Paley–Wiener subspace of vectors in a Hilbert space with applications to integral transforms

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\textbf{Article info}

\textbf{Article history:}
Received 3 March 2008
Available online 25 December 2008
Submitted by R.H. Torres

\textbf{Keywords:}
Paley–Wiener space
Bandlimited functions
Self-adjoint operators
Bernstein inequality
Riesz interpolation formula
Sampling
Variational splines
Hilbert frame
Dual frame
Sturm–Liouville operators
Integral transforms

\textbf{Abstract}

The goal of this article is to introduce an analogue of the Paley–Wiener space of bandlimited functions, $PW_\omega$, in Hilbert spaces and then apply the general result to more specific examples. Guided by the role that the differentiation operator plays in some of the characterizations of the Paley–Wiener space, we construct a space of vectors using a self-adjoint operator $D$ in a Hilbert space $H$, and denote this space by $PW_\omega(D)$. The article can be virtually divided into two parts. In the first part we show that the space $PW_\omega(D)$ has similar properties to those of the space $PW_\omega$, including an analogue of the Bernstein inequality and the Riesz interpolation formula. We also develop a new characterization of the abstract Paley–Wiener space in terms of solutions of Cauchy problems associated with abstract Schrödinger equations. Finally, we prove two sampling theorems for vectors in $PW_\omega(D)$, one of which uses the notion of Hilbert frames and the other is based on the notion of variational splines in $H$. In the second part of the paper we apply our abstract results to integral transforms associated with singular Sturm–Liouville problems. In particular we obtain two new sampling formulas related to one-dimensional Schrödinger operators with bounded potential.

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1. Introduction

Let $\sigma > 0$ and $1 \leq p < \infty$. The Bernstein space $B^p_\sigma$ is a Banach space consisting of all entire functions $f$ of exponential type with type at most $\sigma$ that belong to $L^p(\mathbb{R})$ when restricted to the real line. It is known [8, p. 98] that $f \in B^p_\sigma$ if and only if $f$ is an entire function satisfying

$$
\|f(x+iy)\|_p \leq \|f\|_p \exp(\sigma |y|), \quad z = x + iy,
$$

where the norm on the left is taken with respect to $x$ for any fixed $y$ and

$$
\|f\|_p = \left( \int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty
$$

and

$$
\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)| < \infty, \quad \text{if } p = \infty.
$$
Unlike the spaces \( L^p(\mathbb{R}) \), the spaces \( B^p_{\sigma} \) are closed under differentiation and the differentiation operator plays a vital role in their characterization. The Bernstein spaces have been characterized in a number of different ways and one can prove that the following are equivalent:

(A) A function \( f \in L^p(\mathbb{R}) \) belongs to \( B^p_{\sigma} \) if and only if its distributional Fourier transform has support \([-\sigma, \sigma]\) in the sense of distributions.

(B) Let \( f \in C^\infty(\mathbb{R}) \) be such that \( f^{(n)} \in L^p(\mathbb{R}) \) for all \( n = 0, 1, \ldots \), and some \( 1 \leq p \leq \infty \), then \( f \in B^p_{\sigma} \) if and only if \( f \) satisfies the Bernstein's inequality \([16, p. 116]\)

\[
\|f^{(n)}\|_p \leq \sigma^n \|f\|_p, \quad n = 0, 1, 2, \ldots; \quad 1 \leq p \leq \infty. \tag{1.1}
\]

(C) Let \( f \in C^\infty(\mathbb{R}) \) be such that \( f^{(n)} \in L^p(\mathbb{R}) \) for all \( n = 0, 1, \ldots \) and some \( 1 \leq p \leq \infty \). Then

\[
\lim_{n \to \infty} \|f^{(n)}\|_p^{1/n} = \sigma.
\]

and \( f \in B^p_{\sigma} \) if and only if \( \lim_{n \to \infty} \|f^{(n)}\|_p^{1/n} = \sigma \).

(D) Let \( f \in C^\infty(\mathbb{R}) \) be such that \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p \leq \infty \). Then \( f \in B^p_{\sigma} \) if and only if it satisfies the Riesz interpolation formula

\[
f^{(1)}(x) = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f \left(x + \frac{\pi}{\sigma}(k - 1/2)\right),
\]

where the series converges in \( L^p(\mathbb{R}) \). Because this characterization is not well known, we will prove it. We have

\[
\|f^{(1)}\|_p = \left\| \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f \left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p \leq \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} \left\| f \left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p. \tag{1.3}
\]

But

\[
\left\| f \left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p = \left\| f(x) \right\|_p,
\]

and \( \sum_{k} \frac{1}{(k-1/2)^2} = \pi^2 \); hence

\[
\|f^{(1)}\|_p \leq \sigma \|f\|_p,
\]

which shows that \( f^{(1)} \in L^p(\mathbb{R}) \). Now by differentiating the Riesz formula once more, we obtain formally

\[
f^{(2)}(x) = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f^{(1)} \left(x + \frac{\pi}{\sigma}(k - 1/2)\right),
\]

but the series on the right-hand side converges because

\[
\left\| \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(k-1/2)^2} f^{(1)} \left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p \leq \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} \left\| f^{(1)} \left(x + \frac{\pi}{\sigma}(k - 1/2)\right) \right\|_p \leq \sigma \|f^{(1)}\|_p. \tag{1.5}
\]

Therefore, it follows that

\[
\|f^{(2)}\|_p \leq \sigma \|f^{(1)}\|_p,
\]

which shows that \( f^{(2)} \in L^p(\mathbb{R}) \) and in addition

\[
\|f^{(2)}\|_p \leq \sigma^2 \|f\|_p.
\]

Now an induction argument shows that

\[
\|f^{(n)}\|_p \leq \sigma^n \|f\|_p, \quad \text{for all } n = 1, 2, \ldots.
\]

That is \( f \) satisfies the Bernstein inequality, hence, \( f \in B^p_{\sigma} \). The converse is shown in [16].
The space $B^2_{\sigma}$ is called the Paley–Wiener space and is denoted by $PW_{\sigma}$. Hence, a function $f$ in $L^2(\mathbb{R})$ belongs to the Paley–Wiener space $PW_{\sigma}(\mathbb{R})$ if and only if

$$f(x) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \hat{f}(t)e^{ikt} \, dt.$$  

In other words, $f \in L^2(\mathbb{R})$ belongs to $PW_{\sigma}(\mathbb{R})$ if it has an extension to the complex plane as an entire function of exponential type not exceeding $\sigma$. The support of the Fourier transform is called the spectrum.

In this article we will add another characterization of the Paley–Wiener space by showing that a function $f$ is in $PW_{\sigma}$ if it is the initial value of a Cauchy problem involving Schrödinger’s equation.

Another important property of the space $B^2_{\sigma}$ is the celebrated Whittaker–Shannon–Kotel’nikov (WSK) sampling theorem, which states that if a function $f$ is band-limited to $[-\sigma, \sigma]$, i.e., $f \in B^2_{\sigma}$, then $f$ can be reconstructed from its samples, $f(k\pi/\sigma)$, that are taken at the equally spaced nodes $k\pi/\sigma$ on the time axis $\mathbb{R}$. The construction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f \left( \frac{k\pi}{\sigma} \right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)} \quad (t \in \mathbb{R}).$$  

(1.8)

The sampling theorem has also been extended to $B^2_{\sigma}$ for $1 \leq p < \infty$, but does not hold for $p = \infty$ as can be seen from the function $f(t) = \sin \sigma t$, which is entire of exponential type $\sigma$ and bounded on the real axis.

It would be interesting to know if the aforementioned characterizations of the Paley–Wiener space could be generalized to more general settings. In particular it is natural to ask if the Paley–Wiener space can be extended to other classes of functions, such as functions represented by integral transforms other than the Fourier transform. Unfortunately, the original proof of the Paley–Wiener theorem [17, p. 13], which depends heavily on complex analysis techniques, is not easily extendible to other transforms. Alternative approaches using real variable techniques have been developed by several people [3–6,13,14,28–31], and for examples of sampling theorems associated with other integral transforms, see [32,33].

The main goal of this article is to introduce an analogue of the Paley–Wiener space in Hilbert spaces and then apply the general result to more specific examples. Since the differentiation operator plays an important role in some of the characterizations of the Paley–Wiener space (see properties (A)–(D) above), we will follow this lead and build our space using a self-adjoint operator $D$ in a Hilbert space $H$, and denote this space by $PW_{\sigma}(D)$.

The article can be virtually divided into two parts. In the first part we will show that the space $PW_{\sigma}(D)$ has similar properties to those of the space $PW_{\sigma}$. We shall

- derive an analogue of the Bernstein inequality and the Riesz interpolation formula;
- develop a new characterization of the space $PW_{\sigma}(D)$ in terms of solutions to Cauchy problems associated with abstract Schrödinger and heat equations. As a special case, we obtain a new characterization of the space $PW_{\sigma}$;
- prove two sampling theorems for vectors in $PW_{\sigma}(D)$, one of which uses the notion of Hilbert frames and the other is based on the notion of variational splines in $H$.

In the second part of the paper we apply our abstract results to integral transforms associated with singular Sturm–Liouville problems. In particular we obtain two new sampling theorems related to one-dimensional Schrödinger operators with bounded potential. In [32] sampling theorems for point-wise sampling associated with different integral transforms were given explicitly. In contrast, the new sampling considered in the present article is more general than point-wise sampling, but the price for this is that the reconstruction algorithm is not as explicit and we basically just show that a construction formula exists. Such general sampling algorithms involving compactly supported distributions were considered in [22] and [23] for regular Paley–Wiener functions. In the particular case, where these distributions are integrals over intervals, the sampling formula for regular Paley–Wiener functions was investigated in [2,12,25,26] where it was called average sampling.

2. Paley–Wiener vectors associated with self-adjoint operators in Hilbert spaces

In this section we continue the work that started in [18–21] and introduce a space of Paley–Wiener vectors in a Hilbert space $H$. As can be seen from (1.1) and (1.2) the differentiation operator plays a vital role in the characterization of classical Paley–Wiener space. In our abstract setting, the differentiation operator will be replaced by a self-adjoint operator $D$ in a Hilbert space $H$. Furthermore, from the abstract setting we will be able to derive a new characterization of the classical Paley–Wiener space that connects Paley–Wiener functions to analytic solutions of a Cauchy problem involving Schrödinger equation.

According to the spectral theory [9], there exist a direct integral of Hilbert spaces $A = \int A(\lambda) \, dm(\lambda)$ and a unitary operator $\mathcal{F}_D$ from $H$ onto $A$, which transforms the domain $D_0$ of the operator $D^k$ onto $A_k = \{a \in A \mid \lambda^k a \in A\}$ with norm
\[ \|a\|_{A_k} = \left( \int_{-\infty}^{\infty} \lambda^{2k} \|a\|^2_{A(\lambda)} \, dm(\lambda) \right)^{1/2} \]

and \( \mathcal{F}_D(Df) = \lambda(\mathcal{F}_Df), \ f \in D_1. \)

**Definition 1.** The unitary operator \( \mathcal{F}_D \) will be called the Spectral Fourier transform and \( a = \mathcal{F}_D f \) will be called the Spectral Fourier transform of \( f \in H. \)

**Definition 2.** We will say that a vector \( f \) in \( H \) belongs to the space \( PW_\omega(D) \) if its Spectral Fourier transform \( \mathcal{F}_D f = a \) has support in \([-\omega, \omega] \).

The next proposition is evident.

**Proposition 3.** The following properties hold true:

(a) The linear set \( \bigcup_{\omega > 0} PW_\omega(D) \) is dense in \( H. \)

(b) The set \( PW_\omega(D) \) is a linear closed subspace in \( H. \)

In the following three theorems we describe some basic properties of Paley–Wiener vectors and show that they share similar properties to those of the classical Paley–Wiener functions. The next theorem, whose proof can be found in [20,21], shows that the space \( PW_\omega(D) \) has properties \( (A) \) and \( (B) \).

**Theorem 4.** The following conditions are equivalent:

(1) \( f \in PW_\omega(D); \)

(2) \( f \) belongs to the set

\[ D_\infty = \bigcap_{k=1}^\infty D_k, \]

and for all \( k \in \mathbb{N}, \) the following Bernstein inequality holds true

\[ \|D^k f\| \leq \omega^k \|f\|; \] (2.1)

(3) for every \( g \in H \) the scalar-valued function \( \langle e^{itD} f, g \rangle \) of the real variable \( t \in \mathbb{R} \) is bounded on the real line and has an extension to the complex plane as an entire function of exponential type \( \omega; \)

(4) the abstract-valued function \( e^{itD} f \) is bounded on the real line and has an extension to the complex plane as an entire function of exponential type \( \omega. \)

To show that the space \( PW_\omega(D) \) has property \( (C), \) we will need the following lemma.

**Lemma 5.** Let \( D \) be a self-adjoint operator in a Hilbert space \( H \) and \( f \in D_\infty. \) If for some \( \omega > 0 \) the upper bound

\[ \sup_{k \in \mathbb{N}} (\omega^{-k} \|D^k f\|) = B(f, \omega) \] (2.2)

is finite, then \( f \in PW_\omega \) and \( B(f, \omega) \leq \|f\|. \)

**Proof.** Condition (2.2) implies that for any complex number \( z \) we have

\[ \|e^{izD} f\| = \sum_{0}^{\infty} \|(iz)^k D^k f\|/k! \leq B(f, \omega) \sum_{0}^{\infty} |z|^k \omega^k /k! = B(f, \omega)e^{\|z\|\omega}; \]

therefore for any vector \( h \in H \) the scalar function \( \langle e^{izD} f, h \rangle \) is an entire function of exponential type \( \omega \) that is bounded on the real axis \( \mathbb{R} \) by the constant \( \|h\|\|f\|. \) An application of the classical Bernstein inequality gives

\[ \left\| \frac{d}{dt} \langle e^{itD} f, h \rangle \right\|_{C(\mathbb{R})} \leq \|e^{itD} (iD)^k f, h \|_{C(\mathbb{R})} \leq \omega^k \|h\|\|f\|, \]

which, for \( t = 0, \) yields

\[ \|D^k f, h\| \leq \omega^k \|h\|\|f\|. \]
Choosing $h$ such that $\|h\| = 1$ and $(D^k f, h) = \|D^k f\|$ we obtain the inequality $\|D^k f\| \leq \omega^k \|f\|$, $k \in \mathbb{N}$, which by Theorem 4 implies that $f \in PW_\omega$ and

$$B(f, \omega) = \sup_{k \in \mathbb{N}} (\omega^{-k} \|D^k f\|) \leq \|f\|. \quad \Box$$

Let us introduce the Favard constants (see [1, Chapter V]) which are defined as

$$K_j = \frac{4}{\pi} \sum_{r=0}^{\infty} (-1)^{j+1} \frac{(2r+1)^{j+1}}{(2r+1)^j}, \quad j, r \in \mathbb{N}.$$ 

It is known [1, Chapter V], that the sequence of all Favard constants with even indices is strictly increasing and belongs to the interval $[1, 4/\pi)$ and the sequence of all Favard constants with odd indices is strictly decreasing and belongs to the interval $(\pi/4, \pi/2]$, i.e.,

$$K_{2j} \in [1, 4/\pi), \quad K_{2j+1} \in (\pi/4, \pi/2]. \quad (2.3)$$

We will need the following generalization of the classical Kolmogorov inequality. It is worth noting that the inequality was first proved by Kolmogorov for $L^\infty(\mathbb{R})$ and later extended to $L^p(\mathbb{R})$ for $1 \leq p < \infty$ by Stein [24] and that is why it is known as the Stein–Kolmogorov inequality.

**Lemma 6.** Let $f \in D_\infty$. Then, the following inequality holds

$$\|D^k f\|^n \leq C_{k,n} \|D^n f\|^k \|f\|^{n-k}, \quad 0 \leq k \leq n, \quad (2.4)$$

where $C_{k,n} = (K_{n-k})^n/(K_n)^{n-k}$.

**Proof.** Indeed, for any $h \in H$ the Kolmogorov inequality [24] applied to the entire function $\langle e^{itD}, h \rangle$ gives

$$\left\| \frac{d}{dt} \langle e^{itD}, h \rangle \right\|^n \leq C_{k,n} \left\| \frac{d^n}{dt^n} \langle e^{itD}, h \rangle \right\|^k \left\| \langle e^{itD}, h \rangle \right\|^{n-k}, \quad 0 \leq k \leq n,$$

or

$$\left\| \langle e^{itD}(iD)^k f, h \rangle \right\|^n \leq C_{k,n} \left\| \langle e^{itD}(iD)^n f, h \rangle \right\|^k \left\| \langle e^{itD}, h \rangle \right\|^{n-k}.$$

Applying the Schwartz inequality to the right-hand side, we obtain

$$\left\| \langle e^{itD} D^k f, h \rangle \right\|^n \leq C_{k,n} \|h\|^k \|D^n f\|^k \|f\|^{n-k} \leq C_{k,n} \|h\|^n \|D^n f\|^k \|f\|^{n-k},$$

which, when $t = 0$, yields

$$\|D^k f\|^n \leq C_{k,n} \|h\|^n \|D^n f\|^k \|f\|^{n-k}.$$

By choosing $h$ such that $\|D^k f, h\| = \|D^k f\|$ and $\|h\| = 1$ we obtain (2.4). \quad \Box

**Definition 7.** Let $f \in PW_\omega(D)$ for some positive number $\omega$. We denote by $\omega_f$ the smallest positive number such that the interval $[-\omega_f, \omega_f]$ contains the support of the Spectral Fourier transform $\mathcal{F}_D f$.

It is easy to see that $f \in PW_{\omega_f}(D)$ and that $PW_{\omega_f}(D)$ is the smallest space to which $f$ belongs among all the spaces $PW_\omega(D)$. For,

$$\|D^k f\| = \left( \int_{-\infty}^{\infty} \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} d\lambda \right)^{1/2} = \left( \int_{-\omega_f}^{\omega_f} \lambda^{2k} \|a(\lambda)\|^2_{A(\lambda)} d\lambda \right)^{1/2} \leq \omega_f^k (\|a\|)_A.$$

Hence, by Theorem 4, $f \in PW_{\omega_f}(D)$. Moreover, if $f \in PW_\omega(D)$ for some $\omega < \omega_f$, then from Definition 2 the Spectral Fourier transform of $f$ has support in $[-\omega, \omega]$ which contradicts the definition of $[-\omega_f, \omega_f]$. The next theorem shows that the space $PW_{\omega_f}(D)$ has property (C).

**Theorem 8.** Let $f \in H$ belong to the space $PW_\omega(D)$, for some $0 < \omega < \infty$. Then

$$d_f = \lim_{k \to \infty} \|D^k f\|^{1/k} \quad (2.5)$$

exists and is finite. Moreover, $d_f = \omega_f$. Conversely, if $f \in D_\infty$ and $d_f = \lim_{k \to \infty} \|D^k f\|^{1/k}$, exists and is finite, then $f \in PW_{\omega_f}$ and $d_f = \omega_f$. 


Proof. From Lemma 6 we have
\[ \|D^k f\|^n \leq C_{k,n} \|D^n f\|^k \|f\|^{n-k}, \quad 0 \leq k \leq n. \]
Without loss of generality, let us assume that \( \|f\| = 1 \). Thus,
\[ \|D^k f\|^{1/k} \leq (\pi/2)^{1/kn} \|D^n f\|^{1/n}, \quad 0 \leq k \leq n. \]
Let \( k \) be arbitrary but fixed. It follows that
\[ \|D^k f\|^{1/k} \leq (\pi/2)^{1/kn} \|D^n f\|^{1/n}, \quad \text{for all } n \geq k, \]
which implies that
\[ \|D^k f\|^{1/k} \leq \lim_{n \to \infty} \|D^n f\|^{1/n}. \]
But since this inequality is true for all \( k > 0 \), we obtain that
\[ \lim_{k \to \infty} \|D^k f\|^{1/k} \leq \lim_{n \to \infty} \|D^n f\|^{1/n}, \]
which proves that \( d_f = \lim_{k \to \infty} \|D^k f\|^{1/k} \) exists.

Now we show that \( d_f \) is finite by showing that \( d_f \leq \omega_f \). Since \( f \in PW_{\omega_f} \), we have by Theorem 4
\[ \|D^k f\|^{1/k} \leq \omega_f \|f\|^{1/k}, \]
and by taking the limit as \( k \to \infty \) we have \( d_f \leq \omega_f \). To show that \( d_f = \omega_f \), let us assume that \( d_f < \omega_f \). Therefore, there exist \( M > 0 \) and \( \sigma \) such that \( 0 < d_f < \sigma < \omega_f \) and
\[ \|D^k f\| \leq M \sigma^k, \quad \text{for all } k > 0. \]
Thus, by Lemma 5 we have \( f \in PW_{\sigma} \), which is a contradiction to the definition of \( \omega_f \).

Conversely, suppose that \( d_f = \lim_{k \to \infty} \|D^k f\|^{1/k} \) exists and is finite. Therefore, there exist \( M > 0 \) and \( \sigma > 0 \) such that \( d_f < \sigma \) and
\[ \|D^k f\| \leq M \sigma^k, \quad \text{for all } k > 0, \]
which, in view of Lemma 5, implies that \( f \in PW_{\sigma} \). Now by repeating the argument in the first part of the proof we obtain
\[ d_f = \omega_f, \quad \text{where } \omega_f = \inf\{\sigma : f \in PW_{\sigma}\}. \]

Now consider the Cauchy problem for the abstract Schrödinger equation
\[ \frac{\partial u(t)}{\partial t} = iDu(t), \quad u(0) = f, \quad i = \sqrt{-1}, \tag{2.6} \]
where \( u : \mathbb{R} \to H \) is an abstract function with values in \( H \).

The next theorem gives another characterization of the space \( PW_{\omega}(D) \), from which we obtain a new characterization of the space \( PW_{\omega} \).

**Theorem 9.** A vector \( f \in H \), belongs to \( PW_{\omega}(D) \) if and only if the solution \( u(t) \) of the corresponding Cauchy problem (2.6) has the following properties:

(1) as a function of \( t \), it has an analytic extension \( u(z) \), \( z \in \mathbb{C} \) to the complex plane \( \mathbb{C} \) as an entire function;
(2) it has exponential type \( \omega \) in the variable \( z \), that is
\[ \|u(z)\|_H \leq e^{\omega|z|} \|f\|_H \]
and it is bounded on the real line.

**Proof.** The solution to (2.6) is given by the formula
\[ u(t) = e^{itD} f, \quad t \in \mathbb{R}, \tag{2.7} \]
where \( e^{itD} \) is the one-parameter group of unitary operators in \( H \). Thus, if \( f \in PW_{\omega}(D) \) the Bernstein inequality holds true and the series
\[ e^{zD} f = \sum_{r=0}^{\infty} \frac{(zD)^r}{r!} f \tag{2.8} \]
is convergent in $H$ and represents an abstract entire function. Since $\|D'f\|_H \leq \omega' \|f\|_H$, we have the estimate

$$\|e^{itD}f\|_H = \left|\lim_{r \to \infty} \sum_{r=0}^{\infty} (e^{itD}f)/r! \right|_H \leq \|f\|_H \sum_{r=0}^{\infty} |z|^r/|r!| = e^{|z|\omega} \|f\|_H,$$

which shows that the function (2.8) has exponential type $\omega$.

Now assume that the solution $u(t) = \exp(itD)f$ of the problem (2.6) has properties (1) and (2) of the theorem. For a fixed vector $g \in H$ we consider the complex-valued function $U_g(z), z \in \mathbb{C}$

$$U_g(z) = \langle e^{itD}f, g \rangle, \quad z \in \mathbb{C}. \tag{2.9}$$

The Schwartz inequality gives

$$|U_g(t)| \leq \|e^{itD}f\|_H \|g\| \leq \|f\|_H \|g\|, \tag{2.10}$$

which shows that $U_g(z)$ is bounded on the real line by $\|f\|_H \|g\|$. On the other hand, if $u(z) = \exp(izD)f$ is an entire function of exponential type $\omega$, then $U_g(z)$ is also an entire function of exponential type $\omega$ that is bounded on the real line. For such functions the classical Bernstein inequality (1.1) holds. Since

$$U_g^{(k)}(z) = \langle e^{itD}(iD)^k f, g \rangle,$$

we have

$$\sup_t |U_g^{(k)}(t)| \leq \omega^k \sup_t |U_g(t)| \leq \omega^k \|f\|_H \|g\|, \quad t \in \mathbb{R}, \tag{2.11}$$

and we obtain

$$|\langle e^{itD}(iD)^k f, g \rangle| \leq \omega^k \|f\|_H \|g\|.$$

When $t = 0$, we have

$$|\langle D^k f, g \rangle| \leq \omega^k \|f\|_H \|g\|.$$

Now choosing $g \in H$ such that $\|g\| = 1$, and $|\langle D^k f, g \rangle| = \|D^k f\|$, we obtain the Bernstein inequality (2.1) which by Theorem 4 implies that $f \in PW_{\omega}(D)$. The theorem is proved. \(\Box\)

As an immediate application of the theorem, consider the following Cauchy problem for the corresponding Schrödinger equation:

$$\frac{\partial u(x, t)}{\partial t} = i\Delta u(x, t), \quad u(x, 0) = f(x), \quad x \in \mathbb{R}^d, \tag{2.12}$$

where $\Delta$ is the Laplacian operator in $L^2(\mathbb{R}^d)$. It is easy to see that the solution $u(x, t)$ to (2.12) is given by the formula

$$u(x, t) = \frac{1}{2\sqrt{\pi it}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4it}\right) f(y) \, dy. \tag{2.13}$$

where $\sqrt{\pi it} = e^{i\pi/4} \sqrt{\pi t}$, if $t > 0$ and $\sqrt{\pi it} = e^{-i\pi/4} \sqrt{\pi |t|}$, if $t < 0$.

The proof of the next theorem is essentially the same as the proof of Theorem 9. Note that when $d = 1$, the theorem gives another characterization of the Paley–Wiener space.

**Theorem 10.** A function $f \in L^2(\mathbb{R}^d)$ is the Fourier transform of a compactly supported function $\hat{f}$ from the space $L^2(\mathbb{R}^d)$ with support in the ball $B(0, \omega) \subset \mathbb{R}^d$ if and only if the solution $u(x, t)$ of the corresponding Cauchy problem (2.12), with $f$ being in the Sobolev space $H^\infty(\mathbb{R}^d)$, has the following properties:

1. as a function in $t$ it has analytic extension $u(x, z)$ to the complex plane $\mathbb{C}$ as an entire function;
2. it has exponential type $\omega$ in the variable $z$

$$\|u(x, z)\|_{L^2(\mathbb{R}^d)} \leq e^{\omega|z|} \|f\|_{L^2(\mathbb{R}^d)}$$

and is bounded on the real line.
Using the same idea we can also give another characterization of the Paley–Wiener space $PW_\omega(\mathbb{R})$. Consider the following Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x}, \quad u(x, 0) = f(x), \quad (2.14)$$

where $f$ belongs to the Sobolev space $H^\infty(\mathbb{R})$.

It is not difficult to prove the following theorem which characterizes Paley–Wiener functions.

**Theorem 11.** A function $f \in L^2(\mathbb{R})$ is the Fourier transform of a compactly supported function $\hat{f}$ from the space $L^2(\mathbb{R})$ with support in $[-\omega, \omega]$ if and only if the solution $u(x, t)$ of the corresponding Cauchy problem (2.14) has the following properties:

1. as a vector-valued function
   $$u(\cdot, t) : \mathbb{R} \rightarrow L^2(\mathbb{R}) \quad (2.15)$$
   in $t$, it has analytic extension $u(x, z), z \in \mathbb{C}$ to the complex plane $\mathbb{C}$ as an entire function;

2. it has exponential type $\omega$ in the variable $z$, meaning that
   $$\|u(\cdot, z)\|_{L^2(\mathbb{R})} \leq e^{\omega |z|} \|f\|_{L^2(\mathbb{R})}, \quad z \in \mathbb{C}, \quad (2.16)$$
   and is bounded on the real line.

3. **Riesz interpolation formula in a Hilbert space**

   In this section, we derive an analogue of the Riesz interpolation formula (1.2) for the Paley–Wiener space $PW_\omega(D)$; hence, showing that the space $PW_\omega(D)$ has property (D).

   Let us introduce the operator

   $$R_\omega^D f = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} (-1)^{k-1} \frac{(k-1/2)^2}{(k-1/2)^2} e^{i\pi \omega (k-1/2)} f, \quad f \in H, \quad \omega > 0. \quad (3.1)$$

Since $\|e^{itD} f\| = \|f\|$ and

$$\frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} = \omega \quad (3.2)$$

it follows that the series in (3.1) is convergent and that $R_\omega^D f$ is a bounded operator in $H$

$$\|R_\omega^D f\| \leq \omega \|f\|, \quad f \in H. \quad (3.3)$$

**Theorem 12.** The following conditions are equivalent:

1. $f \in PW_\omega(D)$;
2. the following Riesz interpolation formula holds true
   $$(iD)^nf = (R_\omega^D)^n f, \quad n \in \mathbb{N}. \quad (3.4)$$

**Proof.** If $f \in PW_\omega(D)$ then according to Theorem 9, for any functional $\psi^*$ the function

$$F_{n-1}(t) = \langle e^{itD} (iD)^{n-1} f, \psi^* \rangle, \quad n \in \mathbb{N},$$

is of exponential type $\omega$ and bounded on $\mathbb{R}$. Thus, the following classical Riesz interpolation formula holds

$$\frac{d}{dt} F_{n-1}(t) = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} (-1)^{k-1} \frac{(k-1/2)^2}{(k-1/2)^2} F_{n-1} \left( t + \frac{\pi}{\omega} (k-1/2) \right).$$

According to a general theory of one-parameter groups of operators

$$\frac{d}{dt} e^{itD} f = iDe^{itD} f.$$

Since

$$\frac{d}{dt} F(t) = \frac{d}{dt} \langle e^{itD} f, \psi^* \rangle = \langle iDe^{itD} f, \psi^* \rangle.$$
we obtain the following formula

$$
(iDe^{iD}(iD)^{-1}f, \psi^*) = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{i((k-1/2)D)}(iD)^{-1}f, \psi^*),
$$

which, for $t = 0$, gives

$$
(iD)^nf, \psi^*) = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{i((k-1/2)D)}(iD)^{-1}f, \psi^*).
$$

(3.5)

Since the last formula holds for any functional $\psi^* \in H^*$, we have the equality

$$
(iD)^nf = R^o_D(iD)^{-1}f, \quad n \in \mathbb{N},
$$

(3.6)

for every function for which property (1) holds. Since the space $PW_\omega(D)$ is invariant under $D$, we obtain

$$
(iD)^2f = (iD)(iD)f = (iD)R^o_Df = (R^o_D)^2f,
$$

and by induction we obtain (3.4).

The implication $(1) \rightarrow (2)$ is proved. Conversely, formulas (3.4) and (3.3) show that (2) implies the Bernstein inequality, which in turn implies part (1), and this completes the proof. □

4. Sampling of Paley–Wiener vectors in Hilbert spaces

The goal of this section is to develop an abstract “irregular” sampling theory for Paley–Wiener vectors associated with a self-adjoint operator in a Hilbert space.

In what follows the notation $D_k$, $k \in \mathbb{N}$, is used for the domain of the operator $D^k$ with the graph norm

$$
\|f\|_k = \|f\| + \|D^k f\|.
$$

We assume that on the space $D_{\lambda}$, a set of continuous functionals $\{\Phi_\nu\}$ is given and they satisfy the following two inequalities. There exist constants $\lambda, C, c > 0$, such that for every $f \in D$

$$
c \left( \sum_\nu |\Phi_\nu(f)|^2 \right)^{1/2} \leq \|f\| \leq C \left( \sum_\nu |\Phi_\nu(f)|^2 \right)^{1/2} + \lambda \|Df\|. \quad (4.1)
$$

Let $Z_\theta$ be the intersection of all kernels $\text{Ker} \Phi_\nu$. We say that a set $M$ is a uniqueness set for the set of functionals $\{\Phi_\nu\}$ if the intersection of $M$ and $Z_\theta$ is trivial. One can easily prove the following uniqueness theorem (see [21]).

**Theorem 13.** If the right-hand side of the inequality (4.1) holds true then $PW_\omega(D)$ is the uniqueness set for the set of functionals $\{\Phi_\nu\}$ as long as

$$
\omega \lambda < 1. \quad (4.2)
$$

Now we are going to introduce a reconstruction method that uses the idea of Hilbert frames. Since by assumption the functionals $\Phi_\nu$ are continuous on a Hilbert space $D_k$, the Riesz theorem about continuous functionals implies the existence of vectors $\psi_\nu \in D_k$ such that for any $f \in D_k$,

$$
\Phi_\nu(f) = \langle f, \psi_\nu \rangle.
$$

If $f \in PW_\omega(D)$ and the assumption (4.2) is satisfied, the inequalities (4.1) along with the Bernstein inequality show that there exist constants $A, B > 0$ such that the following frame inequality

$$
A \left( \sum_\nu |\langle f, \nu \rangle|^2 \right)^{1/2} \leq \|f\| \leq B \left( \sum_\nu |\langle f, \nu \rangle|^2 \right)^{1/2} \quad (4.3)
$$

holds where $\psi_\nu$ is the orthogonal projection of $\nu_\nu$ on the space $PW_\omega(D)$. Thus, by using the classical ideas of Duffin and Schaeffer about dual frames [11] (see also [7]), we obtain the following reconstruction formula.

**Theorem 14.** There exists a frame $\{\theta_\nu\}$ in the space $D_k$ such that the following reconstruction formula holds

$$
f = \sum_\nu \langle f, \psi_\nu \rangle \theta_\nu = \sum_\nu \Phi_\nu(f) \theta_\nu, \quad f \in PW_\omega(D). \quad (4.4)
$$
Now we introduce another reconstruction algorithm that uses the idea of variational splines in Hilbert spaces. To formulate this reconstruction algorithm we have to introduce the following definition of variational splines associated with a self-adjoint operator.

Definition 15. A variational spline interpolating vector \( f \in D_k, k \in \mathbb{N} \), is denoted by \( s_k(f) \) and it is a vector in \( Z_a(D_k) \), \( a = \{ \Phi_\nu(f) \} \), which minimizes the functional \( u \mapsto \| D^k u \|, u \in D_k \).

The following theorem was proved in [21].

Theorem 16. Under the above assumptions the optimization problem has a unique solution for every \( k = 2^l, l \in \mathbb{N} \). Moreover, if the functionals \( \Phi_\nu \) satisfy (4.1), then any \( f \in PW_\omega(D) \) can be reconstructed through the formula

\[
\lim_{k \to \infty} s_k(f) = f
\]

and the error estimate is

\[
\| f - s_k(f) \| \leq 2(\lambda \omega)^k \| f \|, \quad k = 2^l, l = 0, 1, 2, \ldots
\]

5. Applications to Sturm–Liouville operators

In this section we apply the general results obtained in previous sections to specific examples involving differential operators. We specify our characterization of Paley–Wiener functions and formulate our version of the Riesz interpolation formula. In the next section we derive two new sampling formulas for irregular average sampling associated with second order Schrödinger operators with bounded potentials.

5.1. Integral transforms associated with Sturm–Liouville operators on a half-line

Consider the singular Sturm–Liouville problem on the half line

\[
L_x y = L y := - \frac{d^2 y}{dx^2} + q(x)y = \lambda y, \quad 0 \leq x < \infty,
\]

with

\[
y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad \text{for some } 0 \leq \alpha < 2\pi,
\]

and \( q \) is assumed to be real-valued.

Let \( \phi(x, \lambda) \) be the solutions of Eq. (5.1) satisfying the initial conditions \( \phi(0, \lambda) = \sin \alpha, \phi'(0, \lambda) = -\cos \alpha \). Clearly, \( \phi(x, \lambda) \) is a solution of (5.1) and (5.2). It is easy to see that \( \phi(x, \lambda) \) and \( \phi'(x, \lambda) \) are bounded as functions of \( x \) for \( \lambda > 0 \) [27]. It is known [15,27] that if \( f \in L^2(\mathbb{R}^+) \), then

\[
F(\lambda) = \hat{f}(\lambda) = \int_0^\infty f(x)\phi(x, \lambda) \, dx
\]

is well defined (in the mean) and belongs to \( L^2(\mathbb{R}, d\rho) \), and

\[
f(x) = \int_{-\infty}^\infty \hat{f}(\lambda)\phi(x, \lambda) \, d\rho(\lambda),
\]

with

\[
\| f \|_{L^2(\mathbb{R}^+)} = \| \hat{f} \|_{L^2(\mathbb{R}, d\rho)}.
\]

The measure \( \rho(\lambda) \) is called the spectral function of the problem. In many cases of interest the support of \( d\rho \) is \( \mathbb{R}^+ \). In this case the transform (5.4) takes the form

\[
f(x) = \int_0^\infty \hat{f}(\lambda)\phi(x, \lambda) \, d\rho(\lambda),
\]
and the Parseval equality (5.5) becomes \( \| f \|_{L^2(\mathbb{R}^+)} = \| \hat{f} \|_{L^2(\mathbb{R}^+, d\rho)} \). Hereafter, we assume that \( q \) is real-valued, bounded and \( C^\infty(\mathbb{R}^+) \). Because we are interested in the case where the spectrum of the problem is continuous, we shall focus on the case in which the differential equation (5.1) is in the limit-point case at infinity. Restrictions on spectra can be found in [15,27]. The condition \( q \in L^1(\mathbb{R}^+) \) will suffice. In such a case the problem (5.1) and (5.2) is self-adjoint [10, p. 158], i.e., \( \langle Lf, g \rangle = \langle f, Lg \rangle \) for all \( f, g \in \mathcal{D}_L \), where \( \mathcal{D}_L \) consists of all functions \( u \) satisfying:

1. \( u \) is differentiable and \( u' \) is absolutely continuous on \( 0 \leq x \leq b \) for all \( b < \infty \),
2. \( u \) and \( Lu \) are in \( L^2(\mathbb{R}^+) \),
3. \( u(0) \cos \alpha + u'(0) \sin \alpha = 0 \).

Now consider the initial–boundary-value problem involving the Schrödinger equation

\[
i \frac{\partial u(x,t)}{\partial t} = -L_x u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - q(x) u(x,t), \quad 0 \leq x < \infty, \quad t \geq 0,
\]

with

\[
u(x,0) = f(x) \quad (5.8)
\]

and

\[
u(0,t) \cos \alpha + \frac{\partial u(0,t)}{\partial x} \sin \alpha = 0 \quad \text{for all } 0 < t,
\]

where \( f(x) \in L^2(\mathbb{R}^+) \).

Set

\[
u(x,t) = \int_{-\infty}^{\infty} e^{i\lambda t} \hat{f}(\lambda) \phi(x,\lambda) d\rho(\lambda). \quad (5.10)
\]

Formally, if \( f \) and \( Lf \) are in \( L^2(\mathbb{R}^+) \), then

\[
i \frac{\partial u(x,t)}{\partial t} = \int_{-\infty}^{\infty} (-\lambda) e^{i\lambda t} \hat{f}(\lambda) \phi(x,\lambda) d\rho(\lambda) = -L_x u(x,t)
\]

and

\[
u(x,0) = \int_{-\infty}^{\infty} \hat{f}(\rho) \phi(x,\lambda) d\rho = f(x),
\]

and

\[
u(0,t) \cos \alpha + \frac{\partial u(0,t)}{\partial x} \sin \alpha = 0. \quad (5.11)
\]

Therefore, \( u(x,t) \) is a solution of the initial–boundary-value problem (5.7)–(5.9), in the sense of \( L^2(\mathbb{R}^+) \).

**Definition 17.** We say that \( f(x) \in L^2(\mathbb{R}^+) \) is bandlimited with bandwidth \( \omega \) or \( f \in \text{PW}_{\omega}(L) \) if its Spectral Fourier transform \( \hat{f}(\lambda) \) according to Definition 1, has support \([-\omega, \omega]\), where \( L \) is given by (5.1) and (5.2).

It follows from the definition that if \( f \) is bandlimited to \([-\omega, \omega]\), then

\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) \varphi(x,\lambda) d\rho = \int_{-\omega}^{\omega} \hat{f}(\lambda) \varphi(x,\lambda) d\rho,
\]

and hence \( L^n f(x) = \int_{-\omega}^{\omega} \hat{f}(\lambda) (\lambda) \varphi(x,\lambda) d\rho \), which exists for all \( n = 0, 1, 2, \ldots \). Thus, by Parseval’s equality

\[
\| L^n f \|_{L^2(\mathbb{R}^+)}^2 = \int_{-\omega}^{\omega} |\hat{f}(\lambda)|^2 \lambda^{2n} d\rho \leq \omega^{2n} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\rho = \omega^{2n} \| f \|_{L^2(\mathbb{R}^+, d\rho)}^2 = \omega^{2n} \| f \|_{L^2(\mathbb{R}^+)}^2.
\]

That is,
\[
\|L^n f\| \leq \alpha^n \|f\|, \quad n = 0, 1, 2, \ldots, \tag{5.13}
\]

which is a generalization of Bernstein inequality (1.1).

In order to apply Theorem 9, we have to define the domain \(\mathcal{D}^\infty\) on which all iterations of \(L\) are self-adjoint. It is easy to see that \(\mathcal{D}^\infty\) consists of all functions \(u\) satisfying the following conditions:

(i) \(u\) is infinitely differentiable on \(\mathbb{R}^+\);
(ii) \(L^k u\) is in \(L^2(\mathbb{R}^+)\), for all \(k = 0, 1, 2, \ldots\);
(iii) \((L^k u)(0)\) \(\cos \alpha + (\frac{d}{dt} L^k u)(0) \sin \alpha = 0\).

**Theorem 18.** A function \(f \in L^2(\mathbb{R}^+)\) is bandlimited in the sense of Definition 17 with bandwidth \(\omega\) if and only if the solution \(u(x, t)\) of the initial–boundary-value problem (5.7)–(5.9) with \(f \in \mathcal{D}^\infty\) has the following properties:

1. as a function of \(t\) it has analytic extension \(u(x, z)\) to the complex plane as entire function of exponential type \(\omega\);
2. it satisfies the estimate

\[
\|u(\cdot, z)\|_{L^2(\mathbb{R}^+)} \leq e^{\omega|2z|}\|f\|_{L^2(\mathbb{R}^+)} \leq e^{\omega|2z|}\|f\|_{L^2(\mathbb{R}^+)}.
\]

In particular, \(u(x, z)\) is bounded on the real \(t\)-line.

The Riesz interpolation formula for the operator \(L\) now takes the following form

\[
Lf(x) = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} (-1)^{k-1} \frac{(\pi^2/x^2)}{(k - 1/2)^2} u\left(x, \frac{\pi}{\omega} (k - 1/2)\right), \tag{5.14}
\]

where \(f \in PW_\omega(L), \omega > 0\), and \(u(x, t)\) is given by (5.10).

**Example 1 (The Weber transform).** Consider the Bessel differential equation on the half-line \([a, \infty), a > 0\):

\[
Ly = -y'' + \frac{\nu^2 - \frac{3}{4}}{x^2} y = \lambda y, \quad x \in [a, \infty), \quad \nu > -\frac{1}{2}. \tag{5.15}
\]

Let \(\phi(x, \lambda)\) be the solution of Eq. (5.15) under the initial conditions

\[
\phi(a, \lambda) = 0, \quad \phi'(a, \lambda) = -1.
\]

It is known that [27]

\[
\phi(x, \lambda) = \frac{1}{2} \pi \sqrt{ax} \left[J_\nu(xs) Y_\nu(as) \right. - \left. Y_\nu(xs) J_\nu(as)\right],
\]

\[
d\rho(\lambda) = \frac{4}{\pi^2 a} \int_{0}^{\infty} s ds \int_{0}^{\infty} \frac{J_\nu^2(as) + Y_\nu^2(as)}{f_\nu^2(as) + g_\nu^2(as)}.
\]

where \(\lambda = s^2, J_\nu\) and \(Y_\nu\) are the Bessel functions of the first and second kind, respectively. We have the pair of Weber transforms

\[
\hat{f}(s) = \int_{0}^{\infty} \sqrt{x} \left[J_\nu(xs) Y_\nu(as) \right. \left. - Y_\nu(xs) J_\nu(as)\right] f(x) dx,
\]

\[
f(x) = \int_{0}^{\infty} \sqrt{x} \left[J_\nu(x) Y_\nu(as) \right. \left. - Y_\nu(x) J_\nu(as)\right] \frac{s}{f_\nu^2(as) + g_\nu^2(as)} \hat{f}(s) ds, \tag{5.16}
\]

with the Parseval relation

\[
\int_{a}^{\infty} f(x) g(x) dx = \int_{0}^{\infty} \frac{\pi s \hat{f}(s) \hat{g}(s)}{f_\nu^2(as) + g_\nu^2(as)} ds
\]

whenever \(f, g \in L^2(a, \infty)\).

Now consider the initial-value problem involving the Schrödinger equation

\[
\frac{\partial u(x, t)}{\partial t} = iL_x u(x, t), \quad a \leq x < \infty, \quad t \geq 0, \tag{5.17}
\]

with
where $L$ is given by (5.15) and $f(x) \in L^2(a, \infty)$.

Therefore, $\hat{f}$ has support in $[0, \omega]$ if and only if $u(x, t)$ satisfies the conditions given by Theorem 18.

5.2. Integral transforms associated with Sturm–Liouville operators on the whole line

Now we extend the results of the previous section to singular Sturm–Liouville problems on the whole line. Let us consider the following singular Sturm–Liouville problem:

$$Ly := -\frac{d^2 y}{dx^2} + q(x)y = \lambda y, \quad -\infty < x < \infty,$$

with $q$ being real-valued, infinitely differentiable on $\mathbb{R}$ and such that the spectrum is continuous. We restrict ourselves to the case in which $L$ is in the limit-point case at $\pm \infty$. Conditions on $q$ to ensure that can be found in [10,27]. The problem is self-adjoint in the sense that

$$\int_{\mathbb{R}} (Lu)v \, dx = \int_{\mathbb{R}} u(Lv) \, dx,$$

for all $u, v \in D_L$, where $D_L$ consists of all functions $u \in L^2(\mathbb{R})$ with absolutely continuous first derivative in every closed subinterval of $\mathbb{R}$ such that $u, Lu \in L^2(\mathbb{R})$; see [10, p. 260].

Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be the solutions of Eq. (5.19) such that

$$\phi(0, \lambda) = 0, \quad \phi'(0, \lambda) = 1, \quad \theta(0, \lambda) = 1, \quad \theta'(0, \lambda) = 0,$$

so that the Wronskian $W_q(\phi, \theta) = \phi(x, \lambda)\theta'(x, \lambda) - \phi'(x, \lambda)\theta(x, \lambda) = 1$. The functions $\theta(x, \lambda)$ and $\phi(x, \lambda)$ are infinitely differentiable because $q$ is.

Similar to the singular case on the half-line, there exist two functions $m_1(\lambda)$ and $m_2(\lambda)$ analytic in the upper half plane such that, as a function of $\lambda$,

$$\psi_1(x, \lambda) = \theta(x, \lambda) + m_1(\lambda)\phi(x, \lambda)$$

is in $L_2(-\infty, 0)$, and $\psi_2(x, \lambda) = \theta(x, \lambda) + m_2(\lambda)\phi(x, \lambda)$ is in $L_2(0, \infty)$, for $\lambda$ with $\Re \lambda > 0$. Moreover, there exist two non-decreasing functions $\xi(\lambda), \zeta(\lambda)$, and a function of bounded variation $\eta(\lambda)$ such that if $f \in L^2(\mathbb{R})$, then

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\phi(x, \lambda) \, dx \in L^2(\mathbb{R}, d\zeta),$$

$$E(\lambda) = \int_{-\infty}^{\infty} f(x)\theta(x, \lambda) \, dx \in L^2(\mathbb{R}, d\xi)$$

and the following inversion formula

$$f(x) = \int_{\mathbb{R}} E(\lambda)\theta(x, \lambda) d\xi(\lambda) + \int_{\mathbb{R}} F(\lambda)\phi(x, \lambda) d\eta(\lambda) + \int_{\mathbb{R}} E(\lambda)\phi(x, \lambda) d\eta(\lambda) + \int_{\mathbb{R}} F(\lambda)\phi(x, \lambda) d\zeta(\lambda),$$

holds, and in addition if $f$ has components $(E_1, F_1)$ and $g$ has components $(E_2, F_2)$ then

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g}(x) \, dx = \int_{\mathbb{R}} \left\{ E_1(\lambda)\overline{E_2}(\lambda) \, d\xi + E_1(\lambda)\overline{F_1}(\lambda) \, d\eta + F_1(\lambda)\overline{E_2}(\lambda) \, d\eta + F_1(\lambda)\overline{F_2}(\lambda) \, d\xi \right\}.$$

Since $\phi$ and $\theta$ are real for real $\lambda$, we have for real-valued $f$

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |E_1(\lambda)|^2 \, d\xi + 2\int_{\mathbb{R}} E_1(\lambda)F_1(\lambda) \, d\eta + \int_{\mathbb{R}} |F_1(\lambda)|^2 \, d\zeta.$$

If $q$ is an even function, then $d\eta(\lambda) = 0$, and formula (5.22) takes the form

$$f(x) = \int_{-\infty}^{\infty} E(\lambda)\theta(x, \lambda) d\xi(\lambda) + \int_{-\infty}^{\infty} F(\lambda)\phi(x, \lambda) d\zeta(\lambda).$$
with formulas (5.20) and (5.21) remaining the same. The Parseval relation (5.23) becomes
\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |E(\lambda)|^2 \, d\xi(\lambda) + \int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\zeta(\lambda).
\] (5.25)

**Definition 19.** We say that \( f(x) \in L^2(\mathbb{R}) \) is bandlimited with bandwidth \( \omega \), or \( f \in PW_\omega(L) \), if \([-\omega, \omega]\) is the smallest closed interval containing the supports of both \( E(\lambda) \) and \( F(\lambda) \) defined by (5.20) and (5.21), where \( L \) is given by (5.19).

If \( f \) is bandlimited with bandwidth \( \omega \), according to Definition 19, then for all \( n = 0, 1, \ldots \),
\[
L^n f(x) = \int_{-\omega}^{\omega} (-\lambda)^n \left[ E(\lambda)\theta(x, \lambda) \, d\xi(\lambda) + (F(\lambda)\theta(x, \lambda) + E(\lambda)\phi(x, \lambda)) \, d\eta(\lambda) + F(\lambda)\phi(x, \lambda) \, d\zeta(\lambda) \right]
\]
which, in view of Parseval’s equality, yields
\[
\|L^n f\|^2 = \int_{-\omega}^{\omega} \omega^2 \lambda^{2n} \left[ |E(\lambda)|^2 \, d\xi(\lambda) + 2E(\lambda)F(\lambda) \, d\eta(\lambda) + |F(\lambda)|^2 \, d\zeta(\lambda) \right]
\leq \omega^{2n} \int_{-\omega}^{\omega} \left[ |E(\lambda)|^2 + 2E(\lambda)F(\lambda) \, d\eta(\lambda) + |F(\lambda)|^2 \, d\zeta(\lambda) \right]
= \omega^{2n} \|f\|^2.
\] (5.26)

Hence, we have the Bernstein-type inequality \( \|L^n f\| \leq \omega^n \|f\| \) for all \( n = 0, 1, \ldots \).

Let
\[
\frac{\partial u(x, t)}{\partial t} = iL u(x, t), \quad u(x, 0) = f(x).
\] (5.28)

Formally, if \( f \) and \( Lf \) are in \( L^2(\mathbb{R}) \), then \( u(x, t) \) is the solution of the initial-value problem:

Let \( D^\infty \) be the space of all \( C^\infty(\mathbb{R}) \) functions, \( u(x) \), such that \( L^k u \in L^2(\mathbb{R}) \) for \( k = 0, 1, 2, \ldots \).

**Theorem 20.** A function \( f \in L^2(\mathbb{R}) \) is bandlimited in the sense of Definition 19 with bandwidth \( \omega \) if and only if the solution \( u(x, t) \) of the initial-value problem (5.28) with \( f \in D^\infty \) has the following properties:

1. as a function of \( t \) it has analytic extension \( u(x, z) \) to the complex plane as entire function of exponential type \( \omega \);
2. it satisfies the estimate
\[
\|u(\cdot, z)\|_{L^2(\mathbb{R})} \leq e^{\omega|z|} \|f\|_{L^2(\mathbb{R})}
\]
and is bounded on the real \( t \)-line.

The next example demonstrates Definition 19 and also shows that the definition coincides with the classical definition of bandlimitedness as given in the Paley–Wiener theorem.

**Example 2** (The Fourier transform). Consider the singular Sturm–Liouville problem on the whole line:
\[
y'' = -\lambda y, \quad -\infty < x < \infty.
\]
Since \( q(x) = 0 \) is even, we have (5.24). It is easy to see that
\[
m_1(\lambda) = i\sqrt{\lambda}, \quad \psi_1(x, \lambda) = e^{-ix\sqrt{\lambda}},
m_2(\lambda) = -i\sqrt{\lambda}, \quad \psi_2(x, \lambda) = e^{ix\sqrt{\lambda}},
d\xi(\lambda) = \frac{1}{2\pi \sqrt{\lambda}} \, d\lambda = \frac{ds}{\pi}, \quad d\zeta(\lambda) = \frac{\sqrt{\lambda}}{2\pi} \, d\lambda = \frac{s^2}{\pi} \, ds, \quad \text{on } (0, \infty),
\]
where \( \lambda = s^2 \). The transforms (5.20), (5.21), and (5.24) become
\[
E(\lambda) = \int_{-\infty}^{\infty} \cos sx f(x) \, dx, \quad F(\lambda) = -\int_{-\infty}^{\infty} \frac{\sin sx}{s} f(x) \, dx,
\]

\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos sx E(\lambda) \, d\lambda - \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin sx}{s} F(\lambda) s^2 \, d\lambda.
\]

If we denote \(E(\lambda), -sF(\lambda),\) by \(\tilde{E}(s), \tilde{F}(s),\) respectively, then formulas (5.29) become

\[
\tilde{E}(s) = \int_{-\infty}^{\infty} \cos sx f(x) \, dx, \quad \tilde{F}(s) = \int_{-\infty}^{\infty} \sin sx f(x) \, dx,
\]

\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos sx \tilde{E}(s) \, ds + \frac{1}{\pi} \int_{0}^{\infty} \sin sx \tilde{F}(s) \, ds.
\]

and the Parseval’s equation (5.25) takes the form

\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{\pi} \int_{0}^{\infty} |\tilde{E}(s)|^2 \, ds + \frac{1}{\pi} \int_{0}^{\infty} |\tilde{F}(s)|^2 \, ds.
\]

Therefore, a function is bandlimited if and only if its cosine and sine Fourier transforms have compact supports, which agrees with the classical definition.

Following the aforementioned procedure and replacing \(\lambda\) by \(\lambda^2,\) we can write the solution to the initial-value problem (5.28) in the form

\[
u(x, t) = \int_{\mathbb{R}} e^{-i\lambda^2 t} e^{i\lambda x} \tilde{f}(\lambda) \, d\lambda,
\]

where \(\tilde{f}\) is the Fourier transform of \(f.\) By using the inversion formula for the Fourier transform we obtain

\[
u(x, t) = \frac{1}{2\sqrt{\pi} it} \int_{\mathbb{R}} f(y) e^{-(x-y)^2/(4at)} \, dy,
\]

which is (2.13); thus, we have another proof of Theorem 10.

6. Sampling theorem associated with Sturm–Liouville differential operators

In this section we apply previous results to obtain two new sampling formulas associated with the differential operator \(L.\)

Definition 21. A \(\rho\)-lattice \(X(\{x_j\}, \rho), \quad j \in \mathbb{N}, \quad \rho > 0,\) is a set of points \(\{x_j\} \in \mathbb{R}^+\) such that:

1. intervals \([x_j - \rho/2, x_j + \rho/2] = I(x_j, \rho/2)\) do not intersect;
2. intervals \([x_j - \rho, x_j + \rho] = I(x_j, \rho)\) form a cover of \(\mathbb{R}^+\).

In what follows we will always assume that \(0 < \rho < 1.\) With every \(\rho\)-lattice \(X(\{x_j\}, \rho)\) we associate a set of distributions \(\Phi = \{\Phi_j\}\) which is described below. Let \(K_j \subset I(x_j, \rho/2)\) be a compact subset and \(\mu_j\) be a non-negative measure on \(K_j.\) We will always assume that the total measure of \(K_j\) is finite, i.e.

\[
0 < |K_j| = \int_{K_j} d\mu_j < \infty.
\]

We consider the following distribution on \(C_c^\infty(I(x_j, \rho)),\)

\[
\Phi_j(\varphi) = \int_{K_j} \varphi \, d\mu_j,
\]

where \(\varphi \in C_c^\infty(I(x_j, \rho)).\) As a compactly supported distribution of order zero it has a unique continuous extension to the space \(C^\infty(I(x_j, \rho)).\)

We say that a family \(\Phi = \{\Phi_j\}\) is uniformly bounded if there exists a positive constant \(C_\Phi\) such that
\(|K_j| \leq C_\Phi, \text{ for all } j.\)\(\tag{6.2}\)

We will also say that a family \(\Phi = \{\Phi_j\}\) is separated from zero if there exists a constant \(c_\Phi > 0\) such that
\(|K_j| \geq c_\Phi, \text{ for all } j.\)\(\tag{6.3}\)

Some examples of such distributions that are of particular interest to us are the following:

(1) Delta functionals. In this case \(K_j = \{x_j\}\), the measure \(d\mu_j = \) any positive number \(\mu_j\) and \(\Phi_j(f) = \mu_j \delta_{x_j}(f) = \mu_j f(x_j)\).

(2) Finite or infinite sequences of delta functions \(\delta_{x_j}, x_j \in (x_j, \rho/2)\), with corresponding weights \(\mu_{j,k}\). In this case \(K_j = \bigcup \{x_j, k\}\)

\[\Phi_j(f) = \sum_k \mu_{j,k} \delta_{x_j}(f),\]

where we assume that \(0 < |K_j| = \sum_k |\mu_{j,k}| < \infty.\)

(3) \(K_j\) is the closure of \(I(x_j, \rho/2)\) and \(d\mu_j\) is the restriction of a weighted Lebesgue measure \(dx\) on \(\mathbb{R}^+.\)

It was shown in \([22,23]\) that the following inequalities hold for any smooth function \(f\) from the Sobolev space \(H^2(\mathbb{R})\)

\[c \rho^{1/2} \left( \sum_j |\Phi_j(f)|^2 \right)^{1/2} \lesssim \|f\| \lesssim C \left\{ \rho^{1/2} \left( \sum_j |K_j|^{-1} |\Phi_j(f)|^2 \right)^{1/2} + \rho^2 \left\| \frac{d^2 f}{dx^2} \right\| \right\},\]

where \(c\) and \(C\) depend only on the constants \(c_\Phi\) and \(C_\Phi\) in (6.2) and (6.3).

For the rest of this section we assume that the potential \(q\) in (5.1) is bounded, i.e., \(|q| \leq Q\). Under this assumption we obtain

\[\left\| - \frac{d^2 f}{dx^2} \right\| \lesssim \|Lf\| + Q \|f\|.\]

Thus, if
\[C(\Phi) \rho^2 Q < 1\]
then for some \(c_1 = c_1(Q, c_\Phi, C_\Phi), C_1 = C_1(Q, c_\Phi, C_\Phi)\) we have

\[c_1 \rho^{1/2} \left( \sum_j |\Phi_j(f)|^2 \right)^{1/2} \lesssim \|f\| \lesssim C_1 \left\{ \rho^{1/2} \left( \sum_j |K_j|^{-1} |\Phi_j(f)|^2 \right)^{1/2} + \rho^2 \|Lf\| \right\},\]

which shows that our assumption (4.1) is satisfied and we can apply Definition 15 and Theorem 16 to obtain the following theorem.

**Theorem 22.** Suppose that \(q\) is bounded. For the given \(c_\Phi, C_\Phi, \) there exists a constant \(C_1 = C_1(Q, c_\Phi, C_\Phi) > 0\) such that for any family of distributions \(\Phi = \{\Phi_j\}\) defined by (6.1) with (6.2) and (6.3), any sufficiently small \(\rho > 0\), and any

\[\omega < \frac{1}{C_1 \rho^2},\]

every function \(f \in PW_{\omega}(L)\) is uniquely determined by the values \(\{\Phi_j(f)\}\) and can be reconstructed by the formula

\[f = \lim_{k \to \infty} s_k(f)\]

and the error estimate is

\[\|f - s_k(f)\| \leq C(\rho^2 \omega)^k \|f\|. \quad k = 0, 1, 2, \ldots.\]

Moreover, there exists a frame \(\{\Theta_j\}\) in the space \(PW_{\omega}(L)\) such that the following reconstruction formula holds

\[f = \sum_j \Phi_j(f) \Theta_j.\]

For similar results to those in the last theorem, but for shift-invariant spaces, see [2].

**References**