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Quantum superalgebra representations on cohomology groups of non-commutative bundles

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Abstract

Quantum homogeneous supervector bundles arising from the quantum general linear supergroup are studied. The space of holomorphic sections is promoted to a left exact covariant functor from a category of modules over a quantum parabolic sub-supergroup to the category of locally finite modules of the quantum general linear supergroup. The right derived functors of this functor provides a form of Dolbeault cohomology for quantum homogeneous supervector bundles. We explicitly compute the cohomology groups, which are given in terms of well understood modules over the quantized universal enveloping algebra of the general linear superalgebra.

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1. Introduction

We follow the general philosophy of non-commutative geometry [3,13] to study quantum homogeneous supervector bundles arising from the quantum general linear supergroup. Our starting point is the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ (see e.g. [23,26]) of the complex general linear superalgebra $\mathfrak{gl}_{m|n}$ [9,20]. As is well-known, $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ has the structure of a Hopf superalgebra [17,16]. Thus its dual superspace $\mathrm{U}_q(\mathfrak{gl}_{m|n})^*$ acquires a natural associative superalgebraic structure. The subspace $\mathscr{A}(\mathfrak{gl}_{m|n})$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n})^*$ spanned by all the representative functions of the finite-dimensional left $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-modules with integral weights forms a Hopf superalgebra,

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which may be considered as the superalgebra of functions on the quantum general linear supergroup. This Hopf superalgebra is closely related to the multi-parameter quantization of the general linear supergroup of [14], and obviously contains the Hopf superalgebra G_q of [24] as a Hopf sub-superalgebra.

For any given reductive quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{l})$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, we consider the subspace $\mathscr{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$ of $\mathscr{A}(\mathfrak{gl}_{m|n})$ invariant with respect to left translations under $\mathrm{U}_q(\mathfrak{l})$. This subspace forms a sub-superalgebra of $\mathscr{A}(\mathfrak{gl}_{m|n})$. In the spirit of non-commutative geometry [3,13], we shall regard $\mathscr{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$ as (the superalgebra of functions on) a quantum homogeneous superspace, and finite type projective $\mathscr{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$-modules as (spaces of global sections of) quantum supervector bundles on the quantum homogeneous superspace. As we shall see, such $\mathscr{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$-modules also admit a natural $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-action. This provides an interesting link between the non-commutative geometry of the quantum supervector bundles and the representation theory of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. In the context of classical Lie groups, such a link is very well known and constitutes the subject of study of the celebrated Bott–Borel–Weil theory [2,12]. For Lie supergroups in the classical setting, a Bott–Borel–Weil theory was developed in [18,19]. One of the main aims of the present paper is to develop a quantum analogue of the theory of [18,19]. We shall do this by adapting the algebraic theory of induced representations [12,6] developed by Zuckerman, Schmidt, Vogan and others to the context of quantum supergroups. Recall that a Bott–Borel–Weil theory for quantum groups has been developed by Andersen et al. [1]. Our study makes essential use of their results.

We promote the space $\mathscr{S}$ of global sections and the space Γ of holomorphic sections of a quantum homogeneous supervector bundle to covariant functors from appropriate categories of modules over quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ to the category of locally finite $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-modules. These functors are closely related to a quantum analogue of the Zuckerman functor, which will be introduced in Section 4. The derived functors of the 'holomorphic section functor' arising from Γ are the Dolbeault cohomology groups for quantum homogeneous supervector bundles which we seek for. When the quantum homogeneous supervector bundle $\mathscr{S}$ is induced by a finite dimensional irreducible module over a purely even reductive quantum subalgebra, or a finite dimensional dual Kac module (defined by (2.3)) over a general reductive quantum sub-superalgebra, we explicitly compute the cohomology groups in Theorems 5.2 and 5.3. The cohomology is concentrated at one degree, and the non-trivial cohomology groups are isomorphic to dual Kac modules over $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, which are not irreducible unless the highest weights are typical.

Recall that Penkov [18] developed a geometric version of the Bott–Borel–Weil theory for Lie supergroups in the early 80s, whose work also revealed for the first time the intricacies of the subject. (It seems that Ref. [18] is still the most comprehensive treatment available in print on the subject.) Later Santos [19] developed an algebraic theory by adapting concepts and techniques from cohomological induction [12] to Lie superalgebras. Our study here may be considered as a quantum version of their theories for the case of the quantum general linear supergroup. In fact, Theorems 5.2 and 5.3 resemble very much results [18,19] on Lie superalgebras in the classical setting.

Let us now briefly comment on the content of each section in the paper. Section 2 contains some background material on the general linear superalgebra and its quantized universal enveloping algebra. Section 3 introduces the notions of quantum homogeneous superspaces and quantum homogeneous supervector bundles on them, and study some basic properties of theirs. A technical yet important point about quantum supergroups that is drastically different from the case of ordinary quantum groups is that the superspace $\mathscr{S}(\Xi)$ of Definition 3.3 is not always a projective (left or right) module over the superalgebra of functions on a quantum homogeneous superspace. However, when Ξ is a finite dimensional irreducible module over a purely even reductive quantum subalgebra of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, or a finite dimensional dual Kac module (defined by (2.3)) over a general reductive quantum sub-superalgebra, we shall prove that $\mathscr{S}(\Xi)$ is projective. Thus in these cases, we might regard $\mathscr{S}(\Xi)$ as defining some non-commutative bundle [3], and it is these cases which will be studied in detail in this paper.

Section 4 studies induction functors. A generalized Zuckerman functor is introduced, and a version of Frobenius reciprocity is proven. The Frobenius reciprocity in turn enables us to show that the Abelian category of locally finite modules over any parabolic quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ has enough injectives. Furthermore, the induction functors from such categories to the category of locally finite $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-modules are left exact. Therefore it makes good sense to consider the right derived functors of these induction functors. Conceptually, these issues are analogous to some of the general questions addressed in cohomological induction [12] (and also in [19] for Lie superalgebras); it is the technical aspects related to quantum supergroups which need to be straightened out carefully. Now some remarks are in order. For ordinary quantum groups [4,8], a theorem of Drinfeld's [5] derived from deformation theoretical arguments says that a quantum group in some appropriate sense is the 'same' as the corresponding universal enveloping algebra of the associated finite dimensional simple Lie algebra. (An excellent account of the deformation theoretical treatment of ordinary quantum groups can be found in [10].) This gives some assurance that results on universal enveloping algebras of simple Lie algebras may carry over to quantum groups at generic q. However, no analogue of Drinfeld's result is known for quantum supergroups except for some very special cases. More surprisingly, the Jimbo version of the quantum supergroup associated with the best behaved Lie superalgebra $\mathfrak{osp}(1\,|\,2n)$ at generic q has totally different representation theory from that of $\mathfrak{osp}(1\,|\,2n)$ (even though the Drinfeld version of this quantum supergroup is relatively easy to handle). These are indications that quantum supergroups at generic q can differ markedly from universal enveloping algebras of Lie superalgebras. Considerable care needs to be exercised in dealing with quantum supergroups, and certainly we should not take any thing for granted.

In Section 5 we first present the formulation of Dolbeault cohomology for quantum homogeneous supervector bundles in terms of right derived functors of the 'holomorphic section functor', making full use of results of Section 4. Then we compute the Dolbeault cohomology groups for bundles of interest. By applying the Grothendieck spectral sequence for compositions of functors, we are able to reduce the task to a calculation which can be handled by using the Bott–Borel–Weil theorem of Andersen –Polo–Wen [1] for ordinary quantum groups.

2. Preliminaries

This section presents some background material on the general linear superalgebra and its quantized universal enveloping algebra. It also serves to fix notations and conventions.

2.1. The general linear superalgebra $\mathfrak{gl}_{m|n}$

We start by describing the standard Borel subalgebra and the associated root system of the general linear superalgebra. The quantized universal enveloping algebra of the general linear superalgebra will be defined with respect to this root system.

Throughout the paper we shall denote by $\mathfrak{g}$ the complex general linear superalgebra $\mathfrak{gl}_{m|n}$ [9,20]. Let $\mathbf{I}=\{1,2,\dots,m+n\}$, and $\mathbf{I}'=\{1,2,\dots,m+n-1\}$. Purely for the sake of convenience in presentation, we identify $\mathfrak{g}$ with the Lie superalgebra of $(m+n)\times(m+n)$-matrices. (This will not be used anywhere else in the paper.) The $\mathbb{Z}_2$-grading of the matrices is specified as follows. Denote by e_{ab}, $a,b\in\mathbf{I}$, the $(m+n)\times(m+n)$-matrix unit with all entries being zero except that at the (a,b) position which is 1. We declare e_{ab} to be odd if $a\leqslant m<b$ or $a>m\geqslant b$, and even otherwise. Then $\{e_{ab}\,|\,a,b\in\mathbf{I}\}$ forms a homogeneous basis of $\mathfrak{g}$. The maximal even subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{g}_0$, which is equal to $\mathfrak{gl}_m\oplus\mathfrak{gl}_n$. Let $\mathfrak{g}_{+1}=\sum_{i\leqslant m<r}\mathbb{C}e_{ir}$, and $\mathfrak{g}_{-1}=\sum_{i\leqslant m<r}\mathbb{C}e_{ri}$. Then the odd subspace of $\mathfrak{g}$ is $\mathfrak{g}_{+1}\oplus\mathfrak{g}_{-1}$.

We fix the Borel subalgebra $\mathfrak{b}$ consisting of the upper triangular matrices, and take $\mathfrak{h}=\bigoplus_a\mathbb{C}e_{aa}$ as the Cartan subalgebra. Let $\{\varepsilon_a\,|\,a\in\mathbf{I}\}$ be the basis of $\mathfrak{h}^*$ such that $\varepsilon_a(E_{bb})=\delta_{ab}$. The space $\mathfrak{h}^*$ is equipped with a bilinear form $(\ ,\ ):\mathfrak{h}^*\times\mathfrak{h}^*\to\mathbb{C}$ such that

$$(\varepsilon_a,\varepsilon_b)=\begin{cases}\delta_{ab}, & a\leqslant m,\\ -\delta_{ab}, & a>m.\end{cases}$$

We shall denote by $\mathfrak{h}^*_{\mathbb{Z}}$ the $\mathbb{Z}$-span of the ε_a. The set of roots of $\mathfrak{g}$ is $\{\varepsilon_a-\varepsilon_b\,|\,a\neq b\}$, with $\varepsilon_a-\varepsilon_b$ being called odd if $a\leqslant m<b$ or $b\leqslant m<a$, and even otherwise. The set of the positive roots relative to the Borel subalgebra $\mathfrak{b}$ is $\{\varepsilon_a-\varepsilon_b\,|\,a<b\}$, and the set of simple roots is $\{\varepsilon_a-\varepsilon_{a+1}\,|\,a\in\mathbf{I}'\}$. An element $\lambda\in\mathfrak{h}^*$ is called dominant if $2(\lambda,\alpha)/(\alpha,\alpha)\in\mathbb{Z}_+$, for all positive even roots of $\mathfrak{g}$. Denote by 2ρ the signed-sum of the positive roots of $\mathfrak{g}$. A $\lambda\in\mathfrak{h}^*$ is called $\mathfrak{g}$-regular if $(\lambda+\rho,\alpha)\neq0$ for all even roots of $\mathfrak{g}$.

The elements of the following set $\{e_{a,a+1},e_{a+1,a}\,|\,a\in\mathbf{I}'\}\cup\{e_{bb}\,|\,b\in\mathbf{I}\}$ generate $\mathfrak{g}$. We shall call a Lie sub-superalgebra $\mathfrak{r}$ of $\mathfrak{g}$ regular if there exist subsets $\Theta_\pm$ of $\mathbf{I}'$ and a subset Θ_0 of $\mathbf{I}$ such that $\mathfrak{r}$ is generated by the elements of the set $\{e_{aa}\,|\,a\in\Theta_0\}\cup\{e_{b,b+1}\,|\,b\in\Theta_+\}\cup\{e_{c+1,c}\,|\,c\in\Theta_-\}$. The Lie sub-superalgebra $\mathfrak{r}$ is called reductive if $\Theta_0=\mathbf{I}$ and $\Theta_+=\Theta_-$, and is called parabolic if $\Theta_0=\mathbf{I}$ and either Θ_+ or Θ_- is equal to $\mathbf{I}'$. If $\mathfrak{r}$ is a parabolic Lie sub-superalgebra, then it contains the reductive Lie sub-superalgebra, called the Levi factor of $\mathfrak{r}$, generated by the elements of the set $\{e_{aa}\,|\,a\in\Theta_0\}\cup\{e_{b,b+1},e_{b+1,b}\,|\,b\in\Theta_+\cap\Theta_-\}$. Note that a parabolic Lie sub-superalgebra of $\mathfrak{g}$ necessarily contains $\mathfrak{b}$ or the opposite Borel subalgebra $\bar{\mathfrak{b}}$ spanned by the lower triangular matrices.

Obviously our definitions of parabolic and reductive Lie sub-superalgebras are rather restrictive, but they suit well the purpose for studying quantum supergroups.

2.2. The quantized universal enveloping algebra of $\mathfrak{gl}_{m|n}$

Like ordinary quantum groups [4,8], the quantized universal enveloping algebras of Lie superalgebras are also defined by generators and relations, where the generators are related to the choice of a particular root system of the associated Lie superalgebra. As is well known, a simple Lie superalgebra in general admits different root systems which are not Weyl group conjugate. Thus quantizations in different root systems give rise to non-isomorphic quantum supergroups (as Hopf superalgebras). We discussed the problem on other occasions. Here we shall consider only the quantum general linear supergroup related to the standard root system of $\mathfrak{g}$.

Let q be an indeterminate, and denote by $\mathbb{C}(q)$ the field of complex rational functions in q. Set

$$q_a = \begin{cases} q, & a \leqslant m, \\ q^{-1}, & a > m. \end{cases}$$

We define the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ [23,26,24] of the general linear superalgebra $\mathfrak{g}$ to be the unital associative superalgebra over $\mathbb{C}(q)$ with the set of generators

$$\{E_{aa+1}, E_{a+1a} \mid a \in \mathbf{I}'\} \cup \{K_b, K_b^{-1} \mid b \in \mathbf{I}\}$$

subject to the following relations:

$$\begin{aligned} K_a K_a^{-1} = 1, \qquad & K_a^{\pm 1} K_b^{\pm 1} = K_b^{\pm 1} K_a^{\pm 1}, \\ K_a E_{bb\pm 1} K_a^{-1} &= q^{(\varepsilon_a, \varepsilon_b - \varepsilon_{b\pm 1})} E_{bb\pm 1}, \\ [E_{aa+1}, E_{b+1b}\} &= \delta_{ab} \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q_a - q_a^{-1}}, \\ (E_{mm+1})^2 &= (E_{m+1m})^2 = 0, \\ E_{aa+1} E_{bb+1} &= E_{bb+1} E_{aa+1}, \\ E_{a+1a} E_{b+1b} &= E_{b+1b} E_{a+1a}, \quad |a-b| \geqslant 2, \\ \mathscr{S}^{(+)}_{aa\pm 1} &= \mathscr{S}^{(-)}_{aa\pm 1} = 0, \quad a \neq m, \\ \{E_{m-1m+2}, E_{mm+1}\} &= \{E_{m+2m-1}, E_{m+1m}\} = 0, \end{aligned} \tag{2.1}$$

where $[E_{aa+1}, E_{b+1b}\} := E_{aa+1} E_{b+1b} - (-1)^{\delta_{am}\delta_{bm}} E_{b+1b} E_{aa+1}$. The E_{m-1m+2} and E_{m+2m-1} are the $a = m-1$, $b = m+1$, cases of the elements defined by (2.2), and

$$\mathscr{S}^{(+)}_{aa\pm 1} = (E_{aa+1})^2 E_{a\pm 1a+1\pm 1} - (q+q^{-1}) E_{aa+1} E_{a\pm 1a+1\pm 1} E_{aa+1} + E_{a\pm 1a+1\pm 1} (E_{aa+1})^2,$$

$$\mathscr{S}^{(-)}_{aa\pm 1} = (E_{a+1a})^2 E_{a+1\pm 1a\pm 1} - (q+q^{-1}) E_{a+1a} E_{a+1\pm 1a\pm 1} E_{a+1a}$$
$$+ E_{a+1\pm 1a\pm 1}(E_{a+1a})^2.$$

The $\mathbb{Z}_2$ grading of the superalgebra is defined by declaring the elements $K_a^{\pm 1}$, $\forall a \in \mathbf{I}$, and E_{bb+1}, E_{b+1b}, $b \neq m$, to be even and E_{mm+1} and E_{m+1m} to be odd. Throughout the paper, we use $[f]$ to denote the parity of the element f of any $\mathbb{Z}_2$-graded space.

It is well-known that $\mathrm{U}_q(\mathfrak{g})$ has the structure of a Hopf superalgebra [16,17], with a co-multiplication

$$\Delta(E_{aa+1}) = E_{aa+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{aa+1},$$
$$\Delta(E_{a+1a}) = E_{a+1a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1a},$$
$$\Delta(K_a^{\pm 1}) = K_a^{\pm 1} \otimes K_a^{\pm 1},$$

co-unit

$$\varepsilon(E_{aa+1}) = E_{a+1a} = 0, \quad \forall a \in \mathbf{I}',$$
$$\varepsilon(K_b^{\pm 1}) = 1, \quad \forall b \in \mathbf{I},$$

and antipode

$$S(E_{aa+1}) = -E_{aa+1} K_a^{-1} K_{a+1},$$
$$S(E_{a+1a}) = -K_a K_{a+1}^{-1} E_{a+1a},$$
$$S(K_a^{\pm 1}) = K_a^{\mp 1} \otimes K_a^{\mp 1}.$$

Let E_{ab}, E_{ba}, $a < b$, be elements of $\mathrm{U}_q(\mathfrak{g})$ defined by

$$E_{ab} = E_{ac} E_{cb} - q_c^{-1} E_{cb} E_{ac}, \quad a < c < b,$$
$$E_{ba} = E_{bc} E_{ca} - q_c E_{ca} E_{bc}, \quad a < c < b, \tag{2.2}$$

which can be easily shown to be independent of the c chosen [23]. These elements are the generalization to $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ of a similar set of elements for $U_q(\mathfrak{gl}_n)$ constructed by Jimbo [8], and have proven to be very useful in the construction of the universal R-matrix for $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ [11]. The E_{ab} behave very much like the images of e_{ab} in the universal enveloping algebra of $\mathfrak{gl}_{m|n}$. For example, $E_{ab}^2 = E_{ba}^2 = 0$, if $a \leqslant m < b$. More importantly, we have the following Poincaré–Birkhoff–Witt theorem for $\mathrm{U}_q(\mathfrak{g})$.

Theorem 2.1 (Zhang [23], Zou [26]). *The ordered products of non-negative powers of all the E_{ab}, $a \neq b$, and integer powers of all K_c, $c \in \mathbf{I}$, with respect to any linear ordering of the elements of $\{E_{ab}, E_{ba} \mid a < b\} \cup \{K_c \mid c \in \mathbf{I}\}$ form a basis of* $\mathrm{U}_q(\mathfrak{g})$.

Note that if $a \leqslant m < b$ or $b \leqslant m < a$, no square or higher powers of E_{ab} can appear in any of the ordered products.

We shall assume that every $\mathrm{U}_q(\mathfrak{g})$-module to be considered in this paper is $\mathbb{Z}_2$-graded. Let V be a $\mathrm{U}_q(\mathfrak{g})$-module. A weight vector $v \in V$ is the simultaneous eigenvector of all the K_a, $a \in \mathbf{I}$. If V is spanned by its weight vectors, then it is called a weight module. We shall say that v is an integral weight vector with weight $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$ if

$$K_a v = q^{(\mu, \varepsilon_a)} v, \quad \forall a.$$

A weight module with only integral weight vectors will be called an integral weight module.

We shall denote by L_λ, $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the irreducible $\mathrm{U}_q(\mathfrak{g})$-module with a highest weight vector which is an integral weight vector with weight λ. The λ will be referred to as the highest weight of L_λ. It was shown in [23] that L_λ is finite dimensional if and only if its highest weight is dominant.

We shall call a Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ a quantum sub-superalgebra. Corresponding to a regular Lie sub-superalgebra $\mathfrak{r}$ of $\mathfrak{g}$ specified by the sets Θ_0 and $\Theta_\pm$, there exists an associated quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{r})$ generated by the elements of the following set $\{K_a, K_a^{-1} \mid a \in \Theta_0\} \cup \{E_{b,b+1} \mid b \in \Theta_+\} \cup \{E_{c+1,c} \mid c \in \Theta_-\}$. Important quantum sub-superalgebras are $\mathrm{U}_q(\mathfrak{h})$ and the two quantum Borel sub-superalgebras $\mathrm{U}_q(\mathfrak{b})$ and $\mathrm{U}_q(\bar{\mathfrak{b}})$. If $\mathfrak{r}$ is a parabolic (respectively reductive) Lie sub-superalgebra of $\mathfrak{g}$, then $\mathrm{U}_q(\mathfrak{r})$ will be called a parabolic (respectively reductive) quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. If $\mathrm{U}_q(\mathfrak{p})$ is parabolic with the Levi factor $\mathrm{U}_q(\mathfrak{l})$, then we have the Hopf superalgebra inclusions $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}) \subset \mathrm{U}_q(\mathfrak{g})$.

For later use, we consider here a particular module over a reductive quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{l})$. Let $\mathrm{U}_q(\mathfrak{l}_0) = \mathrm{U}_q(\mathfrak{l}) \cap \mathrm{U}_q(\mathfrak{g}_0)$, and denote by $\mathrm{U}_q(\mathfrak{l}_{\leqslant 0})$ the Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{l})$ generated by all the generators of $\mathrm{U}_q(\mathfrak{l})$ but $E_{m,m+1}$. Note that if $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{g}_0)$, then $\mathrm{U}_q(\mathfrak{l}_{\leqslant 0}) = \mathrm{U}_q(\mathfrak{l})$. Let $L_\mu^{(\mathfrak{l}_{\leqslant 0})}$ be the irreducible $\mathrm{U}_q(\mathfrak{l}_{\leqslant 0})$-module with integral $\mathrm{U}_q(\mathfrak{l}_0)$ highest weight μ. Note that the generator $E_{m+1,n}$ of $\mathrm{U}_q(\mathfrak{l}_{\leqslant 0})$ necessarily acts on $L_\mu^{(\mathfrak{l}_{\leqslant 0})}$ by zero. Furthermore, $L_\mu^{(\mathfrak{l}_{\leqslant 0})}$ restricts to an irreducible $\mathrm{U}_q(\mathfrak{l}_0)$-module.

Definition 2.1.

$$K_\mu^{(\mathfrak{l})} := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{l}), L_\mu^{(\mathfrak{l}_{\leqslant 0})}). \tag{2.3}$$

This will be referred to as a dual Kac module over $\mathrm{U}_q(\mathfrak{l})$. The action of any $y \in \mathrm{U}_q(\mathfrak{l})$ on $\zeta \in K_\mu^{(\mathfrak{l})}$ is given by $\langle y\zeta, x\rangle = (-1)^{[y]([x]+[\zeta])}\langle \zeta, xy\rangle$, $\forall x \in \mathrm{U}_q(\mathfrak{l})$.

Let $\mathrm{Ker}\,\varepsilon_{\leqslant 0}$ be the subspace of $\mathrm{U}_q(\mathfrak{l}_{\leqslant 0})$ annihilated by the co-unit ε. It generates a two-sided ideal $J(\mathfrak{l})$ of $\mathrm{U}_q(\mathfrak{l})$. By using the PBW Theorem 2.1 (generalized in the obvious way to $\mathrm{U}_q(\mathfrak{l})$) we can easily show that

$$K_\mu^{(\mathfrak{l})} = (\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l}))^* \otimes_{\mathbb{C}(q)} L_\mu^{(\mathfrak{l}_{\leqslant 0})}. \tag{2.4}$$

It again follows from the PBW theorem that $\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l})$ is finite dimensional. The $\mathrm{U}_q(\mathfrak{h})$-module structure of its dual vector space can be described explicitly. Let $\Phi_1^+(\mathfrak{l})$ be the set of the odd positive roots of $\mathfrak{l}$. Let E be the $\mathrm{U}_q(\mathfrak{h})$-module with a basis $\{v_\gamma \mid \gamma \in \Phi_1^+(\mathfrak{l})\}$ such that $K_a v_\gamma = q^{-(\gamma,\varepsilon_a)} v_\gamma$, for all a. The exterior algebra of E forms a $\mathrm{U}_q(\mathfrak{h})$-module, which we denote by $\Lambda(\mathfrak{l}_{-1})$. Then $(\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l}))^*$ is isomorphic to

$\Lambda(\mathfrak{l}_{-1})$. Therefore, as a $\mathrm{U}_q(\mathfrak{h})$-module (with diagonal action),

$$K_\mu^{(\mathfrak{l})} \cong \Lambda(\mathfrak{l}_{-1}) \otimes_{\mathbb{C}(q)} L_\mu^{(\mathfrak{l}_{\leqslant 0})}. \tag{2.5}$$

Note that when $\mathrm{U}_q(\mathfrak{l})$ is purely even, $K_\mu^{(\mathfrak{l})} = L_\mu^{(\mathfrak{l}_{\leqslant 0})}$.

3. Quantum homogeneous super vector bundles

In this section we introduce quantum homogeneous superspaces and quantum homogeneous supervector bundles in the context of the quantum general linear supergroup. Let us begin by considering the Hopf superalgebra of functions on the quantum general linear supergroup.

3.1. Functions on the quantum general linear supergroup

General references on Hopf (super)algebra are [16,17]. A treatment of the classical general linear supergroup similar to what to be presented here is given in [21]. Let $\mathrm{U}_q(\mathfrak{g})^*$ denote the $\mathbb{Z}_2$-graded dual vector space of $\mathrm{U}_q(\mathfrak{g})$. It has a natural associative superalgebraic structure induced by the co-superalgebraic structure of $\mathrm{U}_q(\mathfrak{g})$. Denote the multiplication of $\mathrm{U}_q(\mathfrak{g})^*$ by $m_\circ$, then $\langle m_\circ(f\otimes g), x\rangle = \langle f\otimes g, \Delta(x)\rangle$, for all $x\in \mathrm{U}_q(\mathfrak{g})$. (We shall use the notations $\phi(v)$ and $\langle \phi, x\rangle$ interchangeably for the image of $v\in V$ under $\phi \in \mathrm{Hom}_{\mathbb{C}(q)}(V, W)$.)

There exist two gradation preserving $\mathrm{U}_q(\mathfrak{g})$-actions on $\mathrm{U}_q(\mathfrak{g})^*$,

$$\mathrm{d}\tilde{L} : \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})^* \to \mathrm{U}_q(\mathfrak{g})^*, \quad x\otimes f \mapsto \mathrm{d}\tilde{L}_x(f),$$
$$\mathrm{d}\tilde{R} : \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})^* \to \mathrm{U}_q(\mathfrak{g})^*, \quad x\otimes f \mapsto \mathrm{d}\tilde{R}_x(f),$$

defined by

$$\langle \mathrm{d}\tilde{L}_x(f), y\rangle = (-1)^{[x][f]}\langle f, S(x)y\rangle,$$
$$\langle \mathrm{d}\tilde{R}_x(f), y\rangle = (-1)^{[x]}\langle f, yx\rangle, \quad \forall y\in \mathrm{U}_q(\mathfrak{g}).$$

It is easy to see that $\mathrm{d}\tilde{L}_{xy} = \mathrm{d}\tilde{L}_x \mathrm{d}\tilde{L}_y$, and $\mathrm{d}\tilde{R}_{xy} = \mathrm{d}\tilde{R}_x \mathrm{d}\tilde{R}_y$, for all $x, y\in \mathrm{U}_q(\mathfrak{g})$. Straightforward calculations show that each of these actions converts $\mathrm{U}_q(\mathfrak{g})^*$ into a (graded) left $\mathrm{U}_q(\mathfrak{g})$-module. Furthermore, with respect to the module structure the product map of $\mathrm{U}_q(\mathfrak{g})^*$ is a $\mathrm{U}_q(\mathfrak{g})$-module homomorphism and the unit element of $\mathrm{U}_q(\mathfrak{g})^*$ is $\mathrm{U}_q(\mathfrak{g})$-invariant. Therefore, each of these actions converts $\mathrm{U}_q(\mathfrak{g})^*$ into a left $\mathrm{U}_q(\mathfrak{g})$-module superalgebra. The fact that the product map of $\mathrm{U}_q(\mathfrak{g})^*$ is a module homomorphism means that the operators $\mathrm{d}\tilde{R}_x$ and $\mathrm{d}\tilde{L}_x$ behave as some sort of generalized super derivations. Indeed, for all $f, g\in \mathrm{U}_q(\mathfrak{g})^*$, we have

$$\mathrm{d}\tilde{R}_x(fg) = \sum_{(x)} (-1)^{[x_{(2)}][f]}\mathrm{d}\tilde{R}_{x_{(1)}}(f)\mathrm{d}\tilde{R}_{x_{(2)}}(g), \tag{3.1}$$

where we have used the standard Sweedler notation for the co-multiplication of x. However, for $\mathrm{d}\tilde{L}$, we have

$$\mathrm{d}\tilde{L}_x(fg) = \sum_{(x)} (-1)^{[x'_{(2)}][f]} \mathrm{d}\tilde{L}_{x'_{(1)}}(f)\mathrm{d}\tilde{L}_{x'_{(2)}}(g), \tag{3.2}$$

with respect to the opposite co-multiplication $\Delta'(x) = \sum_{(x)} x'_{(1)} \otimes x'_{(2)}$ of x. The two actions also super-commute in the sense that $\mathrm{d}\tilde{L}_x \mathrm{d}\tilde{R}_y = (-1)^{[x][y]} \mathrm{d}\tilde{R}_y \, \mathrm{d}\tilde{L}_x$, for all $x, y \in \mathrm{U}_q(\mathfrak{g})$.

Let $\mathrm{U}_q(\mathfrak{g})^\circ := \{f \in \mathrm{U}_q(\mathfrak{g})^* \mid \text{kernel of } f \text{ contains a co-finite ideal of } \mathrm{U}_q(\mathfrak{g})\}$ be the finite dual [17] of the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ of $\mathfrak{g}$. Here we remark again that we only consider $\mathbb{Z}_2$-graded subalgebras and (left, right or two-sided) ideals of $\mathrm{U}_q(\mathfrak{g})$ in this paper. We have the following lemma, which is an adaption of a standard result (see, e.g., Lemma 9.1.1 in [17]) on ordinary associative algebras to $\mathrm{U}_q(\mathfrak{g})$. In fact the result is valid for any associative superalgebra.

Lemma 3.1. *For any $f \in \mathrm{U}_q(\mathfrak{g})^*$, the following conditions are equivalent*:

(1) *f vanishes on a left ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension*;
(2) *f vanishes on a right ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension*;
(3) *f vanishes on an ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension, thus belongs to $\mathrm{U}_q(\mathfrak{g})^\circ$*;
(4) *$\mathrm{d}\tilde{L}_{\mathrm{U}_q(\mathfrak{g})}(f)$ is finite dimensional*;
(5) *$\mathrm{d}\tilde{R}_{\mathrm{U}_q(\mathfrak{g})}(f)$ is finite dimensional*;
(6) *$(\mathrm{d}\tilde{L}_{\mathrm{U}_q(\mathfrak{g})} \otimes \mathrm{d}\tilde{R}_{\mathrm{U}_q(\mathfrak{g})})(f)$ is finite dimensional*;
(7) *$m^*(f) \in \mathrm{U}_q(\mathfrak{g})^* \otimes \mathrm{U}_q(\mathfrak{g})^*$, where $m^* : \mathrm{U}_q(\mathfrak{g})^* \to (\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g}))^*$ is defined by $\langle m^*(f), x \otimes y\rangle = \langle f, xy\rangle$ for all $x, y \in \mathrm{U}_q(\mathfrak{g})$.*

Proof. The proof of Lemma 9.1.1 in [17] can be extended verbatim to superalgebras. □

Therefore, f belongs to $\mathrm{U}_q(\mathfrak{g})^\circ$ if and only if one of the equivalent conditions are satisfied. The lemma in particular enables us to impose a Hopf superalgebra structure on $\mathrm{U}_q(\mathfrak{g})^\circ$, with multiplication $m_\circ$, co-multiplication $\Delta_\circ = m^*|_{\mathrm{U}_q(\mathfrak{g})^\circ}$, unit being ε, and co-unit being the unit $1\!\!1_{\mathrm{U}_q(\mathfrak{g})}$ of $\mathrm{U}_q(\mathfrak{g})$. The antipode $S_\circ$ of $\mathrm{U}_q(\mathfrak{g})^\circ$ is defined by

$$\langle S_\circ(f), x\rangle = \langle f, S(x)\rangle, \quad \forall f \in \mathrm{U}_q(\mathfrak{g})^\circ, \ x \in \mathrm{U}_q(\mathfrak{g}).$$

Recall that the antipode $S_\circ$ is invertible since S is. For convenience, we shall drop the subscript $\circ$ from the notations for all the structure maps but the co-unit of $\mathrm{U}_q(\mathfrak{g})^\circ$.

Every finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module naturally forms a finite dimensional right $\mathrm{U}_q(\mathfrak{g})^\circ$-co-module and vice versa. If V is a finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module, then the associated right $\mathrm{U}_q(\mathfrak{g})^\circ$-co-module structure

$$\delta : V \to V \otimes \mathrm{U}_q(\mathfrak{g})^\circ \tag{3.3}$$

is defined for any $v \in V$ by

$$\delta v(x) = (-1)^{[x][v]} xv, \quad \forall x \in \mathrm{U}_q(\mathfrak{g}).$$

Define a linear map $\phi : V^* \otimes V \to \mathrm{U}_q(\mathfrak{g})^\circ$ by the composition

$$V^* \otimes V \xrightarrow{\mathrm{id}_{V^*} \otimes \delta} V^* \otimes V \otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\langle , \rangle \otimes \mathrm{id}_{\mathrm{U}_q(\mathfrak{g})^\circ}} \mathrm{U}_q(\mathfrak{g})^\circ.$$

We shall refer to the elements of the image of this map as the representative functions of the left $\mathrm{U}_q(\mathfrak{g})$-module V. Clearly, the evaluation of any representative function of V on any element of the annihilator of V (which is a graded two-sided ideal of $\mathrm{U}_q(\mathfrak{g})$) vanishes identically. The annihilator of a finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module has finite co-dimension in $\mathrm{U}_q(\mathfrak{g})$. This re-confirms that the representative functions of any finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module indeed belong to $\mathrm{U}_q(\mathfrak{g})^\circ$. Conversely, $\mathrm{U}_q(\mathfrak{g})^\circ$ is spanned by the representative functions of all the finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-modules. To see this, we consider an arbitrary non-zero element $f \in \mathrm{U}_q(\mathfrak{g})^\circ$. Let K be a graded co-finite ideal of $\mathrm{U}_q(\mathfrak{g})$ contained in the kernel of f. Then $U(\mathfrak{g})/K$ forms a left $\mathrm{U}_q(\mathfrak{g})$-module under left multiplication,

$$\mathrm{U}_q(\mathfrak{g}) \otimes U(\mathfrak{g})/K \to U(\mathfrak{g})/K,$$

$$y \otimes (x + K) \mapsto yx + K.$$

Let $\tilde{f}$ be the element in the dual space of $\mathrm{U}_q(\mathfrak{g})/K$ defined by $\tilde{f}(x + K) = \langle f, x\rangle$ for all $x \in \mathrm{U}_q(\mathfrak{g})$. Note that $\mathbb{1}_{\mathrm{U}_q(\mathfrak{g})} + K$ is not contained in the kernel of f as a set since $f \neq 0$. Let δ denote the right $\mathrm{U}_q(\mathfrak{g})^\circ$-co-module structure map of $\mathrm{U}_q(\mathfrak{g})/K$. Then $f = \langle \tilde{f}, \delta(\mathbb{1}_{\mathrm{U}_q(\mathfrak{g})} + K)\rangle$.

Definition 3.1. Let $\mathscr{A}(\mathfrak{g})$ be the $\mathbb{Z}_2$-graded subspace of $\mathrm{U}_q(\mathfrak{g})^\circ$ spanned by the representative functions of all the finite dimensional left integral weight modules over $\mathrm{U}_q(\mathfrak{g})$.

Lemma 3.2. *$\mathscr{A}(\mathfrak{g})$ is a Hopf sub-superalgebra of* $\mathrm{U}_q(\mathfrak{g})^\circ$.

Proof. The space spanned by the representative functions of any finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module is a sub-co-algebra of $\mathrm{U}_q(\mathfrak{g})^\circ$. Thus $\mathscr{A}(\mathfrak{g})$ forms a sub-co-algebra of $\mathrm{U}_q(\mathfrak{g})^\circ$. Since tensor products and duals of finite dimensional integral weight modules over $\mathrm{U}_q(\mathfrak{g})$ are again finite dimensional integral weight modules, $\mathscr{A}(\mathfrak{g})$ is closed under multiplication, and stable under the antipode of $\mathrm{U}_q(\mathfrak{g})^\circ$. □

Remark 3.1. From the discussion on representative functions one can easily see that $f \in \mathrm{U}_q(\mathfrak{g})^\circ$ belongs to $\mathscr{A}(\mathfrak{g})$ if it satisfies either $\mathrm{d}\tilde{L}_{K_a}(f) = q^{(\mu, \varepsilon_a)} f$, $\forall a \in \mathbf{I}$, or $\mathrm{d}\tilde{R}_{K_a}(f) = q^{(\mu, \varepsilon_a)} f$, $\forall a \in \mathbf{I}$, for some $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$.

Remark 3.2. The representative functions of a finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module are also often referred to as the matrix elements of the $\mathrm{U}_q(\mathfrak{g})$-representation furnished by this module.

3.2. Quantum homogeneous super vector bundles

Recall that in classical geometry, a compact manifold can be recovered from its algebra of continuous functions by the Gelfand–Naimark theorem. Also, the Serre–

Swan theorem establishes a one to one correspondence between the spaces of the continuous sections of vector bundles over a compact manifold and the finite type projective modules of the algebra of continuous functions on the manifold. These results are taken as the starting point for non-commutative geometry [3], where 'manifolds' are replaced by non-commutative algebras, and 'vector bundles' by finitely generated projective modules. The quantum homogeneous superspaces and quantum homogeneous supervector bundles to be studied here are defined in this spirit.

As is well-known, all holomorphic functions on a compact complex manifold are constants. Therefore, the algebra of holomorphic functions contains little information about the manifold itself. This problem persists in classical supergeometry [15] and also quantum geometry [7]. However, as shown in [7] in the context of ordinary quantum groups, we can get around the problem by working with the quantum analogues of smooth functions in a real setting. To do this, we need some basic notions about $*$-Hopf superalgebras [24,25].

A $*$-superalgebraic structure on an associative superalgebra A over $\mathbb{C}(q)$ is a conjugate linear anti-involution $\theta : A \to A$: for all $x, y \in A$, $c, c' \in \mathbb{C}(q)$,

$$\theta(cx + c'y) = \bar{c}\theta(x) + \bar{c}'\theta(y), \quad \theta(xy) = \theta(y)\theta(x), \quad \theta^2(x) = x. \tag{3.4}$$

Here $\bar{c}$ and $\bar{c}'$ are defined in the following way. Let P be a complex polynomial in q. Then $\bar{P}$ is the polynomial obtained by replacing all the coefficients of P by their complex conjugates. Now if there is another polynomial Q in q such that $c = P/Q$, then $\bar{c} = \bar{P}/\bar{Q}$.

Remark 3.3. Note that the second equation in (3.4) does not involve any sign factors as one would normally expect of superalgebras.

We shall sometimes use the notation (A, θ) for the $*$-superalgebra A with the $*$-structure θ. Let (B, θ_1) be another associative $*$-superalgebra. Now $A \otimes B$ has a natural superalgebra structure, with the multiplication defined for any $a, a' \in A$ and $b, b' \in B$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{[b][a']} aa' \otimes bb'.$$

Furthermore, the following conjugate linear map

$$\theta \star \theta_1 : a \otimes b \mapsto (1 \otimes \theta_1(b))(\theta(a) \otimes 1) = (-1)^{[a][b]} \theta(a) \otimes \theta_1(b) \tag{3.5}$$

defines a $*$-superalgebraic structure on $A \otimes B$.

Let us assume that A is a Hopf superalgebra with co-multiplication Δ, co-unit ε and antipode S. If the $*$-superalgebraic structure θ satisfies

$$(\theta \star \theta)\Delta = \Delta\theta, \quad \theta\varepsilon = \varepsilon\theta,$$

then A is called a Hopf $*$-superalgebra. Now

$$\sigma := S\theta$$

satisfies $\sigma^2 = id_A$, as follows from the definition of the antipode.

Let A^0 denote the finite dual of A, which has a natural Hopf superalgebraic structure. If A is a Hopf $*$-superalgebra with the $*$-structure θ, then $\sigma = S\theta$ induces a map

$\omega : A^0 \to A^0$ defined for any $f \in A^0$ by

$$\langle \omega(f), x\rangle = \overline{\langle f, \sigma(x)\rangle}, \quad \forall x \in A. \tag{3.6}$$

As can be easily shown [25], this map ω gives rise to a Hopf $*$-superalgebraic structure on A^0.

In the case of $\mathrm{U}_q(\mathfrak{g})$, the following conjugate anti-involution defines a Hopf $*$-superalgebra structure:

$$\begin{aligned}\theta \quad : \quad & E_{a,a+1} \mapsto E_{a+1,a}K_aK_{a+1}^{-1},\\ & E_{a+1,a} \mapsto K_a^{-1}K_{a+1}E_{a,a+1}, \quad \forall a \in \mathbf{I}',\\ & K_b \mapsto K_b, \quad \forall b \in \mathbf{I}.\end{aligned}$$

The classical counterpart of this map determines a compact real form of the complex general linear superalgebra. Let $\sigma = S\theta$, and define

$$\mathrm{U}_q^{\mathbb{R}}(\mathfrak{g}) := \{x \in \mathrm{U}_q(\mathfrak{g}) \mid \sigma(x) = x\}. \tag{3.7}$$

Clearly $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$ forms an associative superalgebra over $\mathbb{R}(q)$, even though it may not have a Hopf superalgebra structure. We shall refer to it as a real form of $\mathrm{U}_q(\mathfrak{g})$, as $\mathrm{U}_q(\mathfrak{g}) = \mathbb{C}(q) \otimes_{\mathbb{R}(q)} \mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$.

Now $\mathrm{U}_q(\mathfrak{g})^\circ$ acquires a Hopf $*$-superalgebra structure ω which is induced by σ. It is easy to show that the image under ω of any representative function of a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module with integral weights must again be a representative function of a $\mathrm{U}_q(\mathfrak{g})$-module with the same properties. Thus

Lemma 3.3. *$\mathscr{A}(\mathfrak{g})$ forms a Hopf $*$-superalgebra.*

Therefore $\mathscr{A}(\mathfrak{g})$ should be considered as some 'complexification' of the superalgebra of functions on some 'compact real form' of the quantum general linear supergroup.

Let us denote by $\mathrm{d}L_x$ and $\mathrm{d}R_x$, respectively, the restrictions of $\mathrm{d}\tilde{L}_x$ and $\mathrm{d}\tilde{R}_x$ to $\mathscr{A}(\mathfrak{g})$. The following definition will be important for the remainder of the paper. Let $\mathrm{U}_q(\mathfrak{l})$ be a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Set $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l}) := \mathrm{U}_q(\mathfrak{l}) \cap \mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$. We now consider the sub-superalgebra of $\mathscr{A}(\mathfrak{g})$ invariant under the left translation of $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})$.

Definition 3.2. Define

$$\mathscr{A}(\mathfrak{g}, \mathfrak{l}) := \{f \in \mathscr{A}(\mathfrak{g}) \mid \mathrm{d}L_x(f) = \varepsilon(x)f, \ \forall x \in \mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})\}, \tag{3.8}$$

where ε is the co-unit of $\mathrm{U}_q(\mathfrak{g})$.

We shall show presently that $\mathscr{A}(\mathfrak{g}, \mathfrak{l})$ forms a superalgebra. In the philosophy of non-commutative geometry [3], $\mathscr{A}(\mathfrak{g}, \mathfrak{l})$ defines some virtue quantum homogeneous superspace.

Remark 3.4. It is instructive to compare the situation with classical supergeometry [15]. Let Λ be a finite dimensional Grassmann algebra. Take a parabolic subgroup P of $GL(m|n; \Lambda)$ with Lie superalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is some nilpotent ideal of $\mathfrak{p}$.

Let $U(m|n)$ be a compact (in the body) real form of $GL(m|n;\Lambda)$, and set $K = P \cap U(m|n)$. Then we have the symmetric superspace $U(m|n)/K$. The tensor product of Λ with the classical analogue of $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ should capture the essential information of the complexification of the superalgebra of functions on $U(m|n)/K$.

Since $\mathbb{C}(q) \otimes_{\mathbb{R}(q)} \mathrm{U}_q^{\mathbb{R}}(\mathfrak{l}) = \mathrm{U}_q(\mathfrak{l})$, we can show that an element f of $\mathscr{A}(\mathfrak{g})$ belongs to $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ if and only if

$$\mathrm{d}L_x(f) = \varepsilon(x)f, \quad \forall x \in \mathrm{U}_q(\mathfrak{l}).$$

Also, by Remark 3.1, an element g of $\mathrm{U}_q(\mathfrak{g})^\circ$ belongs to $\mathscr{A}(\mathfrak{g})$ if $\mathrm{d}L_k(f) = \varepsilon(k)f$, $\forall k \in \mathrm{U}_q(\mathfrak{h})$. Combining these observations, we arrive at the following result.

Lemma 3.4.

$$\mathscr{A}(\mathfrak{g},\mathfrak{l}) = \{f \in \mathrm{U}_q(\mathfrak{g})^\circ \mid \mathrm{d}L_x(f) = \varepsilon(x)f, \ \forall x \in \mathrm{U}_q(\mathfrak{l})\}.$$

Using the left $\mathrm{U}_q(\mathfrak{g})$-module algebra structure of $\mathscr{A}(\mathfrak{g})$, we immediately show that

Lemma 3.5. *The $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ is a sub-superalgebra of $\mathscr{A}(\mathfrak{g})$.*

Proof. $\mathrm{U}_q(\mathfrak{l})$, being a Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, satisfies $\Delta(\mathrm{U}_q(\mathfrak{l})) \subset \mathrm{U}_q(\mathfrak{l}) \otimes \mathrm{U}_q(\mathfrak{l})$. If $f,\ g \in \mathscr{A}(\mathfrak{g},\mathfrak{l})$, then by (3.2) we have $\mathrm{d}L_x(fg) = \varepsilon(x)fg$, $\forall x \in \mathrm{U}_q(\mathfrak{l})$. Therefore, $fg \in \mathscr{A}(\mathfrak{g},\mathfrak{l})$. □

Since $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ is non-commutative, there is a distinction between left and right $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-modules. However, the two sides of the story are 'mirror images' of each other, thus we shall consider $\mathbb{Z}_2$-graded left $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-modules only. A finitely generated projective module over the superalgebra $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ will be regarded as the space of sections of a quantum supervector bundle over the quantum homogeneous superspace.

Definition 3.3. Let Ξ be a finite dimensional $\mathrm{U}_q(\mathfrak{l})$-module, which naturally restricts to a $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})$-module. Define

$$\mathscr{S}(\Xi) := \{\zeta \in \Xi \otimes \mathscr{A}(\mathfrak{g}) \mid (\mathrm{id} \otimes \mathrm{d}L_x)\zeta = (S(x) \otimes \mathrm{id})\zeta, \ \forall x \in \mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})\}.$$

Again it can be easily shown that

$$\mathscr{S}(\Xi) = \{\zeta \in \Xi \otimes \mathscr{A}(\mathfrak{g}) \mid (\mathrm{id} \otimes \mathrm{d}L_x)\zeta = (S(x) \otimes \mathrm{id})\zeta, \ \forall x \in \mathrm{U}_q(\mathfrak{l})\}. \tag{3.9}$$

This fact will be used to prove the following result.

Proposition 3.1.

(1) *$\mathscr{S}(\Xi)$ forms a left $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-module under the action*

$$\mathscr{A}(\mathfrak{g},\mathfrak{l}) \otimes \mathscr{S}(\Xi) \to \mathscr{S}(\Xi), \quad f \otimes \zeta \mapsto f\zeta, \tag{3.10}$$

defined by $f\zeta := \sum (-1)^{[f][w_i]} w_i \otimes f a_i$ for $\zeta = \sum w_i \otimes a_i$.

(2) *Every* $\mathrm{U}_q(\mathfrak{l})$*-module map* $\phi : \Xi \to \Xi'$ *induces an* $\mathscr{A}(\mathfrak{g},\mathfrak{l})$*-module homomorphism*

$$\mathscr{S}(\phi) = \phi \otimes \mathrm{id} : \mathscr{S}(\Xi) \to \mathscr{S}(\Xi'). \tag{3.11}$$

Proof. For $f \in \mathscr{A}(\mathfrak{g},\mathfrak{l})$ and $\zeta = \sum w_i \otimes a_i \in \mathscr{S}(\Xi)$, we have

$$\begin{aligned}(\mathrm{id}\otimes \mathrm{d}L_x) f\zeta &= \sum (-1)^{([f]+[x])[w_i]} w_i \otimes \mathrm{d}L_x(f a_i)\\ &= \sum (-1)^{([f]+[x])[w_i]+[x][f]} w_i \otimes f\,\mathrm{d}L_x(a_i)\\ &= (S(x)\otimes \mathrm{id}) f\zeta, \quad \forall x \in \mathrm{U}_q(\mathfrak{l}),\end{aligned}$$

where the second step uses (3.2) and the defining property of $\mathscr{A}(\mathfrak{g},\mathfrak{l})$, while the third uses the definition of $\mathscr{S}(\Xi)$. This proves the first claim.

The second claim is quite obvious. □

A quantum homogeneous supervector bundle is called trivial if it is isomorphic to a free left $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-module. Quantum homogeneous supervector bundles induced by $\mathrm{U}_q(\mathfrak{g})$-modules are all trivial.

Proposition 3.2. $\mathscr{S}(\Xi)$ *is freely generated over* $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ *if* Ξ *is the restriction of a finite dimensional* $\mathrm{U}_q(\mathfrak{g})$*-module.*

Proof. The proof is adapted from [7]. Recall that a finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module Ξ has a natural right $\mathrm{U}_q(\mathfrak{g})^\circ$-co-module structure

$$\delta : \Xi \to \Xi \otimes \mathrm{U}_q(\mathfrak{g})^\circ$$

as defined by Eq. (3.3). Let $p : \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ \to \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ$ be defined by $f\otimes g \mapsto (-1)^{[f][g]} g \otimes f$. Define a map $\kappa : \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \to \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ$ by the composition of the following maps:

$$\begin{aligned}&\Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\delta\otimes \mathrm{id}} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\mathrm{id}\otimes p(S^{-1}\otimes \mathrm{id})} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ\\ &\qquad\xrightarrow{\mathrm{id}\otimes m_o} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ,\end{aligned}$$

where m_o is the multiplication of $\mathrm{U}_q(\mathfrak{g})^\circ$. Explicitly,

$$\zeta = \sum v^{(i)} \otimes f^{(i)} \in \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ,$$

$$\kappa(\zeta) = \sum (-1)^{[f^{(i)}][v^{(i)}_{(2)}]} v^{(i)}_{(1)} \otimes f^{(i)} S^{-1}(v^{(i)}_{(2)}), \tag{3.12}$$

where we have used Sweedler's notation for $\delta(v^{(i)})$. The inverse of κ is given by the composition of the following maps

$$\Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\delta\otimes \mathrm{id}} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\mathrm{id}\otimes p} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ \otimes \mathrm{U}_q(\mathfrak{g})^\circ \xrightarrow{\mathrm{id}\otimes m_o} \Xi\otimes \mathrm{U}_q(\mathfrak{g})^\circ.$$

The restriction of κ to $\mathscr{S}(\Xi)$ is a left $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-module map as can be easily seen from (3.12). By using (3.12), we can also show by a direct calculation that for any $\zeta \in \mathscr{S}(\Xi)$,

$$(\mathrm{id}\otimes \mathrm{d}L_u)\kappa(\zeta) = \varepsilon(u)\zeta, \quad \forall u \in \mathrm{U}_q(\mathfrak{l}),$$

that is $\kappa(\zeta)\in\Xi\otimes\mathscr{A}(\mathfrak{g},\mathfrak{l})$ by Lemma 3.4. Since κ is invertible, $\kappa(\mathscr{S}(\Xi)) = \Xi\otimes\mathscr{A}(\mathfrak{g},\mathfrak{l})$. □

As an immediate consequence of the proposition, we obtain the following sufficient condition which renders $\mathscr{S}(\Xi)$ projective over $\mathscr{A}(\mathfrak{g},\mathfrak{l})$.

Corollary 3.1. *The $\mathscr{S}(\Xi)$ is projective over $\mathscr{A}(\mathfrak{g},\mathfrak{l})$ if there exists a $\mathrm{U}_q(\mathfrak{l})$-module $\Xi^\perp$ and a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module V such that $\Xi\oplus\Xi^\perp$ is isomorphic to the restriction of V to a $\mathrm{U}_q(\mathfrak{l})$-module.*

If $\mathrm{U}_q(\mathfrak{g})$ was an ordinary quantized universal enveloping algebra associated with a finite dimensional semi-simple Lie algebra, it was shown in [7] that $\mathscr{S}(\Xi)$ was always projective over $\mathscr{A}(\mathfrak{g},\mathfrak{l})$. Unfortunately this is no longer true for $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. However, if $\mathrm{U}_q(\mathfrak{l})$ is a purely even reductive quantum subalgebra of $\mathrm{U}_q(\mathfrak{g})$, that is, $\mathrm{U}_q(\mathfrak{l})\subset\mathrm{U}_q(\mathfrak{g}_0)$, then $\mathscr{S}(\Xi)$ is a finitely generated projective $\mathscr{A}(\mathfrak{g},\mathfrak{l})$-module. More generally, we have the following result.

Lemma 3.6. *If $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$ is $\mathfrak{g}$-dominant, then $\mathscr{S}(K^{(\mathfrak{l})}_\lambda)$ is projective over $\mathscr{A}(\mathfrak{g},\mathfrak{l})$, where $K^{(\mathfrak{l})}_\lambda$ is the dual Kac module over $\mathrm{U}_q(\mathfrak{l})$ defined by (2.3).*

Proof. To prove this, we let $\bar{L}^{(\mathfrak{g}_{\leqslant 0})}_{-\lambda}$ be the irreducible $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$-module with lowest weight $-\lambda$, which is finite dimensional. Let $\bar{V}_{-\lambda}=\mathrm{U}_q(\mathfrak{g})\otimes_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}\bar{L}^{(\mathfrak{g}_{\leqslant 0})}_{-\lambda}$. Then $\bar{V}_{-\lambda}$ is a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module, which naturally restricts to a $\mathrm{U}_q(\mathfrak{l})$-module. Let $\bar{v}_{-\lambda}\in\bar{V}_{-\lambda}$ be a non-zero vector with weight $-\lambda$, which generates a $\mathrm{U}_q(\mathfrak{l})$-module $\bar{K}=\mathrm{U}_q(\mathfrak{l})\bar{v}_{-\lambda}$. Regard $\bar{V}_{-\lambda}$ as a $\mathrm{U}_q(\mathfrak{h})$-module, we have the decomposition $\bar{V}_{-\lambda}=\bar{K}\oplus\bar{K}^\perp$. This in fact is also a direct sum of $\mathrm{U}_q(\mathfrak{l})$-modules as the weights of $\bar{K}^\perp$ differ from those of $\bar{K}$ by roots not belonging to $\mathfrak{l}$. The dual $\bar{V}^*_{-\lambda}$ of $\bar{V}_{-\lambda}$ has a natural $\mathrm{U}_q(\mathfrak{l})$-module structure, and contains the $\mathrm{U}_q(\mathfrak{l})$-submodule $\bar{K}^*=K^{(\mathfrak{l})}_\lambda$ as a direct summand. Therefore Proposition 3.1 applies to the present situation. □

The space $\mathscr{S}(\Xi)$ has a direct bearing on the representation theory of $\mathrm{U}_q(\mathfrak{g})$.

Lemma 3.7.

(1) *$\mathscr{S}(\Xi)$ forms a $\mathrm{U}_q(\mathfrak{g})$-module under the action*

$$\mathrm{U}_q(\mathfrak{g})\otimes\mathscr{S}(\Xi)\to\mathscr{S}(\Xi),\quad x\otimes\zeta\mapsto(\mathrm{id}\otimes\mathrm{d}R_x)\zeta. \tag{3.13}$$

(2) *for every $\mathrm{U}_q(\mathfrak{l})$-module map $\phi:\Xi\to\Xi'$, the induced map*

$$\mathscr{S}(\phi)=\phi\otimes\mathrm{id}:\mathscr{S}(\Xi)\to\mathscr{S}(\Xi'), \tag{3.14}$$

is $\mathrm{U}_q(\mathfrak{g})$-equivariant.

Proof. The first part follows from the fact that the two actions $\mathrm{d}L$ and $\mathrm{d}R$ of $\mathrm{U}_q(\mathfrak{g})$ on $\mathscr{A}(\mathfrak{g})$ super-commute. To see the second part, let $\zeta=\sum v_i\otimes f_i$ be in $\mathscr{S}(\Xi)$.

Then for all $x \in \mathrm{U}_q(\mathfrak{g})$,

$$x \circ (\mathscr{S}(\phi)\zeta) = \sum (-1)^{([v_i]+[\phi])[x]} \phi(v_i) \otimes \mathrm{d}R_x(f_i)$$
$$= (-1)^{[x][\phi]} \mathscr{S}(\phi)(x \circ \zeta). \qquad \square$$

Of particular interest to us is the case when Ξ is a finite dimensional $\mathrm{U}_q(\mathfrak{p})$-module, where $\mathrm{U}_q(\mathfrak{p})$ is a parabolic quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ with $\mathrm{U}_q(\mathfrak{l})$ as its Levi factor. Then $\mathscr{S}(\Xi)$ contains the following subspace.

Definition 3.4. $\Gamma(\Xi) := \{\zeta \in \mathscr{S}(\Xi) \mid (\mathrm{id} \otimes \mathrm{d}L_x)\zeta = (S(x) \otimes \mathrm{id})\zeta,\ \forall x \in \mathrm{U}_q(\mathfrak{p})\}$.

Again by using the super-commutativity of the $\mathrm{U}_q(\mathfrak{g})$-actions $\mathrm{d}L$ and $\mathrm{d}R$ on $\mathscr{A}(\mathfrak{g})$ we can easily show that

Lemma 3.8. *$\Gamma(\Xi)$ is a $\mathrm{U}_q(\mathfrak{g})$-submodule of $\mathscr{S}(\Xi)$. Also a $\mathrm{U}_q(\mathfrak{p})$-module homomorphism $\phi : \Xi \to \Xi'$ induces a $\mathrm{U}_q(\mathfrak{g})$-equivariant map*

$$\Gamma(\phi) = \phi \otimes \mathrm{id} : \Gamma(\Xi) \to \Gamma(\Xi'). \tag{3.15}$$

Let $X(\mathfrak{p}, \mathfrak{l})$ denote the set of $E_{a+1,a}$ or $E_{a,a+1}$ which are contained in $\mathrm{U}_q(\mathfrak{p})$ but not in $\mathrm{U}_q(\mathfrak{l})$. If the $\mathrm{U}_q(\mathfrak{p})$-module Ξ has the property that every element of $X(\mathfrak{p}, \mathfrak{l})$ acts by zero, then in this case the definition of $\Gamma(\Xi)$ reduces to

$$\Gamma(\Xi) = \{\zeta \in \mathscr{S}(\Xi) \mid (\mathrm{id} \otimes \mathrm{d}L_x)\zeta = 0,\ \forall x \in X(\mathfrak{p}, \mathfrak{l})\}.$$

Thus $\Gamma(\Xi)$ plays a similar role as the space of holomorphic sections in classical geometry. We shall refer to it as the space of holomorphic sections of the homogeneous supervector bundle determined by $\mathscr{S}(\Xi)$.

We shall promote Γ to a covariant functor from the category $\mathscr{C}(\mathfrak{p}, \mathfrak{l})$ of the $\mathrm{U}_q(\mathfrak{l})$-finite modules over the parabolic subalgebra $\mathrm{U}_q(\mathfrak{p})$ to the category $\mathscr{C}(\mathfrak{g}, \mathfrak{g})$ of locally finite $\mathrm{U}_q(\mathfrak{g})$-modules. The resultant functor is shown to be left exact, and its right derived functors will be regarded as the Dolbeault cohomology groups of the homogeneous supervector bundle.

4. Induction functors

We study induction functors and their derived functors in this section, extending results of [19] to quantum supergroups. We shall employ the general methodology of cohomological induction, closely following Refs. [6] and [12]. We shall also frequently adapt results from the seminal paper of Andersen et al. [1]. Some elementary homological algebra will be used, which can be found in any text book on the subject, e.g., [22].

Results of this section will be applied later to develop a representation theoretical formulation of a quantum analogue of Dolbeault cohomology for the quantum homogeneous supervector bundles.

4.1. Categories of modules

We start with a discussion on module categories of $\mathrm{U}_q(\mathfrak{g})$ and its quantum sub-superalgebras. For any quantum superalgebra U, we shall assume that every U-module to be considered in this paper is $\mathbb{Z}_2$-graded. Thus corresponding to each U-module V, there exists another module $\wp V$ which is equal to V as a set, but with $(\wp V)_{\bar{0}} = V_{\bar{1}}$, and $(\wp V)_{\bar{1}} = V_{\bar{0}}$. Let $\wp(v)$ denote the element of $\wp V$ corresponding to $v \in V$. The action of U on $\wp V$ is defined by $x\wp(v) = (-1)^{[x]}\wp(xv)$, for all $x \in \mathrm{U}$.

Let $\mathrm{U}_q(\mathfrak{r})$ be a quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Denote by $\mathscr{C}_{\mathrm{inh}}(\mathfrak{r})$ the category of $\mathrm{U}_q(\mathfrak{r})$-modules, where the space $\mathrm{Hom}_{\mathrm{U}}(V, W)$ of morphisms between any two U-modules V and W is a $\mathbb{Z}_2$-graded subspace of $\mathrm{Hom}_{\mathbb{C}(q)}(V, W)$ consisting of such elements ϕ that for all $x \in \mathrm{U}_q(\mathfrak{r})$ and $v \in V$, $\phi(xv) = (-1)^{[x][\phi]}x\phi(v)$. The parity change map $\wp$ is odd, and becomes a covariant functor on $\mathscr{C}_{\mathrm{inh}}(\mathfrak{r})$ if for any $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ we define $\wp(\phi) \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(\wp V, \wp W)$ to be the same as ϕ on sets. Note that if $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ is an inhomogeneous morphism between objects V and W in $\mathscr{C}_{\mathrm{inh}}(\mathfrak{r})$, the kernel and image of ϕ are not necessarily $\mathbb{Z}_2$-graded in general, thus $\mathscr{C}_{\mathrm{inh}}(\mathfrak{r})$ is not an Abelian category.

Assume $\mathrm{U}_q(\mathfrak{r})$ contains a reductive sub-superalgebra $\mathrm{U}_q(\mathfrak{k})$ of $\mathrm{U}_q(\mathfrak{g})$. Every $\mathrm{U}_q(\mathfrak{r})$-module V naturally restricts to a $\mathrm{U}_q(\mathfrak{k})$-module.

Definition 4.1. The $\mathrm{U}_q(\mathfrak{k})$-finite subspace $V[\mathrm{U}_q(\mathfrak{k})]$ of V is defined to be the $\mathbb{C}(q)$-span of the integral weight vectors $v \in V$ satisfying $\dim(\mathrm{U}_q(\mathfrak{k})v) < \infty$.

Here $\mathrm{U}_q(\mathfrak{k})v := \{xv \mid x \in \mathrm{U}_q(\mathfrak{k})\}$. Elements of $V[\mathrm{U}_q(\mathfrak{k})]$ will be called $\mathrm{U}_q(\mathfrak{k})$-finite. Also, a $\mathrm{U}_q(\mathfrak{r})$-module V is called $\mathrm{U}_q(\mathfrak{k})$-finite if $V = V[\mathrm{U}_q(\mathfrak{k})]$.

Remark 4.1. If V is a $\mathbb{Z}_2$-graded $\mathrm{U}_q(\mathfrak{k})$-finite $\mathrm{U}_q(\mathfrak{r})$-module and $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ a *homogeneous* morphism, then $\phi(V)$ is a $\mathbb{Z}_2$-graded $\mathrm{U}_q(\mathfrak{k})$-finite $\mathrm{U}_q(\mathfrak{r})$-submodule of W.

Let $\mathrm{U}_q(\mathfrak{q})$ be either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. If $\mathrm{U}_q(\mathfrak{q})$ contains the reductive quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{k})$, we shall talk about the pair $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$ of quantum sub-superalgebras. Two pairs of sub-superalgebras $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$ and $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ are said to be compatible if we have the Hopf superalgebra inclusions $\mathrm{U}_q(\mathfrak{q}) \supseteq \mathrm{U}_q(\mathfrak{p})$ and $\mathrm{U}_q(\mathfrak{k}) \supseteq \mathrm{U}_q(\mathfrak{l})$, and in this case, we write $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$. Given a pair $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$, we shall denote by $\mathscr{C}_{\mathrm{inh}}(\mathfrak{q}, \mathfrak{k})$ the full subcategory of $\mathscr{C}_{\mathrm{inh}}(\mathfrak{q})$ with the $\mathrm{U}_q(\mathfrak{k})$-finite $\mathrm{U}_q(\mathfrak{q})$-modules as its objects. Clearly, $\mathscr{C}_{\mathrm{inh}}(\mathfrak{q}, \mathfrak{k})$ is closed under passage to graded sub-modules, graded quotients and finite direct sums. It is also closed under finite tensor products.

Definition 4.2. Let $\mathscr{C}(\mathfrak{q})$ be the subcategory of $\mathscr{C}_{\mathrm{int}}(\mathfrak{q})$ consisting of the same objects and the *even* morphisms of $\mathscr{C}_{\mathrm{inh}}(\mathfrak{q})$. Let $\mathscr{C}(\mathfrak{q}, \mathfrak{k})$ be the full subcategory of $\mathscr{C}(\mathfrak{q})$ with the $\mathrm{U}_q(\mathfrak{k})$-finite objects.

Then $\mathscr{C}(\mathfrak{q})$ is obviously an Abelian category, and it follows from Remark 4.1 that $\mathscr{C}(\mathfrak{q}, \mathfrak{k})$ is also Abelian.

4.2. Induction functors

Let $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ be a pair of quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{g})$, where $\mathrm{U}_q(\mathfrak{p})$ is either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{l})$ is a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ contained in $\mathrm{U}_q(\mathfrak{p})$. We recall that all the objects of the categories $\mathscr{C}(\mathfrak{p})$ and $\mathscr{C}(\mathfrak{p}, \mathfrak{l})$ are $\mathbb{Z}_2$-graded, and all the morphisms of the categories are even.

Definition 4.3. Define a covariant functor $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}} : \mathscr{C}(\mathfrak{p}) \to \mathscr{C}(\mathfrak{p}, \mathfrak{l})$ in the following way: for any object V, let $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(V)$ be the (not necessarily direct) sum of the $\mathrm{U}_q(\mathfrak{l})$-finite $\mathbb{Z}_2$-graded $\mathrm{U}_q(\mathfrak{p})$-submodules of V, and for any morphism, let $\phi \in \mathrm{Hom}_{\mathscr{C}(\mathfrak{p})}(V, W)$, $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(\phi) = \phi|_{Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(V)}$.

$Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$ is well-defined because of Remark 4.1. When $\mathrm{U}_q(\mathfrak{p}) = \mathrm{U}_q(\mathfrak{l}) = \mathrm{U}_q(\mathfrak{g})$, we have an analogue of the Zuckerman functor.

Lemma 4.1. *The functor* $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}} : \mathscr{C}(\mathfrak{p}) \to \mathscr{C}(\mathfrak{p}, \mathfrak{l})$ *is left exact.*

Proof. Even though the proof is straightforward, we nevertheless give the details here because of the importance of the functor $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$. Let us temporarily use Z to denote $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$. Given any exact sequence

$$0 \to U \xrightarrow{i} V \xrightarrow{j} W$$

in $\mathscr{C}(\mathfrak{p})$, we want to show that the following sequence in $\mathscr{C}(\mathfrak{p}, \mathfrak{l})$ is also exact:

$$0 \to Z(U) \xrightarrow{Z(i)} Z(V) \xrightarrow{Z(j)} Z(W).$$

Assume U' is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of U. Then $i(U')$ is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of V. Thus the injectivity of i implies the injectivity of $Z(i)$.

Let V' be a $\mathrm{U}_q(\mathfrak{p})$-submodule of $Z(V)$. If an element $v \in V'$ is in $Ker Z(j)$, then there exists a unique $u \in U$ such that $v = i(u)$. Now $\mathrm{U}_q(\mathfrak{p})v = i(\mathrm{U}_q(\mathfrak{p})u)$ is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of V. The injectivity of i forces $\mathrm{U}_q(\mathfrak{p})u$ to be a $\mathrm{U}_q(\mathfrak{p})$-submodule of $Z(U)$. In particular, $u \in Z(U)$. Thus $Im Z(i) \supseteq Ker Z(j)$. But it is obvious that $Im Z(i) \subseteq Ker Z(j)$. Hence the sequence is also exact at $Z(V)$. □

Let $\mathrm{U}_q(\mathfrak{q})$ either be a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and let $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ be as given above with $\mathrm{U}_q(\mathfrak{q}) \supseteq \mathrm{U}_q(\mathfrak{p})$. We define the covariant functor $I_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}} : \mathscr{C}(\mathfrak{p}, \mathfrak{l}) \to \mathscr{C}(\mathfrak{q})$ by

$$\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}}(V) := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V), \quad \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}}(\phi) := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), \phi) \tag{4.1}$$

for any object V and morphism $\phi \in \mathrm{Hom}_{\mathscr{C}(\mathfrak{p},\mathfrak{l})}(V, W)$. Here $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}}(\phi)$ is defined for any $\zeta \in \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}}(V)$ by

$$\langle \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q}}(\phi)(\zeta), x\rangle = \phi(\langle \zeta, x\rangle), \quad \forall x \in \mathrm{U}_q(\mathfrak{q}).$$

Note that $\langle \zeta, x\rangle \in V$. The $\mathrm{U}_q(\mathfrak{q})$ action on $\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V)$

$$\mathrm{U}_q(\mathfrak{q}) \otimes \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V) \to \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V), \quad y \otimes \zeta \mapsto y \circ \zeta, \tag{4.2}$$

is defined by $\langle y \circ \zeta, x\rangle = (-1)^{[y]([x]+[\zeta])}\langle \zeta, xy\rangle$, for all $x \in \mathrm{U}_q(\mathfrak{q})$. The functor $I^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}$ is the composition of the exact functor of tensoring with $\mathrm{U}_q(\mathfrak{q})^*$ and the left exact functor of taking $\mathrm{U}_q(\mathfrak{p})$ invariant submodules, thus is also left exact.

Definition 4.4. Given compatible pairs $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$, we introduce the covariant functor

$$\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}} := Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}} \circ \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}} : \mathscr{C}(\mathfrak{p},\mathfrak{l}) \to \mathscr{C}(\mathfrak{q},\mathfrak{k}),$$

and call it the induction functor from $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ to $\mathscr{C}(\mathfrak{q},\mathfrak{k})$.

Lemma 4.2. *The induction functor $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}$ is left exact.*

Proof. Since both $\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}$ and $Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}$ are left exact, their composition must also be left exact. □

Let us examine some further properties of the Zuckerman functor and the induction functor.

Let $\mathrm{U}_q(\mathfrak{q})$ be either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and let $\mathrm{U}_q(\mathfrak{k})$ be a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Assume $\mathrm{U}_q(\mathfrak{q}) \supset \mathrm{U}_q(\mathfrak{k})$. Then for any object W of $\mathscr{C}(\mathfrak{q})$,

$$Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W) = W[\mathrm{U}_q(\mathfrak{k})]. \tag{4.3}$$

To prove this, we define the adjoint action of $\mathrm{U}_q(\mathfrak{q})$ on itself

$$\mathrm{ad}_y(x) = \sum_{(y)} (-1)^{[y_{(2)}][x]} y_{(1)} x S(y_{(2)}), \quad x, y \in \mathrm{U}_q(\mathfrak{q}),$$

where Sweedler's notation $\Delta(y) = \sum_{(y)} y_{(1)} \otimes y_{(2)}$ is used for the co-multiplication of y. By using the Poincaré–Birkhoff–Witt (PBW) Theorem 2.1, we can choose a set of y_i each of which is a product of E_{ab} associated with the roots of $\mathfrak{q}$ not contained in $\mathfrak{k}$, such that every element $x \in \mathrm{U}_q(\mathfrak{q})$ can be expressed as a finite sum $x = \sum y_i u_i$ with $u_i \in \mathrm{U}_q(\mathfrak{k})$. By considering the PBW theorem again, we see that there exists a finite set Y_x of the y_i such that every element of the space $\mathrm{ad}_{\mathrm{U}_q(\mathfrak{k})}(x) := \{\mathrm{ad}_u(x) \mid u \in \mathrm{U}_q(\mathfrak{k})\}$ can be expressed in the form $\sum y_i u_i'$ with $y_i \in Y_x$ and $u_i' \in \mathrm{U}_q(\mathfrak{k})$. If $w \in W[\mathrm{U}_q(\mathfrak{k})]$, then

$$u(xw) = \sum_{(u)} (-1)^{[u_{(2)}][x]} (\mathrm{ad}_{u_{(1)}}(x))(u_{(2)} w), \quad x \in \mathrm{U}_q(\mathfrak{q}),\ u \in \mathrm{U}_q(\mathfrak{k}).$$

This implies $u(xw) \in \sum_{y \in Y_x} y(\mathrm{U}_q(\mathfrak{k})(xw))$, for all $u \in \mathrm{U}_q(\mathfrak{k})$. Therefore,

$$\dim(\mathrm{U}_q(\mathfrak{k})(xw)) \leqslant |Y_x| \dim(\mathrm{U}_q(\mathfrak{k})w) < \infty,$$

where Y_x is the cardinality of Y_x. Also, if x carries a fixed weight and w is a weight vector of $W[\mathrm{U}_q(\mathfrak{k})]$, then xw is a weight vector of $W[\mathrm{U}_q(\mathfrak{k})]$ with integral weight. Hence $W[\mathrm{U}_q(\mathfrak{k})]$ is indeed a $\mathrm{U}_q(\mathfrak{q})$-submodule of W.

Lemma 4.3. *Given compatible pairs* $(\mathrm{U}_q(\mathfrak{r}), \mathrm{U}_q(\mathfrak{j})) \supseteq (\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ *of quantum sub-superalgebras of* $\mathrm{U}_q(\mathfrak{g})$, *we have* $\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}$ *as covariant functors from* $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ *to* $\mathscr{C}(\mathfrak{r},\mathfrak{j})$.

Proof. It is clearly true that for any morphism $\phi \in \mathrm{Hom}_{\mathscr{C}(\mathfrak{p},\mathfrak{l})}(V, V')$, we have $\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(\phi) = \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}(\phi)$. To prove that the lemma holds on objects, we need the following technical result which will be proved below: if $(\mathrm{U}_q(\mathfrak{r}), \mathrm{U}_q(\mathfrak{j})) \supseteq (\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$, then for any object W of $\mathscr{C}(\mathfrak{q})$,

$$Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W) = Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W)). \tag{4.4}$$

With (4.4) granted, we have for any object V of $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ that

$$\begin{aligned} \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V) &= Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V)) \\ &= Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)). \end{aligned}$$

The far right-hand side of the equation can be simplified by using the following relation:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)) &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} \mathrm{U}_q(\mathfrak{r}), V) \\ &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{r}), V) \end{aligned}$$

and we arrive at

$$\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V) = Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{r}), V) = \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}(V).$$

Now we turn to the proof of Eq. (4.4), which is equivalent to the statement that for any $\zeta \in Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W)$,

$$\langle \zeta, z \rangle \in Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W), \quad \forall z \in \mathrm{U}_q(\mathfrak{r}). \tag{4.5}$$

By the PBW theorem for $\mathrm{U}_q(\mathfrak{r})$, there exists a set of x_i, each of which is a product of elements associated with roots of $\mathfrak{r}$ not contained in $\mathfrak{q}$, such that every $z \in \mathrm{U}_q(\mathfrak{r})$ can be expressed as a finite sum $\sum y_i x_i$ with $y_i \in \mathrm{U}_q(\mathfrak{q})$. Let v_i be the weight of x_i, which is a sum of roots of $\mathfrak{r}$ thus is integral. We have

$$\langle u \circ \zeta, x_i \rangle = \sum_{(u)} (-1)^{[\zeta][u_{(2)}]} \pi_W(u_{(1)}) \langle \zeta, \mathrm{ad}_{S(u_{(2)})}(x_i) \rangle, \quad u \in \mathrm{U}_q(\mathfrak{k}),$$

where $u \circ \zeta$ is as defined by (4.2), and π_W refers to the $\mathrm{U}_q(\mathfrak{q})$ action on W. From this equation we can deduce that

$$\pi_W(u) \langle \zeta, x_i \rangle = \sum_{(u)} (-1)^{[u_{(2)}][\zeta]} \langle u_{(1)} \circ \zeta, \mathrm{ad}_{u_{(2)}}(x_i) \rangle, \quad u \in \mathrm{U}_q(\mathfrak{k}).$$

Arguing as in the proof of (4.3), we conclude that for every x_i, there exists a finite set X_i of the x_j such that every element of $\mathrm{ad}_{\mathrm{U}_q(\mathfrak{k})}(x_i)$ can be expressed as $\sum x_j u_j$, where $x_j \in X_i$ and $u_j \in \mathrm{U}_q(\mathfrak{k})$. Now $\langle \zeta, \sum x_j u_j \rangle = \sum \langle u_j \circ \zeta, \sum x_j \rangle$, and

$\zeta \in Z_{\mathfrak{r}}^{\mathfrak{r},\mathfrak{j}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W)$ implies that the span of $u \circ \zeta$, for $u \in \mathrm{U}_q(\mathfrak{k})$, is finite dimensional. Therefore,

$$\dim(\pi_W(\mathrm{U}_q(\mathfrak{k}))\langle \zeta, x_i \rangle) < \infty, \quad \text{that is, } \langle \zeta, x_i \rangle \in Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W), \ \forall i.$$

Since $Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$ is $\mathrm{U}_q(\mathfrak{q})$-stable, we have $\sum_i \pi_W(\mathrm{U}_q(\mathfrak{q}))\langle \zeta, x_i \rangle \subset Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$. Now every element of $\mathrm{U}_q(\mathfrak{r})$ can be expressed as $\sum y_i x_i$ with $y_i \in \mathrm{U}_q(\mathfrak{q})$. By the definition of $Z_{\mathfrak{r}}^{\mathfrak{r},\mathfrak{j}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W)$, we have $\sum \langle \zeta, y_i x_i \rangle = \sum (-1)^{[y_i][\zeta]} \pi_W(y_i) \langle \zeta, x_i \rangle$, where the right-hand side has just been shown to belong to $Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$. This proves Eq. (4.5), thus completes the proof of the lemma. □

Denote by $\mathscr{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}} : \mathscr{C}(\mathfrak{q},\mathfrak{k}) \to \mathscr{C}(\mathfrak{p},\mathfrak{l})$ the forgetful functor. We shall refer to the next theorem as Frobenius reciprocity, which in particular implies that the induction functor $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}$ is right adjoint to the forgetful functor $\mathscr{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}$. The theorem plays a crucial role in the study of derived functors of the induction functors.

Theorem 4.1. *There exists the natural even isomorphism*

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathscr{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V), \\ &\phi \mapsto \phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})}), \end{aligned} \tag{4.6}$$

of $\mathbb{Z}_2$-graded vector spaces for all W in $\mathscr{C}(\mathfrak{q},\mathfrak{k})$ and V in $\mathscr{C}(\mathfrak{p},\mathfrak{l})$.

Proof. Since W is an object of $\mathscr{C}(\mathfrak{q},\mathfrak{k})$, the image of any vector of W under an arbitrary $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V))$ belongs to $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. From this we can easily deduce that

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)).$$

The right-hand side can be further rewritten as $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} W, V)$. Now $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} W, V) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathscr{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V)$. Thus

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathscr{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V). \tag{4.7}$$

This establishes the claimed isomorphism between the vector spaces. Let us now show that $\phi \mapsto \phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})})$, $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V))$, is indeed the required map. The $\phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})})$ (We shall write 1 for the identity $\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})}$ of $\mathrm{U}_q(\mathfrak{q})$.) is the evaluation of ϕ at the identity of $\mathrm{U}_q(\mathfrak{q})$. Denote by $\circ$ the action of $\mathrm{U}_q(\mathfrak{q})$ on $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. Then for any $x \in \mathrm{U}_q(\mathfrak{q})$ and $w \in W$, we have

$$\phi(1)(xw) = (-1)^{[x][\phi]}(x \circ \phi)(1)(w) = \phi(x)(w),$$

where the symbol $\circ$ refers to the $\mathrm{U}_q(\mathfrak{q})$-action on $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. If ϕ belongs to the kernel of map (4.6), then $\phi(x) = 0$, $\forall x \in \mathrm{U}_q(\mathfrak{q})$. This forces $\phi = 0$. Thus map (4.6) is injective, and because of isomorphism (4.7), it must be a bijection.

We still need to show that $\phi(1)\in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathscr{F}^{\mathfrak{p},\mathfrak{l}}_{\mathfrak{q},\mathfrak{k}}(W),V)$. But this is clear, as the defining property of $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V)$ implies

$$\phi(x)(w)=(-1)^{[x][\phi]}x(\phi(1)(w)),\quad \forall x\in \mathrm{U}_q(\mathfrak{p}).$$

This completes the proof. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.1. *The induction functor* $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}$ *takes injectives to injectives.*

Proof. If V is an injective object in $\mathscr{C}(\mathfrak{p},\mathfrak{l})$, then $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\cdot,V)$ is an exact functor from $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ to the category of $\mathbb{Z}_2$-graded vector spaces. Thus by the Frobenius reciprocity of Theorem 4.1, $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\cdot,\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V))$ is exact on $\mathscr{C}(\mathfrak{q},\mathfrak{k})$. This in turn implies that $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V)$ is injective in $\mathscr{C}(\mathfrak{q},\mathfrak{k})$. □

Consider the pair $(\mathrm{U}_q(\mathfrak{p}),\mathrm{U}_q(\mathfrak{l}))$ of quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{g})$, where $\mathrm{U}_q(\mathfrak{l})$ is assumed to be reductive as always.

Corollary 4.2. *The category* $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ *has enough injectives.*

Proof. Let $\mathrm{U}_q(\mathfrak{l}_0)=\mathrm{U}_q(\mathfrak{l})\cap\mathrm{U}_q(\mathfrak{g}_0)$. Every $\mathrm{U}_q(\mathfrak{l}_0)$-finite $\mathrm{U}_q(\mathfrak{p})$-module is also $\mathrm{U}_q(\mathfrak{l})$-finite and vice versa, hence the categories $\mathscr{C}(\mathfrak{p},\mathfrak{l}_0)$ and $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ are identical. Since $\mathrm{U}_q(\mathfrak{l}_0)$ is the tensor product of some non-super $\mathrm{U}_q(\mathfrak{gl}_k)$'s, every object of $\mathscr{C}(\mathfrak{l}_0,\mathfrak{l}_0)$ is semi-simple, thus is injective. Let V be an $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-module, which can be restricted to an object of $\mathscr{C}(\mathfrak{l}_0,\mathfrak{l}_0)$. Now $\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\mathfrak{l}_0,\mathfrak{l}_0}(V)$ is injective as follows from the above corollary. By Theorem 4.1 we have the isomorphism

$$F:\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(V,\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\mathfrak{l}_0,\mathfrak{l}_0}(V))\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l}_0)}(V,V).$$

Consider the pre-image of the identity map $\mathrm{id}_V\in\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l}_0)}(V,V)$ under F,

$$\imath:=F^{-1}(\mathrm{id}_V):V\to \mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\mathfrak{l}_0,\mathfrak{l}_0}(V),\quad v\mapsto \imath_v, \tag{4.8}$$

which is an injective $\mathrm{U}_q(\mathfrak{p})$-map. It satisfies $\imath_v(x)=(-1)^{[x][v]}xv$, $\forall x\in\mathrm{U}_q(\mathfrak{p})$. □

Remark 4.2. The $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}$ can be extended to a covariant functor $\mathscr{C}_{\mathrm{inh}}(\mathfrak{q},\mathfrak{k})\to\mathscr{C}_{\mathrm{inh}}(\mathfrak{p},\mathfrak{l})$ in the obvious way. It also takes injectives to injectives. By using Frobenius reciprocity, we can also show that the category $\mathscr{C}_{\mathrm{inh}}(\mathfrak{p},\mathfrak{l})$ has enough injectives.

Now we restrict our attention to the induction functor $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}}:\mathscr{C}(\mathfrak{p},\mathfrak{l})\to\mathscr{C}(\mathfrak{g},\mathfrak{g})$. Since the Abelian category $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ has enough injectives, and $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}}$ is left exact, it makes sense to talk about its right derived functors [22] $(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}})^k$, $k\in\mathbb{Z}_+$. We now give a concrete description of $(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}})^k$. Let V be any object of $\mathscr{C}(\mathfrak{p},\mathfrak{l})$. Then its restriction to a $\mathrm{U}_q(\mathfrak{l}_0)$-module lies in $\mathscr{C}(\mathfrak{l}_0,\mathfrak{l}_0)$ and thus is injective. We construct the following injective resolution of V in $\mathscr{C}(\mathfrak{p},\mathfrak{l})$,

$$0\to V\xrightarrow{\imath}I^0(V)\xrightarrow{\delta^0}I^1(V)\xrightarrow{\delta^1}I^2(V)\xrightarrow{\delta^2}\cdots, \tag{4.9}$$

where the $\mathrm{U}_q(\mathfrak{p})$-modules and maps are defined inductively by

$$I^{k+1}(V) := \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(I^k(V)/\delta^{k-1}(I^{k-1}(V))),$$
$$\delta^k := \imath \circ p : I^k(V) \xrightarrow{p} I^k(V)/\delta^{k-1}(I^{k-1}(V)) \xrightarrow{\imath} I^{k+1}(V). \tag{4.10}$$

Here $\imath$ is similarly defined as in (4.8), p is the canonical projection, and

$$I^0(V) = \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(V), \quad \delta^{-1} = \imath.$$

Sequence (4.9) is clearly a resolution, with all $I^k(V)$ being injective because of Corollary 4.1. We shall call this injective resolution the *standard resolution*. Now apply the left exact covariant functor $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ to it and ignore the first term $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(V)$, we arrive at the following complex in $\mathscr{C}(\mathfrak{g},\mathfrak{g})$:

$$0 \to \Omega^0(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^0} \Omega^1(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^1} \Omega^2(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^2} \cdots, \tag{4.11}$$

where

$$\Omega^k(\mathfrak{p},\mathfrak{l};V) := \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(I^k(V)), \quad d^k := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{g}), \delta^k)|_{\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(I^k(V))}.$$

Let us denote by $\Omega(\mathfrak{p},\mathfrak{l};V)$ the complex (4.11), and by $H^k(\Omega(\mathfrak{p},\mathfrak{l};V))$ its cohomology groups. Then we have [22]

$$(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}})^k(V) = H^k(\Omega(\mathfrak{p},\mathfrak{l};V)), \quad k = 0,1,\ldots,$$

which are independent of the injective resolution (4.9) chosen. Left exactness of the induction functor implies

$$H^0(\Omega(\mathfrak{p},\mathfrak{l};V)) = \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(V).$$

5. Quantum Bott–Borel–Weil theorem

We first formulate the Dolbeault cohomology groups of the quantum homogeneous supervector bundles as the right derived functor of a left exact functor, the 'holomorphic section functor', then compute the cohomology groups. The main results of the section are Theorems 5.2 and 5.3, which might be considered as a form of Bott–Borel–Weil theorem for the quantum general linear supergroup. As we have already mentioned in the Introduction, results reported here generalize the work of Penkov and Santos [18,19] in the classical setting of Lie superalgebras to quantum supergroups.

Throughout this section, we shall assume that $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ is a pair of quantum sub-superalgebras such that $\mathrm{U}_q(\mathfrak{l})$ is a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{p})$ is the parabolic with $\mathrm{U}_q(\mathfrak{l})$ being its Levi factor. For the sake of concreteness, we also assume that $\mathrm{U}_q(\mathfrak{p})$ contains the lower triangular Borel subalgebra $\mathrm{U}_q(\bar{\mathfrak{b}})$ of $\mathrm{U}_q(\mathfrak{g})$. We shall also denote $\mathrm{U}_q(\mathfrak{p}_0) = \mathrm{U}_q(\mathfrak{g}_0) \cap \mathrm{U}_q(\mathfrak{p})$.

5.1. Dolbeault cohomology groups

First note that the domain of $\mathscr{S}$ can be extended to the category $\mathscr{C}(\mathfrak{l},\mathfrak{l})$, and that of Γ to the category $\mathscr{C}(\mathfrak{p},\mathfrak{l})$. Below we shall consider these more generally defined $\mathscr{S}$ and Γ. We have the following result.

Theorem 5.1. *$\mathscr{S}$ coincides with $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ on $\mathscr{C}(\mathfrak{l},\mathfrak{l})$.*

Proof. By examining the second part of Proposition 3.1, we easily see that $\mathscr{S}$ agrees with $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ on maps. Let Ξ be an object of $\mathscr{C}(\mathfrak{l},\mathfrak{l})$. The inclusion $\mathscr{S}(\Xi) \subseteq \mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ is obvious since $\Xi \otimes \mathscr{A}(\mathfrak{g})$ is $\mathrm{U}_q(\mathfrak{g})$-finite with respect to the action $\mathrm{id}_{\Xi} \otimes \mathrm{d}R_{\mathrm{U}_q(\mathfrak{g})}$. Any element $\zeta \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l})}(\mathrm{U}_q(\mathfrak{g}),\Xi)$ can be expressed as $\zeta = \sum \xi_i \otimes f_i$, where $f_i \in \mathrm{U}_q(\mathfrak{g})^*$ and $\zeta_i \in \Xi$. The ζ belongs to $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ only if $\dim(R_{\mathrm{U}_q(\mathfrak{g})}(f_i)) < \infty$, $\forall i$. This is equivalent to the condition that all the f_i belong to $\mathrm{U}_q(\mathfrak{g})^\circ$, as follows from Lemma 3.1. Since Ξ regarded as a $\mathrm{U}_q(\mathfrak{h})$-module is integral, we may assume that the ξ_i are weight vectors with integral weights. The defining property of $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ requires the f_i be $\mathrm{d}L_{\mathrm{U}_q(\mathfrak{h})}$ eigenvectors in $\mathrm{U}_q(\mathfrak{g})^\circ$ with integral weights. Hence by Remark 3.1, the f_i must all belong to $\mathscr{A}(\mathfrak{g})$. $\square$

In exactly the same manner we can show that

Corollary 5.1. *$\Gamma(\Xi) = \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ on $\mathscr{C}(\mathfrak{p},\mathfrak{l})$.*

Remark 5.1. In view of the theorem and this corollary, we regard the right derived functors of $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ as a form of Dolbeault cohomology of the quantum homogeneous super vector bundles. Thus we shall use the more suggestive notation $H^{0,k}(G/P,\mathscr{S}(\Xi))$ to denote $H^k(\Omega(\mathfrak{p},\mathfrak{l};\Xi))$.

5.2. Computation of cohomology groups

The rest of the paper is devoted to the computation of $H^{0,k}(G/P,\mathscr{S}(\Xi))$.

5.2.1. A special case with $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{g}_0)$

Denote by

$$\mathscr{F}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}^{\mathfrak{g}_0,\mathfrak{g}_0} : \mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0) \to \mathscr{C}(\mathfrak{g}_0,\mathfrak{g}_0),$$

$$\mathscr{F}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}^{\mathfrak{g}_0,\mathfrak{l}} : \mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{l}) \to \mathscr{C}(\mathfrak{g}_0,\mathfrak{l}),$$

$$\mathscr{F}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{p}_0,\mathfrak{l}} : \mathscr{C}(\mathfrak{p},\mathfrak{l}) \to \mathscr{C}(\mathfrak{p}_0,\mathfrak{l}),$$

the forgetful functors.

Lemma 5.1. *If* $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{g}_0)$, *then we have the following relations*:

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}, \tag{5.1}$$

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}. \tag{5.2}$$

Proof. The first relation can be confirmed by directly checking the functors involved on objects and morphisms. For any object V of $\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{l})$, $\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(V)=V[\mathrm{U}_q(\mathfrak{g}_0)]$, and thus $\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(V)$ is $V[\mathrm{U}_q(\mathfrak{g}_0)]$ regarded as a $\mathrm{U}_q(\mathfrak{g}_0)$-module. Also, $\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}}\circ\mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(V)= \mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(V)[\mathrm{U}_q(\mathfrak{g}_0)]$, which is again $V[\mathrm{U}_q(\mathfrak{g}_0)]$ regarded as a $\mathrm{U}_q(\mathfrak{g}_0)$-module. Now for any $\phi \in \mathrm{Hom}_{\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{l})}(V,W)$, and all $v\in V[\mathrm{U}_q(\mathfrak{g}_0)]$, we have $\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(\phi)(v) = \phi(v) = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}(v)$, which belongs to $W[\mathrm{U}_q(\mathfrak{g}_0)]$.

Now consider the second relation, which obviously holds on morphisms. By using the quantum PBW theorem, we can easily show that $\mathrm{U}_q(\mathfrak{g}_0)/\mathrm{U}_q(\mathfrak{p}_0) \cong \mathrm{U}_q(\mathfrak{g}_{\leqslant 0})/\mathrm{U}_q(\mathfrak{p})$ under the given conditions on $\mathrm{U}_q(\mathfrak{l})$ and $\mathrm{U}_q(\mathfrak{p})$. Therefore, for any object V of $\mathscr{C}(\mathfrak{p},\mathfrak{l})$, we have the vector space isomorphism

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{g}_{\leqslant 0}),V) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p}_0)}(\mathrm{U}_q(\mathfrak{g}_0),\mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V)). \tag{5.3}$$

Denote by $P : \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{g}_{\leqslant 0}),V) \to \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p}_0)}(\mathrm{U}_q(\mathfrak{g}_0),V)$ the map induced by the inclusion of $\mathrm{U}_q(\mathfrak{g}_0)$ in $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$,

$$\langle P(\zeta),x\rangle = \langle \zeta,x\rangle, \quad \forall x\in \mathrm{U}_q(\mathfrak{g}_0) \subset \mathrm{U}_q(\mathfrak{g}_{\leqslant 0}).$$

This map is $\mathrm{U}_q(\mathfrak{g}_0)$-equivariant, as for any $u\in \mathrm{U}_q(\mathfrak{g}_0)$, we have

$$\langle P(u\circ\zeta),x\rangle = (-1)^{[u]([x]+[\zeta])}\langle \zeta,xu\rangle = \langle u\circ P(\zeta),x\rangle.$$

Now every element in $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$ may be expressed in the form $\sum y_i u_i$ with $y_i\in \mathrm{U}_q(\mathfrak{p})$ and $u_i\in \mathrm{U}_q(\mathfrak{g}_0)$. We have $\langle \zeta, \sum\ y_i u_i\rangle = \sum (-1)^{[y_i][\zeta]}\pi_V(y_i)\langle P(\zeta),u_i\rangle$. Thus $P(\zeta)=0$ if and only if $\zeta=0$. Therefore, the $\mathrm{U}_q(\mathfrak{g}_0)$-map P is injective, which must be bijective because of the vector space isomorphism (5.3).

Since $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}_0) \subset \mathrm{U}_q(\mathfrak{g}_0)$, and $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}) \subset \mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$, by (4.3) we have

$$\mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p}_0)}(\mathrm{U}_q(\mathfrak{g}_0),\mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V))[\mathrm{U}_q(\mathfrak{l})],$$

$$\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{g}_{\leqslant 0}),V)[\mathrm{U}_q(\mathfrak{l})].$$

The restriction of the $\mathrm{U}_q(\mathfrak{g}_0)$-equivariant map P to $\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V)$ now leads to the sought after $\mathrm{U}_q(\mathfrak{g}_0)$-module isomorphism

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) \cong \mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V). \qquad \square \tag{5.4}$$

We also have the following easy result.

Lemma 5.2. *The functor* $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0} : \mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0) \to \mathscr{C}(\mathfrak{g},\mathfrak{g})$ *is exact with*

$$\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}(V) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),V),$$

for any object V *in* $\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0)$.

Proof. We need to show that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),\cdot)$ is exact on $\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0)$, and for any V in $\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0)$, $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),V)$ is $\mathrm{U}_q(\mathfrak{g}_0)$-finite. It is fairly easy to see that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),V)$ is spanned by integral weigh vectors since V is an object of $\mathscr{C}(\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0)$. Let $\mathscr{U}^{+1}$ denote the subspace of $\mathrm{U}_q(\mathfrak{g})$ spanned by the ordered products of $(E_{i\alpha})^{\theta_{i\alpha}}$, $i\leqslant m<\alpha$, $\theta_{i\alpha}=0,1$. Clearly $\dim\mathscr{U}^{+1}=2^{mn}$. Then

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),V)\cong(\mathscr{U}^{+1})^*\otimes V.$$

This in particular implies that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),\cdot)$ is exact. Given any $x\in\mathrm{U}_q(\mathfrak{g}_0)$ and $\eta\in\mathscr{U}^{+1}$, there exist $\eta_j\in\mathscr{U}^{+1}$ and $x_j\in\mathrm{U}_q(\mathfrak{g}_0)$ such that $\eta x=\sum x_j\eta_j$. Let $\zeta=\sum f_i\otimes v_i$ be in $(\mathscr{U}^{+1})^*\otimes V$. We have

$$\begin{aligned}\langle x\circ\zeta,\eta\rangle&=\sum(-1)^{[v_i][\eta]}\langle f_i,\eta x\rangle v_i\\&=\sum(-1)^{[v_i][\eta]}\langle f_i,\eta_j\rangle\pi_V(x_j)v_i.\end{aligned}$$

Since V is $\mathrm{U}_q(\mathfrak{g}_0)$-finite, we can deduce from this equation that $\mathrm{U}_q(\mathfrak{g}_0)\circ\zeta$ is finite dimensional for any $\zeta=\sum f_i\otimes v_i\in(\mathscr{U}^{+1})^*\otimes V$. This completes the proof. $\square$

The following proposition is one of the main results of this paper.

Proposition 5.1. *Let* $\mathrm{U}_q(\mathfrak{l})\subseteq\mathrm{U}_q(\mathfrak{g}_0)$ *be a reductive quantum subalgebra of* $\mathrm{U}_q(\mathfrak{g})$. *Let* $\mathrm{U}_q(\mathfrak{p})\supseteq\mathrm{U}_q(\bar{\mathfrak{b}})$ *be the parabolic quantum sub-superalgebra of* $\mathrm{U}_q(\mathfrak{g})$ *with* $\mathrm{U}_q(\mathfrak{l})$ *as its Levi factor. Let* $L_\lambda^{(\mathfrak{p})}$ *be a finite dimensional irreducible* $\mathrm{U}_q(\mathfrak{p})$*-module with* $\mathrm{U}_q(\mathfrak{l})$*-highest weight* $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$. *Denote by* $L_\lambda^{(\mathfrak{p}_0)}$ *the natural restriction of* $L_\lambda^{(\mathfrak{p})}$ *to a* $\mathrm{U}_q(\mathfrak{p}_0)$*-module. Then*

$$H^{0,k}(G/P,\mathscr{S}(L_\lambda^{(\mathfrak{p})}))=\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})}(\mathrm{U}_q(\mathfrak{g}),(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L_\lambda^{(\mathfrak{p}_0)})),\tag{5.5}$$

where $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L_\lambda^{(\mathfrak{p}_0)})$ *is regarded as a* $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$*-module with* $E_{m+1,m}$ *acting by zero.*

Proof. By Lemma 4.3, $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}}=\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}$. By Lemma 5.1,

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}=\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}}\circ\mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}}\circ\mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}.$$

Using Lemma 4.3 again, we obtain

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}\circ\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{p},\mathfrak{l}}=\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\circ\mathscr{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}.\tag{5.6}$$

Recall the following elementary facts: Let $\mathscr{C}\xrightarrow{G}\mathscr{C}'$ be a left exact covariant functor. (a) Suppose $\mathscr{C}'\xrightarrow{F}\mathscr{C}''$ is an exact covariant functor. Then $F\circ G$ is left exact, and its right derive functors are $(F\circ G)^k=F\circ(G)^k$. (b) Suppose $\tilde{\mathscr{C}}\xrightarrow{F}\mathscr{C}'$ is an exact covariant functor. Then $G\circ F$ is left exact, and its right derived functors are $(G\circ F)^k=(G)^k\circ F$. Applying these results to the situation at hand, we arrive at

$$\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}\circ(\mathrm{I}^{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}_{\mathfrak{p},\mathfrak{l}})^k(L_\lambda^{(\mathfrak{p})})=(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L_\lambda^{(\mathfrak{p}_0)}).$$

The derived functor $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k$ on the right-hand side can be computed by using the quantum Bott–Borel–Weil theorem [1] for $\mathrm{U}_q(\mathfrak{g}_0)=\mathrm{U}_q(\mathfrak{gl}_m)\otimes\mathrm{U}_q(\mathfrak{gl}_n)$. Now $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L_\lambda^{(\mathfrak{p}_0)})$

is either zero or a finite dimensional irreducible $\mathrm{U}_q(\mathfrak{g}_0)$-module. Therefore, its inverse image under the forgetful functor $\mathscr{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}$ must be either zero or $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$-irreducible. In both cases, $E_{m+1,m}$ acts by zero. Thus by using Lemma 5.2, we have

$$(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}})^k(L^{(\mathfrak{p})}_\lambda) = \mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leqslant 0},\mathfrak{g}_0}((\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L^{(\mathfrak{p}_0)}_\lambda)),$$

where $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L^{(\mathfrak{p}_0)}_\lambda)$ is regarded as a $\mathrm{U}_q(\mathfrak{g}_{\leqslant 0})$-module with $E_{m+1,m}$ acting by zero. Another easy application of Lemma 5.2 completes the proof. □

By using the proposition we can easily prove the following result.

Theorem 5.2. *Let* $\mathrm{U}_q(\mathfrak{l}) \subseteq \mathrm{U}_q(\mathfrak{g}_0)$ *be a reductive quantum subalgebra of* $\mathrm{U}_q(\mathfrak{g})$. *Let* $\mathrm{U}_q(\mathfrak{p}) \supseteq \mathrm{U}_q(\bar{\mathfrak{b}})$ *be the parabolic quantum sub-superalgebra of* $\mathrm{U}_q(\mathfrak{g})$ *with* $\mathrm{U}_q(\mathfrak{l})$ *as its Levi factor. Let* $L^{(\mathfrak{p})}_\lambda$ *be a finite dimensional irreducible* $\mathrm{U}_q(\mathfrak{p})$*-module with* $\mathrm{U}_q(\mathfrak{l})$*-highest weight* $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$.

(1) *If* λ *is* $\mathfrak{g}$*-regular, then there exists a unique element* w *of the Weyl group of* $\mathfrak{g}_0$ *rendering* $\mu := w(\lambda+\rho)-\rho$ *dominant with respect to* $\mathfrak{g}$. *In this case,*

$$H^{0,k}(G/P, \mathscr{S}(L^{(\mathfrak{p})}_\lambda)) = \begin{cases} K^{(\mathfrak{g})}_\mu, & k = |w|, \\ 0, & k \neq |w|, \end{cases}$$

where $|w|$ *denotes the length of* w.

(2) *If* λ *is not* $\mathfrak{g}$*-regular, then*

$$H^{0,k}(G/P, \mathscr{S}(L^{(\mathfrak{p})}_\lambda)) = 0, \quad \forall k.$$

Proof. According to the quantum Bott–Borel–Weil theorem for quantized universal enveloping algebras of ordinary Lie algebras [1], the $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L^{(\mathfrak{p}_0)}_\lambda)$ vanishes for all k if λ is not $\mathfrak{g}_0$-regular. If λ is $\mathfrak{g}_0$-regular, then $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L^{(\mathfrak{p}_0)}_\lambda)$ is concentrated at on degree, namely, $(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^k(L^{(\mathfrak{p}_0)}_\lambda)$ is non-vanishing for one k only. We have

$$(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}})^{|w|}(L^{(\mathfrak{p}_0)}_\lambda) = L^{(\mathfrak{g}_0)}_\mu,$$

where $L^{(\mathfrak{g}_0)}_\lambda$ is the irreducible $\mathrm{U}_q(\mathfrak{g}_0)$-module with highest weight μ. Using this result in Proposition 5.1, we arrive at the theorem. □

Remark 5.2. An easy examination will show that the proof for Proposition 5.1 still goes through for $\mathrm{U}_q(\mathfrak{gl}_{m_1|n_1} \oplus \mathfrak{gl}_{m_2|n_2} \oplus \cdots \oplus \mathfrak{gl}_{m_i|n_i})$ for any finite i. The same comment applies to Theorem 5.2.

5.2.2. The general case

We investigate the general case in this subsection. Now $\mathrm{U}_q(\mathfrak{l})$ is an arbitrary reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{p})$ is the parabolic containing $\mathrm{U}_q(\bar{\mathfrak{b}})$ and has the Levi factor $\mathrm{U}_q(\mathfrak{l})$. Let $\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}}) = \mathrm{U}_q(\bar{\mathfrak{b}}) \cap \mathrm{U}_q(\mathfrak{l})$ be the Borel subalgebra of $\mathrm{U}_q(\mathfrak{l})$.

Denote by

$$\mathscr{F}^{\mathfrak{l},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} : \mathscr{C}(\mathfrak{p},\mathfrak{l}) \to \mathscr{C}(\mathfrak{l},\mathfrak{l}), \qquad \mathscr{F}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}} : \mathscr{C}(\bar{\mathfrak{b}},\mathfrak{h}) \to \mathscr{C}(\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h})$$

the forgetful functors. We have the following result.

Lemma 5.3. $\mathscr{F}^{\mathfrak{l},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}} = \mathrm{I}^{\mathfrak{l},\mathfrak{l}}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}} \circ \mathscr{F}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}}$.

Proof. The proof is much the same as that for (5.2). Because of the given conditions on $\mathrm{U}_q(\mathfrak{p})$ and $\mathrm{U}_q(\mathfrak{l})$, Eq. (4.3) gives

$$\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}})}(\mathrm{U}_q(\mathfrak{p}), V)[\mathrm{U}_q(\mathfrak{l})],$$

for any object V of $\mathscr{C}(\bar{\mathfrak{b}},\mathfrak{h})$. We can easily show that there exists the even $\mathrm{U}_q(\mathfrak{l})$-module isomorphism

$$P : \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}})}(\mathrm{U}_q(\mathfrak{p}), V) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}(\mathrm{U}_q(\mathfrak{l}), V)$$

defined by $\langle \zeta, ux\rangle = \pi_V(u)\langle P(\zeta), x\rangle$, for all $u \in \mathrm{U}_q(\bar{\mathfrak{b}})$, $x \in \mathrm{U}_q(\mathfrak{l})$ Therefore,

$$\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}(\mathrm{U}_q(\mathfrak{l}), V)[\mathrm{U}_q(\mathfrak{l})].$$

On the other hand,

$$\mathrm{I}^{\mathfrak{l},\mathfrak{l}}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}} \circ \mathscr{F}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}(\mathrm{U}_q(\mathfrak{l}), \mathscr{F}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}}(V))[\mathrm{U}_q(\mathfrak{l})].$$

Thus the claim of the lemma is indeed true for any object of $\mathscr{C}(\bar{\mathfrak{b}},\mathfrak{h})$. The claim also clearly holds true for morphisms of $\mathscr{C}(\bar{\mathfrak{b}},\mathfrak{h})$. □

Theorem 5.3. *Let* $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ *be* $\mathfrak{l}$*-dominant. Inflate* $K^{(\mathfrak{l})}_\lambda$ *to a* $\mathrm{U}_q(\mathfrak{p})$ *module by requiring that all the generators of* $\mathrm{U}_q(\mathfrak{p})$ *not contained in* $\mathrm{U}_q(\mathfrak{l})$ *act by zero, and denote the resultant* $\mathrm{U}_q(\mathfrak{p})$*-module by* $K^{(\mathfrak{p})}_\lambda$.

(1) *If* λ *is* $\mathfrak{g}$*-regular, then there exists a unique* w *in the Weyl group of* $\mathfrak{g}_0$ *rendering* $\mathfrak{g}$*-dominant the following weight* $\mu := w(\lambda+\rho)-\rho$. *In this case,*

$$H^{0,k}(G/P, \mathscr{S}(K^{(\mathfrak{p})}_\lambda)) = \begin{cases} K^{(\mathfrak{g})}_\mu, & k = |w|, \\ 0, & k \neq |w|. \end{cases}$$

(2) *If* λ *is not* $\mathfrak{g}$*-regular, then* $H^{0,k}(G/P, \mathscr{S}(K^{(\mathfrak{p})}_\lambda)) = 0, \ \forall k$.

Proof. We use Lemma 4.3 to write $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\bar{\mathfrak{b}},\mathfrak{h}} = \mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}$. The functor $\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}$ takes injectives to injectives. Thus for an irreducible $\mathrm{U}_q(\bar{\mathfrak{b}})$-module $\mathbb{C}(q)_\lambda$ with an arbitrary weight $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we have a first quadrant spectral sequence, the Grothendieck spectral sequence (Sections 5.8 and 10.8 of [22]),

$$E^{p,q}_r \ \Rightarrow \ (\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\bar{\mathfrak{b}},\mathfrak{h}})^{p+q}(\mathbb{C}(q)_\lambda)$$

with $E^{p,q}_2$ term

$$E^{p,q}_2 = (\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}})^p(\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}})^q(\mathbb{C}(q)_\lambda),$$

where the differential on $E_r^{p,q}$ has bi-degree $(r, 1-r)$. We shall prove below that $(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^q(\mathbb{C}(q)_\lambda)$ is concentrated at one degree. Let us take this as granted for the moment. Then the spectral sequence collapses at E_2, and we obtain

$$(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{g},\mathfrak{g}})^{p+q}(\mathbb{C}(q)_\lambda) = (\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}})^p(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^q(\mathbb{C}(q)_\lambda). \tag{5.7}$$

Now we consider $(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^q(\mathbb{C}(q)_\lambda)$ for arbitrary $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$. By Lemma 5.3, we have

$$\mathscr{F}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{l},\mathfrak{l}} \circ (\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^q(\mathbb{C}(q)_\lambda) = (\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}})^q \circ \mathscr{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(\mathbb{C}(q)_\lambda). \tag{5.8}$$

Note that $\mathrm{U}_q(\mathfrak{l})$ is the tensor product of the quantized universal enveloping algebras of the direct sum of some general linear algebras and possibly also a general linear superalgebra. By Theorem 5.2 and Remark 5.2, the right-hand side is zero unless λ is $\mathfrak{l}$-regular. When λ is $\mathfrak{l}$-regular, $(\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}})^q \circ \mathscr{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(\mathbb{C}(q)_\lambda)$ is concentrated at one degree. Explicitly, there exists a unique $w_{\mathfrak{l}}$ in the Weyl group of $\mathfrak{l}$ rendering $w_{\mathfrak{l}}(\lambda + \rho_{\mathfrak{l}}) - \rho_{\mathfrak{l}}$ dominant with respect to $\mathfrak{l}$, and we have

$$(\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}})^{|w_{\mathfrak{l}}|} \circ \mathscr{F}_{\mathfrak{b},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(\mathbb{C}(q)_\lambda) = K_{w_{\mathfrak{l}}(\lambda+\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}}^{(\mathfrak{l})}.$$

Here $\rho_{\mathfrak{l}}$ is half of the signed-sum of the positive roots of $\mathfrak{l}$ relative to $\mathfrak{b}_{\mathfrak{l}} = \mathfrak{b} \cap \mathfrak{l}$. Needless to say, the formula remains valid if we replace $\rho_{\mathfrak{l}}$ by ρ.

In order to determine $(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^q(\mathbb{C}(q)_\lambda)$, we consider all the possible objects W_λ of $\mathscr{C}(\mathfrak{p},\mathfrak{l})$ satisfying $\mathscr{F}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{l},\mathfrak{l}}(W_\lambda) = K_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}^{(\mathfrak{l})}$. Any two weights of $K_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}^{(\mathfrak{l})}$ can only differ by an integral combination of the roots of $\mathfrak{l}$. This in particular requires that all the generators of $\mathrm{U}_q(\mathfrak{p})$ not contained in $\mathrm{U}_q(\mathfrak{l})$ act on W_λ by zero. Therefore,

$$(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}})^{|w_{\mathfrak{l}}|}(\mathbb{C}(q)_\lambda) = K_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}^{(\mathfrak{p})}.$$

By using the given condition that λ is $\mathfrak{l}$-dominant, we obtain from (5.7)

$$(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}})^k(K_\lambda^{(\mathfrak{p})}) = (\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{g},\mathfrak{g}})^k(\mathbb{C}(q)_\lambda).$$

Using the special case of Theorem 5.2 with the parabolic being $\mathrm{U}_q(\bar{\mathfrak{b}})$, we complete the proof. □

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