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Eigenvalues and eigenvectors for matrices over distributive lattices

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Abstract

Let $(L, \leqslant, \vee, \wedge)$ be a complete and completely distributive lattice. A vector ξ is said to be an eigenvector of a square matrix A over the lattice L if $A\xi = \lambda\xi$ for some $\lambda \in L$. The elements λ are called the associated eigenvalues. In this paper we characterize the eigenvalues and the eigenvectors and also the roots of the characteristic equation of A.

1. Introduction

The eigenvector–eigenvalue problem (*eigenproblem* for short) of matrices over distributive lattices seems to have appeared firstly in the work of Rutherford [2]. Since then, a number of works in this area were published (see e.g. [3–5]). But the background lattices are usually assumed to be some given Boolean algebras (see e.g. [2–4]) or Bottleneck algebras (see e.g. [5]). Of course this is too restricted to be satisfied.

In the present work, we consider the eigenproblem of matrices over more general lattices, namely in a class of complete and completely distributive lattices. Our main results generalize corresponding results in [2] or [3].

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2. Definitions and preliminary lemmas

Let L be a lattice, $a, b \in L$; the largest $x \in L$ satisfying the inequality $a \wedge x \leqslant b$ is called the *relative pseudocomplement* of a in b, and is denoted by $a \uplus b$. If for any pair of elements $a, b \in L$, $a \uplus b$ exists, then L is said to be a *Brouwerian lattice*. Dually, for $a, b \in L$, the least $x \in L$ satisfying $a \vee x \geqslant b$ is called the relative lower pseudocomplement of a in b, and is denoted by $a \Cap b$. If for any pair of elements $a, b \in L$, $a \Cap b$ exists, then L will be said to be a *dually Brouwerian lattice*.

A lattice L is said to be *completely distributive*, if for any $x \in L$ and any family of elements $\{y_i | i \in I\}$, I being an index set, there are always

$$(CD_1) \qquad x \wedge \left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I}(x \wedge y_i),$$

$$(CD_2) \qquad x \vee \left(\bigwedge_{i \in I} y_i\right) = \bigwedge_{i \in I}(x \vee y_i).$$

It is known ([1], p. 128) that: a *complete lattice L is Brouwerian, iff* (CD_1) *is satisfied in L, and L is dually Brouwerian, iff* (CD_2) *is satisfied in L.*

Therefore, a complete lattice L is both Brouwerian and dually Brouwerian, iff L is completely distributive.

In this paper, L denotes a complete and completely distributive lattice with the greatest element 1 and the least element 0. Unless otherwise specified all matrices and vectors are of order n.

The following notations are used:

$[a, b] = \{x \in L | a \leqslant x \leqslant b\}$ is an interval in L;

$(a] = \{x \in L | x \leqslant a\}$ is the principal ideal generated by $a \in L$;

$[a) = \{x \in L | x \geqslant a\}$ is the principal dual ideal generated by $a \in L$.

From the definition of relative pseudocomplement (relative lower pseudocomplement), we see that inequality $a \wedge x \leqslant b (a \vee x \geqslant b)$ is always solvable and its entire solution set is the ideal $(a \uplus b]$ (the dual ideal $[a \Cap b)$) of L.

The set $V_n(L)$ of all column vectors over L forms a complete and completely distributive lattice isomorphic to the nth direct power of L if we make the following definitions.

$$\xi = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \eta = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow \xi \vee \eta = \begin{bmatrix} x_1 \vee y_1 \\ \vdots \\ x_n \vee y_n \end{bmatrix}, \quad \xi \wedge \eta = \begin{bmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{bmatrix},$$

where $xy = x \wedge y$.

Let

$$o = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The vector o is called the zero vector of $V_n(L)$.

The multiplication of the vector ξ by a scalar a is defined by

$$a\xi = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}.$$

ξ^{T} denotes the row vector whose transpose is ξ and ξ' is defined by

$$\xi' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix},$$

where $x' = x \Psi o$.

A nonempty subset V of $V_n(L)$ is called a vector space in $V_n(L)$ if it is closed under "$\vee$" and under multiplication by scalars (elements of L).

Likewise the set $M_n(L)$ of all $n \times n$ matrices over L forms a complete and completely distributive lattice if we make the following definitions.

For $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij}) \in M_n(L)$,

$$A \vee B = C \Longleftrightarrow a_{ij} \vee b_{ij} = c_{ij} \quad (i,j = 1,2,\ldots,n);$$
$$A \wedge B = C \Longleftrightarrow a_{ij} \wedge b_{ij} = c_{ij} \quad (i,j = 1,2,\ldots,n);$$
$$A \leqslant B \Longleftrightarrow a_{ij} \leqslant b_{ij} \quad (i,j = 1,2,\ldots,n).$$

An additional operation of matrix multiplication in $M_n(L)$ can be introduced by the definition

$$AB = C \Longleftrightarrow \bigvee_{k=1}^{n} a_{ik}b_{kj} = c_{ij} \quad (i,j = 1,2,\ldots,n).$$

It is clear that for any $A, B, C \in M_n(L)$,

$$A(B \vee C) = AB \vee AC, \quad (B \vee C)A = BA \vee CA,$$
$$A(B \wedge C) \leqslant AB \wedge AC, \quad (B \wedge C)A \leqslant BA \wedge CA,$$
$$(AB)C = A(BC); \quad A \leqslant B \Rightarrow AC \leqslant BC \quad \text{and} \quad CA \leqslant CB.$$

The multiplication of a matrix A by a scalar λ is defined by

$$\lambda A = B \Longleftrightarrow \lambda a_{ij} = b_{ij} \quad (i,j = 1,2,\ldots,n).$$

The premultiplication of a vector ξ by A is defined by

$$A\xi = \eta \iff \bigvee_{j=1}^{n} a_{ij}x_j = y_i \quad (i = 1, 2, \ldots, n).$$

A^{T} denotes the transpose of A.

The following lemmas in the next paragraphs are used:

Lemma 2.1 ([6], Lemma 6). *For any $a, b \in L$, we have $a \uplus b = a \uplus (ab)$.*

Lemma 2.2. *If $b \in L$, $\{a_i | i \in I\} \subseteq L$, where I is a certain index set, then*

(1) $\bigwedge_{i\in I}(a_i \uplus b) = \left(\bigvee_{i\in I} a_i\right) \uplus b$;

(2) $\bigwedge_{i\in I}(b \uplus a_i) = b \uplus \left(\bigwedge_{i\in I} a_i\right)$;

(3) $\left(\left(\bigvee_{j\in I} a_j\right) \uplus b\right) \wedge a_i = a_i \wedge b$ *for all $i \in I$.*

Proof. (1) According to the definition of pseudocomplement, $a_i \wedge (a_i \uplus b) \leqslant b$ for all $i \in I$. Let $x = \bigwedge_{i=1}(a_i \uplus b)$. Then $x \leqslant a_i \uplus b$ and so $a_i \wedge x \leqslant a_i \wedge (a_i \uplus b) \leqslant b$ for all $i \in I$. Hence $(\bigvee_{i\in I} a_i) \wedge x \leqslant b$. It follows that $x \leqslant (\bigvee_{i\in I} a_i) \uplus b$. On the other hand, let $y = (\bigvee_{i\in I} a_i) \uplus b$. Then $(\bigvee_{i\in I} a_i) \wedge y \leqslant b$. It follows that $a_i \wedge y \leqslant b$ and so $y \leqslant a_i \uplus b$ for all $i \in I$. Hence, we have $y \leqslant \bigwedge_{i\in I}(a_i \uplus b)$. This proves (1). (2) is proved in Zhao ([6], Lemma 7). (3) Since $(\bigvee_{j\in I} a_j) \uplus b \geqslant b$, we have $((\bigvee_{j\in I} a_j) \uplus b) \wedge a_i \geqslant a_i \wedge b$ for all $i \in I$. On the other hand, since $(\bigvee_{j\in I} a_j) \wedge ((\bigvee_{j\in I} a_j) \uplus b) \leqslant b$, we have $a_i \wedge (\bigvee_{j\in I} a_j) \wedge ((\bigvee_{j\in I} a_j) \uplus b) \leqslant a_i \wedge b$ and so $((\bigvee_{j\in I} a_j) \uplus b) \wedge a_i \leqslant a_i \wedge b$ for all $i \in I$. This proves (3). □

Lemma 2.3. *For any $a, b \in L$, we have $a \uplus b \geqslant a' \vee b$.*

Proof. Since $a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$ and $a \wedge a' = 0$, we have $a \wedge (a' \vee b) = a \wedge b \leqslant b$ and so $a' \vee b \leqslant a \uplus b$. This proves the lemma. □

Lemma 2.4. *For any $a, b \in L$, we have*

(1) $(a \vee b)' = a' \wedge b'$;

(2) $a''' = a'$, *where* $a''' = (a'')'$ *and* $a'' = (a')'$;

(3) $(a \vee a')' = 0$.

Proof. (1) Since $(a \wedge (a \vee b)') \vee (b \wedge (a \vee b)') = (a \vee b) \wedge (a \vee b)' = 0$, we have $a \wedge (a \vee b)' = 0$ and $b \wedge (a \vee b)' = 0$. It follows that $(a \vee b)' \leqslant a'$ and $(a \vee b)' \leqslant b'$. Hence $(a \vee b)' \leqslant a' \wedge b'$. On the other hand, since $a \wedge (a' \wedge b')$

$\leqslant a \wedge a' = 0$ and $b \wedge (a' \wedge b') \leqslant b \wedge b' = 0$, we have $(a \vee b) \wedge (a' \wedge b') = 0$. It follows that $a' \wedge b' \leqslant (a \vee b)'$.This proves (1).

(2) Since $a' \wedge a = 0$, we have $a \leqslant a''$. It follows that $a''' \leqslant a'$. On the other hand, since $a'' \wedge a' = (a')' \wedge a' = 0$, we have $a' \leqslant (a'')' = a'''$. This proves (2).

(3) $(a \vee a')' = a' \wedge a'' = 0$. This proves (3). □

Lemma 2.5. *Let $\lambda \in L$. If $\lambda' \vee \lambda = 1$, then $\lambda'' = \lambda$.*

Proof. It is clear that $\lambda'' \geqslant \lambda$. On the other hand, $\lambda'' = \lambda''(\lambda \vee \lambda') = \lambda''\lambda \vee \lambda''\lambda' = \lambda''\lambda$. It follows that $\lambda \geqslant \lambda''$. Consequently $\lambda'' = \lambda'$. This proves the lemma. □

Lemma 2.6. *For any $a, b \in L$, $(a \uplus b) \wedge b' \leqslant a'$.*

Proof. Since $a \wedge (a \uplus b) \leqslant b$, we have $a \wedge (a \uplus b) \wedge b' \leqslant b \wedge b' = 0$. It follows that $(a \uplus b) \wedge b' \leqslant a'$. This proves the lemma. □

Lemma 2.7. *For any $a, b, c \in L$, we have*

$$(a \uplus (bc)) \wedge a = abc.$$

Proof. It is clear that $(a \uplus (bc)) \wedge a \geqslant abc$. On the other hand, since $a \wedge (a \uplus (bc)) \leqslant bc$, we have $a \wedge (a \uplus (bc)) = a \wedge (a \wedge (a \uplus (bc))) \leqslant abc$. Consequently, $(a \uplus (bc)) \wedge a = abc$. This proves the lemma. □

Lemma 2.8. *For any $a, b \in L$, we have*

(1) $b \vee (a \Cap b) = b$,
(2) $a \vee (a \Cap b) = a \vee b$.

Proof. (1) According to the definition of lower pseudocomplement and using the fact $a \vee b \geqslant b$, we have $b \geqslant a \Cap b$, and so $b \vee (a \Cap b) = b$. This proves (1).

(2) Since $a \vee (a \Cap b) \geqslant b$, we have $a \vee (a \Cap b) = a \vee (a \vee (a \Cap b)) \geqslant a \vee b$. On the other hand, since $b \geqslant a \Cap b$, we have $a \vee b \geqslant a \vee (a \Cap b)$. This proves (2). □

Lemma 2.9. *For any $A = (a_{ij}) \in M_n(L), A^n e = A^{n+1} e$.*

Proof. Since $Ae \leqslant e$, it follows that

$$A^{i+1} e = A^i (Ae) \leqslant A^i e$$

for all i, hence in particular $A^{n+1} e = \bigvee_{i>n} A^i e$. Now, any term t of the kth entry of $A^n e$ is of the form $a_{ki_1} a_{i_1 i_2} \cdots a_{i_{n-1} i_n}$, where $1 \leqslant i_1, i_2, \cdots, i_n \leqslant n$. Since the number of indices in t is greater than n, a repetition among them must occur. Let us call the sequence of entries between two occurrences of one index a *cycle*. If we repeat the cycle twice in a succession, the value of t will not change, due to idempotency, but what we now get, is a term in the kth entry of $A^m e$ for some $m > n$. Hence $A^m e \leqslant \bigvee_{m>n} A^m e = A^{n+1} e$.

Therefore

$$A^n e = A^{n+1} e.$$

This proves the lemma. □

Corollary 2.1. *If $\xi = A^n e$, then $A\xi = \xi$.*

Lemma 2.10. *If $A\xi = \xi$, then $\xi \leqslant A^n e$.*

Proof. Since $\xi \leqslant e$ for any ξ,

$$\xi = A\xi = A^2\xi = \cdots = A^n\xi \leqslant A^n e. \qquad \square$$

Lemma 2.11. *Let $A = (a_{ij}) \in M_n(L)$, $\lambda \in L$ and $\lambda \vee \lambda' = 1$, $\xi = (x_1, \ldots, x_n)^{\mathrm{T}} \in V_n(L)$. If $\lambda A\xi = 0$, then*

$$\xi \leqslant \begin{bmatrix} \left(\bigvee_{i=1}^{n} a_{i1}\right) \Psi \lambda' \\ \vdots \\ \left(\bigvee_{i=1}^{n} a_{in}\right) \Psi \lambda' \end{bmatrix}.$$

Proof.

$$\begin{aligned}
\lambda A\xi = 0 &\Rightarrow \lambda\left(\bigvee_{j=1}^{n} a_{ij}x_j\right) = 0 \ (i = 1, 2, \ldots, n) \\
&\Rightarrow \lambda a_{ij}x_j = 0 \ (i, j = 1, 2, \ldots, n) \\
&\Rightarrow a_{ij}x_j = (\lambda \vee \lambda')a_{ij}x_j = \lambda a_{ij}x_j \vee \lambda' a_{ij}x_j = \lambda' a_{ij}x_j \ (i, j = 1, 2, \ldots, n) \\
&\Rightarrow x_j \leqslant a_{ij} \Psi (\lambda' a_{ij}x_j) \\
&\qquad = (a_{ij} \Psi \lambda') \wedge (a_{ij} \Psi a_{ij}) \wedge (a_{ij} \Psi x_j) \text{ (by Lemma 2.2 (2))} \\
&\qquad \leqslant a_{ij} \Psi \lambda' \ (i, j = 1, 2, \ldots, n) \\
&\Rightarrow x_j \leqslant \bigwedge_{i=1}^{n} (a_{ij} \Psi \lambda') \\
&\qquad = \left(\bigvee_{i=1}^{n} a_{ij}\right) \Psi \lambda' \text{ (by Lemma 2.2 (1))} \ (j = 1, 2, \ldots, n)
\end{aligned}$$

$$\Rightarrow \xi \leqslant \begin{bmatrix} \left(\bigvee_{i=1}^{n} a_{i1}\right) \uplus \lambda' \\ \vdots \\ \left(\bigvee_{i=1}^{n} a_{in}\right) \uplus \lambda' \end{bmatrix}.$$

This proves the lemma. □

Lemma 2.12. *For any* $A \in M_n(L)$, $A(A^{\mathrm{T}}e)' = 0$.

Proof. Let $A = (a_{ij})_{n\times n}$. Then the jth element of $A(A^{\mathrm{T}}e)'$ is

$$\bigvee_{k=1}^{n} a_{jk}\left(\bigvee_{i=1}^{n} a_{ik}\right)' \leqslant \bigvee_{k=1}^{n} a_{jk}a'_{jk} = 0. \qquad \square$$

3. Eigenvectors and eigenvalues

Definition 3.1. Let $A \in M_n(L)$, an *eigenvector* of A is a vector $\xi \in V_n(L)$ such that,

$$A\xi = \lambda\xi$$

for some scalar λ. The element λ is called the associated *eigenvalue*.

It will transpire that every element of L is an eigenvalue of every matrix A and that a given eigenvector may have a variety of eigenvalues. In the classical case only the zero vector has a range of eigenvalues and it is usual to stipulate that an eigenvector is non-zero. In the case of matrices over a lattice there seems to be no advantage in making this restriction and we shall therefore admit the possibility that an eigenvector is the zero vector.

We first consider a given eigenvector ξ of a matrix A in $M_n(L)$ and determine the range of its eigenvalues.

Theorem 3.1. *Let* $A = (a_{ij}) \in M_n(L)$. *If* $\xi = (x_1, \ldots, x_n)^{\mathrm{T}}$ *is an eigenvector of* A, *then the eigenvalues of* ξ *form a sublattice of* L *consisting of the interval* $[\lambda^0, \lambda^*]$, *where*

$$\lambda^0 = e^{\mathrm{T}}A\xi \quad \textit{and} \quad \lambda^* = (e^{\mathrm{T}}\xi) \uplus \lambda^0.$$

Proof. If $A\xi = \lambda\xi = \mu\xi$, then

$$A\xi = \lambda\xi \vee \mu\xi = (\lambda \vee \mu)\xi,$$
$$A\xi = \lambda\xi \wedge \mu\xi = (\lambda \wedge \mu)\xi,$$

from which it appears that the eigenvalues of ξ form a sublattice of L with a greatest element λ^* and a least element λ^0. Since

$$e^{\mathrm{T}}A\xi = e^{\mathrm{T}}(\lambda\xi) = \lambda(e^{\mathrm{T}}\xi) \leqslant \lambda,$$

it follows that $e^{\mathrm{T}}A\xi$ is a lower bound for λ^0. This lower bound is attained, since

$$(e^{\mathrm{T}}A\xi)\xi = \lambda(e^{\mathrm{T}}\xi)\xi = \lambda(x_1 \vee \cdots \vee x_n)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda\xi = A\xi.$$

Hence

$$\lambda^0 = e^{\mathrm{T}}A\xi.$$

From $A\xi = \lambda^0\xi = \lambda^*\xi$, we obtain

$$\lambda^* x_j = \lambda^0 x_j \quad (j = 1, 2, \ldots, n)$$

By Lemma 2.1 we have

$$\lambda^* \leqslant x_j \uplus (\lambda^0 x_j) = x_j \uplus \lambda^0 \quad (j = 1, 2, \ldots, n)$$

and so

$$\lambda^* \leqslant \bigwedge_{j=1}^{n}(x_j \uplus \lambda^0) = \left(\bigvee_{j=1}^{n} x_j\right) \uplus \lambda^0 \text{ (by Lemma 2.2 (1))} = (e^{\mathrm{T}}\xi) \uplus \lambda^0.$$

Thus $(e^{\mathrm{T}}\xi) \uplus \lambda^0$ is an upper bound for λ^*.This upper bound is attained, since

$$((e^{\mathrm{T}}\xi) \uplus \lambda^0)\xi = \begin{bmatrix} ((e^{\mathrm{T}}\xi) \uplus \lambda^0)x_1 \\ \vdots \\ ((e^{\mathrm{T}}\xi) \uplus \lambda^0)x_n \end{bmatrix} = \begin{bmatrix} \left(\left(\bigvee_{i=1}^{n} x_i\right) \uplus \lambda^0\right)x_1 \\ \vdots \\ \left(\left(\bigvee_{i=1}^{n} x_i\right) \uplus \lambda^0\right)x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^0 x_1 \\ \vdots \\ \lambda^0 x_n \end{bmatrix} \text{(by Lemma 2.2 (3))} = \lambda^0\xi = A\xi.$$

Hence

$$\lambda^* = (e^{\mathrm{T}}\xi) \uplus \lambda^0.$$

If $\lambda^0 \leqslant \lambda \leqslant \lambda^*$, then

$$A\xi = \lambda^0\xi \leqslant \lambda\xi \leqslant \lambda^*\xi = A\xi,$$

which demonstrates that λ is also an eigenvalue. This proves the theorem. □

Theorem 3.2. *If an eigenvector ξ has a unique eigenvalue, then* $(e^{\mathrm{T}}\xi)' = 0$.

Proof. From Theorem 3.1, this clearly requires $\lambda^* = \lambda^0$, or $e^{\mathrm{T}}A\xi = (e^{\mathrm{T}}\xi)\uplus\lambda^0$. But $(e^{\mathrm{T}}\xi)\uplus\lambda^0 \geqslant \lambda^0 \vee (e^{\mathrm{T}}\xi)'$ (by Lemma 2.3). It follows that $(e^{\mathrm{T}}\xi)' \leqslant e^{\mathrm{T}}A\xi$. However, $e^{\mathrm{T}}A \leqslant e^{\mathrm{T}}$, so the uniqueness of the eigenvalue demands that $(e^{\mathrm{T}}\xi)' \leqslant e^{\mathrm{T}}\xi$, which is only possible $(e^{\mathrm{T}}\xi)' = 0$. This proves the theorem. □

We next suppose that λ is a given eigenvalue of a matrix A and proceed to determine its eigenvectors.

Theorem 3.3. *Let $A \in M_n(L)$ and λ be a given eigenvalue of A. Then the eigenvectors of λ form a subspace of $V_n(L)$ with the greatest element ξ^*, namely the union of all eigenvectors of λ, and the smallest element o.*

Proof. If $A\xi = \lambda\xi$ and $A\eta = \lambda\eta$, then

$$A(\xi \vee \eta) = A\xi \vee A\eta = \lambda\eta \vee \lambda\eta = \lambda(\xi \vee \eta).$$

$A(a\xi) = a(A\xi) = a(\lambda\xi) = \lambda(a\xi)$ for all $a \in L$. The eigenvectors of λ therefore form a subspace of $V_n(L)$ with the greatest element ξ^* and the smallest element o. This proves the theorem. □

Theorem 3.4. *For A, λ, ξ^* in Theorem 3.3, we have*

(1) $\xi^* \geqslant \lambda A^n e \vee \lambda'(A^{\mathrm{T}}e)'$,

(2) *If λ satisfies $\lambda \vee \lambda' = 1$, then*

$$\xi^* = \lambda A^n e \vee \lambda'(A^{\mathrm{T}}e)'.$$

Proof. (1) By Lemmas 2.9 and 2.12, we have

$$A(\lambda A^n e \vee \lambda'(A^{\mathrm{T}}e)') = \lambda A^{n+1}e \vee \lambda' A(A^{\mathrm{T}}e)' = \lambda A^n e = \lambda(\lambda A^n e \vee \lambda'(A^{\mathrm{T}}e)').$$

Thus,

$$\xi^* \geqslant \lambda A^n e \vee \lambda'(A^{\mathrm{T}}e)'.$$

This proves (1).

(2) If $A\xi = \lambda\xi$, then $\lambda' A\xi = \lambda'\lambda\xi = 0$. Then by Lemmas 2.5 and 2.11

$$\xi \leqslant \begin{bmatrix} \left(\bigvee_{i=1}^{n} a_{i1}\right) \uplus \lambda \\ \vdots \\ \left(\bigvee_{i=1}^{n} a_{in}\right) \uplus \lambda \end{bmatrix},$$

from which we obtain

$$\lambda'\xi \leqslant \begin{bmatrix} \left(\left(\bigvee_{i=1}^{n} a_{i1}\right) \uplus \lambda\right) \wedge \lambda' \\ \vdots \\ \left(\left(\bigvee_{i=1}^{n} a_{in}\right) \uplus \lambda\right) \wedge \lambda' \end{bmatrix}$$

$$\leqslant \begin{bmatrix} \left(\bigvee_{i=1}^{n} a_{i1}\right)' \\ \vdots \\ \left(\bigvee_{i=1}^{n} a_{in}\right)' \end{bmatrix} \quad \text{(by Lemma 2.6)}$$

$$= (A^{\mathrm{T}}e)'$$

and so

$$\lambda'\xi \leqslant \lambda'(A^{\mathrm{T}}e)'.$$

On the other hand,

$$A(\lambda\xi) = \lambda A\xi = \lambda\lambda\xi = \lambda\xi,$$

so by Lemma 2.10, $\lambda\xi \leqslant A''e$ and indeed

$$\lambda\xi \leqslant \lambda A''e.$$

Thus

$$\xi = (\lambda \vee \lambda')\xi = \lambda\xi \vee \lambda'\xi \leqslant \lambda A''e \vee \lambda'(A^{\mathrm{T}}e)'.$$

and so

$$\xi^* \leqslant \lambda A''e \vee \lambda'(A^{\mathrm{T}}e)'.$$

By (1), we have

$$\xi^* = \lambda A''e \vee \lambda'(A^{\mathrm{T}}e)'.$$

This proves (2). □

We now suppose that λ is a given element in L and ξ is a given vector in $V_n(L)$ and proceed to determine the matrix A such that ξ is an eigenvector of it and λ is the associated eigenvalue.

Definition 3.2 [3]. By a *gerbier* we shall mean a semigroup which is also a $\vee$-semilattice in which the multiplication is distributive (on both sides) with respect to $\vee$.

It is readily established that $M_n(L)$ is a gerbier in which the multiplication is the matrix multiplication defined above.

Theorem 3.5. *For any given* $\lambda \in L$ *and* $\xi = (x_1, \ldots, x_n)^T \in V_n(L)$, *define*

$$T(\lambda, \xi) = \{A \in M_n(L), A\xi = \lambda\xi\}.$$

Then $T(\lambda, \xi)$ *is a subgerbier of* $M_n(L)$, *the maximum element of* $T(\lambda, \xi)$ *being the matrix* M *whose* (i, j)*th element is* $m_{ij} = x_j \Psi(\lambda x_i)$.

Proof. Let $A = (a_{ij}), B = (b_{ij}) \in T(\lambda, \xi)$. Then it follows that

$$(AB)\xi = A(B\xi) = A(\lambda\xi) = \lambda(A\xi) = \lambda(\lambda\xi) = \lambda\xi.$$

Hence, $AB \in T(\lambda, \xi)$ and so $T(\lambda, \xi)$ is a subsemigroup of $M_n(L)$. Moreover, for each i,

$$((A \vee B)\xi)_i = \bigvee_{j=1}^{n}((a_{ij} \vee b_{ij}) \wedge x_j) = \left(\bigvee_{j=1}^{n}(a_{ij} \wedge x_j)\right) \vee \left(\bigvee_{j=1}^{n}(b_{ij} \wedge x_j)\right)$$
$$= (A\xi)_i \vee (B\xi)_i = (\lambda\xi)_i \vee (\lambda\xi)_i = (\lambda\xi)_i$$

and so $(A \vee B)\xi = \lambda\xi$, from which it follows that $T(\lambda, \xi)$ is a $\vee$-subsemilattice of $M_n(L)$. We thus have that $T(\lambda, \xi)$ is a subgerbier of $M_n(L)$ since the multiplication in $M_n(L)$ is doubly distributive with respect to $\vee$ and it must necessarily be so in every $\vee$-subsemilattice of $M_n(L)$.

Consider now the matrix M defined by

$$m_{ij} = x_j \Psi(\lambda x_i).$$

we have

$$m_{ij} x_j = (x_j \Psi(\lambda x_i)) x_j = \lambda x_i x_j \text{ (by Lemma 2.7)}$$

so that, for each i,

$$\bigvee_{j=1}^{n}(m_{ij} x_j) = (\lambda x_i) \wedge \left(\bigvee_{j=1}^{n} x_j\right) = \lambda x_i.$$

In other words, $M\xi = \lambda\xi$ and so $M \in T(\lambda, \xi)$.

To show that M is the greatest element of $T(\lambda, \xi)$, we observe that

$$\begin{aligned} A \in T(\lambda, \xi) &\Rightarrow \bigvee_{j=1}^{n}(a_{ij} x_j) = \lambda x_i \text{ for all } i \\ &\Rightarrow a_{ij} x_j \leqslant \lambda x_i \text{ for all } i, j \\ &\Rightarrow a_{ij} \leqslant x_j \Psi(\lambda x_i) \text{ for all } i, j \\ &\Rightarrow A \leqslant M. \end{aligned}$$

This proves the theorem. □

4. The solutions of the characteristic equation

In the classical theory of matrices in a field the eigenvalue problem is closely associated with the characteristic equation of the matrix concerned. In the case of matrices over a lattice the relationship is somewhat obscure but the following remarks can be made. If the positive and negative terms are placed on opposite sides of the equality sign, the classical characteristic equation of a matrix A takes the form

$$\lambda^n + p_2(A)\lambda^{n-2} + q_3(A)\lambda^{n-3} + p_4(A)\lambda^{n-4} + \cdots + b$$
$$= p_1(A)\lambda^{n-1} + q_2(A)\lambda^{n-2} + p_3(A)\lambda^{n-3} + \cdots + c.$$

In 1994, Zhang Kunlun [7] showed that the corresponding lattice equation

$$\lambda^n \vee p_2(A)\lambda^{n-2} \vee q_3(A)\lambda^{n-3} \vee p_4(A)\lambda^{n-4} \vee \cdots \vee b$$
$$= p_1(A)\lambda^{n-1} \vee q_2(A)\lambda^{n-2} \vee p_3(A)\lambda^{n-3} \vee \cdots \vee c \quad (4.1)$$

is satisfied by the matrix $A \in M_n(L)$, thus providing a counterpart to the Cayley–Hamilton theorem. Since, in the lattice case, a *characteristic equation* cannot be defined determinantally it is natural to choose Eq. (4.1) as the defining equation. It should be explained that

$$p_k(A) = \bigvee_{\pi \in S(r_1,\dots,r_k), r_i \leqslant n, \pi \text{ is even}} a_{r_1\pi(r_1)}a_{r_2\pi(r_2)}\cdots a_{r_k\pi(r_k)},$$

$$q_k(A) = \bigvee_{\pi \in S(r_1,\dots,r_k), r_i \leqslant n, \pi \text{ is odd}} a_{r_1\pi(r_1)}a_{r_2\pi(r_2)}\cdots a_{r_k\pi(r_k)},$$

where $S(r_1,\dots,r_k)$ is the symmetric group over the set $\{r_1,\dots,r_k\}, k = 1,2,\dots,n$. $b = p_n(A)$ and $c = q_n(A)$ when n is even, $b = q_n(A)$ and $c = p_n(A)$ when n is odd.

We now assume that the parameter λ in Eq. (4.1) is an element in L and proceed to solve this equation. From the idempotency of λ and the absorption law it follows that Eq. (4.1) takes the form

$$\lambda \vee b = \lambda d \vee c, \quad (4.2)$$

where $d = p_1(A) \vee q_2(A) \vee p_3(A) \vee q_4(A) \vee \cdots$ and either $b = p_n(A), c = q_n(A)$ with n even or $b = q_n(A)$, $c = p_n(A)$ with n odd.

Lemma 4.1. $b \leqslant d$.

Proof. We recall the facts that any permutation can be expressed as a product of independent cycles and that an even cycle involves an odd number of letters, while an odd cycle involves an even number of letters.

(i) Suppose n is odd. Then $b = q_n(A)$ and any term u of b takes the form $a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)}$, where π is odd. Therefore, π must have at least one odd cycle π' as a factor and the cycle π' involves an even number k of letters. It is clear that $k < n$. Let $\pi' = (r_1, \ldots, r_k)$, where $r_1, r_2, \ldots, r_k \in \{1, 2, \ldots, n\}$, and $v = a_{r_1\pi(r_1)}a_{r_2\pi(r_2)} \cdots a_{r_k\pi(r_k)}$. Then v is a term of $q_k(A)$ and $u \leqslant v \leqslant q_k(A)$. Whatever term u is chosen from b, the value taken by k must be one of $n-1, n-3, n-5, \ldots$ Hence for each u we have

$$u \leqslant q_{n-1}(A) \vee q_{n-3}(A) \vee \cdots \vee q_2(A) \leqslant d.$$

Thus $b \leqslant d$.

(ii) If n is even, then $b = p_n(A)$ and any term u of b takes the form $a_{1\pi(1)}a_{2\pi(2)} \cdots a_{n\pi(n)}$, where π is even. There are now two cases to consider. If π has an odd cycle π' as a factor, then, by an argument similar to that employed in cases (i) the cycle π' gives rise to a term v from one of $q_{n-2}(A)$, $q_{n-4}(A), \ldots, q_2(A)$. π' cannot give rise to a term of $q_n(A)$ since this would require π to be odd. Therefore

$$u \leqslant q_{n-2}(A) \vee q_{n-4}(A) \vee \cdots \vee q_2(A) \leqslant d.$$

If, however, π has only even cycles as factors, there must be more than one of them since otherwise π would be odd since n is even. If one of these even cycles involves k letters it would give rise to a term v of $p_k(A)$ with k odd. In this case we would have

$$u \leqslant p_{n-1}(A) \vee p_{n-3}(A) \vee \cdots p_1(A) \leqslant d.$$

Since $u \leqslant d$ in each case, we conclude that $b \leqslant d$. □

We now proceed to solve the characteristic Eq. (4.2).

Theorem 4.1. *The roots of the characteristic Eq.* (4.2) *are those values of* λ *which satisfy*

$$(c \Cap b) \vee (b \Cap c) \leqslant \lambda \leqslant d \vee c.$$

Proof. If λ satisfies the Eq. (4.2), then

$$\lambda \leqslant \lambda \vee b = \lambda d \vee c \leqslant d \vee c \tag{4.3}$$

and similarly

$$b \leqslant \lambda \vee b = \lambda d \vee c \leqslant \lambda \vee c.$$

It follows that

$$\lambda \geqslant c \Cap b. \tag{4.4}$$

We see also from Eq. (4.2) that $\lambda \vee b \geqslant c$. It follows that

$$\lambda \geqslant b \Cap c. \tag{4.5}$$

From Eqs. (4.4) and (4.5)

$$\lambda \geqslant (c ⋒ b) \vee (b ⋒ c).$$

Combining this with Eq. (4.3) we get

$$(c ⋒ b) \vee (b ⋒ c) \leqslant \lambda \leqslant d \vee c$$

and we now verify that these bounds are attained.

First, put $\lambda = d \vee c$, then

$$\lambda \vee b = b \vee d \vee c = d \vee c \text{ (by Lemma 4.1)},$$
$$\lambda d \vee c = (d \vee c)d \vee c = d \vee c.$$

Next, put $\lambda = (c ⋒ b) \vee (b ⋒ c)$, then

$$\begin{aligned}\lambda \vee b &= b \vee ((c ⋒ b) \vee (b ⋒ c))\\ &= (b \vee ((c ⋒ b)) \vee (b ⋒ c)\\ &= b \vee (b ⋒ c) \text{ (by Lemma 2.8(1))}\\ &= b \vee c \text{ (by Lemma 2.8(2))}.\end{aligned}$$

$$\begin{aligned}\lambda d \vee c &= ((c ⋒ b) \vee (b ⋒ c))d \vee c\\ &= (c ⋒ b)d \vee (b ⋒ c)d \vee c\\ &= (c ⋒ b) \vee ((b ⋒ c)d \vee c) \text{ (because } c ⋒ b \leqslant b \leqslant d)\\ &= (c ⋒ b) \vee c \text{ (because } c \geqslant b ⋒ c)\\ &= b \vee c \text{ (by Lemma 2.8(2))}.\end{aligned}$$

This establishes that the bounds are solutions.

Suppose λ is any element in the interval $[(c ⋒ b) \vee (b ⋒ c), d \vee c]$. Then λ must be of the form

$$\lambda = (c ⋒ b) \vee (b ⋒ c) \vee (d \vee c)f, \quad f \in L.$$

Therefore

$$\begin{aligned}\lambda d \vee c &= ((c ⋒ b) \vee (b ⋒ c) \vee (d \vee c)f)d \vee c\\ &= (((c ⋒ b) \vee (b ⋒ c))d \vee c) \vee ((d \vee c)d \vee c)f\\ &= ((c ⋒ b) \vee (b ⋒ c) \vee b) \vee (d \vee c \vee b)f\\ &= ((c ⋒ b) \vee (b ⋒ c) \vee (d \vee c)f) \vee b \text{ (because } b \leqslant d)\\ &= \lambda \vee b.\end{aligned}$$

That is to say, $\lambda = (c ⋒ b) \vee (b ⋒ c) \vee (d \vee c)f$ is also a solution. This proves the theorem. □

Theorem 4.2. *The largest root of the characteristic Eq.* (4.1) *is* $e^{\mathrm{T}}A^{n}e$.

Proof. First, by Theorem 4.1, the largest root of the characteristic Eq. (4.1) or Eq. (4.2) is $\lambda = d \vee c$.

Secondly, we shall prove that $d \vee c = e^{\mathrm{T}}A^{n}e$.

Consider the expression $e^{\mathrm{T}}A^{n}e$ which, using the notation introduced earlier, may be written

$$\bigvee_{1\leqslant i_1,\ldots,i_{n+1}\leqslant n} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_{n+1}}.$$

Any term of $p_1(A)\vee q_2(A)\vee p_3(A)\vee\cdots\vee c(=d\ \vee\ c)$ can be shown to be included in a term of $e^{\mathrm{T}}A^{n}e$. The indices in terms of $p_i(A)$ and $q_i(A)$ are derived from permutations and their form is represented earlier in this paragraph. On the other hand, for any positive integer i, the indices of entries in a term of $e^{\mathrm{T}}A^{i}e$ must be successive. Take any term t of say $p_i(A)$. Since every permutation consists of (several) cycles, we can choose one such cycle in t, drop all the entries outside it and repeat it several times, until a new expression t' with $m\geqslant n$ entries is obtained. Due to properties of the operation $\wedge$, $t\leqslant t'$, but t' is a term of $e^{\mathrm{T}}A^{m}e$.Hence

$$d\vee c\leqslant\bigvee_{m\geqslant n} e^{\mathrm{T}}A^{m}e=e^{\mathrm{T}}A^{n}e. \tag{4.6}$$

As has been mentioned earlier. A satisfies its own characteristic equation. That is to say

$$A^{n}\vee p_2(A)A^{n-2}\vee q_3(A)A^{n-3}\vee\cdots\vee bE=p_1(A)A^{n-1}\vee q_2(A)A^{n-2}\vee\cdots\vee cE.$$

Post-multiplying throughout by $A^{n}e$ and using the fact that by Lemma 2.9

$$A^{n}e=A^{n+1}e=A^{n+2}e=\cdots$$

we obtain

$$A^{n}e\vee(p_2(A)\vee q_3(A)\vee\cdots\vee b)A^{n}e=(p_1(A)\vee q_2(A)\vee\cdots\vee c)A^{n}e.$$

Premultiplying throughout by e^{T} we have

$$e^{\mathrm{T}}A^{n}e\vee(p_2(A)\vee q_3(A)\vee\cdots\vee b)e^{\mathrm{T}}A^{n}e=(p_1(A)\vee q_2(A)\vee\cdots\vee c)e^{\mathrm{T}}A^{n}e$$

or

$$e^{\mathrm{T}}A^{n}e=(p_1(A)\vee q_2(A)\vee\cdots\vee c)e^{\mathrm{T}}A^{n}e,$$

which shows that

$$d\vee c=p_1(A)\vee q_2(A)\vee\cdots\vee c\geqslant e^{\mathrm{T}}A^{n}e. \tag{4.7}$$

Combining Eqs. (4.6) and (4.7) we have finally

$$d\vee c=p_1(A)\vee q_2(A)\vee\cdots\vee c=e^{\mathrm{T}}A^{n}e.$$

Thus the largest root of the characteristic equation is $e^{\mathrm{T}}A^{n}e$.

This completes the proof of Theorem 4.2. □

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