{n_k}) - \sigma^s(P^s \phi(t_{n_k}, p_{n_k}))\| &\leqslant \sqrt{2}\operatorname{dist}(\phi(t_{n_k}, p_{n_k})), \operatorname{graph}\sigma^s) \\ &\leqslant \sqrt{2}\operatorname{dist}(\phi(t_{n_k}, p_{n_k})), W^s_{\mathrm{loc},r}(0) \cap \partial U)\end{aligned}$$

is infinitesimal, so also $(P^u \phi(t_{n_k}, p_{n_k}))$ is compact. We deduce that $(\phi(t_{n_k}, p_{n_k}))$ is compact, concluding the proof of (iv). $\square$

Proof of Theorem 6.5. Let f be a non-degenerate Lyapunov function for F such that (F, f) satisfies (PS). Up to taking a subsequence of (p_n) and changing (s_n) and x, we may assume that for no choice of a sequence $(r_n) \subset]-\infty, 0]$, the sequence $(\phi(r_n, p_n))$ has a subsequence which converges to a rest point z with $f(z) < f(x)$. Indeed, since there are finitely many rest points z such that $f(y) \leqslant f(z) \leqslant f(x)$, the set

$$\mathcal{Z} := \Big\{ z \in \operatorname{rest}(F) \,\Big|\, f(z) \leqslant f(x) \text{ and there exists } (n_k) \subset \mathbb{N} \text{ increasing and } s'_k \leqslant 0 \\ \text{such that } \lim_{k\to\infty} \phi(s'_k, p_{n_k}) = z \Big\}$$

is finite, and non-empty because it contains x. Let $x' = \lim_{k\to\infty} \phi(s'_k, p_{n_k})$ be a point of $\mathcal{Z}$ where f attains its minimum. Then the latter requirement is verified with (p_{n_k}), (s'_k), and x'.

Similarly, by taking a further subsequence of (p_n), and by changing (t_n) and y, we may assume that for no choice of a sequence $(r_n) \subset \mathbb{R}$, the sequence $(\phi(r_n, p_n))$ has a subsequence which converges to a rest point z with $f(y) < f(z) < f(x)$. If either x or y is a cluster point for (p_n) there is nothing to prove, so we may assume that (p_n) is bounded away from x and y.

By Lemma 6.7(iii), there exists a closed neighborhood $U \subset M$ of x such that $p_n \notin U$, and choosing $s'_n \in]s_n, 0[$ such that $\phi(s'_n, p_n) \in \partial U$ (for n large), we have

$$\big\{\phi(s'_n, p_n) \mid n \in \mathbb{N}\big\} \text{ is essentially vertical,} \tag{6.2}$$

$$\limsup_{n\to\infty} f(\phi(s'_n, p_n)) < f(x). \tag{6.3}$$

By Lemma 6.7(iv), there exists a closed neighborhood $V \subset M$ of y such that $p_n \notin V$, and choosing $t_n' \in]0, t_n[$ such that $\phi(t_n', p_n) \in \partial V$ (for n large), we have

$$\left\{\phi(t_n', p_n) \mid n \in \mathbb{N}\right\}$$

$$A \cap \left\{\phi(t_n', p_n) \mid n \in \mathbb{N}\right\} \text{ is pre-compact } \forall A \subset M, A \text{ ess. vert.} \tag{6.4}$$

$$\liminf_{n\to\infty} f(\phi(t_n', p_n)) > f(y). \tag{6.5}$$

(PS) condition implies that $(t_n' - s_n')$ is bounded: otherwise by Remark 6.2, (6.3) and (6.5), we would obtain a sequence $(r_n) \subset \mathbb{R}$ such that $(\phi(r_n, p_n))$ has a subsequence converging to a rest point z, with $f(y) < f(z) < f(x)$, contradicting our previous assumption. Therefore, $t_n' - s_n' \leqslant T$ for every $n \in \mathbb{N}$.

Since the essentially vertical family $\mathcal{F}$ is positively invariant, (6.2) implies that the set

$$\left\{\phi(t_n', p_n) \mid n \in \mathbb{N}\right\} \subset \phi([0, T] \times \left\{\phi(s_n', p_n) \mid n \in \mathbb{N}\right\})$$

is essentially vertical. But then we can choose $A = \left\{\phi(t_n', p_n) \mid n \in \mathbb{N}\right\}$ in (6.4), and we obtain that the sequence $(\phi(t_n', p_n))$ is compact. By the boundedness of t_n' and by the fact that the vector field F is complete, we conclude that also the sequence (p_n) is compact. $\square$

Remark 6.8. An argument similar to the one used above shows that, if F satisfies the assumptions of Theorem 6.5, $x \in \mathrm{rest}\,(F)$ and $a \in \mathbb{R}$, then the set $W^u(x) \cap \{f \geqslant a\}$ is essentially vertical.

6.3. *Examples*

Let us see what Theorem 6.5 says in the cases of finite Morse indices or co-indices.

Example 6.9 (*Vector fields whose rest points have finite Morse index or co-index*). Notice that the trivial subbundle $\mathcal{E} = (0)$ (relevant in the case of rest points with finite Morse index, see Example 2.3) has a strong integrable structure (choose any atlas of M). The family consisting of all pre-compact subsets of M is a family of essentially vertical subsets for $\mathcal{E} = (0)$, and it is obviously closed under the action of the flow.

Similarly, the trivial subbundle $\mathcal{E} = TM$ (relevant in the case of rest points with finite Morse co-index) has a strong integrable structure (again, consider an arbitrary atlas of M). The family consisting of all subsets of M is a family of essentially vertical subsets for $\mathcal{E} = TM$, clearly closed under the action of the flow.

We have already seen that in the case $\mathcal{E} = (0)$ (resp. $\mathcal{E} = TM$) (C1) is equivalent to the fact that all the rest points of F have finite Morse index (resp. co-index).

Therefore the conclusion of Theorem 6.5 holds when (i) the C^1 vector field F is Morse and gradient-like, (ii) either all the rest points of F have finite Morse index, or they have finite Morse co-index, (iii) F satisfies (PS), and (iv) F is complete.

Now let us look back at Example 2.3.

Example 6.10 (*Perturbations of a non-degenerate quadratic form*). Assume that $M = H$ is a Hilbert space, and consider a function of the form

$$f(\xi) = \frac{1}{2}\langle L\xi, \xi\rangle + b(\xi),$$

where $L \in \mathcal{L}(H)$ is self-adjoint invertible, and the gradient of the function $b \in C^2(H)$ is a compact map. Let F be the (negative) gradient vector field of f,

$$F(\xi) = -\mathrm{grad}\, f(\xi) = -L\xi - \mathrm{grad}\, b(\xi).$$

If $\mathrm{grad}\, b$ has linear growth, i.e. $\|\mathrm{grad}\, b(\xi)\| \leqslant c(1 + \|\xi\|)$ for every $\xi \in H$, then F is complete, its flow ϕ maps bounded subsets of $\mathbb{R} \times H$ into bounded subsets of H, and it satisfies

$$\phi(t, \xi) = e^{-tL}\xi - \int_0^t e^{(s-t)L}\mathrm{grad}\, b(\phi(s, \xi))\, ds. \tag{6.6}$$

Consider the constant subbundle $V = V^-(L)$, and the orthogonal projection Q with kernel V. This bundle has the trivial strong integrable structure modeled on (H, V) consisting of the identity map: $\mathcal{A} = \{I\}$. The family $\mathcal{F}$ consisting of all bounded subsets A of H such that QA is pre-compact is an essentially vertical family for $\mathcal{A}$. Moreover identity (6.6) together with the fact that $\mathrm{grad}\, b$ is a compact map implies that $\mathcal{F}$ is invariant for ϕ.

The assumption that $\mathrm{grad}\, b$ has linear growth can be easily dropped. Indeed, the vector field

$$\tilde{F}(\xi) = -h(\xi)\mathrm{grad}\, f(\xi), \quad \text{where } h(\xi) = \frac{1}{1 + \|\mathrm{grad}\, f(\xi)\|^2}$$

is bounded, hence complete, and its flow $\tilde{\phi}$ maps bounded subsets of $\mathbb{R} \times H$ into bounded subsets of H. Notice that f is a non-degenerate Lyapunov function for $\tilde{F}$, and since $Df[\tilde{F}] = -\|\mathrm{grad}\, f\|^2/(1 + \|\mathrm{grad}\, f\|^2)$, the Palais–Smale sequences for $(\tilde{F}, f)$ (in the sense of Definition 6.1) are exactly the Palais–Smale sequences for f (in the usual sense). The flow $\tilde{\phi}$ satisfies

$$\tilde{\phi}(t, \xi) = e^{-\tau(t,\xi)L}\xi - \int_0^t h(\tilde{\phi}(t, \xi))e^{(\tau(s,\xi)-\tau(t,\xi))L}\mathrm{grad}\, b(\tilde{\phi}(s, \xi))\, ds,$$

where $\tau : \mathbb{R} \times H \to \mathbb{R}$ is the function

$$\tau(t, \xi) = \int_0^t h(\tilde{\phi}(s, \xi))\, ds.$$

Then $|\tau(t, \xi)| \leqslant |t|$, and the fact that $\operatorname{grad} b$ is a compact map again implies that the family $\mathcal{F}$ is invariant for $\tilde{\phi}$.

We conclude that the thesis of the compactness Theorem 6.5 holds, when L is invertible and self-adjoint, $b \in C^2(H)$ has compact gradient, and f satisfies the Palais–Smale condition.

In the case of a non-trivial subbundle $\mathcal{E}$ the question is how to find an essentially vertical family which is closed under the action of the flow of F. This question will be addressed in the next section.

7. Flow-invariant essentially vertical families

7.1. Hausdorff measure of non-compactness

We recall that the Hausdorff distance of two subsets A, B of a metric space X is the number

$$\operatorname{dist}_{\mathcal{H}}(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \operatorname{dist}(a, b), \sup_{b \in B} \inf_{a \in A} \operatorname{dist}(a, b) \right\} \in [0, +\infty].$$

We denote by $\mathcal{H}(X)$ the family of all closed subsets of X, and by $\mathcal{H}_b(X)$ the subfamily consisting of bounded subsets, which is a metric space with the Hausdorff distance. A related concept is the notion of measure of non-compactness. If A is a subset of a metric space X, its *Hausdorff measure of non-compactness* is

$$\beta_X(A) := \inf \{ r > 0 \mid A \text{ can be covered by finitely many balls of radius } r \} \in [0, +\infty].$$

Equivalently, $\beta_X(A)$ is the distance from the set of compact subsets of X:

$$\beta_X(A) = \inf \{ \operatorname{dist}_{\mathcal{H}}(A, K) \mid K \subset X \text{ compact} \}. \tag{7.1}$$

It has the following properties (see [Dei85, Section 2.7.3]):

(a) $\beta_X(A) < +\infty$ if and only if A is bounded;
(b) $\beta_X(A) = 0$ if and only if A is totally bounded;
(c) if $A_1 \subset A_2$ then $\beta_X(A_1) \leqslant \beta_X(A_2)$;
(d) $\beta_X(A) \leqslant \beta_A(A) \leqslant 2\beta_X(A)$;

(e) $\beta_X(A_1 \cup A_2) = \max\{\beta_X(A_1), \beta_X(A_2)\}$;
(f) $\beta_X(A) = \beta_X(\overline{A})$;
(g) β_X is continuous with respect to the Hausdorff distance.

If X is a normed vector space, denoting by $\operatorname{co} A$ the convex hull of $A \subset X$, we also have:

(h) $\beta_X(\lambda A) = |\lambda| \beta_X(A)$, and $\beta_X(A_1 + A_2) \leqslant \beta_X(A_1) + \beta_X(A_2)$;
(i) $\beta_X(\operatorname{co} A) = \beta_X(A)$.

7.2. Admissible presentations

We shall require that the essential subbundle $\mathcal{E}$ of TM has a strong presentation (see Definition 1.5) which satisfies the following finiteness and a uniformity conditions.

Definition 7.1. A strong presentation $\{M_i, N_i, \mathcal{Q}_i\}_{i \in I}$ is called an admissible presentation if the Hilbert manifolds N_i are endowed with complete Riemannian metrics, and

(i) the covering $\{M_i\}_{i \in I}$ is star-finite (i.e. every M_i has non-empty intersection with finitely many M_j's);
(ii) there is $r > 0$ such that for every $p \in M$ there exists $i \in I$ such that

$$\overline{\mathcal{Q}_i^{-1}(B_r(\mathcal{Q}_i(p)))} \subset M_i.$$

An admissible presentation for $\mathcal{E}$ determines a strong integrable structure $\mathcal{A}$ (see Proposition 1.4). Moreover, it determines a useful family of essentially vertical subsets of M. Indeed, let $\mathcal{F}$ be the family of subsets $A \subset M$ such that:

$$A \text{ can be covered by finitely many } M_i\text{'s}; \tag{7.2}$$

$$\text{for every } i \in I, \ \mathcal{Q}_i(A \cap M_i) \text{ is pre-compact.} \tag{7.3}$$

Proposition 1.4 implies that this is a family of essentially vertical sets for the strong integrable structure $\mathcal{A}$.

Given an admissible presentation of $\mathcal{E}$ as above, we shall assume the following condition on the vector field F:

(C3) (i) there is $b > 0$ such that $\|D\mathcal{Q}_i \circ F\|_\infty \leqslant b$ for every $i \in I$;
(ii) for every $i \in I$ and $q \in N_i$, there exists $\delta = \delta(q) > 0$ and $c = c(q) \geqslant 0$ such that

$$\beta_{TN_i}(D\mathcal{Q}_i(F(A))) \leqslant c\, \beta_{N_i}(\mathcal{Q}_i(A)) \quad \forall A \subset \mathcal{Q}_i^{-1}(B_\delta(q)). \tag{7.4}$$

Here the tangent bundle TN_i is given the standard metric induced by the Riemannian structure of N_i. Notice that no Riemannian metric on M is involved in this condition.

Remark 7.2. If (C3)-(ii) holds, we can replace the point $q \in N_i$ by a compact set $K \subset N_i$ in (7.4): for every $i \in I$ and every compact set $K \subset N_i$, there exists $\delta = \delta(K) > 0$ and $c = c(K) \geqslant 0$ such that

$$\beta_{TN_i}(D\mathcal{Q}_i(F(A))) \leqslant c\, \beta_{N_i}(\mathcal{Q}_i(A)) \quad \forall A \subset \mathcal{Q}_i^{-1}(N_\delta(K)),$$

where $N_\delta(K)$ denotes the δ-neighborhood of K.

Remark 7.3. If E is a Hilbert space and $\mathcal{Q} : M \to E$ is a C^1 map, one often makes no distinction between the tangential map $D\mathcal{Q} : TM \to TE = E \times E$, $(p, \xi) \mapsto (\mathcal{Q}(p), D\mathcal{Q}(p)[\xi])$, and its second component $D\mathcal{Q} : TM \to E$, $(p, \xi) \mapsto D\mathcal{Q}(p)[\xi]$. When $N_i = E$ is a Hilbert space, we are allowed to replace the tangential map of $\mathcal{Q}_i$ by its second component in (7.4), writing $\beta_E(D\mathcal{Q}_i(F(A)))$ instead of $\beta_{E\times E}(D\mathcal{Q}_i(F(A)))$ on the left-hand side of the inequality. Indeed, if $S \subset TE = E \times E$, and $P_1, P_2 : E \times E \to E$ are the projections onto the first and the second factor, we have

$$\max\{\beta_E(P_1 S), \beta_E(P_2 S)\} \leqslant \beta_{E\times E}(S) \leqslant \beta_E(P_1 S) + \beta_E(P_2 S).$$

The main result of this section is the following proposition.

Proposition 7.4. *Let $\{M_i, N_i, \mathcal{Q}_i\}_{i\in I}$ be an admissible presentation for the essential subbundle $\mathcal{E}$ of TM. Assume that the vector field F is complete and satisfies condition* (C3). *Then the essentially vertical family $\mathcal{F}$ defined by* (7.2) *and* (7.3) *is positively invariant for the flow of F.*

We start with the following local result.

Lemma 7.5. *Let $\mathcal{Q} : M \to E$ be a C^1 map into a Hilbert space. Let $A \subset M$ be such that $\mathcal{Q}(A)$ is pre-compact, and let $t^* \geqslant 0$ be such that $[0, t^*] \times A \subset \Omega(F)$. Assume that there exists $c \geqslant 0$ such that*

$$\beta_E(D\mathcal{Q}(F(A'))) \leqslant c\, \beta_E(\mathcal{Q}(A')) \quad \forall A' \subset \phi([0, t^*] \times A).$$

Then $\mathcal{Q}(\phi([0, t^*] \times A))$ is pre-compact.

Proof. Let $n = \lfloor ct^* \rfloor + 1$, and set $\tau = t^*/n$, so that $\tau c < 1$. For $k \in \mathbb{N}$, $0 \leqslant k \leqslant n$, set $A_k = \phi([0, k\tau] \times A)$. Since

$$\mathcal{Q}(\phi(t, p)) = \mathcal{Q}(p) + t \cdot \frac{1}{t} \int_0^t D\mathcal{Q}(\phi(s, p))[F(\phi(s, p))]\, ds,$$

we have

$$\mathcal{Q}(A_{k+1}) = \mathcal{Q}(\phi([0, \tau] \times A_k) \subset \mathcal{Q}(A_k) + [0, \tau]\overline{\mathrm{co}}\,(D\mathcal{Q}(F(A_{k+1}))).$$

So, by properties (c), (h), and (j) of the Hausdorff measure of non-compactness, for $0 \leqslant k \leqslant n-1$ we have,

$$\begin{aligned}\beta_E(\mathcal{Q}(A_{k+1})) &\leqslant \beta_E(\mathcal{Q}(A_k)) + \tau\beta_E(\overline{\mathrm{co}}\,(D\mathcal{Q}(F(A_{k+1}))))\\ &= \beta_E(\mathcal{Q}(A_k)) + \tau\beta_E(D\mathcal{Q}(F(A_{k+1}))) \leqslant \beta_E(\mathcal{Q}(A_k)) + \tau c\beta_E(\mathcal{Q}(A_{k+1})).\end{aligned}$$

Since $\tau c < 1$,

$$\beta_E(\mathcal{Q}(A_{k+1})) \leqslant \frac{1}{1-\tau c}\beta_E(\mathcal{Q}(A_k)), \quad k = 0, 1, \ldots, n-1$$

and the fact that $\beta_E(\mathcal{Q}(A_0)) = 0$ implies that $\beta_E(\phi([0,t^*] \times A)) = \beta_E(\mathcal{Q}(A_n)) = 0$, as claimed. $\square$

Example 7.6. The conclusion of the above lemma is not implied by the weaker assumption that $\overline{\mathcal{Q}(F(S))}$ should be compact for every set S such that $\overline{\mathcal{Q}(S)}$ is compact, as the following example shows.

Let $H = \ell_2(\mathbb{Z})$, let $\{e_k \mid k \in \mathbb{Z}\}$ be its standard orthonormal basis, let $H^- = \overline{\mathrm{span}}\,\{e_k \mid k \leqslant 0\}$, $H^+ = \overline{\mathrm{span}}\,\{e_k \mid k > 0\}$, and let Q be the orthogonal projector onto H^-. Then there exists a smooth bounded vector field $F : H \to H$ whose restriction to any set of the form

$$\{\xi \in H \mid |\xi - \xi_0| < r\} + H^-, \quad \xi_0 \in H^+,\ r < 1, \tag{7.5}$$

has finite rank, and whose flow ϕ has the property that

$$\overline{Q\phi_1(\{\xi \in H^+ \mid |\xi| \leqslant 1\})}$$

is not compact. In particular, $\overline{F(A)}$ is compact and finite dimensional whenever $\beta_{H^-}(QA) < 1$.

To construct such a vector field, for $k \in \mathbb{N}^*$ choose two functions $f_k, g_k \in C^\infty(\mathbb{R})$ such that $f_k(s) = \sqrt{s + 1/k}$ for $s \in [0,2]$, $\|f_k\|_\infty \leqslant 2$, $g_k(1) = 1$, $g_k(s) = 0$ for $s \leqslant 1 - 1/k$, $0 \leqslant g_k \leqslant 1$. Let $\chi \in C^\infty(\mathbb{R})$ be a function with compact support such that $\chi(s) = 1$ for $|s| \leqslant 2$, and set

$$F(\xi) := \chi(|\xi|)\sum_{k=1}^{\infty} g_k(\xi \cdot e_{-k}) f_k(\xi \cdot e_k) e_k, \quad \xi \in H.$$

The restriction of the vector field F to a set of the kind (7.5) has image contained in the finite dimensional subspace span $\left\{e_k \mid k \in \mathbb{N},\ \overline{\xi} \cdot e_k + 1/k > 1 - r\right\}$. On the other hand, an easy computation shows that

$$\phi(t, e_{-k}) = e_{-k} + \left(\frac{t^2}{4} + \frac{t}{\sqrt{k}}\right) e_k, \quad k \geqslant 1,\ 0 \leqslant t \leqslant 1,$$

so the set $\{Q\phi(t, e_{-k}) \mid k \geqslant 1\}$ does not have compact closure, for any $t \in [0, 1]$.

Lemma 7.7. *Let $\{M_i, N_i, \mathcal{Q}_i\}_{i\in I}$ be an admissible presentation of the essential sub-bundle $\mathcal{E}$ of TM. Let F be a C^1 complete vector field satisfying*

$$\|D\mathcal{Q}_i \circ F\|_\infty \leqslant b \quad \forall i \in I$$

for some $b \geqslant 0$.

(i) *If $\overline{\mathcal{Q}_i^{-1}(B_r(\mathcal{Q}_i(p)))} \subset M_i$, then $\phi(s, p) \in M_i$ for every $|s| \leqslant r/b$, and*

$$\operatorname{dist}(\mathcal{Q}_i(\phi(s, p)), \mathcal{Q}_i(p)) \leqslant b|s|. \tag{7.6}$$

(ii) *If a set $A \subset M$ can be covered by finitely many M_i's, then $A = \bigcup_{i\in I_0} A_i$, where*

$$A_i = \left\{p \in A \cap M_i \mid \overline{\mathcal{Q}_i^{-1}(B_r(\mathcal{Q}_i(p)))} \subset M_i\right\} \tag{7.7}$$

and $I_0 \subset I$ is finite.

(iii) *If a set $A \subset M$ can be covered by finitely many M_i's, then $\phi([0, t] \times A)$ can be covered by finitely many M_i's, for every $t \geqslant 0$.*

Proof. (i) Let J be the maximal interval of numbers s for which $\phi(s, p) \in M_i$. Then

$$\begin{aligned}\operatorname{dist}(\mathcal{Q}_i(\phi(s, p)), \mathcal{Q}_i(p)) &\leqslant \left|\int_0^s \left|\frac{d}{d\sigma}\mathcal{Q}_i(\phi(\sigma, p))\right| d\sigma\right| \\ &= \left|\int_0^s |D\mathcal{Q}_i \circ F(\phi(\sigma, p))|\, d\sigma\right| \leqslant b|s| \quad \forall s \in J.\end{aligned}$$

Together with the fact that the closure of $\mathcal{Q}_i^{-1}(B_r(\mathcal{Q}_i(p)))$ is contained in M_i, this implies that $]-r/b, r/b[\subset J$ and (7.6).

(ii) Since A is covered by finitely many M_i's and the covering $\{M_i\}_{i\in I}$ is star-finite, the indices $i \in I$ for which $A_i \neq \emptyset$ form a finite subset I_0. By the uniformity property of the presentation (Definition 7.1(iv)), $A = \bigcup_{i\in I_0} A_i$.

(iii) If A_i are the sets defined in (7.7), statement (i) implies that $\phi([0, r/b[\times A_i) \subset M_i$ for every $i \in I_0$. Therefore $\phi([0, r/b[\times A)$ is covered by the finite covering $\{M_i\}_{i\in I_0}$, and the conclusion follows by induction. $\square$

Proof of Proposition 7.4. By Lemma 7.7(iii), $\phi(0, t] \times A$ is covered by finitely many $M_i's$, so it is enough to show that the interval

$$\mathcal{T}(A) = \{t \geqslant 0 \mid \mathcal{Q}_i(\phi([0, t] \times A)) \text{ is pre-compact in } N_i, \ \forall i \in I\}$$

coincides with $[0, +\infty[$. Since $0 \in \mathcal{T}(A)$, we can argue by connectedness proving that $\mathcal{T}(A)$ is both open and closed in $[0, +\infty[$.

We claim that $\mathcal{T}(A)$ is open in $[0, +\infty[$. Let $t \in \mathcal{T}(A)$. By Lemma 7.7(ii), $\phi([0, t] \times A) = \bigcup_{i\in I_0} A_i$, where

$$A_i = \left\{ p \in \phi([0, t] \times A) \cap M_i \mid \overline{\mathcal{Q}_i^{-1}(B_r(\mathcal{Q}_i(p)))} \subset M_i \right\} \tag{7.8}$$

and $I_0 \subset I$ is finite. Clearly, $\mathcal{T}(A) = [0, t] + \bigcap_{i\in I_0} \mathcal{T}(A_i)$, so it is enough to prove that $\sup \mathcal{T}(A_i) > 0$, for every $i \in I_0$.

Let $i \in I_0$. Since $\mathcal{Q}_i(A_i)$ is pre-compact, (C3)-(ii), together with Remark 7.2, implies that there exist $c \geqslant 0$ and $\delta > 0$ such that

$$\beta_{TN_i}(D\mathcal{Q}_i \circ F(S)) \leqslant c\beta_{N_i}(\mathcal{Q}_i(S)) \quad \forall S \subset \mathcal{Q}_i^{-1}(N_\delta(\mathcal{Q}_i(A_i))). \tag{7.9}$$

Moreover, $\overline{\mathcal{Q}_i(A_i)}$ is covered by finitely many coordinate neighborhoods: there exist $q_1, \dots, q_n \in \mathcal{Q}_i(A_i)$, $0 < \rho \leqslant \min\{\delta, r\}$, $\mathcal{Q}_i(A_i) \subset \bigcup_{j=1}^n B_{\rho/2}(q_j)$, and local charts

$$\psi_j : \mathrm{dom}\,(\psi_j) \to E,$$

with $\overline{B_\rho(q_j)} \subset \mathrm{dom}\,(\psi_j)$, $\overline{\psi_j(B_\rho(q_j))} \subset \mathrm{dom}\,(\psi_j^{-1})$, and ψ_j^{-1}, $D\psi_j$ Lipschitz. Then $A_i = \bigcup_{j=1}^n A_i^j$, with $A_i^j = A_i \cap \mathcal{Q}_i^{-1}(B_{\rho/2}(q_j))$. Again, it suffices to show that $\sup \mathcal{T}(A_i^j) > 0$.

Let $1 \leqslant j \leqslant n$, and set $U = \mathcal{Q}_i^{-1}(B_\rho(q_j))$. Since $\rho \leqslant r$ and $q_j \in \mathcal{Q}_i(A_i)$, by the definition of A_i we have

$$\overline{U} \subset \overline{\mathcal{Q}_i^{-1}(B_r(q_j))} \subset M_i. \tag{7.10}$$

Let $p \in A_i^j \subset M_i$. Let $[0, \tau(p)[$, $0 < \tau(p) \leqslant +\infty$, be the maximal interval of positive numbers s for which $\phi(s, p) \in U$. By Lemma 7.7(i),

$$\mathrm{dist}\,(\mathcal{Q}_i(\phi(s, p)), q_j) \leqslant \mathrm{dist}\,(\mathcal{Q}_i(\phi(s, p)), \mathcal{Q}_i(p)) + \mathrm{dist}\,(\mathcal{Q}_i(p), q_j) \leqslant bs + \rho/2$$

for every $s \in [0, \tau(p)[$. Together with (7.10) this implies that $\tau(p) \geqslant \rho/(2b)$. Therefore,

$$\phi([0, \rho/(2b)[\times A_i^j) \subset U. \tag{7.11}$$

Let $\mathcal{Q} := \psi_j \circ \mathcal{Q}_i : U \to E$. Since $\rho \leqslant \delta$ and $q_j \in \mathcal{Q}_i(A_i)$, U is contained in $\mathcal{Q}_i^{-1}(N_\delta(\mathcal{Q}_i(A_i)))$. By (7.9), for any $S \subset U$,

$$\begin{aligned} \beta_E((D\mathcal{Q}_i \circ F)(S)) &\leqslant \beta_{E\times E}((D\mathcal{Q} \circ F)(S)) = \beta_{E\times E}(D\psi_j \circ D\mathcal{Q}_i(F(S))) \\ &\leqslant \operatorname{lip}(D\psi_j)\beta_{TN_i}(D\mathcal{Q}_i \circ F(S)) \leqslant c \operatorname{lip}(D\psi_j)\beta_{N_i}(\mathcal{Q}_i(S)) \\ &= c \operatorname{lip}(D\psi_j)\beta_{N_1}(\psi_j^{-1}(\mathcal{Q}(S))) \leqslant c \operatorname{lip}(D\psi_j) \operatorname{lip}(\psi_j^{-1})\beta_E(\mathcal{Q}(S)). \end{aligned}$$

By (7.11), we can take $S = \phi([0, \rho/(2b)[\times A_i^j)$ in the above inequality, and Lemma 7.5 implies that $\mathcal{Q}(\phi([0, \rho/2b[\times A_i^j))$ is pre-compact in E.

Since $\overline{B_\rho(q_j)} \subset \operatorname{dom}(\psi_j)$ and $\overline{\psi_j(B_\rho(q_j))} \subset \operatorname{dom}(\psi_j^{-1})$, the set

$$\overline{\mathcal{Q}_i(\phi([0, \rho/(2b)[\times A_i^j))} = \psi_j^{-1}\overline{(\mathcal{Q}(\phi([0, \rho/(2b)[\times A_i^j)))}$$

is a compact subset of N_i. Therefore, $\sup \mathcal{T}(A_i^j) \geqslant \rho/(2b) > 0$, as we wished to prove.

There remains to show that the interval $\mathcal{T}(A)$ is closed. Let $t = \sup \mathcal{T}(A)$. By Lemma 7.7(ii), $\phi([0, t] \times A) = \bigcup_{i \in I_0} A_i$, where A_i is defined in (7.8) and $I_0 \subset I$ is finite. It is enough to prove that $\mathcal{Q}_i(A_i)$ has compact closure in N_i, for any $i \in I_0$.

Fix $i \in I_0$, and let $q_k = \mathcal{Q}_i(\phi(t, p_k))$, where $p_k \in A$ and $\phi(t, p_k) \in A_i$, be a sequence in $\mathcal{Q}_i(A_i)$. By Lemma 7.7(i), $\phi(t - \tau, p_k) \in M_i$ for any $0 \leqslant \tau < r/b$, and

$$\operatorname{dist}(\mathcal{Q}_i(\phi(t - \tau, p_k)), q_k) \leqslant b\tau. \tag{7.12}$$

Since $t = \sup \mathcal{T}(A)$, the sequence $(\mathcal{Q}_i(\phi(t - \tau, p_k)))_{k \in \mathbb{N}}$ is compact for any $0 < \tau < r/b$. Then (7.12) and the completeness of N_i imply that also the sequence (q_k) is compact, proving that $\mathcal{Q}_i(A_i)$ is pre-compact. $\square$

7.3. *Properties of condition (C3)*

Condition (C3) is stronger than (C2), and like (C2) it is a convex condition. Indeed, the following result holds.

Proposition 7.8. *Let $\{M_i, N_i, \mathcal{Q}_i\}_{i \in I}$ be an admissible presentation of the essential subbundle $\mathcal{E}$. The following facts hold*:

(i) *condition* (C3) *implies condition* (C2);
(ii) *the vector fields F satisfying* (C3) *form a module over the ring $C^1(M) \cap C_b^0(M)$*

Proof. (i) Let $i \in I$, $p \in M_i$, and set $q := \mathcal{Q}_i(p)$. Up to the composition with C^1 local charts

$$\varphi : \mathrm{dom}\,(\varphi) \subset M_i \to H,\ p \in \mathrm{dom}\,(\varphi), \quad \psi : \mathrm{dom}\,(\psi) \subset N_i \to E,\ q \in \mathrm{dom}\,(\psi),$$

such that ψ, and $D\psi$ are bi-Lipschitz, we may assume that $\mathcal{Q}_i$ is a C^1 semi-Fredholm map with $\mathrm{ind}\,\mathcal{Q}_i \geqslant 0$ from an open set of the Hilbert space H into the Hilbert space E. By (C3)-(ii) and Remark 7.3, there exist $\delta > 0$ and $c \geqslant 0$ such that

$$\beta_E(D\mathcal{Q}_i(F(A))) \leqslant c\beta_E(\mathcal{Q}_i(A)) \quad \forall A \subset \mathcal{Q}_i^{-1}(B_\delta(q)). \tag{7.13}$$

Let $T \in \mathcal{L}(H, E)$ be a linear map with finite rank such that $D\mathcal{Q}_i(p) + T$ is surjective. Since T has finite rank,

$$\beta_E(\mathcal{Q}_i(A)) = \beta_E((\mathcal{Q}_i + T)(A)), \quad \beta_E(D\mathcal{Q}_i(F(A))) = \beta_E(D(\mathcal{Q}_i + T)(F(A))). \tag{7.14}$$

The map $\mathcal{Q}_i + T$ is a local submersion at p, so up to considering a change of variable at p, we may assume that the restriction of $\mathcal{Q}_i$ to a neighborhood U of p coincides with a linear surjective map Q from H to E, which by (7.13) and (7.14) verifies

$$\beta_E(QF(A)) \leqslant c\beta_E(QA) \quad \forall A \subset Q^{-1}(B_\delta(q)) \cap U. \tag{7.15}$$

By composing with a linear right inverse of Q, we may also assume that E is a closed subspace of H and that Q is a linear projector onto E. In these coordinates, the essential subbundle $\mathcal{E}$ is locally represented by the constant subbundle $\ker Q$, with projector $P = I - Q$. By (7.15), the map $(I-P)F(p+P)$ is compact in a neighborhood of 0, so its differential at 0,

$$D((I-P)F(p+P))(0) = (I-P)DF(p)P = (L_F P)(p)P,$$

is a compact operator, proving (C2).

(ii) Let F_1 and F_2 be C^1 tangent vector fields on M, and let $h_1, h_2 \in C^1(M) \cap C_b^0(M)$. Let $i \in I$. Clearly, if F_1 and F_2 satisfy (C3)-(i) with constants b_1 and b_2, $h_1F_1 + h_2F_2$ satisfy (C3)-(i) with constant $\|h_1\|_\infty b_1 + \|h_2\|_\infty b_2$.

Let $p \in M_i$, and set $q := \mathcal{Q}_i(p)$. Let $\psi : U \to E$, $q \in U \subset N_i$, be a local chart such that

$$D\psi : TU \to \psi(U) \times E \subset E \times E$$

is bi-Lipschitz of constant 2. By property (h) of the Hausdorff measure of non-compactness, if $A \subset \mathcal{Q}_i^{-1}(U)$,

$$\begin{aligned}\beta_{TN_i}(D\mathcal{Q}_i \circ (h_1F_1 + h_2F_2)(A)) &\leqslant 2\beta_{E\times E}(D\psi \circ D\mathcal{Q}_i \circ (h_1F_1 + h_2F_2)(A)) \\ &\leqslant 2\|h_1\|_\infty \beta_{E\times E}(D\psi \circ D\mathcal{Q}_i \circ F_1(A)) + 2\|h_2\|_\infty \beta_{E\times E}(D\psi \circ D\mathcal{Q}_i \circ F_2(A)) \\ &\leqslant 4\|h_1\|_\infty \beta_{TN_i}(D\mathcal{Q}_i \circ F_1(A)) + 4\|h_2\|_\infty \beta_{TN_i}(D\mathcal{Q}_i \circ F_2(A)).\end{aligned}$$

Therefore, if F_1 and F_2 satisfy (7.4) with constants δ_1, c_1 and δ_2, c_2, then $h_1F_1 + h_2F_2$ satisfies (7.4) with constants $\delta = \min\{\delta_1, \delta_2\}$ and $c = 4\|h_1\|_\infty c_1 + 4\|h_2\|_\infty c_2$. This proves that $h_1F_1 + h_2F_2$ satisfies (C2)-(ii). □

It seems useful to find sufficient conditions implying (C3)-(ii), which do not make use of the measures of non-compactness but are stated only in terms of Lipschitzianity and compactness of some maps. To this purpose, assume that M is endowed with a Riemannian metric.

The following proposition says that under mild Lipschitz assumptions on $\mathcal{Q}_i^{-1}$ and F, condition (C3)-(ii) holds if and only if the maps $D\mathcal{Q}_i \circ F : \mathcal{Q}_i^{-1}(\{q\}) \to T_qN_i$ have pre-compact image. Example 7.6 suggests that without Lipschitz assumptions on F the last condition is not sufficient for the conclusion of Proposition 7.4 to hold.

Proposition 7.9. *Assume that every map $N_i \to \mathcal{H}(M_i)$, $q \mapsto \mathcal{Q}_i^{-1}(\{q\})$, is locally Lipschitz, and that for every $q \in N_i$ there exists $\delta > 0$ such that the map $D\mathcal{Q}_i \circ F$ is Lipschitz on $\mathcal{Q}_i^{-1}(B_\delta(q))$. Then* (C3)-(ii) *holds if and only if $D\mathcal{Q}_i \circ F$ maps every fiber $\mathcal{Q}_i^{-1}(\{q\})$ into a pre-compact set.*

Remark 7.10. The assumption on the local Lipschitzianity of $\mathcal{Q}_i^{-1}$, required in the above proposition involves a uniform lower bound on the non-zero singular values of $D\mathcal{Q}_i$. More precisely, let $\mathcal{Q} : M \to N$ be a C^1 submersion between Riemannian Hilbert manifolds, with M complete and N connected. If there is $\alpha > 0$ such that

$$\inf\left(\sigma(D\mathcal{Q}(p)^* D\mathcal{Q}(p)) \setminus \{0\}\right) \geqslant \alpha \quad \forall p \in M, \tag{7.16}$$

then the map $N \to \mathcal{H}(M)$, $q \mapsto \mathcal{Q}^{-1}(\{q\})$, is $1/\sqrt{\alpha}$-Lipschitz.

In order to prove this statement, let $q_0, q_1 \in N$ and $k > 1/\sqrt{\alpha}$. Let $p_0 \in \mathcal{Q}^{-1}(\{q_0\})$ and let $v : [0,1] \to N$ be a C^1 curve such that $v(0) = q_0$, $v(1) = q_1$. Since $\mathcal{Q}$ is a submersion, for any $t_0 \in [0,1]$ and every $p \in \mathcal{Q}^{-1}(\{v(t_0)\})$, there exists a C^1 local lifting u of v verifying $u(t_0) = p$ and $u'(t_0) \in (\ker T_p\mathcal{Q})^\perp$. Assumption (7.16) easily implies that $|u'(t_0)| \leqslant (1/\sqrt{\alpha})|v'(t_0)|$, so

$$|u'(t)| \leqslant k|v'(t)| \tag{7.17}$$

for any t in a neighborhood of t_0. By a standard maximality argument, it follows that there exists a Lipschitz lifting $u : [0,1] \to M$ of v with $u(0) = p_0$ and verifying (7.17) a.e. in $[0,1]$. Therefore,

$$\mathrm{dist}\,(p_0, \mathcal{Q}^{-1}(\{q_1\})) \leqslant \int_0^1 |u'(t)|\,dt \leqslant k \int_0^1 |v'(t)|\,dt.$$

Hence taking the infimum over k and v we obtain $\mathrm{dist}\,(p_0, \mathcal{Q}^{-1}(\{q_1\})) \leqslant 1/\sqrt{\alpha}\, d(q_0, q_1)$, and by symmetry $\mathrm{dist}_{\mathcal{H}}(\mathcal{Q}^{-1}(\{q_0\}), \mathcal{Q}^{-1}(\{q_1\})) \leqslant 1/\sqrt{\alpha}\, d(q_0, q_1)$, as claimed.

Proof of Proposition 7.9. Let $i \in I$. Since $\beta_{N_i}(\{q\}) = 0$, condition (C3)-(ii) trivially implies that $D\mathcal{Q}_i \circ F$ maps every fiber $\mathcal{Q}_i^{-1}(\{q\})$ into a pre-compact set, for every $q \in N_i$.

Let us prove the converse statement. Let $q \in N_i$ and let $\delta > 0$ be so small that the maps

$$B_\delta(q) \to \mathcal{H}(\mathcal{Q}_i^{-1}(B_\delta(q)), \quad q' \mapsto \mathcal{Q}_i^{-1}(\{q'\}),$$
$$\mathcal{Q}_i^{-1}(B_\delta(q)) \to TN_i, \quad p \mapsto D\mathcal{Q}_i \circ F(p)$$

are Lipschitz. Then also the maps

$$\mathcal{H}(B_\delta(q)) \to \mathcal{H}(\mathcal{Q}_i^{-1}(B_\delta(q)), \quad \Sigma \mapsto \mathcal{Q}_i^{-1}(\Sigma),$$
$$\mathcal{H}(\mathcal{Q}_i^{-1}(B_\delta(q))) \to \mathcal{H}(TN_i), \quad A \mapsto \overline{D\mathcal{Q}_i \circ F(A)}$$

are Lipschitz. Let c be the Lipschitz constant of their composition

$$\mathcal{H}(B_\delta(q)) \xrightarrow{\mathcal{Q}_i^{-1}} \mathcal{H}(\mathcal{Q}_i^{-1}(B_\delta(q)) \xrightarrow{\overline{D\mathcal{Q}_i \circ F}} \mathcal{H}(TN_i).$$

Let $A \subset \mathcal{Q}_i^{-1}(B_\delta(q))$ be the set for which we wish to prove (7.4). We may assume that $A = \mathcal{Q}_i^{-1}(\Sigma)$ for some closed subset $\Sigma \subset B_\delta(q)$. If $\Sigma_0 \subset B_\delta(q)$ is a finite set, our assumption implies that $D\mathcal{Q}_i \circ F(\mathcal{Q}_i^{-1}(\Sigma_0))$ is pre-compact. So by (7.1),

$$\begin{aligned}\beta_{TN_i}(D\mathcal{Q}_i(F(A))) &= \beta_{TN_i}(\overline{D\mathcal{Q}_i(F(\mathcal{Q}_i^{-1}(\Sigma)))})\\ &\leqslant \mathrm{dist}_{\mathcal{H}}(\overline{D\mathcal{Q}_i \circ F} \circ \mathcal{Q}_i^{-1}(\Sigma), \overline{D\mathcal{Q}_i \circ F} \circ \mathcal{Q}_i^{-1}(\Sigma_0)) \leqslant c\, \mathrm{dist}_{\mathcal{H}}(\Sigma, \Sigma_0).\end{aligned}$$

By the density of the space of finite sets in the space of compact sets, by (7.1), and by property (d) of β, we obtain

$$\beta_{TN_i}(D\mathcal{Q}_i(F(A))) \leqslant c \inf_{\substack{\Sigma_0 \subset B_\delta(q)) \\ \Sigma_0 \text{ finite}}} \operatorname{dist}_{\mathcal{H}}(\Sigma, \Sigma_0) = c \inf_{\substack{\Sigma_0 \subset B_\delta(q) \\ \Sigma_0 \text{ compact}}} \operatorname{dist}_{\mathcal{H}}(\Sigma, \Sigma_0)$$
$$= c\,\beta_{B_\delta(q)}(\Sigma) \leqslant 2c\,\beta_{N_i}(\Sigma) = 2c\,\beta_{N_i}(\mathcal{Q}_i(A)),$$

which proves (C3)-(ii). $\square$

Example 7.11 (*Product manifolds*). Let us consider again the situation of Example 2.4: $M = M^- \times M^+$ is given a product complete Riemannian structure, and $\mathcal{V} = TM^- \times (0)$. Consider the projection onto the second factor $\mathcal{Q} : M \to M^+$, $(p^-, p^+) \mapsto p^+$, and the admissible presentation of the subbundle $\mathcal{V}$,

$$\left\{ \mathcal{Q}|_{B^- \times B^+} \mid B^- \times B^+ \subset M^- \times M^+ \ \textit{is bounded} \right\}.$$

Writing the tangent vector field F as $F(p^-, p^+) = (F^-(p^-, p^+), F^+(p^-, p^+)) \in T_{p^-}M^- \times T_{p^+}M^+$, there holds $D\mathcal{Q} \circ F(p^-, p^+) = F^+(p^-, p^+)$. Assume that (i) F^+ is bounded, and (ii) for every $p^+ \in M^+$ and for every bounded set $B^- \subset M^-$ there exists $\delta > 0$ such that F^+ is Lipschitz on $B^- \times B_\delta(p^+)$, and that the map $M^- \to T_{p^+}M^+$, $p^- \mapsto F^+(p^-, p^+)$ is compact. Then Proposition 7.9 implies that F satisfies (C3).

8. Broken flow lines

Let x and y be rest points of the gradient-like Morse vector field F. Let us assume that $W^u(x) \cap W^s(y)$ has compact closure. Consider a sequence of flow lines from the rest point x to the rest point y, and the sequence of their closures:

$$S_n = \overline{\phi(\mathbb{R} \times \{p_n\})} = \phi(\mathbb{R} \times \{p_n\}) \cup \{x, y\}, \quad p_n \in W^u(x) \cap W^s(y).$$

Since $\overline{W^u(x) \cap W^s(y)}$ is compact, up to a subsequence we may assume that $p_n \to p$, and the continuity of ϕ would give us the convergence

$$\phi(\cdot, p_n) \to \phi(\cdot, p)$$

uniformly on compact subsets of $\mathbb{R}$. However, it may happen that $p \notin W^u(x)$, or $p \notin W^s(y)$, so $\phi(\cdot, p)$ could be a flow line connecting two other rest points, and the convergence would not be uniform on $\mathbb{R}$. We will show that in this case a subsequence of (S_n) converges to a broken flow line in the Hausdorff distance. The discussion is independent on conditions (C1-3), and involves only the compactness of $\overline{W^u(x) \cap W^s(y)}$.

Definition 8.1. Let $x, y \in \operatorname{rest}(F)$. A *broken flow line* from x to a y is a set

$$S = S_1 \cup \cdots \cup S_k,$$

where $k \geqslant 1$, S_i is the closure of a flow line from z_{i-1} to z_i, where $x = z_0 \neq z_1 \neq \cdots \neq z_{k-1} \neq z_k = y$ are rest points.

When $k = 1$, a broken flow line is just the closure of a genuine flow line. Let us fix a Lyapunov function f for F. If S is a broken flow line as in the above definition, the following inequalities must hold:

$$f(x) > f(z_1) > \cdots > f(z_{k-1}) > f(y). \tag{8.1}$$

It is easy to check that a compact set $S \subset M$ is a broken flow line from x to y if and only if (i) $x, y \in S$, (ii) S is ϕ-invariant, (iii) the intersection

$$S \cap \{p \in M \mid f(p) = c\}$$

consists of a single point if $c \in [f(y), f(x)]$, and it is empty otherwise. Now we can state the compactness result for the gradient flow lines.

Proposition 8.2. *Assume that the Morse vector field F has a Lyapunov function f, and that x, y are rest points such that $\overline{W^u(x) \cap W^s(y)}$ is compact. Let $(p_n) \subset W^u(x) \cap W^s(y)$, and set $S_n := \phi(\mathbb{R} \times \{p_n\}) \cup \{x, y\}$. Then (S_n) has a subsequence which converges to a broken flow line from x to y, in the Hausdorff distance.*

Proof. The space of compact subsets of a compact metric space is compact with respect to the Hausdorff distance, so (S_n) has a subsequence (S'_n) which converges to a compact set $S \subset \overline{W^u(x) \cap W^s(y)}$. Then $x, y \in S$, and since S'_n is ϕ-invariant, so is S. Since $S'_n \subset f^{-1}([f(y), f(x)])$, we obtain that the set

$$S \cap \{p \in M \mid f(p) = c\} \tag{8.2}$$

is empty for every $c \notin [f(y), f(x)]$.

Let $c \in [f(y), f(x)]$. Then (S'_n) has a subsequence (S''_n) such that

$$S''_n \cap \{p \in M \mid f(p) = c\}$$

converges to some point in (8.2), which is then non-empty. If the set (8.2) contains two points p, q, the fact that (S_n) is a sequence of flow lines allows us to find a sequence (p_n) converging to p, and numbers $t_n \in \mathbb{R}$ such that $\phi(t_n, p_n) \to q$. By reversing if necessary the role of p and q, we may assume that $t_n \geqslant 0$ for every n, and we deduce the convergence:

$$\int_0^{t_n} Df(\phi(t, p_n))[F(\phi(t, p_n))]\, dt = f(\phi(t_n, p_n)) - f(p_n) \to f(q) - f(p) = 0.$$

Then the fact that the rest points of f are isolated easily implies that either $t_n \to 0$, or the sequence of sets $\phi([0, t_n] \times \{p_n\})$ converges to a rest point. In both cases, we obtain that $p = q$. This shows that set (8.2) consists of a single point. Hence S is a broken flow line from x to y. $\square$

9. Intersections of dimension 1 and 2

Assume that the gradient-like Morse vector field F satisfies (C1-2) with respect to a (0)-essential subbundle $\mathcal{E}$ of TM, so that the relative index $m(x, \mathcal{E})$ is a well-defined integer, for any $x \in \text{rest}\,(F)$.

We say that F satisfies the *Morse–Smale property up to order k*, if $W^u(x)$ and $W^s(y)$ have transverse intersection for every pair of rest points x, y such that $m(x, \mathcal{E}) - m(y, \mathcal{E}) \leqslant k$. The Morse–Smale condition up to order 0 implies that, for a broken flow line as in Definition 8.1,

$$m(x, \mathcal{E}) > m(z_1, \mathcal{E}) > \cdots > m(z_{k-1}, \mathcal{E}) > m(y, \mathcal{E}). \tag{9.1}$$

In this section, we shall assume that F has the Morse–Smale property up to order 2, and we shall describe the intersections $W^u(x) \cap W^s(y)$ when $m(x, \mathcal{E}) - m(y, \mathcal{E})$ is either 1 or 2. As in the last section, we shall assume that such an intersection has compact closure. By Theorem 3.3, $W^u(x) \cap W^s(y)$ is a submanifold of dimension 1, respectively 2. The flow ϕ defines a free action of the group $\mathbb{R}$ onto $W^u(x) \cap W^s(y)$, so the quotient, that is the set of the flow lines from x to y, is a manifold of dimension 0, respectively 1.

Assume that x, y are rest points with

$$m(x, \mathcal{E}) - m(y, \mathcal{E}) = 1. \tag{9.2}$$

We claim that $W^u(x) \cap W^s(y)$ consists of *finitely many* connected components. Indeed each connected component is a flow line from x to y, and the set C of their closures is discrete in the Hausdorff distance. On the other hand, (9.1) and (9.2) imply that these are the only broken flow lines from x to y. So by Proposition 8.2, C is also compact, hence finite.

Note that the restriction of the flow ϕ to the closure of a component of $W^u(x) \cap W^s(y)$ is conjugated to the shift flow on $\overline{\mathbb{R}} = [-\infty, +\infty]$:

$$\mathbb{R} \times \overline{\mathbb{R}} \ni (t, u) \mapsto u + t \in \overline{\mathbb{R}}.$$

Now assume that x, z are rest points with

$$m(x, \mathcal{E}) - m(z, \mathcal{E}) = 2. \tag{9.3}$$

The quotient of each connected component W by this action, $W/\mathbb{R}$, being a connected one-dimensional manifold, is either the circle or the open interval. In other words a connected component W of $W^u(x) \cap W^s(z)$ is described by a one-parameter family of flow lines u_λ, where λ ranges in S^1 or in $]0, 1[$.

In the first case one can easily verify that $\overline{W} = W \cup \{x, z\}$ is homeomorphic to a 2-sphere, and that the restriction of ϕ to $\overline{W}$ is conjugated to the exponential flow on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$:

$$\mathbb{R} \times S^2 \ni (t, \zeta) \mapsto e^t \zeta \in S^2.$$

In the second case, by Proposition 8.2, $\overline{W} \setminus W$ contains broken flow lines, which have just one intermediate rest point, by (9.1) and (9.3). Then the flow ϕ restricted to $\overline{W}$ is semi-conjugated to the product of two shift-flows on $\overline{\mathbb{R}}$. More precisely, the situation is described by the following theorem.

Theorem 9.1. *Assume that the gradient-like Morse vector field F satisfies* (C1-2) *with respect to a* (0)*-essential subbundle $\mathcal{E}$ of TM. Assume that F has the Morse–Smale property up to order* 2. *Let x, z be rest points such that $m(x, \mathcal{E}) - m(z, \mathcal{E}) = 2$, and let W be a connected component of $W^u(x) \cap W^s(z)$ such that $\overline{W}$ is compact, and $W/\mathbb{R}$ is an open interval. Then there exists a continuous surjective map*

$$h : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{W}$$

with the following properties:

(i) $\phi_t(h(u, v)) = h(u + t, v + t)$, *for every* $(u, v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, $t \in \mathbb{R}$;

(ii) $h(\mathbb{R}^2) = W$, *and there exist rest points* y, y' *with* $m(y, \mathcal{E}) = m(y', \mathcal{E}) = m(x, \mathcal{E}) - 1$, *and* W_1, W_2, W_1', W_2' *connected components of* $W^u(x) \cap W^s(y)$, $W^u(y) \cap W^s(z)$, $W^u(x) \cap W^s(y')$, $W^u(y') \cap W^s(z)$, *respectively, such that* $W_1 \cup W_2 \neq W_1' \cup W_2'$, *and*

$$h(\mathbb{R} \times \{-\infty\}) = W_1, \quad h(\{+\infty\} \times \mathbb{R}) = W_2,$$
$$h(\{-\infty\} \times \mathbb{R}) = W_1', \quad h(\mathbb{R} \times \{+\infty\}) = W_2'.$$

(iii) *the restrictions of h to $\mathbb{R}^2$, to $\{\pm\infty\} \times \mathbb{R}$, and to $\mathbb{R} \times \{\pm\infty\}$, are diffeomorphisms of class C^1*;

(iv) *if moreover the* (0)*-essential subbundle $\mathcal{E}$ can be lifted to a subbundle $\mathcal{V}$, then*

$$\deg h = -\deg h|_{\{-\infty\}\times\mathbb{R}} \cdot \deg h|_{\mathbb{R}\times\{+\infty\}} = \deg h|_{\mathbb{R}\times\{-\infty\}} \cdot \deg h|_{\{+\infty\}\times\mathbb{R}},$$

where deg *denotes the* $\mathbb{Z}$*-topological degree, referred to the orientations defined in Section* 5.

Concerning (ii), note that it may happen that $y = y'$, and in this case even that $W_1 = W_1'$ or $W_2 = W_2'$, but the last two identities cannot hold simultaneously. When $y \neq y'$, h is injective, so it is a conjugacy. Statement (iv) expresses the coherence we need between the orientations of the one- and two-dimensional intersections of unstable and stable manifolds. The picture is completed by the following proposition.

Proposition 9.2. *Assume that the gradient-like Morse vector field F satisfies* (C1-2) *with respect to a* (0)*-essential subbundle $\mathcal{E}$ of TM. Assume that F has the Morse–Smale property up to order* 2*. Let x, y, z be rest points such that $m(x,\mathcal{E}) = m(y,\mathcal{E}) + 1 = m(z,\mathcal{E}) + 2$, and let W_1, W_2 be connected components of $W^u(x) \cap W^s(y)$, $W^u(y) \cap W^s(z)$, respectively. Then there exists a unique connected component W of $W^u(x) \cap W^s(z)$ such that $\overline{W_1 \cup W_2}$ belongs to the closure of $\left\{\overline{\phi(\mathbb{R} \times \{p\})} \mid p \in W\right\}$ with respect to the Hausdorff distance.*

Both Theorem 9.1 and Proposition 9.2 will be proved in Section 11. The main tool in the proof is the graph transform method, which allows us to study suitable portions of $W^u(x)$ and $W^s(z)$ in a neighborhood of another rest point $y \in \overline{W^u(x) \cap W^s(z)}$.

10. The boundary homomorphism

Let $(M, \mathcal{E})$ be a pair consisting of a complete Riemannian Hilbert manifold M of class C^2, and of a C^1 (0)-essential subbundle of TM having an admissible presentation. Let F be a C^1 Morse vector field on M, admitting a non-degenerate Lyapunov function f. We shall assume (PS), (C1-3), the Morse–Smale property up to order 2, and

(C4) for every $q \in \mathbb{Z}$, f is bounded below on $\mathrm{rest}_q(F) = \{x \in \mathrm{rest}\,(F) \mid m(x,\mathcal{E}) = q\}$.

10.1. Morse complex with coefficients in $\mathbb{Z}$

We first consider the situation in which $\mathcal{E}$ is the (0)-essential class of a subbundle $\mathcal{V}$ of TM. In this case we can fix arbitrary orientations of the Fredholm pairs $(H^s_x, \mathcal{V}(x))$, for every $x \in \mathrm{rest}\,(F)$.

Let x and y be rest points with $m(x,\mathcal{E}) - m(y,\mathcal{E}) = 1$, and let W be a connected component of $W^u(x) \cap W^s(y)$. Then W is a flow line, and it is endowed with the orientation described in Section 5. We can define the number

$$\sigma(W) := \deg[\phi(\cdot, p) : \mathbb{R} \to W]$$

for $p \in W$. In other words $\sigma(W)$ equals $+1$ or -1 depending on whether $F(p) \in T_p W$ is positively or negatively oriented. We define also

$$\sigma(x, y) := \sum_W \sigma(W),$$

where the sum ranges over all the connected components of $W^u(x) \cap W^s(y)$.

Now let x and z be rest points with $m(x,\mathcal{E})-m(z,\mathcal{E})=2$, and let $\mathcal{S}(x,z)$ be the set of broken flow lines from x to z with one intermediate rest point (necessarily unique and of index $m(z,\mathcal{E})+1$). By (PS) and by the Morse assumption there are finitely many rest points y with $f(y) \in]f(z), f(x)[$. By Theorem 6.5 and Proposition 8.2, the set $\mathcal{S}(x,z)$ is finite. By Theorem 9.1 and Proposition 9.2, there is an involution $\overline{W_1 \cup W_2} \mapsto \overline{W_1' \cup W_2'}$ without fixed points on the set $\mathcal{S}(x,z)$, and

$$\sigma(W_1')\sigma(W_2') = -\sigma(W_1)\sigma(W_2). \tag{10.1}$$

Let $q \in \mathbb{Z}$ and let $C_q(F)$ be the free Abelian group generated by the rest points of index q:

$$C_q(F) = \mathrm{span}_{\mathbb{Z}} \mathrm{rest}_q(F).$$

Note that $C_q(F)$ may have infinite rank.

Assumption (C4) allows us to define the homomorphism

$$\partial_q : \ C_q(F) \to C_{q-1}(F)$$

by setting for every $x \in \mathrm{rest}_q(F)$

$$\partial_q x = \sum_{y \in \mathrm{rest}_{q-1}(F)} \sigma(x,y) y. \tag{10.2}$$

The Abelian groups $C_q(F)$ together with the homomorphisms ∂_q are the data of a chain complex. Indeed we have:

Theorem 10.1. *For every $q \in \mathbb{Z}$, $\partial_{q-1} \circ \partial_q = 0$.*

Proof. Let $x \in \mathrm{rest}_q(F)$ and $z \in \mathrm{rest}_{q-2}(F)$. The coefficient of z in $\partial_{q-1}\partial_q x$ is

$$\sum_{y \in \mathrm{rest}_{q-1}(F)} \sigma(x,y)\sigma(y,z) = \sum_{\overline{W_1 \cup W_2} \in \mathcal{S}(x,z)} \sigma(W_1)\sigma(W_2),$$

which is zero by (10.1). □

We will call $\{C_q(F), \partial_q\}_{q \in \mathbb{Z}}$ the Morse complex of F. Clearly, the construction depends on the choice of the subbundle $\mathcal{V}$ and on the orientation of each Fredholm pair $(H_x^s, \mathcal{V}(x))$. Changing the subbundle $\mathcal{V}$ by a compact perturbation changes the Morse complex by a shift of the indices (when M is connected). A change of the orientation of $(H_x^s, \mathcal{V}(x))$ produces an isomorphic Morse complex.

10.2. Morse complex with coefficients in $\mathbb{Z}_2$

In the general case of a (0)-essential subbundle $\mathcal{E}$, statement (iv) of Theorem 9.1 is not available, but we can still define a Morse complex with $\mathbb{Z}_2$ coefficients. Indeed, defining $\sigma(x, y) \in \mathbb{Z}_2$ to be the number of connected components of $W^u(x) \cap W^s(y)$ counted modulo 2, and $C_q(F)$ to be the $\mathbb{Z}_2$-vector space generated by the rest points of index q, (10.2) defines a complex of $\mathbb{Z}_2$-vector spaces.

11. Proof of the conjugacy theorem

11.1. Construction of h near a broken flow line

The main point in the proof of Theorem 9.1 and Proposition 9.2 is to construct h near a broken flow line.

Proposition 11.1. *Assume that the Morse vector field F has a non-degenerate Lyapunov function f, satisfies* (C1-2) *with respect to a* (0)*-essential subbundle $\mathcal{E}$ of TM, and satisfies the Morse–Smale condition up to order* 2*. Let x, y, z be rest points such that $m(x,\mathcal{E}) = m(y,\mathcal{E}) + 1 = m(z,\mathcal{E}) + 2$. Let W_1 and W_2 be connected components of $W^u(x) \cap W^s(y)$ and $W^u(y) \cap W^s(z)$, respectively. Then there exists a continuous injective map*

$$h : \Delta := \left\{(u, v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid v \leqslant u\right\} \to \overline{W^u(x) \cap W^s(z)}$$

with the following properties:

(i) $\phi_t(h(u, v)) = h(u + t, v + t)$, *for every* $(u, v) \in \Delta$, $t \in \mathbb{R}$;
(ii) $h(\Delta \cap \mathbb{R}^2) \subset W^u(x) \cap W^s(z)$, $h(\mathbb{R} \times \{-\infty\}) = W_1$, $h(\{+\infty\} \times \mathbb{R}) = W_2$, *and the restrictions of h to $\Delta \cap \mathbb{R}^2$, to $\mathbb{R} \times \{-\infty\}$, and to $\{+\infty\} \times \mathbb{R}$, are diffeomorphisms of class C^1*;
(iii) *there exists $\delta > 0$ such that for any $p \in W^u(x) \cap W^s(z)$, if $S = \overline{\phi(\mathbb{R} \times \{p\})}$ has Hausdorff distance less than δ from $\overline{W_1 \cup W_2}$, then $S \subset h(\Delta)$*;
(iv) *if moreover the* (0)*-essential subbundle $\mathcal{E}$ can be lifted to a subbundle $\mathcal{V}$, then*

$$\deg h = -\deg h|_{\mathbb{R}\times\{-\infty\}} \cdot \deg h|_{\{+\infty\}\times\mathbb{R}},$$

where deg *denotes the $\mathbb{Z}$-topological degree, referred to the orientations defined in Section 5.*

Let us identify a neighborhood of y in M with a neighborhood of 0 in the Hilbert space H, identifying y with 0. We endow H with an equivalent Hilbert product $\langle\cdot,\cdot\rangle$ which is adapted to $\nabla F(y) = DF(0)$ (see Appendix C), and we set $H^u := H^u_y$, $H^s := H^s_y$, so that $H^u \oplus H^s$ is the splitting of H given by the decomposition of the spectrum of $\nabla F(y)$ into the subset with positive real part and the one with negative real part. Let P^u and P^s denote the corresponding projectors. We shall often identify $H = H^u \oplus H^s$ with $H^u \times H^s$. By $H^u(r)$, resp. $H^s(r)$, we shall denote the closed r-ball centered in 0 of the linear subspace H^u, resp. H^s. We set $Q(r) := H^u(r) \times H^s(r)$. If X and Y are metric spaces and $\theta > 0$, $\mathrm{Lip}_\theta(X, Y)$ will denote the space of θ-Lipschitz maps from X into Y, endowed with the C^0 topology.

Let $p_1 \in W_1$ and $p_2 \in W_2$. Choose $\rho_0 > 0$ so small that the sets

$$X := W^u(x) \cap f^{-1}(f(p_1)) \cap B_{\rho_0}(p_1), \quad Z := W^s(z) \cap f^{-1}(f(p_2)) \cap B_{\rho_0}(p_2),$$

do not contain rest points, and

$$X \cap W^u(x) \cap W^s(y) = \{p_1\}, \quad Z \cap W^u(y) \cap W^s(z) = \{p_2\}.$$

Then X and Z are submanifolds of class C^1, and the Morse–Smale condition implies that X is transverse to $W^s(y)$, and Z is transverse to $W^u(y)$.

Lemma 11.2. *For any $\theta > 0$ there exist $r_0 > 0$, $\rho > 0$, $t_0 \geqslant 0$, and two continuous families*

$$\{\sigma_t\}_{t\in[0,+\infty]} \subset \mathrm{Lip}_\theta(H^u(r_0), H^s(r_0)), \quad \{\tau_t\}_{t\in[-\infty,0]} \subset \mathrm{Lip}_\theta(H^s(r_0), H^u(r_0)),$$

such that each σ_t and each τ_t is C^1, and:

(i) $\phi_{t_0+t}(X \cap B_\rho(p_1)) \cap Q(r_0) = \operatorname{graph} \sigma_t$, *for every* $t \in [0, +\infty[$;
(ii) $W^u(y) \cap Q(r_0) = \operatorname{graph} \sigma_{+\infty}$;
(iii) $\phi_{-t_0+t}(Z \cap B_\rho(p_2)) \cap Q(r_0) = \operatorname{graph} \tau_t$, *for every* $t \in]-\infty, 0]$;
(iv) $W^s(y) \cap Q(r_0) = \operatorname{graph} \tau_{-\infty}$;
(v) *for any* $\theta_1 > 0$ *there exist* $r_1 \in]0, r_0]$ *and* $t_1 \geqslant 0$ *such that*

$$\sigma_t|_{H^u(r_1)} \in \mathrm{Lip}_{\theta_1}(H^u(r_1), H^s(r_1)), \quad \tau_{-t}|_{H^s(r_1)} \in \mathrm{Lip}_{\theta_1}(H^s(r_1), H^s u(r_1))$$

for any $t \geqslant t_1$.

Proof. Let r be as small as required by Propositions C.5 and C.6. Since $T_{p_1}X \oplus T_{p_1}W^s(y) = T_{p_1}M$, the path of subspaces $D\phi_t(p_1)T_{p_1}X$ converges to $T_yW^u(y)$ for $t \to +\infty$, by Theorem B.2(iii). Therefore, we can find $s_1 \geqslant 0$ such that $\phi(s_1, p_1)$ is in the interior of $Q(r)$, and $D\phi_{s_1}(p_1)T_{p_1}X \subset H^u \times H^s$ is the graph of a linear operator from H^u to H^s of norm strictly less than 1. By the implicit function theorem, there exists $\rho > 0$ such that $\phi_{s_1}(X \cap B_\rho(p_1))$ is the graph of a 1-Lipschitz map $\sigma : U \to H^s(r)$, where $U \subset H^u(r)$ is open. Moreover, $\operatorname{graph} \sigma \cap W^s(y) = \{\phi(s_1, p_1)\}$, so by Proposition C.5(v), there exist $s_2 \geqslant 0$ and $\sigma' \in \mathrm{Lip}_1(H^u(r), H^s(r))$ such that

$$\operatorname{graph} \sigma' = \phi_{s_1+s_2}(X \cap B_\rho(p_1)) \cap Q(r),$$

where we have also used Proposition C.6. Let

$$\Gamma : [0, +\infty] \times \mathrm{Lip}_1(H^u(r), H^s(r)) \to \mathrm{Lip}_1(H^u(r), H^s(r))$$

be the graph transform map provided by Proposition C.5. By Proposition C.5(iv), there exist $r_0 \in]0, r]$ and $s_3 \geqslant 0$ such that $\Gamma(t, \sigma') \in \mathrm{Lip}_\theta(H^u(r_0), H^s(r_0))$ for any $t \geqslant s_3$. Setting $t_0 := s_1 + s_2 + s_3$ and $\sigma_t = \Gamma(t - s_3, \sigma')$ for $t \in [0, +\infty]$, statements (i), (ii), and the first part of (v) follow immediately from Propositions C.5 and C.6.

Changing the sign of t and considering the evolution of Z, we obtain a family of maps $\{\tau_t\}$ satisfying (iii), (iv), and the second part of (v). $\square$

Proof of Proposition 11.1. Let θ be a positive number strictly less than 1, and let $r_0, \rho, t_0, \sigma_t, \tau_t$ be as in the lemma above. Since $\theta < 1$, the contracting mapping principle implies the existence of a Lipschitz continuous map

$$\Lambda : \mathrm{Lip}_\theta(H^u(r_0), H^s(r_0)) \times \mathrm{Lip}_\theta(H^s(r_0), H^u(r_0)) \to Q(r_0),$$

which associates to (σ, τ) the unique intersection of the graphs of σ and τ, i.e. the unique fixed point of the contraction

$$H^u(r_0) \times H^s(r_0) \ni (\xi, \eta) \mapsto (\tau(\eta), \sigma(\xi)) \in H^u(r_0) \times H^s(r_0).$$

For $(u, v) \in [0, +\infty] \times [-\infty, 0]$, set

$$X_u = \begin{cases} \phi_{u+t_0}(X \cap B_\rho(p_1)) \cap Q(r_0) & \text{if } u \in [0, +\infty[, \\ W^u(y) \cap Q(r_0) & \text{if } u = +\infty, \end{cases},$$

$$Z_v = \begin{cases} \phi_{v-t_0}(Z \cap B_\rho(p_2)) \cap Q(r_0) & \text{if } v \in]-\infty, 0], \\ W^s(y) \cap Q(r_0) & \text{if } v = -\infty. \end{cases}$$

Then we define $h(u, v)$ to be the unique point of the intersection $X_u \cap Z_v$. The map h is well defined and continuous on $[0, +\infty] \times [-\infty, 0]$ because it can be written as the composition

$$[0, +\infty] \times [-\infty, 0] \ni (u, v) \mapsto (\sigma_u, \tau_v) \xrightarrow{\Lambda} Q(r_0) \hookrightarrow M.$$

Since X and Z are contained in level sets of f, and they contain no rest points, $X_u \cap X_{u'} = \emptyset$ if $u \neq u'$, and $Z_v \cap Z_{v'} = \emptyset$ if $v \neq v'$. So h is injective. By definition, for every $(u, v) \in [0, +\infty] \times [-\infty, 0]$, and $-u \leqslant t \leqslant -v$,

$$\phi_t(h(u, v)) = h(u + t, v + t) \tag{11.1}$$

and we can use formula (11.1) to extend h to a continuous injective map on

$$\Delta = \left\{(u, v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid v \leqslant u\right\},$$

still verifying (11.1), proving (i). Since $X \subset W^u(x)$ and $Z \subset W^s(z)$, for every $(u, v) \in \Delta \cap \mathbb{R}^2$ the point $h(u, v)$ belongs to $W^u(x) \cap W^s(z)$. Since

$$h(u, -\infty) = \phi(u, p_1) \quad \text{and} \quad h(+\infty, v) = \phi(v, p_2),$$

$h|_{\mathbb{R}\times\{-\infty\}}$ and $h|_{\{+\infty\}\times\mathbb{R}}$ are diffeomorphisms of class C^1 onto W_1 and W_2. Since X and Z are of class C^1 and so is ϕ, the implicit function theorem implies that $h|_{\Delta\cap\mathbb{R}^2}$ is a diffeomorphism of class C^1, proving (ii). Notice that, differentiating the identities

$$\begin{aligned}&\phi_t(h(u, v)) = h(u + t, v + t), \quad \phi_t(h(u, -\infty)) = h(u + t, -\infty),\\ &\phi_t(h(+\infty, v)) = h(+\infty, v + t),\end{aligned}$$

with respect to t in $t = 0$, we obtain

$$\begin{aligned}&F(h(u, v)) = \frac{\partial h}{\partial u}(u, v) + \frac{\partial h}{\partial v}(u, v), \quad F(u, -\infty) = \frac{\partial h}{\partial u}(u, -\infty),\\ &F(+\infty, v) = \frac{\partial h}{\partial v}(+\infty, v)\end{aligned} \tag{11.2}$$

for every $u \in \mathbb{R}$, $v \in \mathbb{R}$.

Since y is a rest point, and $p_1 \in W^s(y)$, $p_2 \in W^u(y)$, we can find $\delta \in]0, \rho]$ so small that, if $\phi(t_1, p) \in B_\delta(p_1)$ and $\phi(t_2, p) \in B_\delta(p_2)$, we have

$$t_2 - t_1 \geqslant 2t_0, \quad \phi(t_1 + t_0, p) \in Q(r_0), \quad \phi(t_2 - t_0, p) \in Q(r_0). \tag{11.3}$$

If $p \in W^u(x) \cap W^s(z)$ and $S := \overline{\phi(\mathbb{R} \times \{p\})}$ has Hausdorff distance less than δ from $\overline{W_1 \cup W_2}$, we can find $t_1, t_2 \in \mathbb{R}$ such that $\phi(t_1, p) \in B_\delta(p_1)$ and $\phi(t_2, p) \in B_\delta(p_2)$. By Proposition C.6, the set of $t \in \mathbb{R}$ such that $\phi(t, p) \in Q(r_0)$ is connected, so by (11.3),

$$\phi\left(\frac{t_1 + t_2}{2}, p\right) = \phi\left(\frac{(t_1 + t_0) + (t_2 - t_0)}{2}, p\right) \in Q(r_0). \tag{11.4}$$

Then, setting $u := (t_2 - t_1)/2 - t_0 \geqslant 0$ and $v := -u \leqslant 0$, we obtain

$$\phi\left(\frac{t_1 + t_2}{2}, p\right) \in \phi_{u+t_0}(X \cap B_\rho(p_1)) \cap \phi_{v-t_0}(Z \cap B_\rho(p_2)). \tag{11.5}$$

So, by (11.4) and (11.5), $\phi((t_1+t_2)/2, p) \in X_u \cap Z_v$, that is $\phi((t_1+t_2)/2, p) = h(u, v)$. By (i) the whole flow line through p is in $h(\Delta)$ which, being closed, must contain also S, proving conclusion (iii).

In order to prove (iv), we shall need the following.

Lemma 11.3. *There exist $u_0 \geqslant 0$, $v_0 \leqslant 0$, and a continuous map*

$$\mathcal{W} : [u_0, +\infty] \times [-\infty, v_0] \to \mathrm{Gr}(H),$$

such that $(\mathcal{W}(u, v), \mathcal{V}(h(u, v)))$ is a Fredholm pair for every $(u, v) \in [u_0, +\infty] \times [-\infty, v_0]$, and:

(i) $T_{h(u,v_0)} W^s(z) = \mathbb{R}F(h(u, v_0)) \oplus \mathcal{W}(u, v_0)$, *for every $u \in [u_0, +\infty]$;*
(ii) $\mathcal{W}(u, -\infty) = T_{h(u,-\infty)} W^s(y)$, *for every $u \in [u_0, +\infty]$;*
(iii) $D\phi_{v-v_0}(h(+\infty, v_0))\mathcal{W}(+\infty, v_0) = \mathcal{W}(+\infty, v)$, *for every $v \in]-\infty, v_0]$;*
(iv) $\mathcal{W}(u_0, v) + T_{h(u_0,v)} W^u(x) = H$ *for every $v \in [-\infty, v_0]$, and there exists a non-vanishing continuous vector field $G : [-\infty, v_0] \to H$ along $v \mapsto h(u_0, v)$ such that*

$$\mathcal{W}(u_0, v) \cap T_{h(u_0,v)} W^u(x) = \mathbb{R}G(v), \quad G(-\infty) = \frac{\partial h}{\partial u}(u_0, -\infty),$$

$$G(v_0) = \frac{\partial h}{\partial u}(u_0, v_0),$$

for every $v \in [-\infty, v_0]$.

Proof. Recalling that by (C1) $(H^s, \mathcal{V}(y))$ is a Fredholm pair, we can find $\theta_1 \in]0, \theta]$ so small that

$$\forall S \in \mathcal{L}(H^s, H^u) \text{ such that } \|S\| \leqslant 2\theta_1, \quad (\mathrm{graph}\, S, \mathcal{V}(y)) \text{ is a Fredholm pair.} \tag{11.6}$$

Moreover, we may assume that setting $L := \nabla F(y) = DF(0)$,

$$\frac{4\theta_1^2}{1-\theta_1^2} + 2\theta_1 < \frac{1}{\|L\|^2 \|L^{-1}\|^2}. \tag{11.7}$$

Let r_1 and t_1 be the positive numbers given by Lemma 11.2(v): there holds

$$\|D\sigma_t(\xi)\| \leqslant \theta_1, \quad \|D\tau_t(\eta)\| \leqslant \theta_1 \tag{11.8}$$

for any $t \geqslant t_1$, $\xi \in H^u(r_1)$, $\eta \in H^s(r_1)$. Since

$$F(\xi) = L\xi + o(\xi) \quad \text{for } \xi \to 0, \tag{11.9}$$

the quadratic form

$$g(\xi) = -\tfrac{1}{2}\langle L\xi, \xi\rangle$$

is a Lyapunov function for F in a neighborhood of 0. Therefore there exists $u_1 \geqslant 0$ such that the function $u \mapsto g(h(u, -\infty))$ is strictly decreasing in $[u_1, +\infty[$. So when $v \to -\infty$ the functions

$$[u_1, +\infty[\ni u \mapsto g(h(u, v)) \in \mathbb{R}$$

converge uniformly to a strictly decreasing function. Therefore, using also (11.6), the fact that $W^s(y)$ is tangent to H^s at $y = 0$, (11.7), (11.8), and (11.9), it is easy to check that there are $u_0 \geqslant 0$ and $v_0 \leqslant 0$ such that:

(a) $Dg(h(h_0, v_0))\left[\frac{\partial h}{\partial u}(u_0, v_0)\right] = \frac{\partial}{\partial u} g(h(u, v_0))|_{u=u_0} < 0$;
(b) g is a Lyapunov function for F on $h([u_0, +\infty] \times [-\infty, v_0])$;
(c) if $S \in \mathcal{L}(H^s, H^u)$ has norm $\|S\| \leqslant \theta_1$, and $(u, v) \in [u_0, +\infty] \times [-\infty, v_0]$, then $(\operatorname{graph} S, \mathcal{V}(h(u, v)))$ is a Fredholm pair;
(d) for any $v \in [-\infty, v_0]$ there holds $\|P^u h(u_0, v)\| \leqslant \varepsilon \|P^s h(u_0, v)\|$, and

$$\|F(h(u_0, v))\| \leqslant 2\|L\|\,\|P^s h(u_0, v)\|,$$
$$\|F(h(u_0, v)) - Lh(u_0, v)\| \leqslant \varepsilon \|P^s h(u_0, v)\|,$$

where $\varepsilon > 0$ is so small that

$$\frac{4\theta_1}{1-\theta_1^2}(\theta_1 + \varepsilon) + 2\theta_1 + \varepsilon < \frac{1}{\|L\|^2\|L^{-1}\|^2}; \tag{11.10}$$

(e) for any $(u, v) \in [u_0, +\infty] \times [-\infty, v_0]$ we have

$$\|D\sigma_u(P^u h(u, v))\| \leqslant \theta_1, \quad \|D\tau_v(P^s h(u, v))\| \leqslant \theta_1.$$

We will define $\mathcal{W}(u, v)$ to be the graph of suitable linear maps $S(u, v) \in \mathcal{L}(H^s, H^u)$. We start by defining S on three edges of the square $[u_0, +\infty] \times [-\infty, v_0]$. For $u \in [u_0, +\infty]$ we set

$$S(u, v_0) = D\tau_{v_0}(P^s h(u, v_0)), \quad S(u, -\infty) = D\tau_{-\infty}(P^s h(u, -\infty))$$

and for $v \in]-\infty, v_0]$ we set

$$S(+\infty, v) = D\tau_v(P^s h(+\infty, v)).$$

By (e), $\|S\| \leqslant \theta_1$. The map S is clearly continuous on $[u_0, +\infty] \times \{v_0\}$ and on $[u_0, +\infty] \times \{-\infty\}$. If $v \in]-\infty, v_0]$,

$$\operatorname{graph} S(+\infty, v) = D\phi_{v-v_0}(h(+\infty, v_0))[\operatorname{graph} S(+\infty, v_0)],$$

so S is continuous on $\{+\infty\} \times]-\infty, v_0]$. By Theorem B.2(iii), $S(+\infty, v)$ converges to $S(+\infty, -\infty)$ for $v \to -\infty$, so

$$S : ([u_0, +\infty] \times \{-\infty, v_0\}) \cup (\{+\infty\} \times [-\infty, v_0]) \to \mathcal{L}(H^s, H^u)$$

is a continuous map. We can extend the map S by convexity to a continuous map on $[u_0, +\infty] \times [-\infty, v_0]$ in such a way that $\|S(u, v)\| \leqslant \theta_1$ everywhere, and we set

$$\mathcal{W}(u, v) = \operatorname{graph} S(u, v).$$

By (c), $(\mathcal{W}(u, v), \mathcal{V}(h(u, v)))$ is always a Fredholm pair, and by construction $\mathcal{W}$ satisfies the requirements (i)–(iii).

There remains to check (iv). Let $v \in [-\infty, v_0]$. The tangent space to the unstable manifold of x at $h(u_0, v)$ is

$$T_{h(u_0,v)} W^u(x) = \mathbb{R} F(h(u_0, v)) \oplus \operatorname{graph} D\sigma_{u_0}(P^u h(u_0, v)).$$

Since $S(u_0, v) \in \mathcal{L}(H^s, H^u)$ and $D\sigma_{u_0}(P^u h(u_0, v)) \in \mathcal{L}(H^u, H^s)$ have norm not exceeding $\theta_1 < 1$, we have

$$\mathcal{W}(u_0, v) + T_{h(u_0,v)} W^u(x) = H.$$

Moreover, a simple computation shows that the intersection

$$\begin{aligned}
&\mathcal{W}(u_0, v) \cap T_{h(u_0,v)} W^u(x) \\
&\quad = \operatorname{graph} S(u_0, v) \cap \left(\mathbb{R} F(h(u_0, v)) \oplus \operatorname{graph} D\sigma_{u_0}(P^u h(u_0, v))\right)
\end{aligned}$$

is a one-dimensional space spanned by the vector

$$\tilde{G}(v) = (\tilde{G}^u(v), \tilde{G}^s(v)) \in H^u \times H^s,$$

where

$$\begin{aligned}
\tilde{G}^u(v) &= S(u_0, v)\tilde{G}^s(v), \\
\tilde{G}^s(v) &= (I - T(v)S(u_0, v))^{-1}(P^s F(h(u_0, v)) - T(v)P^u F(h(u_0, v))),
\end{aligned}$$

and $T(v) = D\sigma_{u_0}(P^u h(u_0, v))$. Indeed, $\|TS\| \leqslant \theta_1^2 < 1$, so $I - TS$ is invertible, and

$$\|(I - TS)^{-1}\| \leqslant \frac{1}{1-\theta_1^2}, \quad \|(I - TS)^{-1} - I\| \leqslant \frac{\theta_1^2}{1-\theta_1^2}.$$

By (d) we have the estimates

$$\|\tilde{G}^s\| \leqslant \frac{1}{1-\theta_1^2}(\|P^s F(h)\| + \theta_1 \|P^u F(h)\|) \leqslant \frac{2}{1-\theta_1^2}\|F(h)\| \leqslant \frac{4}{1-\theta_1^2}\|L\| \, \|P^s h\|,$$

$$|\langle LP^u h, \tilde{G}^u\rangle| \leqslant \varepsilon \|L\| \, \|\tilde{G}^u\| \, \|P^s h\| \leqslant \varepsilon\theta_1 \|L\| \, \|\tilde{G}^s\| \, \|P^s h\| \leqslant \frac{4\varepsilon\theta_1}{1-\theta_1^2}\|L\|^2 \|P^s h\|^2,$$

$$\begin{aligned} \|\tilde{G}^s - P^s F(h)\| &\leqslant \|\tilde{G}^s - (P^s F(h) - TP^u F(h))\| + \|TP^u F(h)\| \\ &\leqslant \frac{\theta_1^2}{1-\theta_1^2}\|P^s F(h) - TP^u F(h)\| + 2\theta_1 \|L\| \, \|P^s h\| \\ &\leqslant \left(\frac{4\theta_1^2}{1-\theta_1^2} + 2\theta_1\right)\|L\| \, \|P^s h\|, \end{aligned}$$

$$\begin{aligned} \|\tilde{G}^s - LP^s h\| &\leqslant \|\tilde{G}^s - P^s F(h)\| + \|P^s F(h) - P^s Lh\| \\ &\leqslant \left(\left(\frac{4\theta_1^2}{1-\theta_1^2} + 2\theta_1\right)\|L\| + \varepsilon\right)\|P^s h\|. \end{aligned}$$

Thus

$$\begin{aligned} Dg(h)[\tilde{G}] &= -\langle Lh, \tilde{G}\rangle = -\langle LP^u h, \tilde{G}^u\rangle - \langle LP^s h, \tilde{G}^s\rangle \\ &\leqslant \frac{4\varepsilon\theta_1}{1-\theta_1^2}\|L\|^2\|P^s h\|^2 - \langle LP^s h, LP^s h\rangle + \|L\| \, \|P^s h\| \, \|\tilde{G}^s - LP^s h\| \\ &\leqslant \left(\left(\frac{4\theta_1}{1-\theta_1^2}(\theta_1 + \varepsilon) + 2\theta_1 + \varepsilon\right)\|L\|^2 - \frac{1}{\|L^{-1}\|^2}\right)\|P^s h\|^2 \end{aligned}$$

and by (11.10) we get

$$Dg(h(u_0, v))[\tilde{G}(v)] < 0 \quad \forall v \in [-\infty, v_0]. \tag{11.11}$$

On the other hand, by (a),

$$Dg(h(u_0, v_0))\left[\frac{\partial h}{\partial u}(u_0, v_0)\right] < 0 \tag{11.12}$$

and by (11.2) and (b),

$$Dg(h(u_0, -\infty))\left[\frac{\partial h}{\partial u}(u_0, -\infty)\right] = Dg(h(u_0, -\infty))[F(h(u_0, -\infty))] < 0. \tag{11.13}$$

By construction, $h(u, v_0) \in \operatorname{graph} \tau_{v_0} \cap W^u(x)$, for every $u \geqslant 0$, so

$$\frac{\partial h}{\partial u}(u_0, v_0) \in \operatorname{graph} D\tau_{v_0}(P^s h(u_0, v_0)) \cap T_{h(u_0,v_0)} W^u(x) = \mathcal{W}(u_0, v_0) \cap T_{h(u_0,v_0)} W^u(x).$$

Moreover,

$$\frac{\partial h}{\partial u}(u_0, -\infty) \in T_{h(u_0,-\infty)} W^s(y) \cap T_{h(u_0,-\infty)} W^u(x) = \mathcal{W}(u_0, -\infty) \cap T_{h(u_0,-\infty)} W^u(x).$$

Then (11.11)–(11.13) imply that a vector field G satisfying the requirements of (iv) can be defined by multiplying $\tilde{G}$ by a suitable positive function. $\square$

Remark 11.4. In some particular cases, such as when F is linear in a neighborhood of $y = 0$, a map $\mathcal{W}$ satisfying the requirements of the above lemma can be defined simply as

$$\mathcal{W}(u, v) = \operatorname{graph} D\tau_v(P^s h(u, v)),$$

providing us with a drastic simplification of the proof. However in general, the above expression does not define a continuous map $\mathcal{W}$, the reason being that the graph transform Γ of Proposition C.5 needs not be continuous with respect to the C^1 topology.

We are now ready to prove assertion (iv) of Proposition 11.1. The continuous maps

$$\tau_{W_1} : [u_0, +\infty[\to \mathrm{Gr}_{1,\infty}(TM), \quad u \mapsto T_{h(u,-\infty)} W_1 = \mathbb{R}F(h(u, -\infty)),$$
$$\tau_{W_2} :]-\infty, v_0] \to \mathrm{Gr}_{1,\infty}(TM), \quad v \mapsto T_{h(+\infty,v)} W_2 = \mathbb{R}F(h(+\infty, v)),$$
$$\tau_W : [u_0, +\infty[\times]-\infty, v_0] \to \mathrm{Gr}_{2,\infty}(TM), \quad (u, v) \mapsto T_{h(u,v)} W,$$

have continuous liftings $\widehat{\tau_{W_1}}$, $\widehat{\tau_{W_2}}$ to $\mathrm{Or}(\mathrm{Gr}_{1,\infty}(TM))$, and $\widehat{\tau_W}$ to $\mathrm{Or}(\mathrm{Gr}_{2,\infty}(TM))$, corresponding to the orientations of W_1, W_2, and W, defined in Section 5. Moreover, the continuous map

$$\omega : [u_0, +\infty] \times] - \infty, v_0] \to \mathrm{Fp}(TM), \quad (u, v) \mapsto (T_{h(u,v)} W^s(z), \mathcal{V}(h(u, v))),$$

has a continuous lifting $\hat{\omega}$ to $\mathrm{Or}(\mathrm{Fp}(TM))$, which is determined by the orientation o_z of $(H^s_z, \mathcal{V}(z))$.

By Lemma 11.3(ii), $\mathcal{W}(+\infty, -\infty) = H^s_y$, so the continuous map

$$\alpha : [u_0, +\infty] \times [-\infty, v_0] \to \mathrm{Fp}(TM), \quad (u, v) \mapsto (\mathcal{W}(u, v), \mathcal{V}(h(u, v))),$$

has a unique continuous lifting $\hat{\alpha}$ to $\mathrm{Or}(\mathrm{Fp}(TM))$ such that $\hat{\alpha}(+\infty, -\infty) = o_y$. By Lemma 11.3(iv), there is a continuous curve $\mathcal{X} : [-\infty, v_0] \to \mathrm{Gr}(TM)$ such that $\mathcal{W}(u_0, v) = \mathbb{R}G(v) \oplus \mathcal{X}(v)$ for every $v \in [-\infty, v_0]$, and we can define the continuous map

$$\beta : [-\infty, v_0] \to \mathrm{Fp}(TM), \quad v \mapsto (\mathcal{X}(v), \mathcal{V}(h(u_0, v))).$$

Then $\mathcal{X}(v)$ is a linear supplement of $T_{h(u_0,v)} W^u(x)$ in $T_{h(u_0,v)} M$, so Theorem B.2(iii) implies that

$$\lim_{t \to -\infty} D\phi_t(h(u_0, v))\mathcal{X}(v) = H^s_x,$$

uniformly in $v \in [-\infty, v_0]$. Therefore, the orientation o_x of $(H^s_x, \mathcal{V}(x))$ determines a continuous lifting $\hat{\beta} : [-\infty, v_0] \to \mathrm{Or}(\mathrm{Fp}(TM))$ of β.

Denote by h_1 and by h_2 the restrictions of h to $\mathbb{R} \times \{-\infty\}$ and to $\{+\infty\} \times \mathbb{R}$. If X is an n-dimensional real vector space and ξ is a non-zero element of $\Lambda^n(X)$, the same symbol ξ will also denote the orientation of X induced by ξ. When $n = 1$, we shall identify $\Lambda^1(X)$ with X. If o is an orientation of X, $-o$ will denote the other orientation.

By Lemma 11.3(i), (iv), and by (11.2),

$$T_{h(u_0,v_0)} W^s(z) = \mathbb{R}F(h(u_0, v_0)) \oplus \mathbb{R}G(v_0) \oplus \mathcal{X}(v_0) = T_{h(u_0,v_0)} W \oplus \mathcal{X}(v_0),$$

so by the definition of the orientation of $W \subset W^u(x) \cap W^s(z)$,

$$\hat{\omega}(u_0, v_0) = \widehat{\tau_W}(u_0, v_0) \bigwedge \hat{\beta}(v_0) = (\deg h) \left(\frac{\partial h}{\partial u}(u_0, v_0) \wedge \frac{\partial h}{\partial v}(u_0, v_0) \right) \bigwedge \hat{\beta}(v_0). \quad (11.14)$$

Moreover, $T_{h(+\infty,v_0)}W^s(z) = T_{h(+\infty,v_0)}W_2 \oplus \mathcal{W}(+\infty, v_0)$, so by Lemma 11.3(iii) and by the definition of the orientation of $W_2 \subset W^u(y) \cap W^s(z)$,

$$\begin{aligned}\hat{\omega}(+\infty, v_0) &= \widehat{\tau_{W_2}}(v_0) \bigwedge \hat{\alpha}(+\infty, v_0) = (\deg h_2)\,\frac{\partial h}{\partial v}(+\infty, v_0) \bigwedge \hat{\alpha}(+\infty, v_0)\\ &= (\deg h_2)\, SF(h(+\infty, v_0) \bigwedge \hat{\alpha}(+\infty, v_0),\end{aligned}$$

where we have taken (11.2) into account. By Lemma 11.3(i),

$$\omega(u, v_0) = (\mathbb{R}F(h(u, v_0)) \oplus \alpha_1(u, v_0), \alpha_2(u, v_0))$$

for every $u \in [u_0, +\infty]$, so by the continuity of the product on the orientation bundle we obtain

$$\hat{\omega}(u_0, v_0) = (\deg h_2)\, F(h(u_0, v_0)) \bigwedge \hat{\alpha}(u_0, v_0).$$

Hence by (11.2),

$$\hat{\omega}(u_0, v_0) = (\deg h_2) \left(\frac{\partial h}{\partial u}(u_0, v_0) + \frac{\partial h}{\partial u}(u_0, v_0) \right) \bigwedge \hat{\alpha}(u_0, v_0). \tag{11.15}$$

By Lemma 11.3(ii), (iv), and by (11.2),

$$T_{h(u_0,-\infty)}W^s(y) = \mathcal{W}(u_0, -\infty) = \mathbb{R}G(-\infty) \oplus \mathcal{X}(-\infty) = T_{h(u_0,-\infty)}W_1 \oplus \mathcal{X}(-\infty),$$

so by the definition of the orientation of $W_1 \subset W^u(x) \cap W^s(y)$,

$$\begin{aligned}\hat{\alpha}(u_0, -\infty) = \widehat{\tau_{W_1}}(u_0) \bigwedge \hat{\beta}(-\infty)) &= (\deg h_1)\,\frac{\partial h}{\partial u}(u_0, -\infty) \bigwedge \hat{\beta}(-\infty)\\ &= (\deg h_1)\, G(-\infty) \bigwedge \hat{\beta}(-\infty).\end{aligned}$$

Then by the identity

$$\alpha(u_0, v) = (\mathbb{R}G(v) \oplus \beta_1(v), \beta_2(v)) \quad \forall v \in [-\infty, v_0]$$

and by the continuity of the product on the orientation bundle we obtain

$$\hat{\alpha}(u_0, v_0) = (\deg h_1)\, G(v_0) \bigwedge \hat{\beta}(v_0) = (\deg h_1)\,\frac{\partial h}{\partial u}(u_0, v_0) \bigwedge \hat{\beta}(v_0). \tag{11.16}$$

Identities (11.15) and (11.16), together with the associativity of the product of orientation, imply that

$$\begin{aligned}\hat{\omega}(u_0, v_0) &= (\deg h_2)(\deg h_1)\left(\frac{\partial h}{\partial u}(u_0, v_0) + \frac{\partial h}{\partial v}(u_0, v_0)\right) \bigwedge \left(\frac{\partial h}{\partial u}(u_0, v_0) \bigwedge \hat{\beta}(v_0)\right)\\ &= (\deg h_2)(\deg h_1)\left(\left(\frac{\partial h}{\partial u}(u_0, v_0) + \frac{\partial h}{\partial v}(u_0, v_0)\right) \wedge \frac{\partial h}{\partial u}(u_0, v_0)\right) \bigwedge \hat{\beta}(v_0)\\ &= -(\deg h_2)(\deg h_1)\left(\frac{\partial h}{\partial u}(u_0, v_0) \wedge \frac{\partial h}{\partial v}(u_0, v_0)\right) \bigwedge \hat{\beta}(v_0)\end{aligned}$$

and comparing the above identity with (11.14) we obtain

$$\deg h = -(\deg h_1)(\deg h_2),$$

proving (iv). □

11.2. Conclusion

Proof of Theorem 9.1. Fix a value $c \in]f(z), f(x)[$. By assumption, $W \cap \{f = c\} \cong W/\mathbb{R}$ is an open interval, so it is parameterized by a C^1 diffeomorphism $\gamma : \mathbb{R} \to M$. By Proposition 8.2, there exist an increasing, unbounded sequence (s_n) and a broken gradient flow line S^+ from x to z such that

$$\lim_{n\to\infty} \overline{\phi(\mathbb{R} \times \{\gamma(s_n)\})} = S^+ \tag{11.17}$$

in the Hausdorff distance, and also such that $\gamma(s_n)$ converges to a point $p \in S^+$. Since γ is a homeomorphism, p is not in W, and since W is closed in $W^u(x) \cap W^s(z)$, p is not in $W^u(x) \cap W^s(z)$ either. So S^+ contains a rest point y of intermediate level. As already noticed, the Morse–Smale property and the fact that $m(x, \mathcal{E}) = m(z, \mathcal{E}) + 2$ imply that $m(y, \mathcal{E}) = m(z, \mathcal{E}) + 1$, and that there exist W_1^+ and W_2^+, connected components of $W^u(x) \cap W^s(y)$ and of $W^u(y) \cap W^s(z)$, such that $S^+ = \overline{W_1^+ \cup W_2^+}$. Proposition 11.1 provides us with a map

$$h^+ : \Delta^+ := \left\{(u, v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid v \leqslant u\right\} \to \overline{W^u(x) \cap W^s(z)},$$

verifying properties (i)–(iv). In particular by (iii) and (11.17), $h^+(\Delta^+ \cap \mathbb{R}^2) \subset W$, and by (iv),

$$\deg(h^+) = -\deg(h^+|_{\mathbb{R}\times\{-\infty\}}) \cdot \deg(h^+|_{\{+\infty\}\times\mathbb{R}}). \tag{11.18}$$

By Proposition 11.1(i), for any $t \geqslant 0$,

$$\lim_{s\to-\infty} f(h^+(s+t,s)) = \lim_{s\to-\infty} f(\phi(s,h^+(t,0))) = f(x) > c,$$
$$\lim_{s\to+\infty} f(h^+(s+t,s)) = \lim_{s\to+\infty} f(\phi(s,h^+(t,0))) = f(z) < c,$$

$$\begin{aligned}\frac{\partial}{\partial s}\left[f(h^+(s+t,s))\right] &= \frac{\partial}{\partial s}\left[f(\phi(s,h^+(t,0))\right]\\ &= Df(\phi_s(h^+(t,0)))[F(\phi_s(h^+(t,0))] < 0,\end{aligned}$$

so by the implicit function theorem there exists a function $\eta_+ \in C^1([0,+\infty[,\mathbb{R})$ such that

$$f(h^+(\eta_+(t)+t,\ \eta_+(t))) = c \quad \forall t \geqslant 0.$$

Then $h^+(\eta_+(t)+t,\ \eta_+(t)) \in W \cap \{f = c\}$, and

$$\theta_+(t) := \gamma^{-1}(h^+(\eta_+(t)+t,\ \eta_+(t)))$$

defines a C^1 function $\theta_+ : [0,+\infty[\to \mathbb{R}$. An application of $\phi_{v-\eta_+(u-v)}$ to $h^+(\eta_+(u-v)+t,\ \eta_+(u-v)) = \gamma(\theta_+(u-v))$ yields to the representation

$$h^+(u,v) = \phi(v-\eta_+(u-v),\gamma(\theta_+(u-v))), \quad (u,v) \in \Delta^+ \cap \mathbb{R}^2. \tag{11.19}$$

Since h^+ is a diffeomorphism, the vectors $\partial h^+/\partial u$ and $\partial h^+/\partial v$ are linearly independent, so

$$(\gamma \circ \theta_+)' = (1+\eta_+')\frac{\partial h^+}{\partial u} + \eta_+'\frac{\partial h^+}{\partial v}$$

never vanishes, and from the fact that γ is a diffeomorphism we deduce that $\theta_+'(t) \neq 0$ for every $t \geqslant 0$. Moreover, from Proposition 11.1(iii), $\phi(\mathbb{R}\times\{\gamma(s_n)\}) \subset h_+(\Delta^+)$ for n large, which implies that $\phi(\mathbb{R}\times\{\gamma(s_n)\}) = \phi(\mathbb{R}\times\{h^+(t_n,0)\}) = \phi(\mathbb{R}\times\{\gamma(\theta_+(t_n))\})$ for some $t_n \geqslant 0$. Since γ is injective and meets any flow line at most once, the last equality implies that $\theta_+(t_n) = s_n \to +\infty$. Therefore,

$$\theta_+' > 0, \quad \lim_{t\to+\infty} \theta_+(t) = +\infty. \tag{11.20}$$

The same construction, starting with a sequence $s_n' \to -\infty$, yields to a rest point y' with $m(y',\mathcal{E}) = m(z,\mathcal{E}) + 1$ such that $\overline{\phi(\mathbb{R}\times\{\gamma(s_n')\}}$ converges to $S^- = \overline{W_1^- \cup W_2^-}$, for some connected components W_1^- and W_2^- of $W^u(x) \cap W^s(y')$ and $W^u(y') \cap W^s(z)$, respectively. As before we obtain a map

$$h^- : \Delta^- := \{(u,v) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \ : \ v \geqslant u\} \to \overline{W},$$

where we also used the orientation reversing change of variables $\Delta^- \ni (u,v) \mapsto (v,u) \in \Delta^+$. Hence

$$\deg(h^-) = \deg(h^-|_{\{-\infty\} \times \mathbb{R}}) \cdot \deg(h^-|_{\mathbb{R} \times \{+\infty\}}) \tag{11.21}$$

and we have the representation

$$h^-(u,v) = \phi(v - \eta_-(u-v), \gamma(\theta_-(u-v))), \quad (u,v) \in \Delta^- \cap \mathbb{R}^2. \tag{11.22}$$

for suitable C^1 functions η_- and θ_- on $]-\infty, 0]$, with

$$\theta_-' > 0 \quad \lim_{t \to -\infty} \theta_-(t) = -\infty. \tag{11.23}$$

Proposition 11.1(iii) together with (11.19) and (11.22), implies that $S^- \neq S^+$, as claimed in (ii). Now we can choose two C^1 functions η, $\theta : \mathbb{R} \to \mathbb{R}$, with $\theta' > 0$, coinciding with η_-, θ_- in a neighborhood of $-\infty$ and with η_+, θ_+ in a neighborhood of $+\infty$. The map

$$h(u,v) := \phi(v - \eta(u-v), \gamma(\theta(u-v))), \quad (u,v) \in \mathbb{R}^2,$$

has a continuous extension to $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ and clearly satisfies all requirements (i)–(iv). □

Proof of Proposition 9.2. The conclusion follows immediately from Proposition 11.1(iii). □

Appendix A. Infinite dimensional Grassmannians

The aim of this appendix is to gather the definitions and the relevant properties of some infinite dimensional Grassmannians. Unless otherwise stated, detailed proofs can be found in [AM03a] (but see also [Pal65,Luf67,Qui85,SW85,PS86,CJS95,Shu96]).

A.1. The Hilbert Grassmannian and the space of Fredholm pairs

By $\mathcal{L}(E,F)$, respectively $\mathcal{L}_c(E,F)$, we will denote the space of continuous linear, respectively compact linear, maps from the Banach space E to the Banach space F.

If $F = E$ we will use the abbreviations $\mathcal{L}(E)$ and $\mathcal{L}_c(E)$. The norm of the operator $T \in \mathcal{L}(E, F)$ will be denoted by $\|T\|$. By $\sigma(L)$ and by $\sigma_{\rm ess}(L)$ we will denote the spectrum and the essential spectrum of the operator $L \in \mathcal{L}(E)$, that is the spectrum of $[L]$ in the Calkin algebra $\mathcal{L}(E)/\mathcal{L}_c(E)$.

Let H be a real infinite dimensional separable Hilbert space. The orthogonal projection onto a closed subspace $V \subset H$ will be denoted by P_V, while $V^\perp$ will denote the orthogonal complement of V in H.

Let $\mathrm{Gr}(H)$ be the *Grassmannian of H*, i.e. the set of closed linear subspaces of H. The assignment $V \mapsto P_V$ is an inclusion of $\mathrm{Gr}(H)$ into $\mathcal{L}(H)$, onto the closed subset of the orthogonal projectors of H. We can therefore define, for any $V, W \in \mathrm{Gr}(H)$ the distance

$$\mathrm{dist}\,(W_1, W_2) := \|P_{W_1} - P_{W_2}\|,$$

which makes $\mathrm{Gr}(H)$ a complete metric space. It can be proved that $\mathrm{Gr}(H)$ is an analytic Banach submanifold of the Banach space $\mathcal{L}(H)$: indeed, the subspace of symmetric idempotent elements of a C^*-algebra is always an analytic Banach submanifold.

The connected components of $\mathrm{Gr}(H)$ are the subsets

$$\mathrm{Gr}_{n,k}(H) := \{V \in \mathrm{Gr}(H) \mid \dim V = n,\ \mathrm{codim}\, V = k\}, \quad n, k \in \mathbb{N} \cup \{\infty\},\ n + k = \infty.$$

The orthogonal group $\mathrm{O}(H)$ is contractible, by a well known result by Kuiper [Kui65], and it acts transitively on each of these components. These facts imply that $\mathrm{Gr}_{\infty,\infty}(H)$ is contractible, while $\mathrm{Gr}_{n,\infty}(H)$ and $\mathrm{Gr}_{\infty,n}(H)$ have the homotopy type of $\mathrm{BO}(n)$, the classifying space of the orthogonal group of $\mathbb{R}^n$.

A pair (V, W) of closed subspaces of H is said a *Fredholm pair* if $V \cap W$ is finite dimensional, and $V + W$ is finite codimensional (see also [Kat80, Section IV]). In this situation, the *index* of (V, W) is the number

$$\mathrm{ind}\,(V, W) = \dim V \cap W - \mathrm{codim}(V + W).$$

The set of Fredholm pairs in H will be denoted by $\mathrm{Fp}(H)$: it is open in $\mathrm{Gr}(H) \times \mathrm{Gr}(H)$, and the index is a continuous function on $\mathrm{Fp}(H)$. The connected components of $\mathrm{Fp}(H)$ are the subsets

$$\mathrm{Gr}_{n,\infty}(H) \times \mathrm{Gr}_{\infty,m}(H), \quad \mathrm{Gr}_{\infty,n}(H) \times \mathrm{Gr}_{m,\infty}(H), \quad n, m \in \mathbb{N},$$
$$\mathrm{Fp}^*_k(H) := \left\{(V, W) \in \mathrm{Fp}(H) \mid V, W \in \mathrm{Gr}_{\infty,\infty}(H),\ \mathrm{ind}\,(V, W) = k\right\}, \quad k \in \mathbb{Z}.$$

The space of Fredholm pairs consisting of infinite dimensional spaces will be denoted by

$$\mathrm{Fp}^*(H) := \bigcup_{k \in \mathbb{Z}} \mathrm{Fp}^*_k(H).$$

It can be proved that $\mathrm{Fp}_k^*(H)$ has the homotopy type of $\mathrm{BO}(\infty)$, the classifying space of the infinite real orthogonal group $\mathrm{O}(\infty) = \lim_{n\to\infty} \mathrm{O}(n)$. So $\mathrm{Fp}^*(H)$ is homotopically equivalent to $\mathbb{Z} \times \mathrm{BO}(\infty)$, and by the Bott periodicity theorem we get

$$\pi_i(\mathrm{Fp}^*(H)) = \begin{cases} \mathbb{Z} & \text{for } i \equiv 0, 4 \mod 8, \\ \mathbb{Z}_2 & \text{for } i \equiv 1, 2 \mod 8, \\ 0 & \text{for } i \equiv 3, 5, 6, 7 \mod 8. \end{cases} \tag{A.1}$$

We conclude this section with a result about the existence of hyperbolic rotations, which will be useful in Appendix B.

Proposition A.1. *Let* $V, W \in \mathrm{Gr}_{\infty,\infty}(H)$ *be such that* $\mathrm{dist}\,(V, W) < 1$. *Then there exists* $A \in \mathcal{L}(H)$ *self-adjoint, invertible, with* $\sigma_{\mathrm{ess}}(A) \cap \mathbb{R}^- \neq \emptyset$, $\sigma_{\mathrm{ess}}(A) \cap \mathbb{R}^+ \neq \emptyset$, *such that* $e^A V = W$.

Proof. Since $\mathrm{dist}\,(V, W) < 1$, $W = \mathrm{graph}\, L$, with $L = P_{V^\perp}(P_V|_W)^{-1} \in \mathcal{L}(V, V^\perp)$. Consider the self-adjoint bounded operator

$$S = \begin{pmatrix} \theta & \eta L^* \\ \eta L & 1/\theta \end{pmatrix}, \quad 0 < \theta < 1, \quad \eta \in \mathbb{R},$$

in the splitting $H = V \oplus V^\perp$. Then

$$(S - \theta)(S - 1/\theta) = \eta^2 \begin{pmatrix} L^*L & 0 \\ 0 & LL^* \end{pmatrix}.$$

We fix a $\theta < 1/\|L\|$, so the positive self-adjoint operator on the right-hand side has its spectrum in $[0, 1[$, for every $\eta \in [0, \theta]$. The spectral mapping theorem implies that

$$\{(s - \theta)(s - 1/\theta) \mid s \in \sigma(S)\} \subset [0, 1[,$$

so we have

$$\sigma(S) \subset]0, \theta] \cup [1/\theta, 1/\theta + \theta[\subset]0, 1[\cup]1, +\infty[$$

for any $\eta \in [0, \theta]$. For $\eta = 0$, $\sigma_{\mathrm{ess}}(S) = \{\theta, 1/\theta\}$, so by the semi-continuity of the essential spectrum

$$\sigma_{\mathrm{ess}}(S) \cap]0, 1[\neq \emptyset, \quad \sigma_{\mathrm{ess}}(S) \cap]1, +\infty[\neq \emptyset,$$

for any $\eta \in [0, \theta]$. In particular for $\eta = \theta$, $A = \log S$ is a well-defined operator satisfying the requirements. $\square$

A.2. The determinant and the orientation of Fredholm pairs

Let $n \in \mathbb{N}$. The Grassmannian of n-dimensional linear subspaces $\mathrm{Gr}_{n,\infty}(H)$ is the base space of a non-trivial real line bundle, the determinant bundle

$$\mathrm{Det}(\mathrm{Gr}_{n,\infty}(H)) \to \mathrm{Gr}_{n,\infty}(H),$$

whose fiber at $X \in \mathrm{Gr}_{n,\infty}(H)$ is the line $\mathrm{Det}(X) := \Lambda^{\dim X}(X)$, the component of the exterior algebra of X consisting of tensors of top degree. Such a line bundle has a natural analytic structure. Its $\mathbb{Z}_2$ reduction is the non-trivial double covering

$$\mathrm{Or}(\mathrm{Gr}_{n,\infty}(H)) \to \mathrm{Gr}_{n,\infty}(H),$$

called the orientation bundle, whose fiber at X is the set $\mathrm{Or}(X)$ consisting of the two elements of $\mathrm{Det}(X) \setminus \{0\}/\mathbb{R}^+$. If o_X is an element of $\mathrm{Or}(X)$, the other element will be denoted by $-o_X$. If $n, m \in \mathbb{N}$, the space

$$\mathcal{S}(n,m) = \left\{(X,Y) \in \mathrm{Gr}_{n,\infty}(H) \times \mathrm{Gr}_{m,\infty}(H) \mid X \cap Y = (0)\right\}$$

is the base space of the line bundle

$$\mathrm{Det}(\mathcal{S}(n,m)) \to \mathcal{S}(n,m),$$

whose fiber at (X,Y) is the line $\mathrm{Det}(X) \otimes \mathrm{Det}(Y)$, and the exterior product of tensors of top degree defines an analytic morphism

$$\wedge : \mathrm{Det}(\mathcal{S}(n,m)) \to \mathrm{Det}(\mathrm{Gr}_{n+m,\infty}(H)), \quad \omega_X \otimes \omega_Y \mapsto \omega_X \wedge \omega_Y$$

which lifts the analytic map $(X,Y) \to X+Y$. This operation is associative. The morphism of line bundles $\wedge$ induces a morphism of coverings, denoted by the same symbol, between the orientation bundles:

$$\wedge : \mathrm{Or}(\mathcal{S}(n,m)) \to \mathrm{Or}(\mathrm{Gr}_{n+m,\infty}(H)),$$

where the first space is the total space of the covering over $\mathcal{S}(n,m)$ whose fiber at (X,Y) is $\mathrm{Or}(X) \times \mathrm{Or}(Y)$. The product of orientations satisfies the identity

$$o_X \wedge o_Y = (-o_X) \wedge (-o_Y) = -(-o_X) \wedge o_Y = -o_X \wedge (-o_Y)$$

and it is associative.

These constructions have a natural extension to the space of Fredholm pairs. The *determinant bundle over* $\mathrm{Fp}(H)$ is the line bundle

$$\mathrm{Det}(\mathrm{Fp}(H)) \to \mathrm{Fp}(H),$$

whose fiber at (V, W) is the line

$$\mathrm{Det}(V, W) := \mathrm{Det}(V \cap W) \otimes \mathrm{Det}\left(\left(\frac{H}{V + W}\right)\right)^* .$$

Although the intersection $V \cap W$ and the sum $V + W$ do not depend even continuously on (V, W), it can be shown that the above bundle has an analytic structure. This line bundle is also non-trivial, and its $\mathbb{Z}_2$ reduction is the non-trivial double covering

$$\mathrm{Or}(\mathrm{Fp}(H)) \to \mathrm{Fp}(H),$$

called the *orientation bundle over* $\mathrm{Fp}(H)$, whose fiber at (V, W) is the set $\mathrm{Or}(V, W)$ consisting of the two elements of $\mathrm{Det}(V, W) \setminus \{0\}/\mathbb{R}^+$. If $o_{(V,W)}$ is an element of $\mathrm{Or}(V, W)$, the other element will be denoted by $-o_{(V,W)}$. Note that the fundamental group of each component of $\mathrm{Fp}^*(H)$ is $\mathbb{Z}_2$, so the restriction of the orientation bundle to $\mathrm{Fp}^*(H)$ is the universal covering of $\mathrm{Fp}^*(H)$.

If $n \in \mathbb{N}$, the space

$$\mathcal{S}(n, \mathrm{Fp}) = \left\{(X, (V, W)) \in \mathrm{Gr}_{n,\infty}(H) \times \mathrm{Fp}(H) \mid X \cap V = (0)\right\}$$

is the base space of the line bundle

$$\mathrm{Det}(\mathcal{S}(n, \mathrm{Fp})) \to \mathcal{S}(n, \mathrm{Fp}),$$

whose fiber at $(X, (V, W))$ is the line $\mathrm{Det}(X) \otimes \mathrm{Det}(V, W)$, and there is an analytic morphism

$$\bigwedge : \mathrm{Det}(\mathcal{S}(n, \mathrm{Fp})) \to \mathrm{Det}(\mathrm{Fp}(H)), \quad \omega_X \otimes \omega_{(V,W)} \mapsto \omega_X \bigwedge \omega_{(V,W)}$$

which lifts the analytic map $(X, (V, W)) \to (X + V, W)$. Also this operation is associative, meaning that

$$\omega_X \bigwedge (\omega_Y \bigwedge \omega_{(V,W)}) = (\omega_X \wedge \omega_Y) \bigwedge \omega_{(V,W)}$$

for any $\omega_X \in \mathrm{Det}(X)$, $\omega_Y \in \mathrm{Det}(Y)$, $\omega_{(V,W)} \in \mathrm{Det}(V, W)$, where X, Y are finite dimensional subspaces of H, and (V, W) is a Fredholm pair such that $X \cap Y = (0)$,

$(X+Y)\cap V = 0$. The morphism of line bundles $\bigwedge$ induces a morphism of coverings, denoted by the same symbol, between the orientation bundles:

$$\bigwedge : \mathrm{Or}(\mathcal{S}(n, \mathrm{Fp})) \to \mathrm{Or}(\mathrm{Fp}(H)),$$

where the first space is the total space of the covering over $\mathcal{S}(n, \mathrm{Fp})$ whose fiber at $(X, (V, W))$ is $\mathrm{Or}(X) \times \mathrm{Or}(V, W)$. This map satisfies the identity

$$o_X \bigwedge o_{(V,W)} = (-o_X)\bigwedge(-o_{(V,W)}) = -(-o_X)\bigwedge o_{(V,W)} = -o_X \bigwedge(-o_{(V,W)})$$

and it is associative, meaning that

$$o_X \bigwedge (o_Y \bigwedge o_{(V,W)}) = (o_X \wedge o_Y)\bigwedge o_{(V,W)}$$

for any $o_X \in \mathrm{Or}(X)$, $o_Y \in \mathrm{Or}(Y)$, $o_{(V,W)} \in \mathrm{Or}(V, W)$, where X, Y are finite dimensional subspaces of H, and (V, W) is a Fredholm pair such that $X \cap Y = (0)$, $(X+Y)\cap V = 0$.

A.3. The Grassmannian of compact perturbations

We shall say that the closed linear subspace W is a *compact perturbation* of V if its orthogonal projector P_W is a compact perturbation of P_V. The subspace W is a compact perturbation of W if and only if the operators $P_{V^\perp} P_W$ and $P_{W^\perp} P_V$ are compact. The notion of compact perturbation produces an equivalence relation, and the *Grassmannian of compact perturbations of V*,

$$\mathrm{Gr}(V, H) := \{W \in \mathrm{Gr}(H) \mid W \text{ is a compact perturbation of } V\}$$

is a closed subspace of $\mathrm{Gr}(H)$. If V has finite dimension (respectively, finite codimension), then

$$\mathrm{Gr}(V, H) = \bigcup_{n\in\mathbb{N}} \mathrm{Gr}_{n,\infty}(H) \quad \left(\text{resp.} = \bigcup_{n\in\mathbb{N}} \mathrm{Gr}_{\infty,n}(H)\right).$$

In the more interesting case, V has both infinite dimension and infinite codimension. In such a situation, $\mathrm{Gr}(V, H)$ is a closed proper subset of $\mathrm{Gr}_{\infty,\infty}(H)$. It is an analytic Banach manifold, although just a C^0 Banach submanifold of $\mathrm{Gr}(H)$. This space is also called *restricted Grassmannian* by some authors (see [SW85,PS86,CJS95]).

If W is a compact perturbation of W, then $(V, W^\perp)$ is a Fredholm pair, and the *relative dimension of V with respect to W* is the integer

$$\dim(V, W) := \mathrm{ind}\,(V, W^\perp) = \dim V \cap W^\perp - \dim V^\perp \cap W.$$

When V and W are finite dimensional (resp. finite codimensional), we have $\dim(V, W) = \dim V - \dim W$ (resp. $\dim(V, W) = \operatorname{codim} W - \operatorname{codim} V$).

Proposition A.2 ([AM03a, Proposition 5.1]). *If (V, Z) is a Fredholm pair and W is a compact perturbation of V, then (W, Z) is a Fredholm pair, with*

$$\operatorname{ind}(W, Z) = \operatorname{ind}(V, Z) + \dim(W, V).$$

In particular, if V, W, Y are compact perturbations of the same subspace,

$$\dim(Y, V) = \dim(Y, W) + \dim(W, V). \tag{A.2}$$

Nor the notion of compact perturbation, neither the relative dimension depend on the choice of an equivalent inner product in H.

Proposition A.3 ([AM01, Proposition 2.3]). *Let H_1, H_2 be Hilbert spaces and let $T, S \in \mathcal{L}(H_1, H_2)$ be operators with closed range and compact difference. Then $\ker T$ is a compact perturbation of $\ker S$, $\operatorname{ran} T$ is a compact perturbation of $\operatorname{ran} S$, and*

$$\dim(\operatorname{ran} T, \operatorname{ran} S) = -\dim(\ker T, \ker S).$$

Proposition A.4. *Let $T \in \mathrm{GL}(H)$, $V \in \mathrm{Gr}(H)$, and let P be a projector onto V. Then TV is a compact perturbation of V if and only if the operator $(I - P)TP$ is compact.*

Proof. By choosing a suitable inner product on H, we may assume that $P = P_V$ is an orthogonal projector. The operator $L := TP + T^{*-1}(I - P)$ is invertible, and $P_{TV} = TPL^{-1}$. Therefore, TV is a compact perturbation of V if and only if the operator

$$(P_{TV} - P_V)L = (I - P)TP - PT^{*-1}(I - P) =: S$$

is compact.

Now, if S is compact, so is $(I - P)TP = SP$. On the other hand, since the set

$$\{X \in \mathrm{GL}(H) \mid (I - P)XP \in \mathcal{L}_c(H)\}$$

is a subgroup of $\mathrm{GL}(H)$, if $(I - P)TP$ is compact so is $(I - P)T^{-1}P$. Therefore,

$$S = (I - P)TP - \left((I - P)T^{-1}P\right)^*$$

is compact. $\square$

Let $V \in \mathrm{Gr}_{\infty,\infty}(H)$. The connected components of $\mathrm{Gr}(V, H)$ are the subsets

$$\mathrm{Gr}_n(V, H) := \{W \in \mathrm{Gr}(V, H) \mid \dim(W, V) = n\}, \quad n \in \mathbb{Z}.$$

These components are pairwise diffeomorphic. Each of these components has the homotopy type of $\mathrm{BO}(\infty)$, so the homotopy groups of $\mathrm{Gr}(V, H)$ are those listed in (A.1).

We conclude this section with a result about the kernel of semi-Fredholm operators.

Proposition A.5. *Let $A, B \in \mathcal{L}(H_1, H_2)$ be continuous linear operators between Hilbert spaces, with finite-codimensional range. Assume that the restrictions $A|_{\ker B}$ and $B|_{\ker A}$ are compact. Then* $\ker A$ *is a compact perturbation of* $\ker B$*, the operator $AB^* \in \mathcal{L}(H_2)$ is Fredholm, and*

$$\mathrm{ind}\,(AB^*) = \dim \operatorname{coker} B - \dim \operatorname{coker} A + \dim(\ker A, \ker B). \tag{A.3}$$

Proof. Since A has closed range, there exists $S \in \mathcal{L}(H_2, H_1)$ such that $SA = P_{(\ker A)^\perp}$. Then $P_{(\ker A)^\perp} P_{\ker B} = SAP_{\ker B}$ is compact, and symmetrically so is $P_{(\ker B)^\perp} P_{\ker A}$. Therefore, $\ker A$ is a compact perturbation of $\ker B$. Moreover, $AP_{(\ker B)^\perp} = A - AP_{\ker B}$ is a compact perturbation of A, so it has closed range $\mathrm{ran}\,(AP_{(\ker B)^\perp}) = \mathrm{ran}\,(AB^*)$. Since $(AB^*)^* = BA^*$, the exactness of the sequences

$$0 \to \ker B^* \hookrightarrow \ker(AB^*) \xrightarrow{B^*} \ker A \cap (\ker B)^\perp \to 0,$$

$$0 \to \ker A^* \hookrightarrow \ker(BA^*) \xrightarrow{A^*} \ker B \cap (\ker A)^\perp \to 0,$$

implies that AB^* is Fredholm and that (A.3) holds. $\square$

A.4. Essential Grassmannians

If $m \in \mathbb{N}$, we define the *(m)-essential Grassmannian* $\mathrm{Gr}_{(m)}(H)$ to be the quotient space of $\mathrm{Gr}(H)$ by the equivalence relation

$$\{(V, W) \in \mathrm{Gr}(H) \times \mathrm{Gr}(H) \mid V \text{ is a compact perturbation of } W \text{ and } \dim(V, W) \in m\mathbb{Z}\}.$$

The (1)-essential Grassmannian is simply called *essential Grassmannian*. If $E \in \mathrm{Gr}_{(m)}(H)$ and $V \in \mathrm{Gr}(H)$ is commensurable to the subspaces belonging to the equivalence class E,

$$\dim(V, E) := \dim(V, W), \quad W \in E,$$

defines an element of $\mathbb{Z}/m\mathbb{Z}$.

The essential Grassmannian $\mathrm{Gr}_{(1)}(H)$ is homeomorphic to the space of symmetric idempotent elements of the Calkin algebra $\mathcal{L}(H)/\mathcal{L}_c(H)$, hence it inherits the structure of a complete metric space, and of an analytic submanifold of the Calkin algebra.

Every set $\mathrm{Gr}_{n,\infty}(H)$ or $\mathrm{Gr}_{\infty,n}(H)$, $n \in \mathbb{N}$, represents an isolated point in $\mathrm{Gr}_{(0)}(H)$, which has thus countably many isolated points. If $m \geqslant 1$, the sets

$$\bigcup_{n\in m\mathbb{Z}+k} \mathrm{Gr}_{n,\infty}(H) \quad \text{and} \quad \bigcup_{n\in m\mathbb{Z}+k} \mathrm{Gr}_{\infty,n}(H), \quad k = 0, 1, \dots, m-1,$$

represent $2m$ isolated points in $\mathrm{Gr}_{(m)}(H)$. The remaining part of $\mathrm{Gr}_{(m)}(H)$ is connected, being the quotient space of $\mathrm{Gr}_{\infty,\infty}(H)$, and it is denoted by $\mathrm{Gr}^*_{(m)}(H)$.

The space $\mathrm{Gr}^*_{(0)}(H)$ is simply connected, while the fundamental group of $\mathrm{Gr}^*_{(m)}(H)$ for $m \geqslant 1$ is infinite cyclic. If $m \geqslant 1$ divides $k \in \mathbb{N}$, the natural projection

$$\mathrm{Gr}^*_{(k)}(H) \to \mathrm{Gr}^*_{(m)}(H)$$

is a covering map. It is the universal covering of $\mathrm{Gr}^*_{(m)}(H)$ if $k = 0$, it induces the homomorphism $q \mapsto (k/m)q$ between fundamental groups if $k \neq 0$. For $m = 1$ we obtain a covering map with a basis having the structure of an analytic Banach manifold and of a complete metric space, hence the same structures can be lifted to $\mathrm{Gr}_{(k)}(H)$, for any $k \neq 1$.

Finally, the natural projection

$$\mathrm{Gr}_{\infty,\infty}(H) \to \mathrm{Gr}^*_{(m)}(H) \tag{A.4}$$

is a C^0 fiber bundle.[6] Its total space is contractible, and its typical fiber is

$$\bigcup_{[n]\in\mathbb{Z}/m\mathbb{Z}} \mathrm{Gr}_n(V, H), \quad \text{where } V \in \mathrm{Gr}_{\infty,\infty}(H),$$

a disjoint union all of whose components have the homotopy type of $\mathrm{BO}(\infty)$. Therefore, the exact homotopy sequence of a fibration yields to the isomorphisms

$$\pi_i(\mathrm{Gr}^*_{(m)}(H)) \cong \pi_{i-1}(\mathrm{Gr}(V, H)) = \begin{cases} \mathbb{Z} & \text{for } i \equiv 1, 5 \mod 8, \\ \mathbb{Z}_2 & \text{for } i \equiv 2, 3 \mod 8, \\ 0 & \text{for } i \equiv 0, 4, 6, 7 \mod 8 \end{cases}$$

for $i \geqslant 2$.

[6] Although map (A.4) is analytic, it has no differentiable trivializations.

Appendix B. Linear ordinary differential operators in Hilbert spaces

This appendix summarizes some results about linear ordinary differential operators in Hilbert spaces. See [AM03b] for a detailed exposition (see also [RS95,LT03] for related results in the framework of unbounded operators, and for an extensive bibliography).

Let H be a real Hilbert space. A bounded linear operator $L \in \mathcal{L}(H)$ is said *hyperbolic* if its spectrum does not meet the imaginary axis. In such a case, let $H = V^+(L) \oplus V^-(L)$ be the L-invariant splitting of L into closed subspaces, corresponding to the decomposition of the spectrum of L into positive and negative real part.

Proposition B.1 (*[AM01, Proposition 2.2]*). *Let $L, L' \in \mathcal{L}(H)$ be hyperbolic operators. If L' is a compact perturbation of L, then $V^+(L')$ is a compact perturbation of $V^+(L)$, and $V^-(L')$ is a compact perturbation of $V^-(L)$.*

Let $A : [0, +\infty] \to \mathcal{L}(H)$ (resp. $A : [-\infty, 0] \to \mathcal{L}(H)$) be a piecewise continuous path such that $A(+\infty)$ (resp. $A(-\infty)$) is hyperbolic. We shall denote by $X_A : [0, +\infty[\to \mathrm{GL}(H)$ (resp. $X_A :]-\infty, 0] \to \mathrm{GL}(H)$) the solution of the linear Cauchy problem $X_A'(t) = A(t)X_A(t)$, $X_A(0) = I$. The *linear stable space* of A (resp. the *linear unstable space* of A) is the linear subspace of H

$$W_A^s = \left\{ \xi \in H \mid \lim_{t \to +\infty} X_A(t)\xi = 0 \right\}$$

$$\left(\text{resp. } W_A^u = \left\{ \xi \in H \mid \lim_{t \to -\infty} X_A(t)\xi = 0 \right\} \right).$$

The main properties of the linear stable space are listed in the following:

Theorem B.2 *([AM03b, Proposition 1.2 and Theorems 2.1, 3.1]) Let $A : [0, +\infty] \to \mathcal{L}(H)$ be a piecewise continuous path such that $A(+\infty)$ is hyperbolic. Then W_A^s is a closed subspace of H, which depends continuously on A in the $L^\infty([0, +\infty[, \mathcal{L}(H))$ topology. The following convergence results for $t \to +\infty$ hold*:

(i) *W_A^s is the only closed subspace W such that $X_A(t)W$ converges to $V^-(A(+\infty))$;*
(ii) *$\|X_A(t)|_{W_A^s}\|$ converges to 0 exponentially fast.*

The above limits are locally uniform in A, with respect to the L^∞ topology. Moreover, if $V \in \mathrm{Gr}(H)$ is a linear supplement of W_A^s,

(iii) *$X_A(t)V$ converges to $V^+(A(+\infty))$;*
(iv) *$\inf_{\substack{\xi \in V \\ |\xi|=1}} |X_A(t)\xi|$ diverges exponentially fast.*

The above limits are locally uniform in $V \in \mathrm{Gr}(H)$, and in A, with respect to the L^∞ topology. Finally:

(v) $W_{-A^*}^s = (W_A^s)^\perp$.

The analogous statements for the linear unstable space can be deduced from the above theorem, taking into account the identity $X_A(t) = X_B(-t)$ for $B(t) = -A(-t)$. The following proposition characterizes those paths A for which the evolution of the linear stable space remains in a fixed essential class:

Proposition B.3 (*[AM03b, Proposition 3.8]*). *Let $A : [0, +\infty] \to \mathcal{L}(H)$ be a piecewise continuous path such that $A(+\infty)$ is hyperbolic. Let V be a closed subspace of H, and let P be a projector onto V. Then the following statements are equivalent*:

(i) $X_A(t)W_A^s$ *is a compact perturbation of V for any* $t \geqslant 0$;
(ii) $V^-(A(+\infty))$ *is a compact perturbation of V and $[A(t), P]P$ is compact for any* $t \geqslant 0$.

The proof of the above proposition is based on the following fact: if V is a closed linear subspace of H, then the orthogonal projector $P(t)$ onto $X_A(t)V$ solves the Riccati equation

$$P'(t) = (I - P(t))A(t)P(t) + P(t)A(t)^*(I - P(t)), \tag{B.1}$$

as shown in [AM03b, formula (35)].

Now let $A : [-\infty, +\infty] \to \mathcal{L}(H)$ be a continuous path such that $A(-\infty)$ and $A(+\infty)$ are hyperbolic. If $C_0^0(\mathbb{R}, H)$ (resp. $C_0^1(\mathbb{R}, H)$) denotes the Banach space of continuous curves $u : \mathbb{R} \to H$ such that $u(t)$ is infinitesimal (resp. $u(t)$ and $u'(t)$ are infinitesimal) for $t \to \pm\infty$, we can consider the bounded linear operator

$$F_A : C_0^1(\mathbb{R}, H) \to C_0^0(\mathbb{R}, H), \quad (F_A u)(t) = u'(t) - A(t)u(t).$$

Its main properties are listed in the following:

Theorem B.4 (*[AM03b, Theorem 5.1 and Remark 5.1]*). *Let $A : [-\infty, +\infty] \to \mathcal{L}(H)$ be a continuous path such that $A(-\infty)$ and $A(+\infty)$ are hyperbolic. Then*:

(i) F_A *has closed range if and only if the linear subspace* $W_A^s + W_A^u$ *is closed*;
(ii) F_A *is surjective if and only if* $W_A^s + W_A^u = H$;
(iii) F_A *is injective if and only if* $W_A^s \cap W_A^u = (0)$;
(iv) F_A *is a Fredholm operator if and only if* (W_A^s, W_A^u) *is a Fredholm pair, and in this case* $\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u)$.

It is easy to build examples of paths A having two arbitrary closed linear subspaces as linear stable space and linear unstable space, so the above theorem shows that in general F_A may not have closed range, and its kernel and cokernel may be infinite dimensional. Even F_A is Fredholm, $A(t)$ is self-adjoint and invertible for any t, and $A(-\infty) = A(+\infty)$, the operator F_A may still have any index. In the following proposition we exhibit such an example with positive index. By Theorem B.2(v), we obtain an example with negative index by considering the path $B(t) = -A(t)$.

Proposition B.5. *Let H be a separable infinite dimensional real Hilbert space. Let $H = H^- \oplus H^+$ be an orthogonal splitting, with $H^-, H^+ \in \mathrm{Gr}_{\infty,\infty}(H)$. For any $k \in \mathbb{N}$ there exists $A \in C^\infty(\mathbb{R}, \mathrm{GL}(H) \cap \mathrm{Sym}(H))$ such that $A(t) = P_{H^+} - P_{H^-}$ for $t \notin (0,1)$, and $W_A^s + W_A^u = H$, $\dim W_A^s \cap W_A^u = k$. In particular, F_A is a surjective Fredholm operator of index k.*

Proof. Let $W \subset H^+$ be a linear subspace of dimension k. Since $\mathrm{Gr}_{\infty,\infty}(H)$ is connected, there exist closed subspaces $V_0 = H^- \oplus W, V_1, \dots, V_{m-1}, V_m = H^-$ in $\mathrm{Gr}_{\infty,\infty}(H)$ with $\mathrm{dist}\,(V_{j-1}, V_j) < 1$ for any $j \in \{1, \dots, m\}$ (in fact it is possible to take $m = 4$). Denote by $\mathcal{S}$ the open subset of $\mathrm{Sym}(H)$ consisting of the invertible operators A with $\sigma_{\mathrm{ess}}(A) \cap \mathbb{R}^\pm \neq \emptyset$. By Proposition A.1 we can find operators $A_1, \dots, A_m$ in $\mathcal{S}$ such that $e^{A_j/m} V_{j-1} = V_j$. Define the piecewise constant path $B : \mathbb{R} \to \mathcal{S}$ as

$$B(t) = \begin{cases} P_{H^+} - P_{H^-} & \text{for } t < 0 \text{ or } t \leqslant 1, \\ A_j & \text{for } \frac{j-1}{m} \leqslant t < \frac{j}{m}, \quad j \in \{1, \dots, m\}. \end{cases}$$

Since $X_B(t) = e^{tA_j/m} e^{A_{j-1}/m} \dots e^{A_1/m}$ for $(j-1)/m \leqslant t \leqslant j/m$, there holds

$$X_B(1)(H^- \oplus W) = e^{A_m/m} \dots e^{A_1/m} V_0 = V_m = H^-.$$

Since $\mathcal{S}$ is connected, there is a sequence $(B_n) \subset C^\infty(\mathbb{R}, \mathcal{S})$ with $B_n(t) = P_{H^+} - P_{H^-}$ for $t \notin (0,1)$, (B_n) bounded in $L^\infty(\mathbb{R}, \mathcal{L}(H))$, and $B_n \to B$ in $L^1(\mathbb{R}, \mathcal{L}(H))$. By the identity

$$X_A(t) = X_B(t) + \int_0^t X_B(t) X_B(\tau)^{-1} (A - B)(\tau) X_A(\tau)\, d\tau,$$

the sequence $(X_{B_n}(1))$ converges to $X_B(1)$, hence

$$\begin{aligned} W_{B_n}^s &= X_{B_n}(1)^{-1} W_{B_n(\cdot+1)}^s = X_{B_n}(1)^{-1} W_{P_{H^+} - P_{H^-}}^s \\ &= X_{B_n}(1)^{-1} H^- \to X_B(1)^{-1} H^- = H^- \oplus W. \end{aligned}$$

Moreover, $W_{B_n}^u = W_{P_{H^+} - P_{H^-}}^u = H^+$, so for n large enough $A = B_n$ satisfies $W_A^u + W_A^s = H$ and $\dim W_A^s \cap W_A^u = k$. □

Appendix C. Hyperbolic rest points

This appendix summarizes some well known results about hyperbolic dynamics. See [Shu87].

C.1. Local statements

Let F be a vector field of class C^1 defined on a neighborhood U of 0 in the real Hilbert space H. We denote by $\Omega(F)$ the maximal subset of $\mathbb{R} \times U$ where the local flow of F, i.e. the solution of

$$\partial_t \phi(t, p) = F(\phi(t, p)), \quad \phi(0, p) = p$$

is defined. We assume that 0 is a hyperbolic rest point for F, meaning that $F(0) = 0$ and $L := DF(0)$ is a hyperbolic operator, that is $\sigma(L) \cap i\mathbb{R} = \emptyset$. Let $H^u \oplus H^s$ be the splitting of H corresponding to the partition of the spectrum of L into the closed subsets $\sigma(L) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and $\sigma(L) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. By P^u and $P^s = I - P^u$ we shall denote the projections onto H^u and H^s, and we shall often identify $H = H^u \oplus H^s$ with $H^u \times H^s$.

There exists an equivalent inner product $\langle \cdot, \cdot \rangle$ on H with associated norm $\| \cdot \|$ which is *adapted* to L, meaning that H^u and H^s are orthogonal, and

$$\langle L\xi, \xi \rangle \geqslant \lambda \|\xi\|^2 \quad \forall \xi \in H^u, \quad \langle L\xi, \xi \rangle \leqslant -\lambda \|\xi\|^2 \ \ \forall \xi \in H^s \tag{C.1}$$

for some $\lambda > 0$. Indeed, we may choose any positive λ which is strictly less than $\min |\operatorname{Re} \sigma(L)|$, as shown by the following lemma, applied to $L|_{H^s}$ and to $-L|_{H^u}$.

Lemma C.1. *Let L be a bounded linear operator on H and let λ be a real number such that $\lambda > \max \operatorname{Re} \sigma(L)$. Then there exists an equivalent inner product $\langle \cdot, \cdot \rangle$ on H such that*

$$\langle L\xi, \xi \rangle \leqslant \lambda \langle \xi, \xi \rangle \quad \forall \xi \in H.$$

Proof. Up to replacing L by $L - \lambda I$, we may assume that $\lambda = 0$. Let $\langle \cdot, \cdot \rangle_*$ be any Hilbert product on H, and denote by $\| \cdot \|_*$ both the associated norm on H and the induced norm on $\mathcal{L}(H)$. By the spectral radius formula and by the spectral mapping theorem,

$$\lim_{n \to \infty} \|e^{nL}\|_*^{1/n} = \max |\sigma(e^L)| = \max |e^{\sigma(L)}| < 1.$$

Let $k \in \mathbb{N}$ be so large that $\|e^{kL}\|_* \leqslant 1$, and set

$$\langle \xi, \eta \rangle := \int_0^k \langle e^{tL}\xi, e^{tL}\eta \rangle_* \, dt \quad \forall \xi, \eta \in H.$$

Then $\langle\cdot,\cdot\rangle$ is an equivalent inner product on H, and for any $\xi \in H$

$$\langle L\xi, \xi\rangle = \int_0^k \langle e^{tL} L\xi, e^{tL}\xi\rangle_* \, dt = \frac{1}{2}\int_0^k \frac{d}{dt}\|e^{tL}\xi\|_*^2 \, dt$$
$$= \frac{1}{2}\left(\|e^{kL}\xi\|_*^2 - \|\xi\|_*^2\right) \leqslant \frac{1}{2}\left(\|e^{kL}\|_*^2 - 1\right)\|\xi\|_*^2 \leqslant 0,$$

concluding the proof. $\square$

If V is a closed linear subspace of H and $r > 0$, $V(r)$ will denote the closed ball of V centered in 0 with radius r. Moreover, we set

$$Q(r) := \left\{\xi \in H \mid \|P^u\xi\| \leqslant r, \ \|P^s\xi\| \leqslant r\right\}.$$

If $A \subset X \subset H$, the set A is said *positively* (*negatively*) *invariant with respect to* X if for every $\xi \in A$ and for every $t > 0$, $\phi([0,t] \times \{\xi\}) \subset X$ implies $\phi([0,t] \times \{\xi\}) \subset A$ (resp. for every $\xi \in A$ and for every $t < 0$, $\phi([t,0] \times \{\xi\}) \subset X$ implies $\phi([t,0] \times \{\xi\}) \subset A$).

Lemma C.2. *For any $r > 0$ small enough, the set*

$$\left\{\xi \in Q(r) \mid \|P^s\xi\| \leqslant \|P^u\xi\|\right\} \quad (resp. \ \left\{\xi \in Q(r) \mid \|P^u\xi\| \leqslant \|P^s\xi\|\right\})$$

is positively (*resp. negatively*) *invariant with respect to $Q(r)$. Moreover, if ξ belongs to the set*

$$\left\{\xi \in Q(r) \mid \|P^u\xi\| = r\right\} \quad (resp. \ \left\{\xi \in Q(r) \mid \|P^s\xi\| = r\right\}),$$

then $\phi(t,\xi) \notin Q(r)$ (*resp. $\phi(-t,\xi) \notin Q(r)$*) *for every $t > 0$ small enough.*

Proof. By a first-order expansion of F at 0 and by (C.1),

$$\frac{d}{dt}\left\|P^u\phi(t,\xi)\right\|^2\Big|_{t=0} \geqslant 2\lambda\|P^u\xi\|^2 + o(\|P^u\xi\|^2) \quad \text{if } \|P^s\xi\| \leqslant \|P^u\xi\|,$$
$$\frac{d}{dt}\left\|P^s\phi(t,\xi)\right\|^2\Big|_{t=0} \leqslant -2\lambda\|P^s\xi\|^2 + o(\|P^s\xi\|^2) \quad \text{if } \|P^u\xi\| \leqslant \|P^s\xi\|.$$

All the statements follow from the above inequalities. $\square$

Given $r > 0$, the *local unstable manifold* and the *local stable manifold* of 0 are the sets

$$W^u_{\mathrm{loc},r}(0) = \{\xi \in Q(r) \mid]-\infty, 0] \times \{\xi\} \subset \Omega(F) \text{ and } \phi(]-\infty, 0] \times \{\xi\}) \subset Q(r)\},$$
$$W^s_{\mathrm{loc},r}(0) = \{\xi \in Q(r) \mid [0, +\infty[\times \{\xi\} \subset \Omega(F) \text{ and } \phi([0, +\infty[\times \{\xi\}) \subset Q(r)\}.$$

Then the local stable manifold theorem (see [Shu87, Chapter 5]) states that:

Theorem C.3. *For any $r > 0$ small enough, $W^u_{\mathrm{loc},r}(0)$ (respectively, $W^s_{\mathrm{loc},r}(0)$) is the graph of a C^1 map $\sigma^u : H^u(r) \to H^s(r)$ such that $\sigma^u(0) = 0$ and $D\sigma^u(0) = 0$ (resp. of a C^1 map $\sigma^s : H^s(r) \to H^u(r)$ such that $\sigma^s(0) = 0$ and $D\sigma^s(0) = 0$). Moreover, for any $\xi \in W^u_{\mathrm{loc},r}(0)$ (resp. for any $\xi \in W^s_{\mathrm{loc},r}(0)$), there holds*

$$\lim_{t\to-\infty} \phi(t, \xi) = 0 \quad (resp. \lim_{t\to+\infty} \phi(t, \xi) = 0).$$

We recall that a non-degenerate local Lyapunov function for the vector field F at the rest point 0 is a C^1 real function defined on a neighborhood of 0 in H, such that $Df(\xi)[F(\xi)] < 0$ for $\xi \neq 0$, and which is twice differentiable at 0, with the quadratic form $D^2 f(0)$ coercive on H^u, and the quadratic form $-D^2 f(0)$ coercive on H^s (necessarily, $Df(0) = 0$). A first order expansion of F at 0 shows that the restriction of the function

$$f(\xi) = -\tfrac{1}{2}\langle L\xi, \xi\rangle,$$

to a suitably small neighborhood of 0 is a non-degenerate local Lyapunov function for F at 0.

Lemma C.4. *For any $r > 0$ small enough, for every sequence $(\xi_n) \subset H$ converging to 0 and for every sequence $(t_n) \subset [0, +\infty[$ such that $\phi([0, t_n] \times \{\xi_n\}) \subset Q(r)$ and $\phi(t_n, \xi_n) \in \partial Q(r)$, there holds*

$$\operatorname{dist}\left(\phi(t_n, \xi_n), W^u_{\mathrm{loc},r}(0) \cap \partial Q(r)\right) \to 0.$$

Furthermore, if f is a non-degenerate local Lyapunov function for F at 0, there holds

$$\limsup_{n\to\infty} f(\phi(t_n, \xi_n)) < f(0).$$

Finally, there exists $r' < r$ such that

$$\sup\left\{f(\xi) \mid \xi \in \partial Q(r) \; and \; \exists t < 0 \; such \; that \; \phi(-t, \xi) \in Q(r')\phi([-t, 0] \times \{\xi\}) \subset Q(r)\right\} < \inf\left\{f(\xi) \mid \xi \in Q(r')\right\}.$$

Proof. If the vector field is linear, $F(\xi) = L\xi$, the first conclusion is immediate: actually for any $(\xi_n) \subset H$ converging to 0 and any $(t_n) \subset [0, +\infty[$, there holds

$$\lim_{n\to\infty} \operatorname{dist}\left(e^{t_n L}\xi_n, H^u\right) = 0. \tag{C.2}$$

By the Grobman–Hartman theorem, if $r_1 > 0$ is small enough the local flow ϕ restricted to $Q(r_1)$ is conjugated to its linearization $(t, \xi) \mapsto e^{tL}\xi$, by a bi-uniformly continuous homeomorphism. [7] By Theorem C.3, we may also assume that r_1 is so small that $W^u_{\mathrm{loc},r_1}(0)$ is the graph of a uniformly continuous map $\sigma^u : H^u(r_1) \to H^s(r_1)$.

Let $r < r_1$ and set $\eta_n := \phi(t_n, \xi_n) \in \partial Q(r)$, with $\xi_n \to 0$ and $t_n \geqslant 0$. By the linear case (C.2) and by the uniform continuity of the conjugacy, there exists $(\eta'_n) \subset W^u_{\mathrm{loc},r_1}(0)$ such that $\|\eta_n - \eta'_n\|$ is infinitesimal. Setting $\eta''_n = (P^u\eta_n, \sigma^u(P^u\eta_n)) \in W^u_{\mathrm{loc},r}(0) \cap \partial Q(r)$, by the uniform continuity of σ^u we have

$$\begin{aligned}
&\operatorname{dist}\left(\eta_n, W^u_{\mathrm{loc},r}(0) \cap \partial Q(r)\right)\\
&\quad\leqslant \|\eta_n - \eta''_n\| \leqslant \|\eta_n - \eta'_n\| + \|P^u\eta'_n - P^u\eta''_n\| + \|P^s\eta'_n - P^s\eta''_n\|\\
&\quad= \|\eta_n - \eta'_n\| + \|P^u\eta'_n - P^u\eta_n\| + \|\sigma^u(P^u\eta'_n) - \sigma^u(P^u\eta_n)\| \to 0,
\end{aligned}$$

proving the first claim. Since the local unstable manifold is tangent to H^u at 0, since $Df(0) = 0$ and $-D^2f(0)$ is coercive on H^u, by $o(r)$ considerations we have

$$\sup\left\{f(\xi) \mid \xi \in W^u_{\mathrm{loc},r}(0) \cap \partial Q(r)\right\} < f(0),$$

if $r > 0$ is small enough. Since f is uniformly continuous on $Q(r)$ for r small enough, the second claim follows from the first one. The last claim is an immediate consequence of the second one, arguing by contradiction. □

Given two metric spaces X and Y and a positive number θ, $\mathrm{Lip}_\theta(X, Y)$ will denote the space of θ-Lipschitz maps from X to Y, endowed with the C^0 topology. The following version of the graph transform theorem is proved in [AM01, Proposition A.3 and Addendum A.5] (see also [Shu87, Chapter 5]).

Proposition C.5. *For any $r > 0$ small enough there is a continuous (nonlinear) semi-group*

$$\Gamma : [0, +\infty] \times \mathrm{Lip}_1(H^u(r), H^s(r)) \to \mathrm{Lip}_1(H^u(r), H^s(r))$$

such that for every $\sigma \in \mathrm{Lip}_1(H^u(r), H^s(r))$ there holds:

(i) $\Gamma(0, \sigma) = \sigma$, *and* $\Gamma(t + s, \sigma) = \Gamma(t, \Gamma(s, \sigma))$, *for every* $t, s \in [0, +\infty]$;

[7] We recall that this conjugacy is found as a fixed point of a contraction T on a suitable space of continuous maps (see [Shu87, Chapter 7]). Since for $\alpha \in]0, 1[$ small enough, the space of α-Hölder continuous maps is T-invariant, such a conjugacy turns out to be Hölder continuous together with its inverse. In general, it needs not be even Lipschitz continuous.

(ii) *for every* $t \in [0, +\infty[$, *the restriction of* ϕ_t *to* $Q(r)$ *maps the graph of* σ *onto the graph of* $\Gamma(t, \sigma)$, *that is*

$$\operatorname{graph} \Gamma(t, \sigma) = \{\phi(t, \xi) \mid \xi \in \operatorname{graph} \sigma \textit{ and } \phi([0, t] \times \{\xi\}) \subset Q(r)\}\,;$$

(iii) $\operatorname{graph} \Gamma(+\infty, \sigma) = W^u_{\mathrm{loc},r}(0)$;

(iv) *for any* $\theta > 0$ *there exists* $r_0 \in]0, r]$ *and* $t_0 \geqslant 0$ *such that the restriction of* $\Gamma(t, \sigma)$ *to* $H^u(r_0)$ *is in* $\mathrm{Lip}_\theta(H^u(r_0), H^s(r_0))$, *for any* $t \in [t_0, +\infty]$ *and any* $\sigma \in \mathrm{Lip}_1(H^u(r), H^s(r))$.

Furthermore:

(v) *if* $V \subset H^u(r)$ *is open and* $\sigma \in \mathrm{Lip}_1(V, H^s(r))$ *is such that* $\operatorname{graph} \sigma \cap W^s_{\mathrm{loc},r}(0) \neq \emptyset$, *then there exists* $t \geqslant 0$ *and* $\sigma' \in \mathrm{Lip}_1(H^u(r), H^s(r))$ *such that the restriction of* ϕ_t *to* $Q(r)$ *maps the graph of* σ *onto the graph of* σ', *that is*

$$\operatorname{graph} \sigma' = \{\phi(t, \xi) \mid \xi \in \operatorname{graph} \sigma \textit{ and } \phi([0, t] \times \{\xi\}) \subset Q(r)\}\,.$$

C.2. Global statements

Now let F be a C^1 vector field on the real Hilbert manifold M, and let $\phi : \Omega(F) \to M$, $\Omega(F) \subset \mathbb{R} \times M$, denote its local flow. Let x be a hyperbolic rest point of F. We can identify a neighborhood of x in M with a neighborhood of 0 in the Hilbert space H, identifying x with 0. We denote by $H = H^u \oplus H^s$ the splitting of H associated to the hyperbolic operator $\nabla F(x) = DF(0)$, and we endow H with an equivalent inner product adapted to $\nabla F(x)$, as in the previous section. For $r > 0$ small enough, we set

$$Q(r) = H^u(r) \times H^s(r), \quad Q^+(r) = \partial H^u(r) \times H^s(r), \quad Q^-(r) = H^u(r) \times \partial H^s(r).$$

Lemma C.2 and the last statement of Lemma C.4 have the following consequence.

Proposition C.6. *For any* $r > 0$ *small enough there holds*:

(i) *if* $p \in Q(r)$ *and* $\phi(t, p) \notin Q(r)$ *for some* $t > 0$, *then there exists* $s \in [0, t[$ *such that* $\phi(s, p) \in Q^+(r)$;

(ii) *if* $p \in Q(r)$ *and* $\phi(t, p) \notin Q(r)$ *for some* $t < 0$, *then there exists* $s \in]t, 0]$ *such that* $\phi(s, p) \in Q^-(r)$.

Moreover, if F admits a global C^1 *Lyapunov function which is twice differentiable and non-degenerate at* x:

(iii) *if* $p \in Q^+(r)$, *then* $\phi(t, p) \notin Q(r)$ *for any* $t > 0$;

(iv) *if* $p \in Q^-(r)$, *then* $\phi(t, p) \notin Q()$ *for any* $t < 0$.

The unstable and stable manifolds of x are the ϕ-invariant subsets of M

$$W^u(x) = \left\{ p \in M \mid]-\infty, 0] \times \{p\} \subset \Omega(F) \text{ and } \lim_{t \to -\infty} \phi(t, p) = x \right\},$$

$$W^s(x) = \left\{ p \in M \mid [0, +\infty[\times \{p\} \subset \Omega(F) \text{ and } \lim_{t \to +\infty} \phi(t, p) = x \right\}.$$

The local stable manifold theorem (Theorem C.3) and Proposition C.6 imply:

Theorem C.7. *The sets $W^u(x)$ and $W^s(x)$ are images of C^1 injective immersions*

$$e^u : H^u \to M, \quad e^s : H^s \to M,$$

such that $e^u(0) = e^s(0) = x$, and $De^u(0)$ and $De^s(0)$ are the identity mappings. If moreover F admits a global C^1 Lyapunov function which is twice differentiable and non-degenerate at x, then for any $r > 0$ small enough

$$W^u(x) \cap Q(r) = W^u_{\mathrm{loc},r}(0), \quad W^s(x) \cap Q(r) = W^s_{\mathrm{loc},r}(0)$$

and the maps e^u, e^s are embeddings, so that $W^u(x)$ and $W^s(x)$ are C^1 submanifolds of M.

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