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Graphs with Magnetic Schrödinger Operators of Low Corank[1]

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Colin de Verdière (1998, *J. Combin. Theory, Ser. B.* **74**, 121–146) introduced the graph parameter $\nu(G)$, which is defined as the maximal corank of any positive semidefinite magnetic Schrödinger operator fulfilling a certain transversality condi-

$\nu(G) \leq n$. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Let $G=(V,E)$ be a graph with vertex set $V=\{1,\ldots,n\}$, which is allowed to have parallel edges, but which does not have loops. Let $\mathcal{M}_G$ be defined as the space of all Hermitian $n\times n$ matrices $M=(m_{i,j})$ with

(i) $m_{i,j}\neq 0$ if i and j are connected by only one edge, and with
(ii) $m_{i,j}=0$ if $i\neq j$ and i and j are not adjacent.

So, $m_{i,i}\in\mathbb{R}$, and if i and j are connected by at least two edges, then we allow $m_{i,j}=0$. A matrix $M\in\mathcal{M}_G$ is said to fulfill the *strong Arnol'd property* (SAP) if there is no nonzero Hermitian $n\times n$ matrix $X=(x_{i,j})$ such that $MX=0$, and $x_{i,j}=0$ if $i=j$ or if i and j are adjacent. The maximum corank of any positive semidefinite Hermitian $n\times n$ matrix $M\in\mathcal{M}_G$ fulfilling the strong Arnol'd property is denoted by $\nu(G)$. This invariant $\nu(G)$ was studied by Colin de Verdière [5], although only for connected simple graphs G. He showed that if G' is a connected minor of a connected simple

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graph G, then $\nu(G') \leqslant \nu(G)$, and that $\nu(G) \leqslant 1$ if and only if G is a tree. Furthermore, it was shown that, in contrast to the graph invariant $\mu(G)$ introduced in [4] (see [3] for an English translation), the invariant $\nu(G)$ is not bounded by some formula in terms of the genus of the surface in which the graph can be embedded. It turns out that $\nu(G)$ can be arbitrarily large on the class of planar graphs; see [5] for a description of planar graphs for which $\nu(G)$ can be arbitrarily large. However, $\nu(G)$ was shown to be bounded from above by some kind of tree-width, $\mathrm{la}(G)$, of a graph G, and $\mathrm{la}(G)-1$ was shown to be bounded from above by the tree-width of G. Using the result of Robertson and Seymour (see for example [6] for a short proof) that for every planar graph H there is a number k such that if G has tree-width at least k, then G has a minor isomorphic to H, one can conclude that $\nu(G)$ is large if and only if the tree-width of the graph G is large.

In this paper we characterize the class of graphs G for which $\nu(G) \leqslant k$, $k=2, 3$. For $k=2$, this class has K_4 and C_3^2 (this is the graph obtained from K_4 by applying one $Y\Delta$-transformation) as excluded minors; see Section 5. This was also shown in [7]; here we give a different proof. For $k=3$, the excluded minors are K_5 and all those graphs that can be obtained from the 3-cube by repeatedly applying $Y\Delta$-transformation; see Section 9.

What happens if instead of the strong Arnol'd property we look at another property? For example, what is the maximum corank of any positive semidefinite $M \in \mathcal{M}_G$ such that if k is the corank of M, then every $(n-k)\times(n-k)$ principal submatrix of M is nonsingular? This question was solved by Lovász *et al.* in [10]. Modulo the fact that they used real-valued orthogonal representations instead of Hermitian matrices $M \in \mathcal{M}_G$, the answer is that the maximum corank of any such matrix is at least k if and only if G is k-connected. In Section 6 we show that any $M \in \mathcal{M}_G$ of corank k for which every principal $(n-k)\times(n-k)$ matrix is nonsingular fulfills the strong Arnol'd property. This implies that $\nu(G)$ is at least the connectivity of G. In that section we also show that if G is a graph whose underlying simple graph is V_8, then $\nu(G) \leqslant 3$ unless G has a minor isomorphic to one of the graphs obtained from the 3-cube by repeatedly applying $Y\Delta$-transformations.

2. PRELIMINARIES

Basic graph theory. Let $G=(V, E)$ be a graph which we allow to have parallel edges but no loops. The underlying simple graph of a graph G is the graph obtained by suppressing multiple edges. If $S \subseteq V$, then

$G-S$ denotes the subgraph of G induced by the vertices in $V \setminus S$. If $S \subseteq V$, then $G[S]$ denotes the induced subgraph of G on S. If H is a subgraph of G, then $N(H)$ denotes the set of neighbors in $V(G) \setminus V(H)$ of vertices in H. If $e \in E$ (by assumption e is not a loop), then G/e denotes the graph obtained from G by deleting e and identifying the ends of e. We say that G/e is obtained from G by contracting edge e. A graph that is obtained from a subgraph of G by contracting a series of edges is called a minor of G. Let G and H be graphs. We say that G has an H-minor if G has a minor isomorphic to H. We say that a class $\mathscr{C}$ of graphs is closed under taking minors and isomorphism if it has the property that if G belongs to $\mathscr{C}$, then every graph isomorphic to G belongs to $\mathscr{C}$, and if G' is a minor of a graph G which belongs to $\mathscr{C}$, then G' belongs to $\mathscr{C}$. Let $\mathscr{C}$ be a class of graphs closed under taking minors and isomorphism. Then a graph H is called an excluded minor for $\mathscr{C}$ if H does not belong to $\mathscr{C}$, but each proper minor of H belongs to $\mathscr{C}$. The well-quasi-ordering theorem of Robertson and Seymour [11] says that any class of graphs closed under taking minors and isomorphisms has a finite collection of excluded minors.

A pair (G_1, G_2) with $G_1 \cup G_2 = G$, $E(G_1) \cap E(G_2) = \emptyset$ is called a separation of G; its order is $|V(G_1) \cap V(G_2)|$. A subset S of the vertices of G is called a vertex cut if $G-S$ is disconnected. If $X, Y, Z \subseteq V$, then Z *separates* X and Y if every path of G between X and Y has a vertex in common with Z.

The degree of a vertex v of a graph G is the number of incident edges. The neighborhood of v is the set of vertices which are adjacent to v. Since we allow parallel edges, it may happen that the degree of a vertex is larger than the number of vertices in its neighborhood. A graph G' is obtained from G by a *$Y\Delta$-transformation* (at v) if v is a vertex of G of degree 3 which has three vertices in its neighborhood, G' is obtained from G by deleting vertex v and its incident edges, and by adding an edge between each pair of vertices of the neighborhood of v. A graph G' is obtained from G by a *ΔY-transformation* if G' can be obtained by deleting the edges of a triangle of G and by adding a new vertex and edges of this vertex to all vertices of the triangle.

The complete graph on n vertices is denoted by K_n. By C_3^2 we denote the graph obtained from K_4 by one $Y\Delta$-transformation.

By $K_n^=$ we denote the graph obtained from K_n by adding to each edge an edge in parallel. So $K_3^= = C_3^2$. Let $Q_3 := K_2 \times K_2 \times K_2$; that is, the graphs with vertex set all binary vectors of length 3 and two vertices are connected if their vectors differ only in one coordinate. We denote by K_4^2 the graph $K_4^=$. Let the K_4^2-family be the collection of all graphs that can be obtained from Q_3 by a series of $Y\Delta$-transformations. We denote by $Q_3Y\Delta$ the graph obtained from Q_3 by applying one $Y\Delta$-transformation. Note that the only graphs of the K_4^2-family to which we cannot apply a $Y\Delta$-transformation are $K_{2,2,2}$ and K_4^2. In Fig. 1 the graphs of the K_4^2-family are depicted.

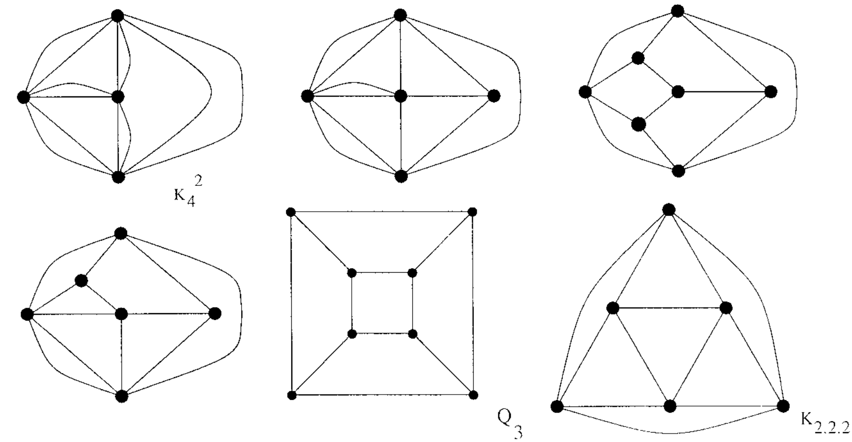

FIGURE 1

Matrix theory. An $n \times n$ matrix $M = (m_{i,j})$ with complex entries is Hermitian if $m_{i,j} = \overline{m_{j,i}}$ for all $i, j \in \{1, ..., n\}$; $\overline{m_{j,i}}$ denotes the complex conjugate of $m_{i,j}$. We denote the kernel of M by $\ker(M)$; this is the space of all vectors $x \in \mathbb{C}^n$ satisfying $Mx = 0$. The corank of M is the dimension of the kernel of M; we use the notation corank(M) for the corank of M.

Let M be a Hermitian matrix. If there is a nonzero $x \in \mathbb{C}^n$ with $Mx = \lambda x$, then λ is an eigenvalue of M, and x is called an eigenvector of M belonging to λ. Since M is Hermitian, all eigenvalues of M are real, and hence we can order the eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$.

Sylvester's law of inertia states that if A is a nonsingular $n \times n$ matrix and M is a Hermitian $n \times n$ matrix, then $A^{\mathrm{H}}MA$ has the same number of negative and positive eigenvalues (counting multiplicities) and the same corank as M.

Let

$$R(x) := \frac{x^{\mathrm{H}}Mx}{x^{\mathrm{H}}x}.$$

The quotient $R(x)$ is called the *Rayleigh quotient*. For all nonzero vectors x, $\lambda_1 \leqslant R(x)$, and $\lambda_1 = R(x)$ if and only if x is an eigenvector belonging to λ_1. A Hermitian $n \times n$ matrix M is positive semidefinite if all its eigenvalues are nonnegative; that is, $x^{\mathrm{H}}Mx \geqslant 0$ for all $x \in \mathbb{C}^n$. If $x^{\mathrm{H}}Mx = 0$ in this case, then the Rayleigh quotient tells us that $x \in \ker(M)$. A Hermitian $n \times n$ matrix M is positive definite if $x^{\mathrm{H}}Mx > 0$ for all nonzero $x \in \mathbb{C}^n$. If M is a positive semidefinite Hermitian $n \times n$ matrix of corank k, then there is a $(n-k) \times n$ matrix C of rank $n-k$ such that $M = C^{\mathrm{H}}C$.

Let $V = \{1, ..., n\}$. If $x \in \mathbb{C}^n$ then $\mathrm{supp}(x) := \{i \in V \mid x_i \neq 0\}$. If $S \subseteq V$ and $x \in \mathbb{C}^n$, then x_S denotes the subvector of x induced by the indices in S. If M

is an $n \times n$ matrix and $S, R \subseteq V$, then M_S denotes the principal submatrix of M induced by row and column indices in S, and $M_{S,R}$ denotes the submatrix of M induced by row indices in S and column indices in R.

PROPOSITION 2.1. *Let M be positive semidefinite. Let $x \in \ker(M)$ be nonzero and let $S := \mathrm{supp}(x)$. Then M_S is singular. Conversely, if M_S is singular, then there is a nonzero $x \in \ker(M)$ with $\mathrm{supp}(x) \subseteq S$.*

Proof. Since $x \in \ker(M)$, $x^{\mathrm{H}} M x = x_S^{\mathrm{H}} M_S x_S = 0$. As M_S is a principal submatrix of M, M_S is positive semidefinite. The Rayleigh quotient tells us that $M_S x_S = 0$, and hence M_S is singular.

Conversely, let $y \in \ker(M_S)$ be nonzero, and let x be defined by $x_S = y$ and $x_i = 0$ if $i \notin S$. Then $x^{\mathrm{H}} M x = y^{\mathrm{H}} M_S y = 0$. As M is positive semidefinite, $Mx = 0$, by the Rayleigh quotient. ∎

For further definitions in matrix theory, we refer to [9]. For the basic definitions in graph theory, we refer to [2].

3. CERTAIN SCHRÖDINGER OPERATORS

Let $G = (V, E)$ be a graph with vertex set $V = \{1, ..., n\}$ and with no loops. Let $\mathcal{M}_G$ be the set of all Hermitian $n \times n$ matrices $M = (m_{i,j})$ with

(i) $m_{i,j} \neq 0$ if i and j are connected by only one edge, and with

(ii) $m_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent.

So, $m_{i,i} \in \mathbb{R}$ and if i and j are connected by at least two edges then we allow $m_{i,j} = 0$.

THEOREM 3.1. *Let $G = (V, E)$ be a graph and let $M \in \mathcal{M}_G$ be positive semidefinite. Let $k := \mathrm{corank}(M)$. If $x \in \ker(M)$, $x \neq 0$, then $G[\mathrm{supp}(x)]$ has at most k components. Furthermore, if C is a component of $G[\mathrm{supp}(x)]$ and y is defined by $y_i = x_i$, $i \in V(C)$, and $y_i = 0$, $i \notin V(C)$, then $y \in \ker(M)$.*

Proof. Let $C_1, ..., C_t$ be the components of $G[\mathrm{supp}(x)]$. Let $x(C_l)$, for $l = 1, ..., t$, be the vector with $x(C_l)_j = x_j$ if $j \in C_l$ and $x(C_l)_j = 0$ otherwise. Note that $x(C_l)^H Mx(C_l) = 0$ and that $x(C_k)^H Mx(C_l) = 0$, as the support of $Mx(C_l)$ is a subset of $V \setminus \mathrm{supp}(x)$. For each $(\alpha_1, ..., \alpha_t) \in \mathbb{C}^t$, let

$$z_{(\alpha_1, ..., \alpha_t)} := \sum_{i=1}^{t} \alpha_i x(C_i).$$

Since

$$z^H_{(\alpha_1, \ldots, \alpha_t)} M z_{(\alpha_1, \ldots, \alpha_t)} = 0,$$

we see, from the Rayleigh quotient that $z_{(\alpha_1, \ldots, \alpha_t)}$ belongs to $\ker(M)$. Hence $t \leqslant k$. Furthermore $y = x(C) \in \ker(M)$. ▮

PROPOSITION 3.2. *Let $M \in \mathcal{M}_G$ be positive semidefinite. Let $x \in \ker(M)$ be nonzero with $G[\operatorname{supp}(x)]$ connected. If $v \notin \operatorname{supp}(x)$ is adjacent to a vertex of $\operatorname{supp}(x)$, then there are at least two edges connecting v to $\operatorname{supp}(x)$.*

Proof. If there is only one edge connecting v to $\operatorname{supp}(x)$, then the vth column of Mx is nonzero. ▮

PROPOSITION 3.3. *Let $M = (m_{i,j}) \in \mathcal{M}_G$ be positive semidefinite. Then $m_{i,i} = 0$ only if i is connected to each of its neighbors by at least two edges.*

Proof. Since M is positive semidefinite $M = A^{\mathrm{H}} A$ for some matrix A, where A^{H} denotes the conjugate of A. If $m_{i,i} = 0$, then the ith column of A would be zero, and hence in M all entries $m_{i,j}$ and $m_{j,i}$ would be zero for $j \in V(G)$. This is, by definition, only possible if i is connected to each of its neighbors by at least two edges. ▮

4. THE STRONG ARNOL'D PROPERTY

Let $\mathcal{M}_{n,k}$ denote the manifold of all Hermitian $n \times n$ matrices with corank k. A matrix $M \in \mathcal{M}_G$ of corank k is said to fulfill the *strong Arnol'd property* (w.r.t. G) if $\mathcal{M}_G$ and $\mathcal{M}_{n,k}$ intersect transversally in M. This means that the span of the tangent space of $\mathcal{M}_G$ at M and the tangent space of $\mathcal{M}_{n,k}$ at M is equal to the space of all Hermitian $n \times n$ matrices. In Theorem 4.3 the equivalence of this definition with the definition given in the Introduction is shown. If it is clear what graph G we use, then we omit G, and we say that M fulfills the SAP. To check whether a matrix fulfills the SAP, we have the following theorem [3, 5]. Although stated for simple connected graphs the theorem also holds for graphs with parallel edges.

THEOREM 4.1. *Let $G = (V, E)$ be a graph with n vertices. A matrix $M \in \mathcal{M}_G$ fulfills the SAP if and only if, for every Hermitian $n \times n$ matrix A, there is a Hermitian $n \times n$ matrix $B = (b_{i,j})$, with $b_{i,j} = 0$ if $i \neq j$ and i and j are not connected by an edge, such that for all $x \in \ker(M)$, $x^{\mathrm{H}} A x = x^{\mathrm{H}} B x$.*

This criterion shows that the SAP only depends on $\ker(M)$ (and of course G.)

Let $M \in \mathscr{M}_G$ and let L be a matrix whose rows consist of the vectors of a basis of $\ker(M)$ (viewed as row vectors). Let l_i, $i \in V$ be the columns of L. Then Theorem 4.1 says that M fulfills the SAP if and only if the linear span of all the matrices $l_i l_i^{\mathrm{H}}$, $i \in V$, and $l_i l_j^{\mathrm{H}} + l_j l_i^{\mathrm{H}}$, $i, j \in V$ adjacent, is equal to the space of all Hermitian $d \times d$ matrices, where $d = \operatorname{corank}(M)$. Looking to the normal space of the linear span, we get the following criterion.

THEOREM 4.2. *Let $G=(V, E)$ be a graph. Let $M \in \mathscr{M}_G$ and let l_i, $i \in V$, be the columns of the matrix whose rows are the vectors of a basis of* $\ker(M)$. *Let $d := \operatorname{corank}(M)$. Then M fulfills the SAP if and only if there is no nonzero Hermitian $d \times d$ matrix A such that $l_i^{\mathrm{H}} A l_i = 0$ for all $i \in V$ and $l_i^{\mathrm{H}} A l_j = 0$ for all $i, j \in V$ adjacent.*

By using, for example, Gaussian elimination one can check if a matrix $M \in \mathscr{M}_G$ fulfills the SAP. Another useful criterion to check if a matrix M fulfills the SAP is stated in the following theorem. In [8] it is stated for real-valued symmetric matrices, but a proof for Hermitian matrices goes along the same lines as in the real-valued case.

THEOREM 4.3 [8]. *Let $G=(V, E)$ be a graph. Then $M \in \mathscr{M}_G$ fulfills the SAP if and only if there is no nonzero Hermitian matrix $X=(x_{i,j})$ with $x_{i,j}=0$ if $i=j$ or if i and j are adjacent, such that $MX=0$.*

This theorem follows from Theorem 4.2, as we can take $X := L^{\mathrm{H}} A L$ if M does not fulfill the SAP, and by the spectral decomposition theorem any such X can be written as $K^{\mathrm{H}} B K$, where each row of K belongs to the span of the rows of X and where B is a diagonal matrix.

The following proposition allows us to get bounds on $\nu(G)$.

PROPOSITION 4.4. *Let $G=(V, E)$ be a graph. Let $M \in \mathscr{M}_G$ be positive semidefinite and such that M fulfills the SAP. Then $G[\operatorname{supp}(x)]$ is a connected graph for each nonzero $x \in \ker(M)$.*

Proof. If there exists a nonzero $x \in \ker(M)$ with $G[\operatorname{supp}(x)]$ disconnected, then, by Theorem 3.1, there are nonzero vectors $y, z \in \ker(M)$ with $\operatorname{supp}(y) \cap \operatorname{supp}(z) = \varnothing$. Let $X = (x_{i,j}) := yz^{\mathrm{H}} + zy^{\mathrm{H}}$. Then $MX=0$, X is nonzero, and $x_{i,j}=0$ if $i=j$ or if i and j are adjacent. ∎

In this proposition, the SAP is essential. It is easy to give an example of a graph G such that there is a positive semidefinite matrix $M \in \mathscr{G}$ and a nonzero $x \in \ker(M)$, with $G[\operatorname{supp}(x)]$ disconnected. Let G be the graph consisting of two isolated vertices, and let $M \in \mathscr{M}_G$ be the 2×2 all-zero matrix. Then $x=(1,1)^{\mathrm{H}} \in \ker(M)$ has $G[\operatorname{supp}(x)]$ disconnected. Thus M does not fulfill the SAP. Any positive semidefinite matrix $N=(n_{i,j}) \in \mathscr{M}_G$

fulfilling the SAP will have $n_{i,i} = 0$ for at most one vertex of G. In the more general case where G is an arbitrary but disconnected graph, any positive semidefinite matrix $M \in \mathscr{M}_G$ will have $M_{V(C)}$ singular for at most one component C of G.

Let G be the graph as depicted in Fig. 2. The matrix

$$M := (m_{i,j}) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

belongs to $\mathscr{M}_G$. A representation of the kernel of M is given by the row vectors of the following matrix

$$L := (l_i) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

The Hermitian matrix

$$A := (a_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

shows that M does not fulfill the SAP. However, for each $x \in \ker(M)$, $G[\mathrm{supp}(x)]$ is connected. So the converse of Proposition 4.4 is not true in general.

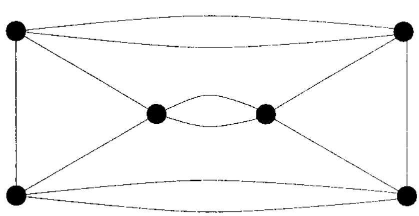

FIGURE 2

5. THE PARAMETER ν OF A GRAPH

In this section we recall the definition of $\nu(G)$, given by Colin de Verdière in [5]. We give an extension to graphs which are allowed to have parallel edges and which are not necessarily connected.

The invariant $\nu(G)$ is defined as the largest corank k of any positive semidefinite matrix $M \in \mathcal{M}_G$ fulfilling the SAP.

THEOREM 5.1. *If G' is a minor of G then $\nu(G') \leqslant \nu(G)$.*

See [5] for a proof for connected simple graphs; see [7] for a proof for graphs where parallel edges are allowed. Using Theorems 5.4 and 5.7, the proof of Theorem 5.1 for graphs with parallel edges follows from the proof of Theorem 5.1 for connected simple graphs; in the proofs of Theorems 5.4 and 5.7, one does not need to use Theorem 5.1.

Let $\mathscr{C}_k$ be the class of graphs G with $\nu(G) \leqslant k$. By Theorem 5.1, $\mathscr{C}_k$ is closed under taking minors and isomorphism. By the well-quasi-ordering theorem of Robertson and Seymour [11], for each fixed k, there exists a finite collection of excluded minors for $\mathscr{C}_k$. The following two propositions give some obvious excluded minors for $\mathscr{C}_k$.

PROPOSITION 5.2. $\nu(K_n) = n-1$.

See [5]. As each proper minor H of K_n has $\nu(H) < n-1$, K_n is an excluded minor for the class of graphs G with $\nu(G) \leqslant n-2$. (To see that $\nu(H) < n-1$ if H arises from K_n by contracting an edge, one can use the facts that $K_n^{=}$ is an excluded minor for $\nu(G) \leqslant n-1$ and that H is a proper minor of $K_n^{=}$. See the following proposition.)

PROPOSITION 5.3. $\nu(K_n^{=}) = n$.

Proof. It is clear that $\nu(K_n^{=}) \leqslant n$. To see that $\nu(K_n^{=}) \geqslant n$, take $M := 0$. Then M is positive semidefinite and M fulfills the SAP. ∎

As each proper minor H of $K_n^{=}$ has $\nu(H) < n$, $K_n^{=}$ is an excluded minor for the class of graphs G with $\nu(G) \leqslant n-1$. Hence we have that K_4 and $C_3^2 = K_3^{=}$ are excluded minors for the class of graphs G with $\nu(G) \leqslant 2$. We shall see that K_4 and C_3^2 are the only excluded minors for the class of graphs G with $\nu(G) \leqslant 2$.

The following two propositions show that, for any integer $t > 1$, an excluded minor of the class of graphs G with $\nu(G) \leqslant t$ has no ($\leqslant 2$)-vertex cut.

PROPOSITION 5.4. *Let G_1 and G_2 be graphs. If G is a ($\leqslant 1$)-sum of G_1 and G_2, then $\nu(G) = \max\{\nu(G_1), \nu(G_2)\}$.*

A proof of this proposition can be found in [7]. The proof is similar to the proof of the following proposition.

PROPOSITION 5.5. *Let G be a 2-connected graph. Let $S=\{s_1, s_2\}$ be a 2-vertex cut of G and let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If G_1 or G_2 is a path connecting the vertices of S, say G_1 is a path, then $\nu(G)=\nu(G_2')$, where G_2' is the graph obtained from G_2 by adding a new edge between the vertices of S. Otherwise $\nu(G)=\max\{\nu(G_1'), \nu(G_2')\}$, where G_i', $i=1,2$ is the graph obtained from G_i by adding two edges between the vertices of S.*

Proof. The case where G_1 or G_2 is a path follows from Theorem 5.7. So we may assume that neither G_1 nor G_2 is a path connecting the vertices of S.

Since G_i', $i=1,2$ is a minor of G, $\nu(G) \geqslant \max\{\nu(G_1'), \nu(G_2')\}$.

To prove $\nu(G) \leqslant \max\{\nu(G_1'), \nu(G_2')\}$, let $M \in \mathcal{M}_G$ be a matrix with $\operatorname{corank}(M)=\nu(G)$ and such that M fulfills the SAP. Let $C := V(G_1)-S$ and let $D := V(G_2)-S$. We may write

$$M=\begin{pmatrix} M_C & M_{C,S} & 0 \\ M_{S,C} & M_S & M_{S,D} \\ 0 & M_{D,S} & M_D \end{pmatrix}.$$

Either M_C or M_D is positive definite, by Propositions 2.1 and 4.4. We assume that M_C is positive definite.

Let

$$P := \begin{pmatrix} 1 & -M_C^{-1}M_{C,S} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then

$$P^{\mathrm{H}}MP = \begin{pmatrix} M_C & 0 \\ 0 & M' \end{pmatrix},$$

where

$$M' = (m'_{i,j}) := \begin{pmatrix} M_S - M_{S,C}M_C^{-1}M_{C,S} & M_{S,D} \\ M_{D,S} & M_D \end{pmatrix}.$$

Sylvester's law of inertia tells us that M' is positive semidefinite and that M' has corank $\nu(G)$. It is clear that $M' \in \mathcal{M}_{G_2'}$.

Suppose to the contrary that M' does not fulfill the SAP. Then there is a nonzero Hermitian matrix $X' = (x'_{i,j})$ with $x_{i,j}=0$ if $i=j$ or if i and j are adjacent and with $M'X'=0$. Then $X'_{S,S}=0$, as the vertices of S are adjacent in G_2'. Let

$$X := \begin{pmatrix} 0 & 0 & -M_C^{-1} M_{C,S} X'_{S,D} \\ 0 & 0 & X'_{S,D} \\ -X'_{D,S} M_{S,C} M_C^{-1} & X'_{D,S} & X'_D \end{pmatrix}.$$

Then $X = (x_{i,j})$ is a nonzero Hermitian matrix, with $x_{i,j} = 0$ if $i = j$ or if i and j are connected by an edge in G, such that $MX = 0$. Hence M would not fulfill the SAP. ∎

With Propositions 5.4 and 5.5 we get

THEOREM 5.6. *$\nu(G) \leqslant 2$ if and only if G has no K_4- and no C_3^2-minor.*

Proof. Since $\nu(K_4) = 3$ and $\nu(C_3^2) = 3$, a graph G with a K_4- or a C_3^2-minor has $\nu(G) \geqslant 3$. For the converse, let G be a graph with $\nu(G) \geqslant 3$ and with no K_4-minor. As each 3-connected graph has a K_4-minor, G is not 3-connected. By Propositions 5.4 and 5.5, we may assume G has no $(\leqslant 2)$-vertex cuts. So G has at most three vertices. Now only the all-zero 3×3 matrix has corank equal to 3. Therefore G has a C_3^2-minor (indeed it has a subgraph isomorphic to C_3^2). ∎

THEOREM 5.7. *If G' is a subdivision of G, then $\nu(G') = \nu(G)$.*

A proof of this theorem can be found in [7]. The reader should be able to provide a proof after reading the proof of the next theorem.

THEOREM 5.8. *If G' is obtained from G by a $Y\Delta$-transformation, then $\nu(G') \geqslant \nu(G)$.*

Proof. Let $M = (m_{i,j}) \in \mathcal{M}_G$ be a positive semidefinite Hermitian matrix with corank $\nu(G)$ and such that M fulfills the SAP. Let v be the vertex of degree 3 of the Y. By Proposition 3.3, $m_{v,v} > 0$. Let S be the set of three vertices adjacent to v and let $C := V(G) - (S \cup \{v\})$. We may write

$$M = \begin{pmatrix} m_{v,v} & M_{v,S} & 0 \\ M_{S,v} & M_{S,S} & M_{S,C} \\ 0 & M_{C,S} & M_{C,C} \end{pmatrix}.$$

Let

$$M' := \begin{pmatrix} M_{S,S} - \dfrac{1}{m_{v,v}} M_{S,v} M_{v,S} & M_{S,C} \\ M_{C,S} & M_{C,C} \end{pmatrix}.$$

Then $M' \in \mathcal{M}_{G'}$. Let

$$P := \begin{pmatrix} 1 & -\dfrac{1}{m_{v,v}} M_{v,S} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then

$$P^{\mathrm{H}} M P = \begin{pmatrix} m_{v,v} & 0 \\ 0 & M' \end{pmatrix}.$$

From Sylvester's law of inertia it follows that M' is positive semidefinite and that the corank of M' is equal to the corank of M.

To see that M' fulfills the SAP, suppose that there exists a nonzero Hermitian matrix $X' := (x'_{i,j})$, with $x'_{i,j} = 0$ if $i = j$ or if i and j are connected by an edge, such that $M'X' = 0$. Then $X'_{S,S} = 0$. Let $X_{v,C} := -(1/m_{v,v})\, M_{v,S} X'_{S,C}$, $X_{v,v} = 0$, $X_{v,S} = 0$, $X_{S,S} = 0$, $X_{S,C} = X'_{S,C}$, and $X_{C,C} = X'_{C,C}$. Then $X = (x_{i,j})$ is a nonzero Hermitian matrix, with $x_{i,j} = 0$ if $i = j$ or if i and j are connected by an edge in G, such that $MX = 0$. Hence M would not fulfill the SAP. ∎

Note that it may happen that $\nu(G') > \nu(G)$ if G' is obtained from G by a $Y\Delta$-transformation, as $\nu(K_{1,3}) = 1$, while $\nu(K_3) = 2$.

We now give some excluded minors for the class of graphs G with $\nu(G) \leqslant 3$.

PROPOSITION 5.9. $\nu(Q_3) \geqslant 4$.

Proof. Let

$$M := \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 3 \end{pmatrix}.$$

Then $M \in \mathcal{M}_{Q_3}$, and M is positive semidefinite and has corank 4. Furthermore M fulfills the SAP, as can easily be checked by using Theorem 4.3. ∎

PROPOSITION 5.10. *All graphs G of the K_4^2-family have $\nu(G) = 4$.*

Proof. Since all graphs of the K_4^2-family can be obtained from Q_3 by applying $Y\Delta$-transformations, we have that $\nu(G) \geqslant 4$ for all graphs G in the K_4^2-family. If one of the graphs G of the K_4^2-family has $\nu(G) > 4$, then, as $K_{2,2,2}$ or K_4^2 can be obtained from G by applying $Y\Delta$-transformations, $\nu(K_{2,2,2}) > 4$ or $\nu(K_4^2) > 4$. With Proposition 5.3 and Corollary 6.5a we derive a contradiction. Hence $\nu(G) = 4$ for all graphs G in the K_4^2-family. ∎

Since the proper minors of H of the graphs of the K_4^2-family all have $\nu(H) \leqslant 3$, the graphs of the K_4^2-family are excluded minors for the class of graphs G with $\nu(G) \leqslant 3$. In Section 9 we shall see that K_5 and the graphs of the K_4^2-family are the only excluded minors for the class of graphs G with $\nu(G) \leqslant 3$.

6. ORTHOGONAL REPRESENTATIONS

A different characterization of $\nu(G)$ can be given, by using orthogonal representations. The main results of this section are Proposition 6.6 and Theorem 6.5.

An *orthogonal representation* of $G = (V, E)$ in $\mathbb{C}^d$ is a function $\phi: V \to \mathbb{C}^d$ such that $\phi(u)$ and $\phi(v)$ are orthogonal if the vertices u and v are nonadjacent in G. If, moreover, $\phi(u)$ and $\phi(v)$ are orthogonal only if u and v are nonadjacent or if u and v are connected by at least two edges, then the orthogonal representation ϕ is called *faithful*. This notion of faithful is slightly different from the one given in [10], where an orthogonal representation ϕ is called faithful if $\phi(u)$ and $\phi(v)$ are orthogonal if and only if u and v are nonadjacent. However, for simple graphs these two notions coincide.

An orthogonal representation $\phi: V \to \mathbb{C}^d$ of $G = (V, E)$ is in *general position* if for every set of d vertices $\{v_1, ..., v_d\}$, the set of vectors $\{\phi(v_1), ..., \phi(v_d)\}$ is linearly independent.

Orthogonal representations of G were studied by Lovász *et al.* in [10], where they showed the following theorem (using $\mathbb{R}^d$ instead of $\mathbb{C}^d$, but that does not matter here).

THEOREM 6.1. *A graph G with n vertices has a faithful general-position orthogonal representation in $\mathbb{C}^d$ if and only if G is $(n-d)$-connected.*

Each faithful orthogonal representation $\phi: V \to \mathbb{C}^d$ gives rise to a positive semidefinite matrix $M = (m_{i,j}) \in \mathcal{M}_G$ of corank $n-d$ defined by $m_{i,j} = \phi(i)^{\mathrm{H}} \phi(j)$. Conversely, from each positive semidefinite matrix $M = (m_{i,j}) \in \mathcal{M}_G$ of corank $n-d$ we can make a faithful orthogonal representation $\phi: V \to \mathbb{C}^d$. This follows from a standard result from matrix theory saying that for each positive semidefinite Hermitian $n \times n$ matrix M of corank $n-d$, there exists a $d \times n$ matrix Q of rank d such that $M = Q^{\mathrm{H}}Q$.

Let us say that an orthogonal representation is *stable* if the corresponding matrix $M = (m_{i,j}) \in \mathcal{M}_G$ defined by $m_{i,j} = \phi(i)^{\mathrm{H}} \phi(j)$ fulfills the SAP.

THEOREM 6.2. *Let $G = (V, E)$ be a graph with n vertices. Then $n - \nu(G)$ is equal to the smallest dimension d such that there is a stable faithful orthogonal representation of G in $\mathbb{C}^d$.*

LEMMA 6.3. *Let $G = (V, E)$ be a graph. Let $\phi: V \to \mathbb{C}^d$ be an orthogonal representation of G. Let $M = (m_{i,j})$ with $m_{i,j} = \phi(i)^{\mathrm{H}} \phi(j)$. Then ϕ is in general position if and only if each nonzero $x \in \ker(M)$ has at most $(n-d-1)$ entries equal to zero.*

Proof. Let $\{x_1, ..., x_{n-d}\}$ be a basis of $\ker(M)$, and let $l_1, ..., l_n$ be defined by

$$L = (l_1 \ \dots \ l_n) = \begin{pmatrix} x_1^{\mathrm{H}} \\ \vdots \\ x_{n-d}^{\mathrm{H}} \end{pmatrix}.$$

Then ϕ is in general position if and only if $\{l_i \mid i \in V\}$, is in general position. This is equivalent to the statement that each nonzero $x \in \ker(M)$ has at most $(n-d-1)$ entries equal to zero. ∎

PROPOSITION 6.4. *Every general-position orthogonal representation is stable.*

Proof. Let $\phi: V(G) \to \mathbb{C}^d$ be a general-position orthogonal representation of G. Let $M = (m_{i,j})$ with $m_{i,j} = \phi(i)^{\mathrm{H}} \phi(j)$. Let $X = (x_{i,j})$ with $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent, such that $MX = 0$. By Theorem 6.1, every vertex of G has degree at least $(n-d)$. Hence every column of X contains at least $(n-d)$ zeroes. By Lemma 6.3, $X = 0$. So M fulfills the SAP, and hence ϕ is stable. ∎

From this proposition and Theorem 6.1, the following lower bound for $\nu(G)$ follows.

THEOREM 6.5. *If G has a k-connected minor then $\nu(G) \geqslant k$.*

COROLLARY 6.5a. $\nu(K_5) = 4$ *and* $\nu(K_{2,2,2}) = 4$.

Proof. By Proposition 5.2, $\nu(K_5) = 4$. As $K_{2,2,2}$ is 4-connected, $\nu(K_{2,2,2}) \geqslant 4$ by Theorem 6.5. If $\nu(K_{2,2,2}) > 4$, then there exists a matrix $M \in \mathcal{M}_{K_{2,2,2}}$ with $\operatorname{corank}(M) > 4$ and fulfilling the SAP. As $\operatorname{corank}(M) > 4$, there is a nonzero vector $x \in \ker(M)$ with $x_{v_1} = x_{v_2} = x_{v_3} = x_{v_4} = 0$, where v_1, v_2, v_3 are vertices of $K_{2,2,2}$ in different color classes and $v_4 \neq v_1, v_2, v_3$. But for each vertex v for which $x_v \neq 0$, there is a vertex w with $x_w = 0$ such that w is only adjacent to v. This is a contradiction, for w should be adjacent to at least two vertices of $\operatorname{supp}(x)$. ∎

If U is a unitary matrix (this means that $U^{\mathrm{H}}U = I$) then the function $U\phi$ is an orthogonal representation of G for which $(U\phi(i))^{\mathrm{H}}\, U\phi(j) = \phi(i)^{\mathrm{H}}\phi(j)$. Hence, if ϕ is a stable orthogonal representation of $G = (V, E)$ and U is a unitary matrix then also $U\phi$ is a stable orthogonal representation of G. If $d\colon V \to \mathbb{C}$ with $d(v) \neq 0$ for all $v \in V$, then ϕd is a stable orthogonal representation of G.

The four-rung Mobius ladder is denoted by V_8 (see Fig. 3).

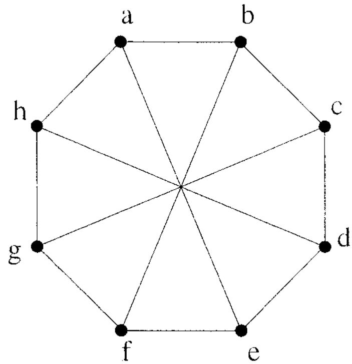

FIGURE 3

PROPOSITION 6.6. *Let G be a graph whose underlying simple graph is isomorphic to V_8. If there exists a positive semidefinite matrix $M = (m_{i,j}) \in \mathcal{M}_G$ with* $\operatorname{corank}(M) \geqslant 4$ *fulfilling the SAP, then G has a minor isomorphic to a graph in the K_4^2-family.*

Proof. In this proof we label the vertices of V_8 as in Fig. 3. For every vertex v of G, $m_{v,v} > 0$; or equivalently, there is no $x \in \ker(M)$ with $|\operatorname{supp}(x)| = 1$. For if not, then v is connected to each of its neighbors by at least two edges, which implies that G has a $K_4^2 \Delta Y$-minor.

We next show that

$$(1) \qquad \operatorname{corank}(M) \leqslant 4 \qquad \text{for all} \quad M \in \mathcal{M}_G.$$

Suppose $\operatorname{corank}(M) > 4$. Then there are nonzero vectors $x, y \in \ker(M)$ with $x_h = x_e = x_c = x_f = 0$ and $y_a = y_f = y_d = y_g = 0$. Since there are no vectors $z \in \ker(M)$ with $|\operatorname{supp}(z)| = 1$, $\operatorname{supp}(x) = \{a, b\}$ and $\operatorname{supp}(y) = \{b, c\}$. Let $z := x_b y - y_b x$. Then $\operatorname{supp}(z) = \{a, c\}$, and hence $G[\operatorname{supp}(z)]$ is disconnected. So $\operatorname{corank}(M) \leqslant 4$.

We next show that

$$\text{(2)} \qquad \operatorname{corank}(M) \leqslant 3 \quad \text{for all} \quad M \in \mathcal{M}_{V_8}.$$

For this we use Theorem 6.2. So there is a faithful orthogonal representation $\phi\colon V(V_8) \to \mathbb{C}^4$. Hence we may take $\phi(a) = (1, 0, 0, 0)$, $\phi(c) = (0, 1, 0, 0)$, $\phi(f) = (0, 0, 1, 0)$. Then we can write $\phi(d) = (0, 1, 0, d_4)$, $\phi(e) = (e_1, 0, 1, e_4)$, $\phi(g) = (0, g_2, g_3, 1)$, and $\phi(h) = (1, 0, 0, h_4)$. If $\phi(b)$ would be of the form $(b_1, b_2, b_3, 0)$, then, since $\phi(b)$ and $\phi(d)$ are orthogonal, $b_2 = 0$ and $\phi(c)$ is orthogonal to $\phi(b)$, which implies that b and c are connected by at least two edges. Hence we can write $\phi(b) = (b_1, b_2, b_3, 1)$. Since $\phi(b)$ and $\phi(d)$ are orthogonal, $\overline{b_2} = -d_4$. Since $\phi(b)$ and $\phi(e)$ are orthogonal, $\overline{b_1} e_1 + \overline{b_3} = -e_4$. Since $\phi(b)$ and $\phi(h)$ are orthogonal, $\overline{b_1} = -h_4$. So $\phi(b) = (-\overline{h_4}, -\overline{d_4}, -\overline{e_4} - b_1 \overline{e_1}, 1)$ Since $\phi(d)$ and $\phi(g)$ are orthogonal, $g_2 = -\overline{d_4}$. Since $\phi(e)$ and $\phi(g)$ are orthogonal, $g_3 = -\overline{e_4}$. Hence $\phi(g) = (0, -\overline{d_4}, -\overline{e_4}, 1)$. Since $\phi(e)$ and $\phi(h)$ are orthogonal, $\overline{e_1} + \overline{e_4} h_4 = 0$; hence $\overline{b_1} = \overline{e_1}/\overline{e_4}$. So $\phi(b) = (-\overline{h_4}, -\overline{d_4}, -\overline{e_4} - (|e_1|^2/|e_4|^2)\, \overline{e_4}, 1)$.

But $\phi(b)$ and $\phi(g)$ are orthogonal, so $|d_4|^2 - |e_4|^2 + |e_1|^2 + 1 = 0$, which gives a contradiction. Hence $\operatorname{corank}(N) \leqslant 3$ for each positive semidefinite $N \in \mathcal{M}_{V_8}$.

Therefore, if $M = (m_{i,j}) \in \mathcal{M}_G$ has $\operatorname{corank}(M) = 4$, then $m_{i,j} = 0$ for at least one pair of adjacent vertices. Thus G has at least one parallel edge. We distinguish, up to symmetry, two cases. Namely, a and h are connected by at least two edges, or a and e are connected by at least two edges.

We look to the case where $m_{a,h} = 0$. Then, since $\operatorname{corank}(M) = 4$, there is a nonzero vector $x \in \ker(M)$ with $x_b = x_d = x_e = 0$. Then $x_a = 0$ as $m_{a,a} > 0$. Now, either $x_f \neq 0$ and $m_{e,f} = 0$, or $x_f = 0$ and $x_g \neq 0$, for otherwise $x_h = 0$ and $x_c = 0$, which implies that $x = 0$. But, if $x_f = 0$ and $x_g \neq 0$, then $m_{f,g} = 0$.

Suppose that $m_{f,g} = 0$. Then we show that also $m_{b,c} = 0$, $m_{d,e} = 0$, and hence b and c are connected by at least two edges and d and e are connected by at least two edges. This graph G clearly has a K_4^2-minor. Again using Theorem 6.2, there is a faithful orthogonal representation $\phi\colon VG \to \mathbb{C}^4$. We may assume that $\phi(a) = (1, 0, 0, 0)$, $\phi(c) = (0, 1, 0, 0)$, $\phi(f) = (0, 0, 1, 0)$, and $\phi(h) = (0, 0, 0, 1)$. This implies that we may write $\phi(b) = (b_1, b_2, b_3, 0)$, $\phi(d) = (0, d_2, 0, d_4)$, $\phi(e) = (e_1, 0, e_3, 0)$, and $\phi(g) = (0, g_2, 0, g_4)$ (as $m_{f,g} = 0$). Hence $\phi(d)$ and $\phi(e)$ are orthogonal, which means that d and e must be connected by at least two edges. Because $\phi(b)$ and $\phi(d)$ are orthogonal, $\overline{b_2}d_2 = 0$. Hence $b_2 = 0$ or $d_2 = 0$. Suppose $d_2 = 0$. Then $g_4 = 0$, as $\phi(g)$ and $\phi(d)$ are orthogonal, and hence $\phi(g) = (0, g_2, 0, 0)$. But since $\phi(g)$ and $\phi(b)$ are orthogonal, $b_2 = 0$. If $b_2 = 0$ then $\phi(b)$ and $\phi(c)$ are orthogonal, which means that b and c are connected by at least two edges.

Suppose that $m_{e,f} = 0$. Then using Theorem 6.2, there is a faithful orthogonal representation $\phi: VG \to \mathbb{C}^4$. We may assume that $\phi(a) = (1, 0, 0, 0)$, $\phi(c) = (0, 1, 0, 0)$, $\phi(f) = (0, 0, 1, 0)$, and $\phi(h) = (0, 0, 0, 1)$. This implies that we may write $\phi(b) = (b_1, b_2, b_3, 0)$, $\phi(d) = (0, d_2, 0, d_4)$, $\phi(e) = (e_1, 0, 0, 0)$, and $\phi(g) = (0, g_2, g_3, g_4)$. Since e and b are not adjacent, $\overline{e_1}b_1 = 0$, so $b_1 = 0$, for $e_1 = 0$ implies that $m_{e,e} = 0$. Since b and d are not adjacent, $\overline{b_2}d_2 = 0$, so $b_2 = 0$ or $d_2 = 0$. If $b_2 = 0$, then $\phi(b) = (0, 0, b_3, 0)$ and $\phi(g) = (0, g_2, 0, g_4)$. Then b and c are connected by at least two edges, as are f and g, and e and d. Hence G has a K_4^2-minor. So we may assume that $d_2 = 0$. Then $\phi(g) = (0, g_2, g_3, 0)$. Then c and d are connected by at least two edges, as are e and f and h and g. But also in this case G has a K_4^2-minor.

Hence we may assume that a and e are connected by at least two edges. We may, furthermore, assume that there are no parallel edges on the Hamilton circuit of G, as that case was handled above. Since $\operatorname{corank}(M) = 4$, there is a nonzero vector $x \in \ker(M)$ with $x_b = x_h = x_f = 0$. This implies that $x_a = 0$, which implies $x_c = 0$, and this implies $x_g = 0$. Then $x_d = x_e = 0$ follows, and hence $x = 0$, which gives a contradiction. ∎

7. ROOTED GRAPHS

A *rooted graph* $(G, (s_1, \ldots, s_t))$ is a pair, where G is a graph and $s_1, \ldots, s_t \in V(G)$.

Let G and H be graphs. The following few definitions are taken from [13]. A *model* ϕ of H in G assigns to each edge e of H an edge $\phi(e)$ of G and to each vertex v of H a nonnull connected subgraph $\phi(v)$ of G, such that

(i) the graphs $\phi(v)$, $v \in V(H)$, are mutually vertex-disjoint, the edges $\phi(e)$, $e \in E(H)$, are all distinct, and for $v \in V(H)$ and $e \in E(H)$, $\phi(e) \notin E(\phi(v))$;

(ii) for $e \in E(H)$, if e has ends u and v, then $\phi(e)$ has one end in $V(\phi(u))$ and the other in $V(\phi(v))$.

So, if ϕ is a model of H in G, then H is isomorphic to a minor of G.

Let $(G, (s_1, ..., s_t))$ and $(H, (r_1, ..., r_t))$ be rooted graphs. A *model* ϕ of $(H, (r_1, ..., r_t))$ in $(G, (s_1, ..., s_t))$ is a model of H in G such that $s_i \in V(\phi(r_i))$, $1 \leqslant i \leqslant k$. We say that $(H, (r_1, ..., r_t))$ is isomorphic to a minor of $(G, (s_1, ..., s_t))$ if there is a model of $(H, (r_1, ..., r_t))$ in $(G, (s_1, ..., s_t))$. We say that $(G, (s_1, ..., s_t))$ has an $(H, (r_1, ..., r_t))$-minor if $(H, (r_1, ..., r_t))$ is isomorphic to a minor of $(G, (s_1, ..., s_t))$.

Let (A, B) be a separation of G with $V(A \cap B) = \{s_1, ..., s_k\}$. If $(A', (r_1, ..., r_k))$ is isomorphic to a rooted minor of $(A, (s_1, ..., s_k))$, then, after identifying r_i with s_i for $i = 1, ..., k$, $A' \cup B$ is isomorphic to a minor of G. So if (A', B) is a separation of a graph containing an excluded minor and $(A', (r_1, ..., r_k))$ is isomorphic to a rooted minor of $(A, (s_1, ..., s_k))$, then, after identifying the roots of $(A', (r_1, ..., r_k))$ with those of $(A, (s_1, ..., s_k))$, $A \cup B$ contains the excluded minor as well. So only separations (A, B) are allowed where $(A', (r_1, ..., r_k))$ is not isomorphic to a minor of $(A, (s_1, ..., s_k))$.

If $(G, (s_1, ..., s_t))$ is a rooted graph, then a separation (A, B) with $s_i \in B$, for $1 \leqslant i \leqslant t$, and $|A \setminus B| > 0$ is called an *internal separation*; $|V(A \cap B)|$ is called the order of the separation. We say that a rooted graph $(G, (s_1, ..., s_t))$ is *internally t-connected* if there is no internal separation of $(G, (s_1, ..., s_t))$ of order $\leqslant t$.

Let K_4r be the rooted graph $(K_4, (s_1, s_2, s_3))$, where s_1, s_2, and s_3 are three distinct vertices of K_4.

LEMMA 7.1. *Let $(G, (s_1, s_2, s_3))$ be an internally 3-connected rooted graph. Then, $(G, (s_1, s_2, s_3))$ has no K_4r-minor if and only if G has no K_4-minor.*

Proof. The difficult part is to prove that if G has a K_4-minor, then $(G, (s_1, s_2, s_3))$ has a K_4r-minor. So suppose that G has a K_4-minor. Then G has a subdivision of K_4 as a subgraph. Now let K be any subdivision of K_4 in G. If there are only two vertex-disjoint paths between $S = \{s_1, s_2, s_3\}$ and K, then clearly there is a $(\leqslant 2)$-separation (A, B) with $(A \setminus B) \cap S = \emptyset$ and $|A \setminus B| > 0$. So there are three vertex-disjoint paths between S and K. Let $t_1, ..., t_4$ be the vertices of degree 3 in K and let $P_1, ..., P_6$ be the paths of K between the vertices $t_1, ..., t_4$. If the paths from S to K do not end in one path P_i for some i then $(G, (s_1, s_2, s_3))$ clearly has a K_4r-minor.

So, for any subdivision K of K_4 in G, the paths from S to K end in one path, say P_1. We take K such that the length of the path P_1 onto which the paths from S to K end is as short as possible. Let Q_1, Q_2, Q_3 be three vertex-disjoint paths from S to P_1. Let t_1 and t_2 be the ends of P_1. Since there is no $(\leqslant 2)$-separation (A, B) with $(A \setminus B) \cap S = \emptyset$ and $|A \setminus B| > 0$ (so especially there is no such separation (A, B) with $V(A \cap B) = \{t_1, t_2\}$), there must be a path P from $K - V(P_1)$ to $Q_1 \cup Q_2 \cup Q_3 \cup P_1$. We may assume that P has no internal vertices in $Q_1 \cup Q_2 \cup Q_3 \cup K$. If P has one end in $(Q_1 \cup Q_2 \cup Q_3) \setminus V(P_1)$, then we can find three vertex-disjoint paths which do not all end in P_1. Hence P has one end in P_1. Let u_1, u_2 be the vertices of two paths of Q_1, Q_2, Q_3 ending in P_1 so that the other path of Q_1, Q_2, Q_3 ends in P_1 between the vertices u_1 and u_2. If P has one end in P_1 between u_1 and u_2, not including u_1 and u_2, then we can find a subdivision K' of K_4 in G such that not all paths from S to K' end in one path between the vertices of degree 3 of K'. So P has one end in P_1 which does not lie between u_1 and u_2 (but we allow that the end of P is u_1 or u_2). Then we can find a subdivision K' of K_4 in G such that the path of K' onto which all paths from S to K' end has shorter length. This gives a contradiction, so $(G, (s_1, s_2, s_3))$ has a K_4r-minor. ∎

Let C_3^2r be the rooted graph $(C_3^2, (s_1, s_2, s_3))$, where s_1, s_2, and s_3 are three distinct vertices of C_3^2.

THEOREM 7.2. *Let $(G, (s_1, s_2, s_3))$ be an internally 3-connected rooted graph,. Then G has no C_3^2- and no K_4-minor if and only if $(G, (s_1, s_2, s_3))$ has no C_3^2r- and no K_4r-minor.*

Proof. If G has no C_3^2- and no K_4-minor, then clearly $(G, (s_1, s_2, s_3))$ has no C_3^2r- and no K_4r-minor.

For the converse we may assume, by Lemma 7.1, that G has no K_4-minor. Suppose to the contrary that G has a C_3^2-minor. We assume that G is a minimal counterexample, that is, G' has no C_3^2-minor for each proper minor G' of G.

It is clear that we may assume that G is connected.

Suppose G is not 2-connected. Let r be a cut-vertex of G. Since $(G, (s_1, s_2, s_3))$ is internally 3-connected, each component C of $G-\{r\}$ either contains at least two vertices of $\{s_1, s_2, s_3\}$ (and hence exactly two vertices) or consists of only one vertex, and this vertex is one of the vertices of $\{s_1, s_2, s_3\}$. Let C_1 be a component of $G-\{r\}$ consisting of one vertex, which we may assume is s_1 without loss of generality. Then $(G-s_1, (r, s_2, s_3))$ has no C_3^2r-minor, and hence G has no C_3^2-minor. Thus G is 2-connected.

Suppose that G has a $K_{2,3}$-minor. Then there are two distinct vertices p_1 and p_2, three openly vertex-disjoint paths P_1, P_2, and P_3 connecting p_1 to p_2, and each of these paths has more than one edge. If there is a path from an internal vertex of one path to an internal vertex of another path, then G has a K_4-minor. Hence $\{p_1, p_2\}$ is a vertex cut of G, and each path in $\{P_1, P_2, P_3\}$ belongs to a component of $G-\{p_1, p_2\}$. Let C_1, C_2, and C_3 be the components of $G-\{p_1, p_2\}$. Since $(G, (s_1, s_2, s_3))$ is internally 3-connected, each component C_i, $i = 1, 2, 3$, contains a vertex of $\{s_1, s_2, s_3\}$; we may assume that $s_i \in V(C_i)$, $i = 1, 2, 3$. Let G_i, for $i = 1, 2, 3$, be the graph obtained from $G[V(C_i) \cup \{p_1, p_2\}]$ by adding two edges connecting p_1 and p_2. Then $(G_1, (s_1, p_2, p_3))$ is isomorphic to a minor of $(G, (s_1, s_2, s_3))$.

Similar statements hold for $(G_2, (p_1, s_2, p_3))$ and $(G_3, (p_1, p_2, s_3))$. By minimality, G_i, $i = 1, 2, 3$ has no C_3^2-minor. But then G has no C_3^2-minor.

Hence G is outerplanar; we assume that G is embedded into the plane. Let C be the circuit bounding the infinite face. If C has no chord, then G has at most three vertices. But then clearly G has no C_3^2-minor if $(G, (s_1, s_2, s_3))$ has no C_3^2r-minor. So we may assume that C has a chord. Let p_1 and p_2 be the ends of a chord. Let C_1 and C_2 be the components of $G - \{p_1, p_2\}$, and let G_i for $i = 1, 2$ be the graph obtained from $G[V(C_i) \cup \{p_1, p_2\}]$ by adding an edge between p_1 and p_2. So, in G_i, $i = 1, 2$, at least two edges are connecting p_1 and p_2. Suppose that C_1 contains exactly one vertex of $\{s_1, s_2, s_3\}$, say it contains s_1. Then $(G_1, (s_1, p_1, p_2))$ or $(G_1, (s_1, p_2, p_1))$ is isomorphic to a minor of $(G, (s_1, s_2, s_3))$, and hence, by minimality, G_1 has no C_3^2-minor. Almost the same argument applies to the case when C_1 contains two vertices of $\{s_1, s_2, s_3\}$, and similarly for C_2. So G_1 and G_2 have no C_3^2-minor. But then G has no C_3^2-minor, a contradiction. ∎

Let $K_3^r p := (G, (u, v, w))$, where G is the graph with vertex set $\{u, v, w, x\}$ and edge set $\{uv, ux, vx, xw\}$; see Fig. 4, where bold vertices are the roots of $K_3^r p$.

Proposition 7.3. *Let $(G, (s_1, s_2, s_3))$ be an internally 3-connected rooted graph. Suppose $K_3^r p$ is not isomorphic to a minor of $(G, (s_1, s_2, s_3))$. Then the underlying simple graph of $G - s_3$ is a subgraph of a path.*

Proof. Add to $(G, (s_1, s_2, s_3))$ an edge connecting s_1 and s_3, and one connecting s_2 and s_3; let the rooted graph obtained be $(G', (s_1, s_2, s_3))$. As $(G, (s_1, s_2, s_3))$ has no $K_3^r p$-minor, $(G', (s_1, s_2, s_3))$ has no $K_4 r$-minor, and hence, by Lemma 7.1, G' has no K_4-minor.

Suppose that there is no path from s_1 to s_2 disjoint from s_3. Then, as $(G, (s_1, s_2, s_3))$ is internally 3-connected, $V(G) = \{s_1, s_2, s_3\}$, and the proposition is clear.

So we may assume that there is a path P from s_1 to s_2 disjoint from s_3; we take this path as short as possible. Suppose that $G - (\{s_3\} \cup V(P))$ is

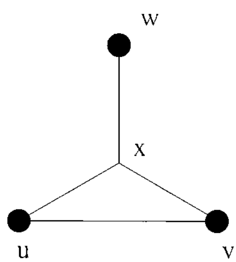

FIGURE 4

nonempty; let C be a component of $G-(\{s_3\} \cup V(P))$. If $N(C)$ has more than two vertices, then G' has a K_4-minor; so $N(C)$ has at most two vertices. But this contradicts the fact that $(G, (s_1, s_2, s_3))$ is internally 3-connected. Hence $V(G) = V(P) \cup \{s_3\}$. Since P is as short as possible, there is no edge connecting two nonadjacent vertices of P. So deleting s_3 from G gives a graph whose underlying simple graph is a path. ∎

8. TREE-DECOMPOSITIONS

A *tree-decomposition* of a graph $G = (V, E)$ is a pair (T, W) where T is a tree and $W = (W_t \mid t \in V(T))$ is a family of subsets of V with the following properties.

(i) $\bigcup \{W_t \mid t \in V(T)\} = V$,

(ii) every edge of G has both ends in some W_t, and

(iii) if $t_1, t_2, t_3 \in V(T)$ and t_2 lies on a path from t_1 to t_3, then $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$.

The subsets W_t are called the *bags* of the tree-decomposition. The width of a tree-decomposition is $\max(|W_t| - 1 \mid t \in V(T))$, and the *tree-width* of G is the minimum width of any tree-decomposition of G. See [12].

Let (T, W) be a tree-decomposition of G with $W_s \subseteq W_t$ for adjacent vertices s and t of T. Let T' be the tree obtained from T by contracting the edge connecting s and t; let the new vertex be r. Let $W_r = W_t$. Then (T', W) is a tree-decomposition of G with width equal to the width of the tree-decomposition (T, W). We call a tree-decomposition (T, W) such that there are no adjacent vertices s, t with $W_s \subseteq W_t$, a *nice* tree-decomposition. By the construction given above it is possible to find a nice tree-decomposition for every tree-decomposition.

If G' is a minor of G, then the tree-width of G' is at most the tree-width of G. Hence the class of graphs G with tree-width at most k can be characterized by a finite family of excluded minors. For $k = 1$ the only excluded minor is K_3. For $k = 2$ the only excluded minor is K_4. For $k = 3$ the excluded minors are given in the following theorem.

THEOREM 8.1. *A graph G has tree-width $\leqslant 3$ if and only if G has no K_5, $K_{2,2,2}$, $C_5 \times K_2$, or V_8-minor.*

See [1] for a proof of the excluded minors characterization of the class of graphs with tree-width $\leqslant 3$. We use this characterization of graphs with tree-width $\leqslant 3$ in the proof of the characterization of the graphs G with $\nu(G) \leqslant 3$. As a matter of fact the tree-decompositions we use are very special as Lemma 8.3 shows. We first state a lemma.

LEMMA 8.2 [12, Lemma 3.4]. *Let (T, W) be a tree-decomposition of G. Let rs be an edge of T and let T_1 and T_2 be the two components of $T \setminus rs$. Then $W_s \cap W_r$ separates $\bigcup \{W_t \mid t \in V(T_1)\}$ and $\bigcup \{W_t \mid t \in V(T_2)\}$.*

LEMMA 8.3. *Let G be a 3-connected graph of tree-width 3. If G has no Q_3- or $Q_3 Y\Delta$-minor, then there is a nice tree-decomposition (T, W) of width 3 of G, where for each $W_t \in W$, there are at most two sets A_t and B_t of size 3, such that for each component D of $G - W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$.*

Proof. The 3-connectivity of G implies that if (T, W) is any nice tree-decomposition of width 3 of G, then $|W_t| = 4$ for all $t \in V(T)$. For suppose that $|W_t| \leqslant 3$ for some $t \in V(T)$. Let f be an edge of T one end of which is t, and let s be the other end of f. Let T_1, T_2 be the components of $T \setminus f$. By Lemma 8.2, $W_t \cap W_s$ separates $B_1 := \bigcup \{W_r \mid r \in V(T_1)\}$ and $B_2 := \bigcup \{W_r \mid r \in V(T_2)\}$. Since (T, W) is a nice tree-decomposition, $B_1 \setminus (W_t \cap W_s) \neq \emptyset$ and $B_2 \setminus (W_t \cap W_s) \neq \emptyset$. Hence $W_t \cap W_s$ is a vertex cut of G. But $|W_t \cap W_s| \leqslant 2$, contradicting the 3-connectivity of G.

Suppose to the contrary that there is no nice tree-decomposition (T, W) of width 3 of G, where, for each $W_t \in W$, there are at most two sets A_t and B_t of size 3 such that for each component D of $G - W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$. Call a bag W_t for which there are no two sets A_t and B_t of size 3 such that for each component D of $G - W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$ a *bad bag*. Take a nice tree-decomposition (T, W) such that the number of bad bags is minimal. Let W_s be a bad bag of (T, W).

Let $D_1, \ldots, D_k$ be the components of $G - W_s$. Since G is 3-connected, $|N(D_i)| = 3$ for $i = 1, \ldots, k$. Suppose that among the family of sets $N(D_i)$, $i = 1, \ldots, k$, there are four distinct sets A_s, B_s, C_s, D_s. Let D_1, D_2, D_3, D_4 be components of $G - W_s$ such that $N(D_1) \subseteq A_s$, $N(D_2) \subseteq B_s$, $N(D_3) \subseteq C_s$, and $N(D_4) \subseteq D_s$. Contracting these components to a point and deleting all other components of $G - W_s$ shows that G has a Q_3-minor in this case. Hence there are at most three distinct sets among $N(D_i)$, $i = 1, \ldots, k$.

Suppose there are three distinct sets A_s, B_s, C_s among $N(D_i)$, $i = 1, \ldots, k$. Let A be the subgraph of G induced by A_s and by all components D of $G - W_s$ for which $N(D) \subseteq A_s$. Define in the same way the subgraphs B and C. Let w be the common vertex of A_s, B_s, and C_s; we write $\{w, w_1, w_2\} = A_s$, $\{w, w_2, w_3\} = B_s$, and $\{w, w_3, w_1\} = C_s$. If the rooted graphs $(A, (w_1, w_2, w))$, $(B, (w_2, w_3, w))$, and $(C, (w_3, w_1, w))$ all have a $K_3^r p$-minor, then G has a $Q_3 Y\Delta$-minor. So at least one of the rooted graphs has no $K_3^r p$-minor; we may assume that $(A, (w_1, w_2, w))$ has no $K_3^r p$-minor. Let $T_1, \ldots, T_r$ be the components of $T - s$ such that for each vertex $t \in V(T_i)$, $i = 1, \ldots, r$, $W_t \subseteq B$. Let t_i, $i = 1, \ldots, r$, be the vertex of T_i adjacent to s. Define similarly $T'_1, \ldots, T'_{r'}$ and $t'_1, \ldots, t'_{r'}$, except for C instead of B. By Proposition 7.3, A has a nice tree-decomposition (P, U) of width $\leqslant 2$ with

P a path. Let p_i, $i=1,2$ be the vertex of P such that $U_{p_i} \in U$ contains w_i and w. Let $W'_p = U_p \cup \{w\}$, $p \in V(P)$, and $p \neq p_1, p_2$; let $W'_{p_1} = B_s \cup U_{p_1}$ and $W'_{p_2} = C_s \cup U_{p_2}$. Let $W'_t = W_t$, $t \in V(T)$ and $t \neq s$. Let S be the tree obtained from $T_1, \ldots, T_r, T'_1, \ldots, T'_{r'}$ and P by connecting the vertices t_i, $i=1,\ldots,r$ to p_1, and t'_i, $i=1,\ldots,r'$ to p_2. Then (S, W') is a nice tree-decomposition of G with one bad bag less, which contradicts the assumption that (T, W) is a tree-decomposition with a minimum number of bad bags. Hence there is a nice tree-decomposition (T, W) of width 3 of G, where, for each $W_t \in W$, there are at most two sets A_t and B_t of size 3 such that for each component D of $G-W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$. ▌

We use the following lemma on tree-width in Lemma 8.5.

LEMMA 8.4 [12, Lemma 3.5]. *Let (T, W) be a tree-decomposition of G with $|V(T)| \geqslant 2$. For each $t \in V(T)$ let G_t be a connected subgraph of G with $V(G_t) \cap W_t = \varnothing$. Then there exist $t, t' \in V(T)$, adjacent in T, such that $W_t \cap W_{t'}$ separates $V(G_t)$ and $V(G_{t'})$ in G.*

LEMMA 8.5. *Let $G=(V,E)$ be a graph, and let (T, W) be a tree-decomposition of G. Let $M \in \mathscr{M}_G$ be positive semidefinite matrix fulfilling the SAP. Then there is a bag $W_t \in W$ such that either $W_t = V(G)$ or $M_{V(G)\backslash W_t}$ is positive definite.*

Proof. Suppose to the contrary that there is no bag $W_t \in W$ such that $W_t \neq V(G)$ or $M_{V(G)\backslash W_t}$ is positive definite. Then $|V(T)| \geqslant 2$, and we can use Lemma 8.4. For each W_t we take the component G_t of $G-W_t$ with $M_{V(G_t)}$ singular. By Lemma 8.4, there exist $t, t' \in V(T)$, adjacent in T, such that $W_t \cap W_{t'}$ separates $V(G_t)$ and $V(G_{t'})$. Hence G_t and $G_{t'}$ belong to different components of $G-(W_t \cap W_{t'})$. By Propositions 2.1 and 4.4, M does not fulfill the SAP, a contradiction. ▌

LEMMA 8.6. *Let $G=(V,E)$ be a graph, and let (T, W) be a tree-decomposition of G of width 3, where for each $W_t \in W$, there are at most two sets A_t and B_t of size at most 3, such that for each component D of $G-W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$. Let $M \in \mathscr{M}_G$ be positive semidefinite with corank at least 4. Let $W_t \in W$ such that $M_{V(G)\backslash W_t}$ is positive definite. Let w_1 be the vertex in $A_t \backslash (A_t \cap B_t)$, let w_2 be the vertex in $B_t \backslash (A_t \cap B_t)$, and assume that w_1 and w_2 are connected by at least one edge. Then M has corank 4 and w_1 and w_2 are connected by at least two edges.*

Proof. Suppose to the contrary that there is exactly one edge connecting w_1 to w_2. Let D_A be the set of vertices of all components D of $G-W_t$ with $N(D) \subseteq A_t$, and let D_B be the set of vertices of all components D of $G-W_t$ with $N(D) \subseteq B_t$. We leave the cases where D_A or D_B is empty to the reader; they can be done similarly. Since $M_{V(G)\backslash W_t}$ is positive definite, M_{D_A} and M_{D_B} are positive definite. We may write

$$M = \begin{pmatrix} M_{D_A} & M_{D_A, W_t} & 0 \\ M_{W_t, D_A} & M_{W_t} & M_{W_t, D_B} \\ 0 & M_{D_B, W_t} & M_{D_B} \end{pmatrix}.$$

Let

$$P := \begin{pmatrix} I & -M_{D_A}^{-1} M_{D_A, W_t} & 0 \\ 0 & I & -M_{D_B}^{-1} M_{D_B, W_t} \\ 0 & 0 & I \end{pmatrix}.$$

Sylvester's law of inertia tells us that

$$P^{\mathrm{H}} M P = \begin{pmatrix} M_{D_A} & 0 & 0 \\ 0 & M_{W_t} - M_{W_t, D_A} M_{D_A}^{-1} M_{D_A, W_t} - M_{W_t, D_B} M_{D_B}^{-1} M_{D_B, W_t} & 0 \\ 0 & 0 & M_{D_B} \end{pmatrix}$$

is positive semidefinite and that it has the same corank as M. So

$$L = (l_{i,j}) := M_{W_t} - M_{W_t, D_A} M_{D_A}^{-1} M_{D_A, W_t} - M_{W_t, D_B} M_{D_B}^{-1} M_{D_B, W_t} \tag{3}$$

has corank at least 4. Note that $M_{W_t, D_A} M_{D_A}^{-1} M_{D_A, W_t}$ can have nonzero entries only for those row and column indices in A_t and that $M_{W_t, D_B} M_{D_B}^{-1} M_{D_B, W_t}$ can have nonzero entries only for those row and column indices in B_t. As there is only one edge connecting w_1 to w_2, $l_{w_1, w_2} = m_{w_1, w_2} \neq 0$. This is absurd as the only matrix of corank at least 4 with at most four rows and four columns is the 4×4 all-zero matrix. It also follows that M has corank 4. ∎

9. EXCLUDED MINORS

In this section we first give the excluded minors for the class of graphs G with $\nu(G) \leqslant 3$ if G has tree-width at most 3. Then using the fact that V_8 is a splitter for the class of graphs with no K_5-minor, we can give the complete family of excluded minors for the class of graphs with $\nu(G) \leqslant 3$.

LEMMA 9.1. *Let G be a graph with tree-width $\leqslant 3$. If G has no minor isomorphic to a graph in the K_4^2-family, then $\nu(G) \leqslant 3$.*

Proof. Suppose to the contrary that $\nu(G) > 3$. Let $M = (m_{i,j}) \in \mathcal{M}_G$ be a positive semidefinite matrix with corank$(M) > 3$ and which fulfills the SAP. By Proposition 5.5, we may assume that G is 3-connected. By Lemma 8.3, there is a nice tree-decomposition (T, W) of width 3 of G, where, for each $W_t \in W$, there are two sets A_t and B_t of size 3 such that for each component D of $G - W_t$ either $N(D) \subseteq A_t$ or $N(D) \subseteq B_t$. By Lemma 8.5, there is a bag W_t of the tree-decomposition (T, W) such that for each component D of $G - W_t$, $M_{V(D)}$ is positive definite. Let w_1 be the vertex in $A_t \setminus (A_t \cap B_t)$, and let w_2 be the vertex in $B_t \setminus (A_t \cap B_t)$. Let $\{u_1, u_2\} := A_t \cap B_t$. By Lemma 8.6, w_1 and w_2 are connected by at least two edges, and $\operatorname{corank}(M) = 4$.

Let D_A be the set of vertices of all components D of $G - W_t$ with $N(D) \subseteq A_t$, and let D_B be the set of vertices of all components D of $G - W_t$ with $N(D) \subseteq B_t$. Let A be the graph induced by $A_t \cup D_A$ and let B be the graph obtained from the subgraph of G induced by $B_t \cup D_B$ by deleting all edges connecting the vertices u_1 and u_2. So A and B have no edges in common.

In some of the following formulas we assume that D_A and D_B are nonempty. We leave it to the reader to provide the formulas when D_A or D_B is empty.

Let

$$N := (n_{i,j}) = M_{A_t} - M_{A_t, D_A} M_{D_A} M_{D_A, A_t}. \tag{4}$$

From (3) it follows that the only possible nonzero entries of N are n_{u_1, u_1}, n_{u_2, u_2}, n_{u_1, u_2}. Let

$$P = (p_{i,j}) := \begin{pmatrix} M_{D_A} & M_{D_A, w_1} & M_{D_A, u_1} & M_{D_A, u_2} \\ M_{w_1, D_A} & m_{w_1, w_1} & m_{w_1, u_1} & m_{w_1, u_2} \\ M_{u_1, D_A} & m_{u_1, w_1} & m_{u_1, u_1} - n_{u_1, u_1} & m_{u_1, u_2} - n_{u_1, u_2} \\ M_{u_2, D_A} & m_{u_2, w_1} & m_{u_2, u_1} - n_{u_2, u_1} & m_{u_2, u_2} - n_{u_2, u_2} \end{pmatrix},$$

and let

$$Q = (q_{i,j}) := \begin{pmatrix} n_{u_1, u_1} & n_{u_1, u_2} & m_{u_1, w_2} & M_{u_1, D_B} \\ n_{u_2, u_1} & n_{u_2, u_2} & m_{u_2, w_2} & M_{u_2, D_B} \\ m_{w_2, u_1} & m_{w_2, u_2} & m_{w_2, w_2} & M_{w_2, D_B} \\ M_{D_B, u_1} & M_{D_B, u_2} & M_{D_B, w_2} & M_{D_B} \end{pmatrix}.$$

Then P and Q are positive semidefinite matrices, each of corank 3, which follows from (3).

We distinguish several cases. The first case is where $n_{u_1, u_2} \neq 0$.

Let A' be the graph obtained from A by adding a new edge between u_1 and u_2. We claim that $\nu(A') \geqslant 3$. To see this, we take the matrix P. The matrix P is a positive semidefinite matrix and has corank 3. As A' has an additional edge between u_1 and u_2, $P \in \mathcal{M}_{A'}$. So it remains to show that P fulfills the SAP (w.r.t. A'). Suppose to the contrary that P does not fulfill the SAP. Then there is a nonzero Hermitian matrix

$$Y = (y_{i,j}) := \begin{pmatrix} Y_{D_A} & y_{w_1} & y_{u_1} & y_{u_2} \\ y_{w_1}^{\mathrm{H}} & 0 & y_{w_1,u_1} & y_{w_1,u_2} \\ y_{u_1}^{\mathrm{H}} & y_{u_1,w_1} & 0 & 0 \\ y_{u_2}^{\mathrm{H}} & y_{u_2,w_1} & 0 & 0 \end{pmatrix},$$

with $y_{i,j} = 0$ if $i = j$ or if i and j are adjacent, such that $PY = 0$. Let

$$X = (x_{i,j}) := \begin{pmatrix} Y_{D_A} & y_{w_1} & y_{u_1} & y_{u_2} & 0 & X_{D_A,D_B} \\ y_{w_1}^{\mathrm{H}} & 0 & y_{w_1,u_1} & y_{w_1,u_2} & 0 & X_{w_1,D_B} \\ y_{u_1}^{\mathrm{H}} & y_{u_1,w_1} & 0 & 0 & 0 & 0 \\ y_{u_2}^{\mathrm{H}} & y_{u_2,w_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X_{D_B,D_A} & X_{D_B,w_1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the Hermitian matrix where

$$X_{D_B,D_A} := -M_{D_B}^{-1} M_{D_B,\{u_1,u_2\}} \begin{pmatrix} y_{u_1}^{\mathrm{H}} \\ y_{u_2}^{\mathrm{H}} \end{pmatrix},$$

$$X_{D_B,w_1} := -M_{D_B}^{-1} M_{D_B,\{u_1,u_2\}} \begin{pmatrix} y_{u_1,w_1} \\ y_{u_2,w_1} \end{pmatrix},$$

and $X_{D_A,D_B} := X_{D_B,D_A}^{\mathrm{H}}$, $X_{w_1,D_B} := X_{D_B,w_1}^{\mathrm{H}}$. Then $MX = 0$. Since $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent, M does not fulfill the SAP. This contradiction shows that P fulfills the SAP. Thus $\nu(A') = 3$, and hence A' has a K_4- or a C_3^2-minor. Since $(A', (w_1, u_1, u_2))$ is internally 3-connected, $(A', (w_1, u_1, u_2))$ has a K_4r- or a C_3^2r-minor by Theorem 7.2. Hence $(A, (w_1, u_1, u_2))$ has an H_1r- or an H_2r-minor, where H_1r is a rooted graph obtained from K_4r by deleting one edge connecting s_2 and s_3 and where H_2r is a rooted graph obtained from C_3^2r by deleting one edge connecting s_2 and s_3. Let B' be the graph obtained from B by adding an edge connecting u_1 and u_2. We claim that $\nu(B') \geqslant 3$. To see this, we take the matrix Q. Then Q is a positive semidefinite matrix with $\operatorname{corank}(Q) = 3$, and $Q \in \mathscr{M}_{B'}$. With the same argument as above one shows that Q fulfills the SAP (w.r.t. B'). Hence $\nu(B') \geqslant 3$, which implies that $(B, (w_2, u_1, u_2))$ has an H_1r- or an H_2r-minor. But then G has a minor isomorphic to a graph in the K_4^2-family.

So we may assume that $n_{u_1,u_2}=0$. Suppose that $y_{u_1,u_2}=0$ for every Hermitian matrix $Y=(y_{i,j})$ with $y_{i,j}=0$ if $i=j$ or if i and j are adjacent in A and with $PY=0$. Then the argument given above shows that P fulfills the SAP (w.r.t. A), and hence $\nu(A)\geqslant 3$, which implies that $(A,(w_1,u_1,u_2))$ has a K_4r- or a C_3^2r-minor. Add two edges in parallel to B between the vertices u_1 and u_2, and denote the graph obtained by B'. Then in the same way as above it can be shown that $(B',(w_2,u_1,u_2))$ contains a K_4r- or a C_3^2r-minor. Hence $(B,(w_2,u_1,u_2))$ contains a Fr-minor, where Fr is the rooted graph obtained from C_3^2r by deleting the two parallel edges between s_2 and s_3. But then it can clearly be seen that G has a minor isomorphic to a graph in the K_4^2-family.

Suppose that $z_{u_1,u_2}=0$ for every Hermitian matrix $Z=(z_{i,j})$ with $z_{i,j}=0$ if $i=j$ or if i and j are adjacent in B and with $QZ=0$. Then with the same argument as above, Q fulfills the SAP (w.r.t. B), and hence $(B,(w_2,u_1,u_2))$ has a K_4r- or a C_3^2r-minor. Add two edges in parallel to A between the vertices u_1 and u_2, and denote the graph obtained by A'. Then in the same way as above it can be shown that $(A',(w_1,u_1,u_2))$ contains a K_4r- or a C_3^2r-minor. Hence $(A,(w_1,u_1,u_2))$ contains a Fr-minor. But then it can clearly be seen that G has a minor isomorphic to a graph in the K_4^2-family.

So we may assume that there is a Hermitian matrix $Y=(y_{i,j})$ with $y_{i,j}=0$ if $i=j$ or if i and j are adjacent in A and with $PY=0$, such that $y_{u_1,u_2}\neq 0$, and that there is a Hermitian matrix $Z=(z_{i,j})$ with $z_{i,j}=0$ if $i=j$ or if i and j are adjacent in B and with $QZ=0$, such that $z_{u_1,u_2}\neq 0$. We may, furthermore, assume that $y_{u_1,u_2}=z_{u_1,u_2}$. Let

$$X=(x_{i,j}):=\begin{pmatrix} Y_{D_A} & y_{w_1} & y_{u_1} & y_{u_2} & x_{w_2} & X_{D_A,D_B} \\ y_{w_1}^{\mathrm{H}} & 0 & y_{w_1,u_1} & y_{w_1,u_2} & 0 & x_{w_1}^{\mathrm{H}} \\ y_{u_1}^{\mathrm{H}} & y_{u_1,w_1} & 0 & y_{u_1,u_2} & z_{u_1,w_2} & z_{u_1}^{\mathrm{H}} \\ y_{u_2}^{\mathrm{H}} & y_{u_2,w_1} & y_{u_2,u_1} & 0 & z_{u_2,w_2} & z_{u_2}^{\mathrm{H}} \\ x_{w_2}^{\mathrm{H}} & 0 & z_{w_2,u_1} & z_{w_2,u_2} & 0 & z_{w_2}^{\mathrm{H}} \\ X_{D_B,D_A} & x_{w_1} & z_{u_1} & z_{u_2} & z_{w_2} & Z_{D_B} \end{pmatrix},$$

where

$$x_{w_2}:=-M_{D_A}^{-1}M_{D_A,\{u_1,u_2\}}\begin{pmatrix} z_{u_1,w_2} \\ z_{u_2,w_2}\end{pmatrix},$$

$$x_{w_1}:=-M_{D_B}^{-1}M_{D_B,\{u_1,u_2\}}\begin{pmatrix} y_{u_1,w_1} \\ y_{u_2,w_1}\end{pmatrix},$$

$$X_{D_B, D_A} := -M_{D_B}^{-1} M_{D_B, \{w_1, u_1, u_2, w_2\}} \begin{pmatrix} y_{w_1}^{\mathrm{H}} \\ y_{u_1}^{\mathrm{H}} \\ y_{u_2}^{\mathrm{H}} \\ x_{w_2}^{\mathrm{H}} \end{pmatrix}$$

$$= -(x_{w_1} \quad z_{u_1} \quad z_{u_2} \quad z_{w_2})\, M_{\{w_1, u_1, u_2, w_2\}, D_A} M_{D_A}^{-1}$$

$$= M_{D_B}^{-1} M_{D_B, \{w_1, u_1, u_2, w_2\}} X_{\{w_1, u_1, u_2, w_2\}} M_{\{w_1, u_1, u_2, w_2\}, D_A} M_{D_A}^{-1},$$

and $X_{D_A, D_B} = X^{\mathrm{H}}_{D_B, D_A}$. Note that

$$(y_{w_1} \quad y_{u_1} \quad y_{u_2} \quad x_{w_2}) = -M_{D_A}^{-1} M_{D_A, \{w_1, u_1, u_2, w_2\}} X_{\{w_1, u_1, u_2, w_2\}},$$

and

$$(x_{w_1} \quad z_{u_1} \quad z_{u_2} \quad z_{w_2}) = -M_{D_B}^{-1} M_{D_B, \{w_1, u_1, u_2, w_2\}} X_{\{w_1, u_1, u_2, w_2\}}.$$

Then $MX = 0$ and $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent. So M does not fulfill the SAP. This contradiction concludes the proof. ∎

Recall that a graph H is called a *splitter* for a class $\mathscr{C}$ of graphs if each graph G of $\mathscr{C}$ which has H as a proper minor has a 2-vertex cut. For example, V_8 is a splitter of the class of graphs with no K_5-minor.

THEOREM 9.2. *$\nu(G) \leqslant 3$ if and only if G has no K_5-minor and no minor isomorphic to a graph in the K_4^2-family.*

Proof. We already know that a graph G with a K_5-minor or a minor isomorphic to a graph in the K_4^2-family has $\nu(G) > 3$.

For the converse, let G have no K_5-minor and no minor isomorphic to a graph in the K_4^2-family. By Proposition 5.5, we may assume that G is 3-connected. Since $K_{2,2,2}$ belongs to the K_4^2-family, and Q_3 is a minor of $C_5 \times K_2$, G either has a V_8-minor or G has tree-width $\leqslant 3$. Since V_8 is a splitter for the class of graphs with no K_5-minor, either the underlying simple graph of G is isomorphic to V_8 or G has tree-width $\leqslant 3$. From Proposition 6.6 and Lemma 9.1 it follows that $\nu(G) \leqslant 3$. ∎

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