Evaluation-functional-preserving maps

Nizar Jaoua\textsuperscript{a,}\textsuperscript{*}, Haïkel Skhiri\textsuperscript{b}

\textsuperscript{a}Faculté des Sciences de Gabès, Département de Mathématiques, Cité Erriadh, 6072 Zrig Gabès, Tunisia
\textsuperscript{b}Faculté des Sciences de Monastir, Département de Mathématiques, Avenue de l’environnement, 5019 Monastir, Tunisia

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Abstract

Given any space of holomorphic functions in the open unit disc \(D\), satisfying certain conditions, we characterize the self-mappings of its algebraic dual space which preserve the set of all evaluation functionals \(\delta_z\). Among these maps, we give a description of those which contract the norm and those which preserve it. In the case where the norm \(\|\delta_z\|\) depends strictly increasingly on \(|z|\), we show that the first ones arise exactly from the self-maps of \(D\) vanishing at 0. When this dependence is only injective, we prove that the second ones are precisely induced by the rotations of \(D\). We provide a nice generalization of those results in the case where \(\|\delta_z\|\) grows with \(|\theta(z)|\), for a given automorphism \(\theta\) of \(D\).

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1. Introduction

Let \(X = \mathcal{H}(D)\) be the algebra of holomorphic functions on the open unit disc \(D\). Equipped with the topology of the uniform convergence in every compact subset of \(D\) (KUC topology),

\textsuperscript{*} Corresponding author.

E-mail addresses: nizar.jaoua@gmail.com (N. Jaoua), haikel.skhiri@gmail.com (H. Skhiri).
X appears as a Banach space. For a given point $z \in D$, the evaluation functional at $z$, denoted by $\delta_z$, is defined on $X$ by $\delta_z(f) = f(z)$. Clearly, such a map is in the topological dual space $X'$. For a given subspace $Y$ of $X$, we denote by $Y^*$ the algebraic dual space; i.e. the space of all linear functionals on $Y$ and by $\mathcal{F}_e(Y)$ the subset of $Y^*$ consisting of all evaluation functionals. Several estimates and identities involving members of $Y$ can be seen as formulations of specific properties about members of $\mathcal{F}_e(Y)$. To quote a simple example, one can think about the famous Schwarz lemma, providing an inequality together with its equality case, and consider the subspace $Y$ of all bounded members of $X$ vanishing at 0. Such formulations generate an interplay between complex-function theory and operator theory and therefore can explain in part our concern with the set $\mathcal{F}_e(Y)$.

Note that most of the works about this class have been carried out in the same direction. Indeed, their authors have focused on boundedness, estimation and possible computation of the norm, for well-known Banach spaces (see e.g. [2, p. 18]). Here we take a new direction, but we make use of some of those results. Our contribution is motivated by the main idea of some other works: characterizing the maps preserving specific topics related to the operator theory. For example, in [5,1,11] and more recently in [8], the authors have studied this problem, respectively, for the spectrum, the spectral radius, the invertibility and the generalized spectrum. Too recently in [10], the second author has tackled this question for the reduced minimum modulus.

Here, we target two basic questions:

**Question 1.** What are the maps from $Y^*$ into itself that leave $\mathcal{F}_e(Y)$ invariant? Such maps will be called $\mathcal{F}_e(Y)$-preservers.

**Question 2.** Among those maps, in the case where $\mathcal{F}_e(Y) \subset Y'$, what are those which restrictions to $\mathcal{F}_e(Y)$ contract (respectively, preserve) the norm? Such preservers will be denoted by $\mathcal{F}_e(Y)$-nc (respectively, $\mathcal{F}_e(Y)$-np).

We find it reasonable to investigate Question 1 with a practical identification of $\mathcal{F}_e(Y)$. This will be the aim of Section 2 in which we shall describe this class by the multiplicativity property under well-specified conditions on $Y$.

On the other hand, by considering the map $C_\varphi : f \mapsto f \circ \varphi$ where $\varphi$ is a given holomorphic self-map of $D$, we get an operator taking $X$ into itself, called composition operator with symbol $\varphi$. Therefore, any member of $\mathcal{F}_e(Y)$ can be seen as a special $C_\varphi$ leaving $Y$ invariant. Under those same conditions on $Y$, those $C_\varphi$ will be described the same way as $\mathcal{F}_e(Y)$.

In Section 3, using the adjoint operators of those $C_\varphi$, we will provide a complete answer to Question 1, for the maps $\Phi$ with associated operators $T_\Phi$ leaving $Y$ invariant ($(T_\Phi f)(z) = (\Phi \delta_z)f$ for all $z \in D$). In particular, we will see that, restricted to $\mathcal{F}_e(Y)$, each of them is nothing else but the restriction to $\mathcal{F}_e(Y)$ of one $C_{\varphi}^*$. That is why $\varphi$ will be also considered as the symbol of the $\mathcal{F}_e(Y)$-preserver $\Phi$.

The final section will be devoted to the study of the Question 2. We shall start by considering the Hardy spaces on which every $\delta_z$ is bounded with a well-determined expression for the norm. This will be outlined in Theorem A. The $\mathcal{F}_e(Y)$-nc’s will be described as the $\mathcal{F}_e(Y)$ preservers with symbols fixing 0 and the $\mathcal{F}_e(Y)$-np’s as those with rotation symbols.
We will then extend those results to all subspaces \( Y \) with \( \mathcal{F}_e(Y) \subset Y' \) such that \( \|\delta_z\| \) depends on \( |z| \) strictly increasing for the \( \mathcal{F}_e(Y) \)-nc’s and injectively for the \( \mathcal{F}_e(Y) \)-np’s. For more general subspaces on which \( \|\delta_z\| \) depends injectively on \( |\theta(z)| \) where \( \theta \in X \), we shall handle all the \( \mathcal{F}_e(Y) \)-np’s by using the analytic extension principle. In particular, when \( \theta \) is an automorphism, we will obtain a nice generalization of the corresponding result given in the rotation case. For the \( \mathcal{F}_e(Y) \)-nc’s, we will provide an analogous extension when \( \|\delta_z\| \) grows with \( |\theta(z)| \). At the end, we will present a general case where \( \|\delta_z\| \) depends on \( |z| \) but not injectively. Also there, we will find that, among all symbols fixing 0, only rotations can induce the \( \mathcal{F}_e(Y) \)-np. This will be an easy consequence of a more general result we will show, giving “innerness” of the symbol as a necessary condition for it to induce an \( \mathcal{F}_e(Y) \)-np. In addition, under a slight restriction of that general case, we will determine all the \( \mathcal{F}_e(Y) \)-nc’s the same way as when the dependence is strictly increasing.

Throughout this paper, the monomial \( z \rightarrow z^n \), for all integer \( n \geq 1 \), will be denoted by \( p_n \) and the constant function, taking 1 as value, will be denoted by \( p_0 \).

### 2. Characterization of \( \mathcal{F}_e(Y) \)

We open this study with a characterization of \( \mathcal{F}_e(X) \) as a part of \( X' \).

**Theorem 2.1.** Let \( \tau \in X' \). The following are equivalent.

1. \( \tau \in \mathcal{F}_e(X) \);
2. \( \tau(p_0) = 1 \) and \( \tau(fg) = \tau(f)\tau(g) \) for all \( f, g \in X \);
3. \( \tau \neq 0 \) and \( \tau(fg) = \tau(f)\tau(g) \) for all \( f, g \in X \).

**Proof.** (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are immediate.

(3) \( \Rightarrow \) (2) Apply (3) with \( f = g = p_0 \) to get \( \tau(p_0) = (\tau(p_0))^2 \) which gives \( \tau(p_0) = 1 \), since otherwise \( \tau(p_0) = 0 \) and this leads to the contradiction \( \tau = 0 \), by writing \( \tau(f) = \tau(fp_0) = \tau(f)\tau(p_0) \).

(2) \( \Rightarrow \) (1) Set \( a = \tau(p_1) \). So one has \( \tau(p_n) = a^n \) for all \( n \geq 1 \). Since \( p_n \overset{\text{KUC}}{\to} 0 \), by continuity of \( \tau \), one must have \( \lim_{n \to +\infty} a^n = 0 \) and hence \( a \in D \). On the other hand, \( \tau \) coincides with \( \delta_a \) on the subspace of polynomials which is dense in \( X \). Therefore, as both of them are continuous, one obtains \( \tau = \delta_a \in \mathcal{F}_e(X) \). \( \square \)

**Remark 1.** Actually, any linear functional (not supposed to be continuous) which satisfies (2) or equivalently (3) is necessarily in \( \mathcal{F}_e(X) \) (see just below) and then continuous. So, Theorem 2.1 can be extended to \( X^* \). Here is another proof of (2) \( \Rightarrow \) (1) without assuming the continuity of \( \tau \).

We first verify that \( a := \tau(p_1) \in D \). If the opposite were true, then the function \( g := 1/(p_1 - ap_0) \) would be in \( X \). But this would lead to the contradiction \( \tau(p_0) = \tau(g(1/g)) = \tau(g)\tau(1/g) = 0 \). On the other hand, for any \( f \in X \), consider \( f_a \in X \) defined by \( f_a(z) = (f(z) - f(a))/(z - a) \) if \( z \in D \setminus \{a\} \) and \( f_a(a) = f'(a) \). Since \( f = (p_1 - ap_0)f_a + f(a)p_0 \), one gets

\[
\tau(f) = \tau(p_1 - ap_0)f_a + f(a)\tau(p_0) = f(a) = \delta_a(f),
\]

and we are done.
Now, a natural question is: what is the generalization of Theorem 2.1 to $Y^a$ where $Y$ is any subspace of $X$?

First observe that, in the last proof, we used the fact that $p_0, p_1 \in X$, $1/(p_1 - ap_0) \in X$ for all $a \in \mathbb{C} \setminus D$ and $f_a, p_1 f_a \in X$ whenever $f \in X$ and $a \in D$. This leads to the suggestion of the following conditions on $Y$:

1. $Y$ is invariant under the multiplication by $p_1$ and under the maps $T_a : f \mapsto f_a$, with $a \in D$.
2. For all $a \in \mathbb{C} \setminus D$, $Y$ contains an $N$th root ($N \geq 1$) of $1/(p_1 - ap_0)$.
3. $Y$ contains all bounded analytic functions.

In fact, condition (2) is a general version of the fact that $1/a_1$ is invariant under the multiplication by $a_1$.

Examples. 1. The whole space $X$. All of those conditions are definitely satisfied.

2. The Hardy spaces $H^p$ with $1 \leq p < \infty$. We recall that the Hardy space $H^p$ with $0 < p < \infty$ is the subspace of $X$ consisting of all functions $f$ such that

$$
\| f \|_p := \left( \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.
$$

When $1 \leq p \leq \infty$, such an amount defines a norm for which $H^p$ is a Banach space. (3) is clearly satisfied. For any $a \in \mathbb{C} \setminus D$, $1/(p_1 - ap_0) \in H^p$, for all $0 < q < 1$, in particular for $q = p/N$ with $N = [p] + 1$ where $[p]$ denotes the integer part of $p$. With this $N$, (2) clearly holds. The first part of (1) follows from the inequality $|p_1 f| \leq |f|$. To show the second part, let $a \in D$ and $r$ be arbitrary such that $(1 + \lvert a \rvert)/2 < r < 1$. One has

$$
|f_a(re^{i\theta})| \leq \frac{1}{r - |a|} |f(re^{i\theta}) - f(a)|
$$

so that

$$
\frac{1}{2\pi} \int_0^{2\pi} |f_a(re^{i\theta})|^p \, d\theta \leq \left( \frac{2}{1 - |a|} \right)^p \| f - f(a) \|^p.
$$

As $f_a \in X$, this implies that $f_a \in H^p$ with $\| f_a \| \leq 2/(1 - |a|)\| f - f(a) \|$.

3. The standard and weighted Bergman spaces $A^p$ and $A^{p,x}$ with $1 \leq p < \infty$ and $\alpha > -1$. Let $dA$ denote the normalized Lebesgue measure on $D$; i.e., $dA(z) = (1/\pi) \, dx \, dy = (1/r) \, dr \, d\theta$. $A^p$ is the space of all $f \in X$ such that

$$
\| f \|^p := \int_D |f(z)|^p \, dA(z) < \infty.
$$
This gives a norm for which $A^p$ is a Banach space. More generally, $A^p_\mathbb{D}$ is the Banach subspace of $X$ where the norm is given by

$$
\|f\|_{(p)} := \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z).
$$

Such a space satisfies (3) as the weight function $z \mapsto (1 - |z|^2)^\alpha$ is integrable on $\mathbb{D}$. The first part of (1) clearly holds. To get the second part, for any $a \in \mathbb{D}$, take one $0 < r < 1 - |a|$. We denote by $\overline{D}(a, r)$ the closed disc centered at $a$ with radius $r$. From

$$
|f_a(z)|^p \leq \frac{1}{r^p} |f(z) - f(a)|^p \quad \text{for all } z \in \mathbb{D} \setminus \overline{D}(a, r),
$$

we get the integrability of the function $z \mapsto |f_a(z)|^p (1 - |z|^2)^\alpha$ on $\mathbb{D} \setminus \overline{D}(a, r)$ and then its integrability on the whole disc $\mathbb{D}$ since it is bounded on $\overline{D}(a, r)$. As $f_a \in X$, this means that $f_a \in A^p_\mathbb{D}$.

To show (2), observe that for all $a \in \mathbb{C} \setminus \overline{D}$, $1/(p_1 - ap_0)$ is in $A^p_\mathbb{D}$ since it is in $X$ and bounded. Now, let $a$ on the unit circle. In the case where $\alpha \geq 0$, use the polar coordinates and take any $N \geq [p] + 2$ to ensure the integrability of $(1/(p_1 - ap_0))^{p/N}$ and then to have the $N$th root of $1/(p_1 - ap_0)$ in $A^p_\mathbb{D}$. In the remaining case $-1 < \alpha < 0$, there exists $0 < r < 1$ small enough and $c_\alpha > 0$ such that

$$
|z - a| \leq c_\alpha (1 - |z|) \quad \text{for all } z \in \overline{D}(a, r).
$$

Hence, for all integer $N \geq 1$, it follows that

$$
\frac{1}{|z - a|^{p/N}} \leq c_\alpha^{-\alpha} \frac{1}{|z - a|^{(p/N) - \alpha}} \quad \text{for all } z \in \overline{D}(a, r).
$$

From this, one can deduce the integrability of the left side of the last inequality on $\overline{D}(a, r)$, whenever $N \geq [p/(\alpha + 2)] + 1$. On the remaining part of $\mathbb{D}$, this same function is also integrable since $1/(p_1 - ap_0)^{p/N}$ is bounded and the weight function is integrable. Therefore, for those $N$, the $N$th root of $1/(p_1 - ap_0)$ is in $A^p_\mathbb{D}$. Consequently, each weighted Bergman space satisfies (2).

In the sequel, $Y$ denotes any subspace of $X$ satisfying (1)–(3). The following result provides a generalization of Theorem 2.1 to the space $Y^*$.

**Theorem 2.2.** Let $\tau \in Y^*$. The following are equivalent.

1. $\tau \in \mathcal{F}_\tau(Y)$;
2. $\tau(p_0) = 1$ and $\tau(fg) = \tau(f)\tau(g)$ for all $f, g \in Y$ such that $fg \in Y$;
3. $\tau \neq 0$ and $\tau(fg) = \tau(f)\tau(g)$ for all $f, g \in Y$ such that $fg \in Y$.

**Proof.** Here, we only need to show $(2) \implies (1)$. We get $(1) \implies (2) \iff (3)$ the same way as previously. Once again, by defining $a = \tau(p_1)$, one gets a point of $\mathbb{D}$. Indeed, suppose the contrary, i.e. $|a| \geq 1$. Then, according to (2), there exists $N \geq 1$ and $h \in Y$ such that
There is a unique $C$ such a situation, we have to consider the composition operator $p_2$. Since
\[ \frac{1}{h^N} \] and then $1/h$ are bounded in $D$ so that $Y$ contains them according to (3), one can write
\[ \tau \left( \frac{1}{h^N} \right) = \tau(p_1 - a p_0) = 0. \]
This implies that $\tau(1/h) = 0$. Consequently,
\[ \tau(p_0) = \tau \left( \frac{1}{h} \right) = \tau(h) \tau \left( \frac{1}{h} \right) = 0, \]
which is in contradiction with (2). We achieve the proof as in the second one of Theorem 2.1, since $p_0$, $f_a$ and $p_1 f_a$ belong to $Y$ for all $f \in Y$, according to (1) and (3).

\[ \square \]

**Remark 2.** It is not difficult to see that the evaluation functionals on a given subspace $Y$ of $X$ are continuous if and only if $Y$ is equipped with a topology stronger than the kuc one. In this case, by the previous theorem, any linear functional on $Y$ satisfying (1) or (2) is necessarily continuous.

Now, one can restrict the evaluation to the range of a given analytic self-map $\varphi$ of $D$. In such a situation, we have to consider the composition operator $C_\varphi$ sending every $f \in X$ into $f \circ \varphi : z \mapsto f(\varphi(z)) = \delta_{\varphi(z)} f$. Notice that evaluation functionals are constant-symbol composition operators. For the general ones, we have the following characterization.

**Theorem 2.3.** Let $T : Y \rightarrow Y$ be a linear map. The following are equivalent.

1. There is a unique $\varphi \in Y$ with $\varphi(D) \subseteq D$ such that $T = C_\varphi$.
2. $T(p_0) = p_0$ and $T(fg) = T(f)T(g)$ for all $f, g \in Y$ such that $fg \in Y$.
3. $T \neq 0$ and $T(fg) = T(f)T(g)$ for all $f, g \in Y$ such that $fg \in Y$.

**Proof.** (1) $\implies$ (2) and (2) $\implies$ (3) are obvious.

(3) $\implies$ (2) This can be shown exactly the same way as (3) $\implies$ (2) in Theorem 2.1 by replacing $\tau$ with $T$.

(2) $\implies$ (1) Let $z$ be arbitrary in $D$. By defining $\tau = \delta_z T$, one gets a linear functional on $Y$ satisfying (2) of Theorem 2.2. So, according to this theorem, there is a unique $w \in D$ such that $\tau = \delta_w$. Now let $\varphi$ be the self-map of $D$ sending $z$ into $w$. Since

$\varphi(z) = w = \delta_w(p_1) = \delta_z T(p_1) = T(p_1)(z)$

and $z$ is arbitrary in $D$, it follows that $\varphi = T(p_1) \in Y$. On the other hand, for all $f \in Y$ and $z \in D$, one has

$T f(z) = \delta_z T f = \tau f = \delta_w f = f(w) = f(\varphi(z)) = C_\varphi f(z)$.

This means that $T = C_\varphi$. To show the uniqueness of $\varphi$, assume that $T = C_{\psi}$ with $\psi \in Y$ and $\psi(D) \subseteq D$. Since $p_1 \in Y$, it follows that $\psi = C_{\psi}(p_1) = C_\varphi(p_1) = \varphi$. This completes the proof. \[ \square \]
3. $\mathcal{F}_e(Y)$-preservers

We recall that any operator $T : Y \rightarrow Y$, has an adjoint operator denoted by $T^*$ and defined on $Y^*$ by $T^*(z) = zT$. Conversely, for a given map $\Phi : Y^* \rightarrow Y^*$ (not necessarily linear), we introduce the associated operator $T_\Phi$ on $Y$ by $(T_\Phi f)(z) = \Phi(\delta_z f)$ for all $z \in D$. In contrast with $T^*$, leaving the source space invariant, $T_\Phi$ may not send $Y$ into itself nor into $X$, even though $T_{T^*} = T$. However, in the case where $Y$ is left invariant under $T_\Phi$, one can observe that $T_\Phi^*$ agrees with $\Phi$ on $\mathcal{F}_e(Y)$. The following formula says how $T^*$ acts on $\mathcal{F}_e(Y)$ when $T = C_\phi$.

**Proposition 3.1.** Let $\phi \in Y$ with $\phi(D) \subseteq D$ and $C_\phi(Y) \subseteq Y$. For all $z \in D$, we have

$$C_\phi^*(\delta_z) = \delta_{\phi(z)}.$$  

**Proof.** For all $f \in Y$, one has

$$(C_\phi^*\delta_z)(f) = \delta_z(C_\phi f) = \delta_z(f \circ \phi) = f(\phi(z)) = \delta_{\phi(z)}(f)$$

and this gives the desired formula. \qed

The following result characterizes the composition operators on $Y$ in terms of their adjoint operators.

**Theorem 3.2.** Let $T : Y \rightarrow Y$ be a linear map. The following are equivalent.

1. There is a unique $\phi \in Y$ with $\phi(D) \subseteq D$ such that $T = C_\phi$.
2. $T^*(\mathcal{F}_e(Y)) \subseteq \mathcal{F}_e(Y)$.

**Proof.** (1) $\Rightarrow$ (2) This follows immediately from Proposition 3.1.

(2) $\Rightarrow$ (1) We will use the characterization of $C_\phi$ by (2) of Theorem 2.3. First of all, notice that, for every $z \in D$, one has $T_\Phi^*\delta_z \in \mathcal{F}_e(Y)$ so that $T_\Phi^*\delta_z(p_0) = 1$.

But, for all $z \in D$, $T(p_0)(z) = \delta_z(T(p_0)) = T_\Phi^*\delta_z(p_0) = 1$ and hence $T(p_0) = p_0$. On the other hand, for all $f, g \in Y$ such that $fg \in Y$ and all $z \in D$, one has

$$T(fg)(z) = \delta_z T(fg) = T_\Phi^*\delta_z(fg) = (T_\Phi^*\delta_z f)(T_\Phi^*\delta_z g)$$

$$= (\delta_z T f)(\delta_z T g) = (T f)(T g)(z) = (T f T g)(z).$$

The third equality is due to the fact that $T_\Phi^*\delta_z \in \mathcal{F}_e(Y)$. It follows that $T(fg) = T f T g$ and we conclude by Theorem 2.3. \qed

Now we reach the main result of this section, giving a characterization of all maps from $Y^*$ into itself, leaving $\mathcal{F}_e(Y)$ invariant, with associated operators on $Y$ leaving $Y$ invariant too.

**Theorem 3.3.** Let $\Phi$ be a map from $Y^*$ into itself such that $T_\Phi(Y) \subseteq Y$. The following are equivalent.

There is a unique \( \phi \in Y \) such that \( \phi(D) \subseteq D \), \( C_\phi(Y) \subseteq Y \) and \( \Phi = C_\phi^* + \Psi \) with \( \Psi : Y^* \to Y^* \) such that \( \Psi(F_e(Y)) = \{0\} \).

There is a unique \( \phi \in Y \) such that \( \phi(D) \subseteq D \), \( C_\phi(Y) \subseteq Y \) and \( \Phi|_{F_e(Y)} = (C_\phi^*)|_{F_e(Y)} \).

(3) \( \Phi(F_e(Y)) \subseteq F_e(Y) \).

**Proof.** (1) \( \iff \) (2) This is immediate.

(2) \( \iff \) (1) By setting \( \Psi := \Phi - C_\phi^* \), we define a map from \( Y^* \) into itself sending \( F_e(Y) \) onto \( \{0\} \).

(2) \( \iff \) (3) According to Theorem 3.2, \( C_\phi^*(F_e(Y)) \subseteq F_e(Y) \). As \( \Phi \) agrees with \( C_\phi^* \) on \( F_e(Y) \), one also has the same inclusion with \( \Phi \) instead of \( C_\phi^* \).

(3) \( \iff \) (2) As \( T_\Phi(Y) \subseteq Y \), one can see that \( T_\Phi^* \) coincides with \( \Phi \) on \( F_e(Y) \), so that (3) yields \( T_\Phi^*(F_e(Y)) \subseteq F_e(Y) \). Therefore, again by Theorem 3.2, applied with the linear map \( T_\Phi \), there is a unique \( \phi \in Y \) with \( \phi(D) \subseteq D \) such that \( T_\Phi = C_\phi \). This completes the proof. \( \square \)

**Remark 3.** In Theorem 3.3, \( \Psi \) is not necessarily zero everywhere. To give an example, define \( \Phi \) by \( \Phi(\tau) = \tau(I + C_\phi) - \delta_{\epsilon(p_1)} \) with \( \phi \) as in this theorem and \( \Psi \) by \( \Psi(\tau) = \tau - \delta_{\epsilon(p_1)} \). Since \( \Phi \) can be written as in (1) of this theorem, it leaves \( F_e(Y) \) invariant. Notice here that \( \Psi \neq 0 \) as \( \Psi(0) = -\delta_0 \neq 0 \).

In the sequel, \( \Phi \) is supposed to be a self-mapping of \( Y^* \) such that \( T_\Phi(Y) \subseteq Y \) and \( \Phi(F_e(Y)) \subseteq F_e(Y) \). According to Theorem 3.3, \( \Phi \) is determined by a unique \( \phi \in Y \) such that \( \phi(D) \subseteq D \), \( C_\phi(Y) \subseteq Y \) and \( \Phi|_{F_e(Y)} = (C_\phi^*)|_{F_e(Y)} \). For this reason, \( \Phi \) will be called an \( F_e(Y) \)-preserver induced by the symbol \( \phi \).

**Corollary 3.4.** The following are equivalent.

(1) \( \Phi \) has a fixed point in \( F_e(Y) \);

(2) \( \phi \) has a fixed point in \( D \).

**Proof.** Since \( \delta_{\phi(a)} = C_\phi^*(\delta_a) = \Phi(\delta_a) \) for all \( a \in D \), one can deduce that \( \Phi(\delta_a) = \delta_a \) if and only if \( \phi(a) = a \). \( \square \)

**4.** \( F_e(Y) \)-nc’s and \( F_e(Y) \)-np’s

Next, in the case where \( F_e(Y) \subseteq Y' \), we say that a given \( F_e(Y) \)-preserver \( \Phi \) is \( F_e(Y) \)-norm-contracting (respectively, \( F_e(Y) \)-norm-preserving) and we write \( F_e(Y) \)-nc (respectively, \( F_e(Y) \)-np), if

\[
\|\Phi(\delta_z)\| \leq \|\delta_z\| \quad \text{(respectively, } \|\Phi(\delta_z)\| = \|\delta_z\|) \quad \text{for all } z \in D.
\]

We are going to determine those maps for the Hardy spaces and for more general ones. For the sake of simplicity, we enumerate the following statements as follows:

(\( C_1 \)) \( \Phi \) is an \( F_e(Y) \)-nc.

(\( C_2 \)) \( \|\Phi(\delta_0)\| \leq \|\delta_0\| \).
\((C_3)\) \(\varphi\) fixes 0.
\((P_1)\) \(\Phi\) is an \(\mathcal{F}_+(\mathcal{Y})\)-np.
\((P_2)\) \(\|\Phi(\delta_0)\| = \|\delta_0\|\) and \(\|\Phi(\delta_a)\| = \|\delta_a\|\) for one \(a \in D\setminus\{0\}\).
\((P_3)\) \(\varphi\) is a rotation of \(D\).

The following result about the evaluation functionals on \(\mathcal{H}^p\) can be found in [2, p. 18].

**Theorem A.** Let \(1 \leq p < \infty\). For all \(z \in D\), \(\delta_z\) is bounded on \(\mathcal{H}^p\) with

\[
\|\delta_z\| = (1 - |z|^2)^{-1/p}.
\]

**Theorem 4.1.** In the case where \(\mathcal{Y} = \mathcal{H}^p\) with \(1 \leq p < \infty\), we have equivalence among \((C_1)\), \((C_2)\) and \((C_3)\) and among \((P_1)\), \((P_2)\) and \((P_3)\).

**Proof.** \((C_1) \Rightarrow (C_2)\) and \((P_1) \Rightarrow (P_2)\) are obvious.

\((P_3) \Rightarrow (P_1)\) This follows from a straight verification using Theorem A.

\((C_2) \Rightarrow (C_3)\) Thanks again to this theorem, one has

\[
(1 - |\varphi(0)|^2)^{-1/p} = \|\delta_{\varphi(0)}\| = \|\Phi(\delta_0)\| \leq \|\delta_0\| = 1.
\]

But \(0 \leq |\varphi(0)| < 1\). Thus \(\varphi(0) = 0\).

\((C_3) \Rightarrow (C_1)\) Since \(\varphi(D) \subseteq D\), by Schwarz lemma, one has \(|\varphi(z)| \leq |z|\) for all \(z \in D\).

Therefore,

\[
\|\Phi(\delta_z)\| = \|\delta_{\varphi(z)}\| = (1 - |\varphi(z)|^2)^{-1/p} \leq (1 - |z|^2)^{-1/p} = \|\delta_z\|.
\]

\((P_2) \Rightarrow (P_3)\) As \((C_2)\) follows from the first part of \((P_2)\), one necessarily has \(\varphi(0) = 0\).

Moreover, thanks to Theorem A, the second part of \((P_2)\) implies that \(|\varphi(a)| = |a|\) for one \(a \in D\setminus\{0\}\). Therefore, \((P_3)\) follows from the equality case in Schwarz lemma. \(\square\)

**Remark 4.** 1. Theorem 4.1 also holds for the spaces \(A^2_{\mathcal{H}}\). A similar proof can be made as \(\delta_z\) is bounded and its norm has a similar expression: \(\|\delta_z\|^2 = (\alpha + 1)(1 - |z|^2)^{-\alpha-2}\). This assertion can be deduced from an exercise in [2, p. 27]. For the other weighted Bergman spaces, although boundedness of \(\delta_z\) can be shown thanks to the reproducing kernels of the Hilbert space \(A^2_{\mathcal{H}}\), the exact value of its norm seems to be still unknown.

2. We recall that \((C_3)\) is sufficient for \(C_{\varphi}\) to be a contraction on \(\mathcal{H}^p\) with \(1 \leq p < \infty\). This is a special case of Littlewood subordination principle (see [3, p. 10, 7]). Using the same argument, one can confirm this for all the spaces \(A^p_{\mathcal{H}}\). So one can deduce, from Theorem 4.1 (with \(C^p_{\varphi}\) instead of \(\Phi\)), the sufficiency of \((C_2)\) for \(C^p_{\varphi}\) to be a contraction on \((\mathcal{H}^p)'\) and \((A^2_{\mathcal{H}})'\) as well. Note that even though \(A^2_{\mathcal{H}}\) is a special weighted version of \(\mathcal{H}^2\), for general ones called weighted Hardy spaces, the sufficiency of \((C_3)\) may not occur. Too recently in [6], the first author has carried out a large study of this problem.

3. As rotations induce onto isometric \(C_{\varphi}\)‘s on \(\mathcal{H}^p\) and \(A^p_{\mathcal{H}}\), it follows from Theorem 4.1 (with \(C^p_{\varphi}\) instead of \(\Phi\)) that \((P_2)\) is sufficient for \(C^p_{\varphi}\) to be an isometry on \((\mathcal{H}^p)'\) and \((A^2_{\mathcal{H}})'\). On the other hand, one can easily see, from this theorem, that any onto isometric \(C_{\varphi}\) on \(\mathcal{H}^p\) or \(A^2_{\mathcal{H}}\) is necessarily induced by a rotation, avoiding therefore using the characterization of bijective \(C_{\varphi}\)‘s which can be found in [4].
Now as each functional $\delta_z$ is bounded on the spaces given in Theorem 4.1 and its remarks, with norm depending strictly increasingly on $|z|$, one can expect the extension of that theorem to more general normed spaces sharing this property. But first, in the following, we characterize such spaces among those which norm topology is stronger than the KUC one, or equivalently, those $Y$ such that $\mathcal{F}_e(Y) \subset Y'$, without paying attention to the injectivity of this dependence. In the sequel, $Y$ is supposed to be endowed with this kind of norm topology.

**Theorem 4.2.** The following are equivalent.

1. There is a non-decreasing positive (respectively, positive) function $h$ on $[0,1)$ such that $\|\delta_z\| = h(|z|)$, for all $z \in D$.
2. Any symbol fixing 0 (respectively, rotation of $D$) induces an $\mathcal{F}_e(Y)$-nc (respectively, $\mathcal{F}_e(Y)$-np).

**Proof.** (1) $\implies$ (2) The first part follows from Schwarz lemma and the non-decrease of $h$. Whereas the second is due to the fact that any rotation of $D$ preserves the modulus.

(2) $\implies$ (1) Given two arbitrary distinct points $a$ and $b$ in $D$ such that $|a| = |b|$. One necessarily has $a \neq 0$ and $b \neq 0$. Set $\lambda = b/a$ and consider the rotations of $D : \varphi_1 = \lambda p_1$ and $\varphi_2 = \bar{z} p_1$. We denote by $\Phi_1$ and $\Phi_2$ the induced $\mathcal{F}_e(Y)$-preservers of $\varphi_1$ and $\varphi_2$, respectively. Applying the second part of (2) with $\varphi_1$ gives

$$\|\delta_b\| = \|\delta_{\varphi_1(a)}\| = \|\Phi_1(\delta_a)\| = \|\delta_a\|$$

and thus the second part of (1). However, the first part of (2) applied, respectively, with $\varphi_1$ and $\varphi_2$ gives

$$\|\delta_b\| = \|\Phi_1(\delta_a)\| \leq \|\delta_a\| \quad \text{and} \quad \|\delta_a\| = \|\Phi_2(\delta_b)\| \leq \|\delta_b\|.$$ 

This provides the desired equality and hence the dependence of $\|\delta_z\|$ on $|z|$ also occurs under the assumption of the first part of (2). To show the non-decrease of the function $h$ representing this dependence in this case, suppose the contrary, then consider two points $a$ and $b$ in $D$ such that $|a| < |b|$ and $h(|a|) > h(|b|)$. Definitely, this would be in contradiction with the norm-contracting character of the $\mathcal{F}_e(Y)$-nc induced by the symbol $(a/b)p_1$. This achieves the proof. \qed

Now, here is how Theorem 4.1 can be extended to much more spaces.

**Theorem 4.3.** Let $h$ be a strictly increasing, (respectively, one-to-one) positive function on $[0,1)$. Assume that $\|\delta_z\| = h(|z|)$, for all $z \in D$. Then we have equivalence among $(C_1)$, $(C_2)$ and $(C_3)$, (respectively, $(P_1)$, $(P_2)$ and $(P_3)$).

**Proof.** $(C_1) \implies (C_2)$ and $(P_1) \implies (P_2)$ are obvious.

$(P_3) \implies (P_1)$ This is due to Theorem 4.2 as $\|\delta_z\|$ depends on $|z|$.

$(C_2) \implies (C_3)$ One has

$$h(|\varphi(0)|) = \|\delta_{\varphi(0)}\| = \|\Phi(\delta_0)\| \leq \|\delta_0\| = h(0).$$

Since $h$ is strictly increasing, it follows that $|\varphi(0)| \leq 0$, and this gives $(C_3)$. 


(C3) \implies (C1) Using the growth of \( h \), the inequality \( |\varphi(z)| \leq |z| \), ensured by Schwarz lemma everywhere in \( D \), gives
\[
\| \Phi(\delta_z) \| = \| \delta_{\varphi(z)} \| = h(|\varphi(z)|) \leq h(|z|) = \| \delta_z \|
\]
which means \((C1)\).

(P2) \implies (P3) The injectivity of \( h \) imposes both equalities \( \varphi(0) = 0 \) and \( |\varphi(a)| = |a| \) which are sufficient for \( \varphi \) to be a rotation, according to Schwarz lemma. This achieves the proof. \( \square \)

Next, we investigate the \( \mathcal{F}_e(Y) \)-norm-preserving character in a more general setting where \( \| \delta_z \| \) rather depends on \( |\theta(z)| \) with \( \theta \) holomorphic in \( D \). But first, let us tackle the dependence without the modulus nor the holomorphicness.

**Proposition 4.4.** Let \( h \) be a one-to-one positive function on \( \mathbb{C} \) and let \( \theta : D \to \mathbb{C} \). Assume that \( \| \delta_z \| = h(\theta(z)) \), for all \( z \in D \). Then the following are equivalent.

1. \( \Phi \) is an \( \mathcal{F}_e(Y) \)-np;
2. \( \theta \circ \varphi = \theta \).

**Proof.** (2) \implies (1) Let \( z \in D \), we have
\[
\| \Phi(\delta_z) \| = \| \delta_{\varphi(z)} \| = h(\theta(\varphi(z))) = h(\theta(z)) = \| \delta_z \|.
\]

(1) \implies (2) Since \( h \) is one-to-one, a necessary condition for \( \varphi \) to induce an \( \mathcal{F}_e(Y) \)-np is the identity
\[
\theta \circ \varphi(z) = \theta(z) \quad \text{for all} \quad z \in D. \quad \square
\]

**Corollary 4.5.** Let \( \omega, c \in \mathbb{C} \), \( n \in \mathbb{N} \setminus \{0\} \) and \( h \) be a one-to-one positive function on \( \mathbb{C} \). Assume that \( \| \delta_z \| = h((z - \omega)^n + c) \), for all \( z \in D \). Then, the following are equivalent.

1. \( \Phi \) is an \( \mathcal{F}_e(Y) \)-np;
2. either \( \omega = 0 \) and \( \varphi = up_1 \) where \( u \) is a complex nth root of 1, or \( \omega \neq 0 \) and \( \varphi = p_1 \).

**Proof.** (2) \implies (1) This is obvious.

(1) \implies (2) By applying Proposition 4.4 with \( \theta(z) = (z - \omega)^n + c \), we get \( (\varphi(z) - \omega)^n + c = (z - \omega)^n + c \), for all \( z \in D \). This implies that \( \varphi = up_1 + (1 - u)\omega p_0 \) and this gives (2) since \( \varphi(D) \subseteq D \). \( \square \)

**Theorem 4.6.** Let \( h \) be a one-to-one positive function on \([0, \infty)\) and \( \theta \in X \). Assume that \( \| \delta_z \| = h(|\theta(z)|) \), for all \( z \in D \). Then the following are equivalent.

1. \( \Phi \) is an \( \mathcal{F}_e(Y) \)-np;
2. there exists \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) such that \( \theta \circ \varphi = \lambda \theta \).
Proof. (2) \(\implies\) (1) Let \(z \in D\), we have
\[
\|\Phi(\delta_z)\| = \|\delta_{\phi(z)}\| = h(|\theta(\phi(z))|) = h(|\theta(z)|) = \|\delta_z\|.
\]
(1) \(\implies\) (2) Since \(h\) is one-to-one, a necessary condition for \(\phi\) to induce an \(\mathcal{F}_e(Y)\)-map is the identity
\[
|\theta \circ \phi(z)| = |\theta(z)| \quad \text{for all} \quad z \in D.
\]
If \(\theta\) is identically null, then (2) clearly holds. In the alternative, there exists an open disc \(\Delta \subset D\) in which \(\theta\) has no zero. By considering the holomorphic function \(\psi = (\theta \circ \phi)/\theta\) in \(\Delta\), we deduce from (*) that \(\psi\) has a constant modulus equal to 1. So there exists \(\lambda \in \mathbb{C}\) with \(|\lambda| = 1\) such that \(\theta(\phi(z)) = \lambda \theta(z)\), for all \(z \in \Delta\). Therefore, (2) occurs since \(\theta \circ \phi\) and \(\lambda \theta\) are holomorphic in \(D\). \(\Box\)

Corollary 4.7. Let \(\omega \in \mathbb{C} \setminus \{0\}\) and \(h\) be a one-to-one positive function on \([0, \infty)\). Assume that \(\|\delta_z\| = h(|z - \omega|)\), for all \(z \in D\). Then the following are equivalent.

(1) \(\Phi\) is an \(\mathcal{F}_e(Y)\)-map;
(2) \(\phi = p_1\).

Proof. (2) \(\implies\) (1) is clear.

(1) \(\implies\) (2) Applying Theorem 4.6, with \(\theta(z) = z - \omega\), we get \(\phi(z) - \omega = \lambda(z - \omega)\) for some \(\lambda \in \mathbb{C}\) and \(|\lambda| = 1\). This means that \(\phi\) is a rotation with \(\omega\) as a center. But \(\phi\) maps \(D\) into itself and \(\omega \neq 0\). Therefore, \(\phi\) is exactly the identity map. \(\Box\)

The equivalence between (\(P_1\)) and (\(P_3\)), in Theorem 4.3, is clearly the special case of Theorem 4.6 where \(\theta\) is a rotation. Moving to any automorphism of \(D\), the following result provides two more general formulations of (\(P_1\)).

Proposition 4.8. Let \(h\) be a one-to-one positive function on \([0, 1)\) and \(\theta\) be an automorphism of \(D\). Assume that \(\|\delta_z\| = h(|\theta(z)|)\), for all \(z \in D\). Then the following are equivalent.

(1) \(\Phi\) is an \(\mathcal{F}_e(Y)\)-map;
(2) \(\phi\) is an automorphism fixing \(\theta^{-1}(0)\);
(3) there exists a rotation \(\rho\) such that \(\phi = \theta^{-1} \circ \rho \circ \theta\).

Proof. (1) \(\implies\) (2) According to Theorem 4.6, there is \(\lambda \in \mathbb{C}\) with \(|\lambda| = 1\) such that \(\phi = \theta^{-1} \circ \lambda \theta\). This clearly gives (2).

(2) \(\implies\) (3) By composition of automorphisms of \(D\), the map \(\rho = \theta \circ \phi \circ \theta^{-1}\) is also an automorphism of \(D\). Moreover, (2) ensures that it fixes 0. Hence, \(\rho\) is a rotation and this gives (3) as \(\phi = \theta^{-1} \circ \rho \circ \theta\).

(3) \(\implies\) (1) As rotations preserve the modulus, it follows from (3) that
\[
|\theta(\phi(z))| = |\theta(z)| \quad \text{for all} \quad z \in D.
\]
(1) follows then by applying \(h\) to both sides of the last equality. This achieves the proof. \(\Box\)
From the previous proposition, one can deduce the following.

**Corollary 4.9.** Assume that the norm on \( \mathcal{F}_e(Y) \) is as previously. If moreover, \( \theta \) is not a rotation, then \( p_1 \) is the only one rotation inducing an \( \mathcal{F}_e(Y) \)-np.

Moving to the \( \mathcal{F}_e(Y) \)-nc’s, one can expect, as follows, an analogous version of Proposition 4.8, providing therefore two formulations of \((C_1)\) in a more general case compared to that given in Theorem 4.3.

**Proposition 4.10.** Let \( h \) be a strictly increasing positive function on \([0,1)\) and \( \theta \) be an automorphism of \( D \). Assume that \( \|\delta_z\| = h(|\theta(z)|) \), for all \( z \in D \). Then the following are equivalent.

1. \( \Phi \) is an \( \mathcal{F}_e(Y) \)-nc;
2. \( \phi \) fixes \( \theta^{-1}(0) \);
3. there exists \( \psi \in X \) with \( \psi(D) \subseteq D \) and \( \psi(0) = 0 \) such that \( \varphi = \theta^{-1} \circ \psi \circ \theta \).

**Proof.** (1) \( \Rightarrow \) (2) By writing \( \|\Phi(\delta_a)\| \leq \|\delta_a\| \) with \( a = \theta^{-1}(0) \), one obtains

\[
h(|\theta(\phi(a))|) \leq h(|\theta(a)|) = h(0).
\]

Since \( h \) is strictly increasing, this implies that \( \theta(\phi(a)) = 0 \) and hence \( \phi(a) = \theta^{-1}(0) = a \).

(2) \( \Rightarrow \) (3) Take \( \psi = \theta \circ \phi \circ \theta^{-1} \).

(3) \( \Rightarrow \) (1) For all \( z \in D \), one has

\[
|\theta(\phi(z))| = |\psi(\theta(z))| \leq |\theta(z)|,
\]

where the last estimate is due to the Schwarz lemma. We complete the proof by applying to each side the increasing function \( h \). \( \square \)

**Remark 5.** Actually, the previous proposition can extend to any biholomorphic function in \( D \) having one zero. Just replace \([0,1)\) with \([0, \infty)\) in the hypothesis and \( D \) with \( \theta(D) \) in (3). In particular, one can derive the following analogous version of Corollary 4.7, in which \( D(-\omega, 1) \) denotes the open disc centered at \( -\omega \) with radius 1.

**Corollary 4.11.** Let \( \omega \in D \setminus \{0\} \) and \( h \) be a strictly increasing function on \([0, \infty)\). Assume that \( \|\delta_z\| = h(|z - \omega|) \), for all \( z \in D \). Then the following are equivalent.

1. \( \Phi \) is an \( \mathcal{F}_e(Y) \)-nc;
2. \( \varphi(\omega) = \omega \);
3. there exists \( \psi \in H(D(-\omega, 1)) \) leaving this disc invariant such that \( \psi(0) = 0 \) and \( \varphi(z) = \psi(z - \omega) + \omega \).

In the sequel we denote by \( \overline{D} \) the closure of \( D \) and by \( \partial D \) the unit circle. We recall that \( \varphi \) is said to be *inner* if \( |\varphi^*| = 1 \) almost everywhere on \( \partial D \), where \( \varphi^* \) denotes the radial limit of \( \varphi \). For the existence of \( \varphi^* \), see [3,9].
Theorem 4.12. Let $\omega \in \mathbb{C} \setminus \{0\}$, $h$ be a one-to-one positive function on $[0, \infty)$ and $\theta : \overline{D} \to \overline{D}$ continuous and satisfying the following:

(i) $\theta(D) \subset D$, $\theta(\partial D) \subset \partial D$ and $\theta(0) = 0$;
(ii) for all $z_1, z_2 \in D$ such that $|z_1| \neq |z_2|$, we have $|\theta(z_1)| \neq |\theta(z_2)|$.

Assume that $\|\delta_z\| = h(||\theta(z)|| - \omega)$, for all $z \in D$. If $\Phi$ is an $\mathcal{F}_e(Y)-np$, then $\phi$ is inner. If moreover, $\phi(0) = 0$, then $\phi$ is a rotation.

Proof. Since $h$ is one-to-one, a necessary condition for $\Phi$ to be an $\mathcal{F}_e(Y)-np$ is the identity

\[(*) \quad |\theta(\phi(z))| - \omega = |\theta(z)| - \omega \quad \text{for all } z \in D.\]

Let us consider first the case $\phi(0) = 0$. If $\phi$ were not a rotation, then by Schwarz lemma, it would follow that

\[|\phi(z)| < |z| \quad \text{for all } z \in D \setminus \{0\}.\]

Thus, from $(*)$, one would deduce that

\[(**) \quad \text{Re} \omega = \frac{1}{2}(|\theta(\phi(z))| + |\theta(z)|) \quad \text{for all } z \in D \setminus \{0\}.\]

and then

\[\text{Re} \omega \geq \frac{|\theta(z)|}{2} \quad \text{for all } z \in D \setminus \{0\},\]

from which it would follow that $\text{Re} \omega > 0$. But taking the limit in $(**)$, when $z \to 0$, would force $\text{Re} \omega = 0$, which would be in contradiction with the previous condition. Consequently, $\phi$ is nothing else but a rotation.

We consider now the other case; i.e. $\phi(0) \neq 0$. By writing $(*)$ with $z = 0$ and taking (ii) in count together with the last condition of (i), one can see that

\[0 \leq \text{Re} \omega = \frac{1}{2}(|\theta(\phi(0))| + |\theta(0)|) = \frac{1}{2}|\theta(\phi(0))| < \frac{1}{2}.\]

On the other hand, it follows from $(*)$, by taking (ii) in count, that

\[(***) \quad \text{Re} \omega = \frac{1}{2}(|\theta(\phi(z))| + |\theta(z)|)\]

for all $z \in D$ such that $|\phi(z)| \neq |z|$. If $\phi$ were not inner, then there would be $\zeta \in \partial D$ such that $|\phi^*(\zeta)| < 1$. Hence, $(***)$ would hold for all $z = r\zeta$ with $r(0 < r < 1)$ close enough to 1 so that $|\phi(r\zeta)| < r$. Then, by passing to the limit as $r \to 1$ and taking the second condition of (i) in count, one would deduce that

\[\text{Re} \omega = \frac{1}{2}(|\theta(\phi^*(\zeta))| + 1) \geq \frac{1}{2}.\]

This would be in contradiction with the above estimate. Therefore, $\phi$ is necessarily inner, and we are done. $\square$

From Theorem 4.12, one can easily deduce the following.
Corollary 4.13. Let \( \omega \in \mathbb{C}\setminus\{0\} \), \( n \in \mathbb{N}\setminus\{0\} \) and \( h \) be a one-to-one positive function on \([0, \infty)\). Assume that \( \|\delta_z\| = h(||z^n| - |\omega||) \), for all \( z \in D \). If \( \Phi \) is an \( \mathcal{F}_e(Y) \)-nc, then \( \varphi \) is inner. If moreover, \( \varphi(0) = 0 \), then \( \varphi \) is a rotation.

Turning back to the \( \mathcal{F}_e(Y) \)-nc’s, here is what we can say about them for the spaces \( Y \) involved in Theorem 4.12 but with less restriction on \( \theta \).

Proposition 4.14. Let \( \omega \in \mathbb{C}\setminus\{0\} \), \( h \) be a strictly increasing positive function on \([0, \infty)\) and \( \theta \) be any function from \( D \) into \( \mathbb{C} \) fixing \( \theta \). Assume that \( \|\delta_z\| = h(||\theta(z)| - |\omega||) \), for all \( z \in D \). If \( \Phi \) is an \( \mathcal{F}_e(Y) \)-nc, then either
\[
\Re(\omega) \leq 0 \quad \text{and} \quad \theta(\varphi(0)) = 0
\]
or
\[
\Re(\omega) > 0 \quad \text{and} \quad 0 \leq |\theta(\varphi(0))| \leq 2\Re(\omega).
\]

Proof. Thanks to the growth of \( h \), one can formulate the property of \( \Phi \) by the inequality
\[
||\theta(\varphi(z))| - |\omega|| \leq ||\theta(z)| - |\omega|| \quad \text{for all} \quad z \in D.
\]
In particular, for \( z = 0 \), and since \( \theta(0) = 0 \), one obtains
\[
||\theta(\varphi(0))| - |\omega|| \leq |\omega|,
\]
or equivalently, by taking the square of each side,
\[
r(r - 2a) \leq 0,
\]
where \( r = |\theta(\varphi(0))| \) and \( a = \Re(\omega) \). Clearly, this gives \( r = 0 \) if \( a \leq 0 \) and \( 0 \leq r \leq 2a \) if \( a > 0 \). \( \square \)

Applying the previous proposition with \( \theta(z) = z^n \), one can easily deduce the following.

Corollary 4.15. Let \( \omega \in \mathbb{C}\setminus\{0\} \), \( n \in \mathbb{N}\setminus\{0\} \) and \( h \) be a strictly increasing positive function on \([0, \infty)\). Assume that \( \|\delta_z\| = h(||z^n| - |\omega||) \), for all \( z \in D \). If \( \Phi \) is an \( \mathcal{F}_e(Y) \)-nc, then either
\[
\Re(\omega) \leq 0 \quad \text{and} \quad \varphi(0) = 0
\]
or
\[
\Re(\omega) > 0 \quad \text{and} \quad 0 \leq |\varphi(0)| \leq 2\Re(\omega).
\]

More precisely, according to the Proof of Proposition 4.14 and thanks to Schwarz lemma, one can obtain the following.

Corollary 4.16. Let \( \omega \in \mathbb{C}\setminus\{0\} \), \( n \in \mathbb{N}\setminus\{0\} \) with \( \Re(\omega) \leq 0 \) and \( h \) be a strictly increasing, (respectively, one-to-one) positive function on \([0, \infty)\). Assume that \( \|\delta_z\| = h(||z^n| - |\omega||) \), for all \( z \in D \). Then we have equivalence among \( (C_1) \), \( (C_2) \) and \( (C_3) \), (respectively, \( (P_1) \), \( (P_2) \) and \( (P_3) \)).
Remark 6. Corollary 4.16 provides a range of spaces \( Y \) on which \( \|\delta_z\| \) depends non-injectively on \( |z| \) and for which the \( \mathcal{F}_E(Y)-nc \)’s and the \( \mathcal{F}_E(Y)-np \)’s are characterized the same way as in the opposite case handled in Theorem 4.3. Here, one can ask whether there exists a normed subspace \( Y \) on which \( \delta_z \) is bounded with \( \|\delta_z\| = h(|z|) \) for all \( z \in D \) and which dual supports an \( \mathcal{F}_E(Y)-nc \) (respectively, \( \mathcal{F}_E(Y)-np \)) with a symbol not fixing 0 (respectively, other than a rotation). If there is one, \( h \) is necessarily non-strictly-increasing (respectively, non-injective). In such a case, the following question imposes itself: how could one describe any \( \mathcal{F}_E(Y)-nc \) (respectively, \( \mathcal{F}_E(Y)-np \))?

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References