Panconnectivity and edge-pancycliclicity of faulty recursive circulant $G(2^m, 4)$

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Abstract

In this paper, we investigate a problem on embedding paths into recursive circulant $G(2^m, 4)$ with faulty elements (vertices and/or edges) and show that each pair of vertices in recursive circulant $G(2^m, 4)$, $m \geq 3$, are joined by a fault-free path of every length from $m + 1$ to $|V(G(2^m, 4) \setminus F)| - 1$ inclusive for any fault set $F$ with $|F| \leq m - 3$. The bound $m - 3$ on the number of acceptable faulty elements is the maximum possible. Moreover, recursive circulant $G(2^m, 4)$ has a fault-free cycle of every length from 4 to $|V(G(2^m, 4) \setminus F)|$ inclusive excluding 5 passing through an arbitrary fault-free edge for any fault set $F$ with $|F| \leq m - 3$.

Keywords: Panconnected; Edge-pancyclic; Embedding; Linear arrays; Rings; Recursive circulants; Fault tolerance; Interconnection networks

1. Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnection network is an important issue in parallel processing [5,11,19,21–24]. An interconnection network is often modelled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modelled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges.

In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. A graph $G$ is called $f$-fault hamiltonian (resp. $f$-fault hamiltonian-connected) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \setminus F$ for any set $F$ of faulty elements with $|F| \leq f$. On the other hand, if the paths joining each pair of vertices of every length shorter than or equal to a hamiltonian path are required the problem is concerned with panconnectivity of the graph. If the cycles of arbitrary size (up to a hamiltonian cycle) are required the problem is concerned with pancyclicity of the graph.

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Definition 1. A graph $G$ is called $f$-fault l-panconnected if each pair of fault-free vertices are joined by a path in $G \setminus F$ of every length from $l$ to $|V(G \setminus F)| - 1$ inclusive for any set $F$ of faulty elements with $|F| \leq f$.

Definition 2. A graph $G$ is called $f$-fault almost edge-pancyclic (resp. $f$-fault nearly edge-pancyclic) if for any set $F$ of faulty elements with $|F| \leq f$, there exists a cycle of every length from 4 to $|V(G \setminus F)|$ inclusive (resp. from 4 to $|V(G \setminus F)|$ inclusive excluding 5) that passes through an arbitrary fault-free edge.

Panconnectivity of some interconnection networks without faulty elements was reported in the literature. A graph $G$ is said to be panconnected (resp. almost panconnected) if each pair of vertices $s$ and $t$ in $G$ are joined by an $s$–$t$ path of every length from $d(s, t)$ to $V(G) - 1$ (resp. from $d(s, t) + 2$ to $V(G) - 1$) inclusive. Here, $d(s, t)$ denotes the distance between $s$ and $t$. Recursive circulant $G(2^m, 2)$ [16], alternating group graphs [5], and augmented cubes [13] are panconnected, and recursive circulant $G(2^m, 4)$ [16], locally twisted cubes [14], and twisted cubes [7] are almost panconnected. Recently, fault-panconnectivity of a family of hypercube-like interconnection networks called restricted HL-graphs was investigated in [20]. It was shown that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault $2m - 3$-panconnected. The family includes many interconnection networks proposed in the literature such as twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes.

Edge-pancyclicity of some fault-free interconnection networks such as recursive circulants, crossed cubes, twisted cubes was studied in [1], [9], and [8]. A graph $G$ is called $f$-fault l-edge-pancyclic if for any fault set $F$ with $|F| \leq f$, there exists a cycle of every length from $l$ to $|V(G \setminus F)|$ inclusive that passes through an arbitrary fault-free edge. An $f$-fault l-panconnected graph is obviously $f$-fault l-1-edge-pancyclic. In the presence of faulty elements, the fault-pancyclicity result in [20] implies that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault $2m - 2$-edge-pancyclic.

Pancyclicity and fault-pancyclicity of various interconnection networks were investigated. A graph $G$ is called $f$-fault pancyclic (resp. $f$-fault almost pancyclic) if $G \setminus F$ contains a cycle of every length from 3 to $|V(G \setminus F)|$ inclusive (resp. 4 to $|V(G \setminus F)|$ inclusive) for any fault set $F$ with $|F| \leq f$. The works on fault-pancyclicity can be summarized as saying that many interconnection networks of degree $\delta$ are $\delta - 2$-fault pancyclic or $\delta - 2$-fault almost pancyclic depending on the existence of length 3 cycles in the network; for example, augmented cubes [13], recursive circulants [2,17], Möbius cubes [11], crossed cubes [23], twisted cubes [24], and restricted HL-graphs [20].

A recursive circulant is an interconnection network proposed in [18]. Recursive circulant $G(N, d)$, $d \geq 2$, is defined as follows: the vertex set $V = \{v_0, v_1, v_2, \ldots, v_{N-1}\}$, and the edge set $E = \{(v_i, v_j) \mid$ there exists $k, 0 \leq k \leq \lfloor \log_d N \rfloor - 1, \text{ such that } i + d^k \equiv j \pmod{N}\}$. $G(N, d)$ is a circulant graph with $N$ vertices and jumps of powers of $d, d^0, d^1, \ldots, d^{\lfloor \log_d N \rfloor - 1}$. Examples of $G(N, d)$ are shown in Fig. 1.

In this work, our attention is restricted to $G(N, d)$ with $N = 2^m$ and $d = 4$. $G(2^m, 4)$, whose degree is $m$, compares favorably to the hypercube $Q_m$. While retaining attractive properties of hypercube $Q_m$ such as node symmetry, recursive structure, the maximum connectivity, etc., it achieves noticeable improvements in diameter [18] and possesses a complete binary tree with $2^m - 1$ vertices as a subgraph [12]. A recursive circulant has a cycle-based construction, and thus it is expected to have nice properties concerned with cycles. $G(N, d)$ with degree 3 or higher is hamiltonian-connected [6]. $G(N, d)$ with $N = cd^m$ and $1 \leq c < d$ is hamiltonian decomposable [3,10,15], that is, the set of edges can be partitioned into edge-disjoint hamiltonian cycles (and a 1-factor when the degree is odd). In [10], the edge forwarding index and bisection width for recursive circulants were also analyzed.
In this paper, we investigate panconnectivity and edge-pancyclicity of recursive circulant $G(2^m, 4)$ with faulty elements. It will be shown that $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault $m + 1$-panconnected and $m - 3$-fault nearly edge-pancyclic. The bound $m - 3$ on the number of acceptable faulty elements for $G(2^m, 4)$ to be $l$-panconnected for any fixed $l$ (less than the number of fault-free vertices) is the maximum possible in a sense that no graph of degree $m$ is $m - 2$-fault $l$-panconnected as well as Hamilton-connected.

In the rest of this paper, we will use standard terminology for graphs (see Ref. [4]). This paper is organized as follows. In the next section, we will present some basic properties of recursive circulant $G(2^m, 4)$. Panconnectivity and edge-pancyclicity of faulty recursive circulant $G(2^m, 4)$ will be proved in Sections 3 and 4, respectively. Finally in Section 5, the concluding remarks of this paper will be given.

2. Recursive circulant $G(2^m, 4)$

Recursive circulant $G(N, d)$ can also be defined as the Cayley graph of the cyclic group $\mathbb{Z}_N$ with the generating set $\{d^0, d^1, \ldots, d^{\lfloor \log_d N \rfloor - 1}\}$. Every Cayley graph over a general group is vertex symmetric, and thus regular. Recursive circulant $G(N, d)$ has a recursive structure when $N = cd^m, 1 \leq c < d$ [18]. In other words, $G(cd^m, d)$ can be defined recursively by utilizing the following property.

**Property 1** ([18]). Let $V_i$ be a subset of vertices in $G(cd^m, d)$ such that $V_i = \{v_j | j \equiv i \pmod{d}\}$, $m \geq 1$. For $0 \leq i \leq d - 1$, the subgraph of $G(cd^m, d)$ induced by $V_i$ is isomorphic to $G(cd^{m-1}, d)$.

$G(cd^m, d)$, $m \geq 1$, can be constructed recursively on $d$ copies of $G(cd^{m-1}, d)$ as follows. Let $G_i(V_i, E_i), 0 \leq i \leq d - 1$, be a copy of $G(cd^{m-1}, d)$. We assume that $V_i = \{v_0^i, v_1^i, \ldots, v_{cd^{m-1}-1}^i\}$, and $G_i$ is isomorphic to $G(cd^{m-1}, d)$ with the isomorphism mapping $v_j^i$ to $v_{jd+i}$. The vertex set $V$ of $G(cd^m, d)$ is $\bigcup_{0 \leq i \leq d - 1} V_i$, and the edge set $E$ is $\bigcup_{0 \leq j \leq d - 1} E_i \cup X$, where $X = \{(v_j^i, v_{j'}^i) | j + 1 \equiv j' \pmod{cd^m}\}$. The construction of $G(32, 4)$ on four copies of $G(8, 4)$ is illustrated in Fig. 2. Note that recursive circulant $G(2^m, 4)$ has a recursive structure when $m \geq 2$. In the recursive structure, $G(2^m, 4)$ consists of four components $G_0$, $G_1$, $G_2$, and $G_3$; each of them is isomorphic to $G(2^{m-2}, 4)$. A vertex in $G_i$ is represented by $v_j^i, 0 \leq j < 2^{m-2}, 0 \leq i \leq 3$, as well as $v_j^0, 0 \leq j < 2^m$, without saying in which $G_i$ the vertex is contained.

Hereafter in this paper, we denote by $G_i \oplus G_j$ and $G_i \oplus G_j \oplus G_k$ for some $0 \leq i, j, k \leq 3$ the subgraphs of $G(2^m, 4)$ induced by $V_i \cup V_j$ and $V_i \cup V_j \cup V_k$, respectively. Let $F$ be the set of faulty elements in $G(2^m, 4)$. $F_i$ denotes the set of faulty elements in $G_i, i = 0, 1, 2, 3$, and $F_{i,i+1 \mod 4}$ denotes the set of faulty edges joining vertices in $G_i$ and vertices.
in $G_{i+1 \mod 4}$, so that $F = \bigcup_{0 \leq i \leq 3} (F_i \cup F_{i,i+1 \mod 4})$. Let $f_i = |F_i|$ and $f_{i,i+1 \mod 4} = |F_{i,i+1 \mod 4}|$. We denote by $f_i'$ the number of faulty vertices in $G_i$, and by $f$ the total number of faulty vertices, so that $f = \sum_{0 \leq i \leq 3} f_i'$.

From now on, all arithmetic on the indices of vertices will be assumed to be done modulo $2^m$. Some properties of recursive circulant $(2^m, 4)$ explored to establish our main results are listed below, where the diameter $D_m$ of $G(2^m, 4)$ is defined as the maximum distance between any two vertices in the graph.

**Lemma 1 (Shortest Path [18]).** Let $G_0$, $G_1$, $G_2$, and $G_3$ be the components of $(2^m, 4)$. (a) Every shortest path joining a pair of vertices $v_0^i$ and $v_j^i$ passes through only vertices in $G_i$. (b) There exists a shortest path between $v_0^i$ and $v_j^i$ passing through $v_0^i$ when $i = 1$, and passing through $v_0^i$ when $i = 3$. In the case $i = 2$, there exists a shortest path between $v_0^i$ and $v_j^i$ passing through $v_0^i$ when $d(v_0^i, v_j^i) \leq d(v_0^i, v_j^i+1)$, and passing through $v_j^i$ when $d(v_0^i, v_j^i) \leq d(v_0^i, v_j^i)$.

**Lemma 2 (Diameter [18]).** (a) $D_{m-2} + 1 \leq D_m \leq D_{m-2} + 2$ for $m \geq 2$. (b) $D_m = 3m/4$.

**Lemma 3 (Fault-Hamiltonicity [22,19]).** (a) $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian. (b) The product $G(2^m, 4) \times K_2$ of $G(2^m, 4)$ and $K_2$, $m \geq 3$, is $m-2$-fault hamiltonian-connected and $m-1$-fault hamiltonian.

Lemma 3(a) implies that $G(2^m, 4)$, $m \geq 3$, with at most $m-1$ faulty elements has a hamiltonian path joining some pair of fault-free vertices.

3. Panconnectivity of faulty $G(2^m, 4)$

In this section, we will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected. Throughout this paper, a path in a graph is represented as a sequence of vertices. A path joining a pair of vertices $s$ and $t$ is called an $s$-$t$ path.

Panconnectivity of fault-free recursive circulants $G(2^m, 2^k)$ was investigated in [16]. It was shown that between any pair of vertices $s$ and $t$, there exists a path of every length $d(s, t) + \Delta$ or longer for some $\Delta$. One of the results is given in the following, which will be utilized for our purpose.

**Lemma 4 ([16]).** $G(2^m, 4)$ is almost panconnected. That is, between any pair of vertices $s$ and $t$ in $G(2^m, 4)$, there exists a path of every length $l$, $d(s, t) + 2 \leq l \leq 2m - 1$.

A concatenation of two paths $(x_1, x_2, \ldots, x_p)$ and $(y_1, y_2, \ldots, y_q)$ is defined to be the path $(x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$.

**Lemma 5.** (a) $G(2^m, 4)$, $m \geq 3$, is 0-fault $D_m + 1$-panconnected.

(b) $G(2^m, 4)$, $m \geq 5$, is 0-fault $D_m$-panconnected.

**Proof.** We prove (a) by induction on $m$. Due to Lemma 4, it suffices to show that for any pair of vertices $s$ and $t$ with $d(s, t) = D_m$, there exists a path of length $D_m + 1$ between them. For $m = 3, 4$, the construction is immediate by inspection. Let $m \geq 5$. We assume $s = v_0^0$ without loss of generality. There are two cases up to symmetry. If $t = v_j^i$ for some $j \neq 1$, we first find a $v_1^0$-$t$ path $P'$ in $G_1$ of length $D_m$. The path $P'$ exists since $D_{m-2} + 1 \leq D_m$. Then, $(s, P')$ is a desired path of length $D_m + 1$. Now, let $t = v_2^j$ for some $j \neq 1$. By Lemma 1, $D_m$ is equal to $d(v_1^0, t) + 2$ or $d(v_0^0, t) + 2$. We assume w.l.o.g. that $D_m = d(v_1^0, t) + 2$. Letting $P''$ be a shortest $v_1^0$-$t$ path in $G_2$, we have a path $(s, v_0^3, v_1^3, P'')$ of length $d(v_1^0, t) + 3 = D_m + 1$.

To prove (b), we assume that each $G_i$ is $D_{m-2} + 1$-panconnected and furthermore, whenever $m - 2 \geq 5$, it is $D_{m-2}$-panconnected. It suffices to construct a path of length $D_m$ joining every pair of vertices $s$ and $t$ with $d(s, t) = D_m - 1$. Let $s = v_0^0$. If $t = v_j^0$ for some $j \neq 1$, there exists an $s$-$t$ path in $G_0$ of every length $D_{m-2} + 1$ or longer, and thus we are done. When $t = v_1^0$ for some $j \neq 1$, there exists a $v_1^0$-$t$ path $P'$ of length $D_{m-2} + 1$, and $(s, P')$ is an $s$-$t$ path of length $D_{m-2} + 2$. If $D_m = D_{m-2} + 2$, we are done. Suppose otherwise ($D_m = D_{m-2} + 1$); observe $m \geq 7$. Note that $D_3, D_4, D_5$, and $D_6$ are 2, 3, 4, 5, respectively. Employing the assumption that $G_1$ is $D_{m-2}$-panconnected, we have an $s$-$t$ path $(s, P'')$ of length $D_m$, where $P''$ is a $v_1^0$-$t$ path in $G_1$ of length $D_{m-2}$. Finally when $t = v_2^0$ for some $j \neq 1$, assuming w.l.o.g. that $d(s, t) = d(v_1^0, t) + 2$, a concatenation of $(s, v_0^3, v_3^3)$ and a shortest $v_1^0$-$t$ path in $G_2$ results in an $s$-$t$ path of length $d(s, t) + 1 = D_m$. Thus, the proof is completed. □
Now, we are to investigate panconnectivity of faulty recursive circulants. We will show that $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault $m + 1$-panconnected. For $m = 3, 4$, we have the following lemma.

**Lemma 6.** (a) $G(8, 4)$ is 0-fault 3-panconnected.
(b) $G(16, 4)$ is 0-fault 4-panconnected and 1-fault 5-panconnected.

**Proof.** Lemma 5(a) says that $G(8, 4)$ is 0-fault 3-panconnected and $G(16, 4)$ is 0-fault 4-panconnected. To show that $G(16, 4)$ is 1-fault 5-panconnected, we need to construct an $s$-$t$ path of every length 5 or longer for any pair of fault-free vertices $s$ and $t$ in $G(16, 4)$ with one faulty element. When the faulty element is a vertex $v_j$, the construction of an $s$-$t$ path in $G(16, 4) \backslash v_j$ is by a case analysis and omitted here. Suppose there exists a faulty edge $(x, y)$. If $\{x, y\} = \{s, t\}$, letting $(x, y)$ be a virtual fault-free edge, Lemma 5(a) is applied. Otherwise, letting $x \notin \{s, t\}$ be a virtual faulty vertex, an $s$-$t$ path of every length up to 14 is constructed. An $s$-$t$ hamiltonian path of length 15 also exists due to Lemma 3(a). \(\square\)

To prove the main result for $m \geq 5$, we exploit the recursive structure of $G(2^m, 4)$ and a technique: so called “strong induction”. In other words, assuming that each component $G_i$ which is isomorphic to $G(2^{m-2}, 4)$ is not only $m - 5$-fault $m - 1$-panconnected but also $\frac{m - 2}{2}$-fault $m - 2$-panconnected and $\frac{m - 5}{2}$-fault $m - 3$-panconnected and so on, we show that $G(2^m, 4)$ is $m - 3$-fault $m + 1$-panconnected and $\frac{m - 3}{2}$-fault $m$-panconnected and so on.

**Theorem 1.** $G(2^m, 4)$, $m \geq 3$, is $\lfloor \frac{m - 3}{2k} \rfloor$-fault $m - k + 1$-panconnected for any integer $k$, $0 \leq k \leq L(m - 3) + 1$, where $L(m) = \lfloor \log_2 n \rfloor$ for $n \geq 1$ and $L(0) = 0$.

**Proof.** By Lemma 6, the theorem holds for $m = 3, 4$. Hereafter, we assume $m \geq 5$. Observe that $\lfloor \frac{m - 3}{2k} \rfloor = 0$ if $k = L(m - 3) + 1$, and that $\lfloor \frac{m - 2}{2} \rfloor = 1$ if $k = L(m - 3)$. When $k = L(m - 3) + 1$, due to Lemma 5(b), the theorem holds. We claim that $D_m = \lfloor \frac{3m - 1}{4} \rfloor \leq m - L(m - 3)$ for any $m \geq 5$. The inequality can be checked for small $m$ in the following table. For $m \geq 19$, it suffices to show that $\frac{3m - 1}{4} + 1 \leq m - \lfloor \log_2 (m - 3) \rfloor$ or equivalently $4 \lfloor \log_2 (m - 3) \rfloor + 3 \leq m$. Let $k \geq 4$ be an integer such that $2^k \leq m - 3 < 2^k + 1$. Then, we have $4 \lfloor \log_2 (m - 3) \rfloor + 3 = 4k + 3 + 2^k \leq m$. Obviously $4k + 3 \leq 2^k + 3$ for any $k \geq 4$, and thus the claim is proved.

| $m$   | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $m - L(m - 3)$ | 4  | 5  | 5  | 6  | 7  | 8  | 8  | 9  | 10 | 11 | 11 | 12 | 11 | 14 |

Thus, assuming $f_0 \geq f_j$ for any $j = 1, 2, 3$, there are two cases.

Case 1. $f_i \leq \lfloor \frac{(m-2)-j}{2} \rfloor$ for every $i = 0, 1, 2, 3$. It is straightforward to see that $k \leq L((m - 2) - 3) + 1$. Thus, each $G_i \backslash F_i$ is $m - k - 1$-panconnected. Furthermore when $m = 5$, $G_i$ is fault-free and, by Lemma 6(a), it is 3-panconnected. We first consider panconnectivity of $G_0 \oplus G_1$.

**Claim 1.** Each pair of vertices $x$ and $y$ in $G_0 \oplus G_1$ are joined by an $x$-$y$ path of every length 1, $m - k + 1 \leq l \leq 2^{m-1} - f_0 - f_1 - 1$.

To prove the claim, let $x$ and $y$ be vertices in $G_0$ first. There exists an $x$-$y$ path $P_0$ in $G_0$ of every length $l_0$, $m - k - 1 \leq l_0 \leq 2^{m-2} - f_0 - 1$. To construct a longer path $P_1$ that passes through vertices in $G_1$ as well as vertices in $G_0$, let $P'$ be an $x$-$y$ path in $G_0$ of every length $l'$, $2m - 5 \leq l' \leq 2^{m-2} - f_0 - 1$. Then, there is an edge $(v_0', v_0')$ on $P'$ such that all of $(v_0', v_0')$, $(v_0', v_1')$, and $(v_0, v_1')$ are fault-free since each faulty element can “block” at most two such candidate edges and the number of faulty elements is at most $m - 3$. The path $P_1$ can be obtained from merging $P'$ and a $v_1' - v_1$ path $P''$ in $G_1$ with the edges $(v_0', v_1')$ and $(v_0', v_1')$. When $m \geq 6$, the length $l''$ of $P''$ is any integer in the range $m - k - 1 \leq l'' \leq 2^{m-2} - f_1 - 1$ and thus the length $l_1$ of $P_1$ is in the range $(2m - 5) + (m - k - 1) + 1 \leq l_1 \leq 2^{m-1} - f_0 - f_1 - 1$. It is straightforward to see that $(2m - 5) + (m - k - 1) + 1 \leq (2^{m-2} - f_0 - 1) + 1$ since $3m - 5 \leq 2^{m-2} - (m - 5)$ for every $m \geq 6$. When
Since we have $m = 5$, observing $f_i = 0$ for each $i$, we have $5 \leq l' \leq 7$. Furthermore, by Lemma 6(a), we have $3 \leq l'' \leq 7$. Thus, $9 \leq l_1 \leq 15$. It remains to construct an $x$-$y$ path of length 8. Let $v^0_p$ (resp. $v^0_q$) be a vertex in $G_0$ which is either $x$ (resp. $y$) or at least adjacent to it such that (i) $v^0_p \neq y$ and $v^0_q \neq x$, (ii) $(v^0_p, v^0_p)$ and $(v^0_q, v^0_q)$ are fault-free, and (iii) $v^0_p \neq v^0_q$. Since there exists a $v^0_q$-$v^0_p$ path $P''$ of every length $l''$, $3 \leq l'' \leq 7$, we have an $x$-$y$ path $P_1 = (s, v^0_q, v^0_q, P'', v^0_q, G_0, d, v^0_q, y)$ of length 8. Therefore, we have an $x$-$y$ path of every length $l, m - k - 1 \leq l \leq 2m - 1 - f_0 - f_1 - 1$.

Now, let $x$ be a vertex in $G_0$ and $y$ be a vertex in $G_1$. Let $v^1_p$ be a vertex in $G_1$ which is either $y$ or at least adjacent to it such that (i) $v^1_y \neq x$ and (ii) $(v^1_p, v^1_p)$, and $(v^1_q, v^1_q)$ are fault-free. The existence of such a vertex $v^1_p$ is due to there being $m - 1$ candidates and at most $m - 2$ blocking elements (the source $x$ and at most $m - 3$ faulty elements). Letting $P'$ be an $x$-$v^1_q$ path in $G_0$ of every length $l', m - k - 1 \leq l' \leq 2m - 2 - f_0 - 1$, we have an $x$-$y$ path $P_0 = (P', v^1_p, y)$ of every length $l_0, m - k + 1 \leq l_0 \leq 2m_0 - 2 - f_0 - 1$. To construct a longer path, we let $(v^0_q, v''_q)$ be an edge such that (i) $v^0_q \neq x$ and $v^0_q \neq y$, and (ii) $v^0_q, v^1_q$, and $(v^0_q, v^1_q)$ are fault-free. Letting $P'$ be an $x$-$v^0_q$ path in $G_0$ of every length $l', m - k - 1 \leq l' \leq 2m - 2 - f_0 - 1$, and letting $P''$ be a $v''_q$-$y$ path in $G_1$ of every length $l''$, $m - k - 1 \leq l'' \leq 2m - 2 - f_0 - 1$, we have an $x$-$y$ path $P_1 = (P', P'')$ of every length $l_1, 2m - 2k - 1 \leq l_1 \leq 2m - 2 - f_0 - f_1 - 1$. We have $2m - 2k - 1 \leq 2m - 2 - f_0 - 1$ since $2m_1 \leq 2m - 2 - (m - 5) + 1$ for every $m \geq 5$. Therefore, we have an $x$-$y$ path of every length $m - k + 1$ or more. This completes the proof of Claim 1.

Note that for each of $G_1 \oplus G_2, G_2 \oplus G_3$, and $G_3 \oplus G_0$, we can establish the same statement as Claim 1 since we do not use the assumption of $f_0 \geq 2, 3, f_3$ in the proof. From now on, we will construct an $s$-$t$ path of every length $l, m - k + 1 \leq l \leq 2m - f_0 - 1$. We assume w.l.o.g. that $s$ is contained in $G_0$.

Subcase 1.1. $t$ is a vertex in $G_0, G_1$, or $G_3$.

We assume w.l.o.g. that $t$ is contained in $G_0 \oplus G_1$. By Claim 1, there exists an $s$-$t$ path $P_0$ in $G_0 \oplus G_1$ of every length $l_0, m - k + 1 \leq l_0 \leq 2m - 1 - f_0 - f_1 - 1$. Let $P'$ be an $s$-$t$ path in $G_0 \oplus G_1$ of every length $l'$, $2m - 5 \leq l' \leq 2m - 1 - f_0 - f_1 - 1$. There is an edge $(x, y)$ on $P'$ such that $\bar{x}, \bar{y}$ are fault-free, where $\bar{x}$ and $\bar{y}$ are the vertices in $G_2 \oplus G_3$ adjacent to $x$ and $y$, respectively. Letting $P''$ be an $\bar{x}$-$\bar{y}$ path in $G_2 \oplus G_3$ of every length $l''$, $m - k + 1 \leq l'' \leq 2m - 1 - f_0 - f_1 - 1$, an $s$-$t$ path $P_1$ can be obtained from merging $P'$ and $P''$ with edges $(x, \bar{x})$ and $(y, \bar{y})$. The length $l_1$ of $P_1$ is any integer in the range $3m - k - 3 \leq l_1 \leq 2m - f_0 - 1$. It holds true that $3m - k - 3 \leq 2m - 1 - f_0 - f_1 - 1 + 1$ since $3m - 3 \leq 2m - 1 - (m - 3)$ for any $m \geq 5$. Thus, we have an $s$-$t$ path of every length $m - k + 1$ or more.

Subcase 1.2. $t$ is a vertex in $G_2$.

We let $s = v^1_j$ and $t = v^1_j$ for some $j$. First, we will construct an $s$-$t$ path $P_0$ of every length $l_0, m - k + 1 \leq l_0 \leq 2m - 2 - \frac{m - 5}{2}$. Let us consider the subcase when $|F| - 1 \leq f_0 + f_2 \leq |F|$. In this subcase, we assume w.l.o.g. that $j \neq 1$. (Suppose otherwise; we can construct an $s$-$t$ path $P_0$ with the roles of $G_1$ and $G_3$ being interchanged in a symmetric way.) If all of $v^1_j, (s, v^1_j), (1, v^1_j)$ are fault-free, letting $P'$ be a $v^1_j$-$v^1_j$ path in $G_1$ of every length $l'$, $m - k - 1 \leq l' \leq 2m - 2 - f_0 - f_1 - 1$, we have an $s$-$t$ path $P_0 = (s, P', t)$ of every length $l_0, m - k + 1 \leq l_0 \leq 2m - 2 - f_0 - f_1 - 1$. Obviously, $2m - 2 - \frac{m - 5}{2} \leq 2m - 1 - f_0 - f_1 - 1$. Suppose otherwise; exactly one among the four elements $v^1_j, (s, v^1_j), v^0_j$, and $(t, v^1_j)$ is faulty. If $j \neq 0$, an $s$-$t$ path $P_0$ passing through vertices in $G_3$ can be constructed symmetrically. Let $j = 0$ and let $v^1_j$ be a vertex in $G_2$ adjacent to $t$ such that $v^0_j$ and $(t, v^1_j)$ are fault-free. There is a $v^0_j$-$v^1_j$ path $P'$ in $G_3$ of every length $l', 3 \leq l' \leq 2m - 2 - 1$, by Lemma 4. Note that $G_3$ is fault-free and $v^0_j$ is adjacent to $v^1_j$. Thus, the length $l_0$ of $P_0 = (s, P', v^1_j, t)$ is any integer in the range $6 \leq l_0 \leq 2m - 2 + 2$. Observe that $6 \leq m - k + 1$ for any $m$ and $k$ with $m \geq 5$ and $0 \leq k \leq \lfloor (m - 3) \rfloor$ except only when $m = 5$ and $k = 1 (|F| = 1)$. For the exceptional case, regarding the faulty element as a virtual fault-free one, we will construct two vertex-disjoint $s$-$t$ paths of length 5. Letting $P'$ be an $s$-$v^0_j$ path of length 3 in $G_0$ and $P''$ be a $v^1_j$-$t$ path of length 3 in $G_2$, we have two paths $(P', v^0_j, t)$ and $(s, v^1_j, P'')$ at least one of the two is a fault-free path since $|F| = 1$.

Now we will construct an $s$-$t$ path $P_0$ of every length $l_0, m - k + 1 \leq l_0 \leq 2m - 2 - \frac{m - 5}{2}$, when $f_0 + f_2 \leq |F| - 2 (|F| \geq 2)$. Remember that $f_2 \leq f_0$. Then, in the following claim, we can obtain a result stronger than that $G_2 \backslash F_2$ is $m - k - 1$-panconnected.

Claim 2. $G_2 \backslash F_2$ is $m - k - 2$-panconnected.

To prove the claim, it suffices to show that $f_2 \geq \lceil \frac{(m-2)-3}{2^k+1} \rceil$ and $k+1 \leq L((m-2)-3) + 1$. Suppose $f_2 \geq \lceil \frac{(m-2)-3}{2^k+1} \rceil$; we have $f_0 + f_2 \geq 2 \lceil \frac{(m-2)-3}{2^k+1} \rceil + 2 \geq \lceil \frac{(m-2)-3}{2^k} \rceil + 1 \geq \lceil \frac{m-3}{2^k} \rceil - 1 \geq \lceil \frac{m-3}{2^k} \rceil - |F| - 1$, which is a contradiction. Suppose $k \geq L((m-2)-3) + 1$; we have $|F| = \lceil \frac{m-3}{2^k} \rceil \leq \frac{m-3}{2^k} \leq 1$ since $m - 3 < 2^2, 2L((m-2)-3)+1$ for any $m \geq 5$. This is a contradiction to $|F| \geq 2$. Thus, we have the claim.

In the subcase of $f_0 + f_2 \leq |F| - 2$, we assume w.l.o.g. that $p \neq j$ for each vertex $v_p^0$ adjacent to $s$. (Suppose otherwise; we can construct an $s$–$t$ path $P_0$ passing through a vertex in $G_2$ instead of a vertex in $G_1$ in a symmetric way. Note that for any pair of vertices $v_i$ and $v_{i+1}$, there exists no vertex adjacent to both $v_i$ and $v_{i+1}$ since $G(2^m, 4)$ does not have a cycle of length 3.) There exists a vertex $v_p^0$ adjacent to $s$ such that $(s, v_p^0, v_p^1, v_p^2)$ is a fault-free path (and $v_p^2 \neq t$). Letting $P'$ be a $v_p^2$–$t$ path in $G_2$ of every length $l'$, $m - k - 2 \leq l' \leq 2^{m-2} - f^2_v - 1$, we have an $s$–$t$ path $P_0 = (s, v_p^0, v_p^1, P')$ of every length $l_0, m - k + 1 \leq l_0 \leq 2^{m-2} - f^2_v + 2$. Obviously, $2^{m-2} - \lceil \frac{m-5}{2^k} \rceil \leq 2^{m-2} - f^2_v + 2$.

We are to construct a longer path $P_1$ that passes through vertices in $G_0, G_1, G_2$. There exists a fault-free vertex $v_i^1$ in $G_2$ adjacent to $t$ such that all of $v_i^1$, $(v_i^2, t)$, and $(v_i^3, v_i^4)$ are fault-free. Letting $P'$ be an $s$–$v_i^1$ path in $G_0 \oplus G_1$ of every length $l'$, $m - k + 1 \leq l' \leq 2^{m-1} - f^0_v - f^1_v - 1$, we have an $s$–$t$ path $P_1 = (P', v_i^1, t)$ of every length $l_1, m - k + 3 \leq l_1 \leq 2^{m-1} - f^0_v - f^1_v + 1$. Observe that $m - k + 3 \leq 2^{m-2} - \lceil \frac{m-5}{2^k} \rceil + 1$ since $m \geq 3 \geq 2^{m-2} - \lceil \frac{m-5}{2^k} \rceil + 1$ for any $m \geq 5$. Finally, it remains to verify the path $P_2$ longer than $P_1$. $P_2$ is constructed from $P_1$ by replacing the edge $(v_i^1, t)$ with a $v_i^2$–$t$ path in $G_2 \oplus G_1$ of every length $l''$, $m - k + 1 \leq l'' \leq 2^{m-2} - f^0_v - f^1_v - 1$. Then, the length $l_2$ of $P_2$ is any integer in the range $2m - 2k + 3 \leq l_2 \leq 2m - f_1$. Observe that $2m - 2k + 3 \leq 2^{m-1} - f^0_v - f^1_v + 2$ since $2m - 3 \leq 2^{m-1} - (m - 3) + 2 \leq 2^{m-1} - f^0_v - f^1_v + 2$ for any $m \geq 5$.

Case 2. Either $k \geq 1$ and $F_0 = F$ or $k = 0$ and $|F_0| \geq |F| - 1$.

In this case, we have $f_0 \geq 1$. Let us consider pancyclicity of $G_0, G_1 \oplus G_2, G_2 \oplus G_3,$ and $G_1 \oplus G_2 \oplus G_3$ first in the following Claim 3 through 5.

**Claim 3.** $G_1 \setminus F_i$ is $m - k - 1$-panconnected for every $i = 1, 2, 3$, except only when $m = 5, k = 0, f_0 = 1,$ and $f_j = 1$ for some $j = 1, 2, 3$.

Recall that $G_1$ is $\lceil \frac{(m-2)-3}{2^k} \rceil$-fault $m - k - 1$-panconnected. The claim holds for $k \geq 1$ or $m \geq 6$ and $k = 0$ since $|F_i| = 0$ for $k \geq 1$ and $\lceil \frac{(m-2)-3}{2^k} \rceil = m - 5 \geq |F_i|$ for $m \geq 6$ and $k = 0$. If $m = 5$ and $k = 0$, we have $|F| = 2$, and thus the claim holds only when $f_1 = f_2 = f_3 = 0$. This completes the proof of the claim.

For the exceptional case of Claim 3, it will be proved later in Lemma 7 that $G(2^5, 4) \setminus F$ with $|F| = 2$ and $f_0 = f_j = 1$ for some $j = 1, 2, 3$ is 6-panconnected. Hereafter in this proof, we will exclude the exceptional case. Then, we have $|F_i| \leq \lceil \frac{(m-2)-3}{2^k} \rceil$ for every $i = 1, 2, 3$. By virtue of Claim 1, we have Claim 4.

**Claim 4.** $G_1 \oplus G_2 \setminus F$ and $G_2 \oplus G_3 \setminus F$ are $m - k - 1$-panconnected.

**Claim 5.** $G_1 \oplus G_2 \oplus G_3 \setminus F$ is $m - k + 1$-panconnected with an exception of $m = 5$ and $k = 1$.

To prove the claim, between any pair of vertices $x$ and $y$, an $x$–y path of every length $m - k + 1$ or more will be constructed. First, we consider the case where $x$ and $y$ are contained in $G_1 \oplus G_2$. There exists an $x$–y path $P_0$ in $G_0 \oplus G_1$ of every length $l_0, m - k + 1 \leq l_0 \leq 2^{m-1} - f^0_v - f^1_v - 1$, by Claim 4. To construct a longer path, we assume w.l.o.g. that $F_3 = F_{2,3} = \emptyset$ if both $x$ and $y$ are contained in $G_2$. Let $P'$ be an $x$–y path in $G_1 \oplus G_2$ of every length $l' \geq 2^{m-2} + 4$. Then, there exists an edge $(v^2_p, v^2_q)$ on $P'$ such that $v^2_p, v^2_q$ are fault-free. Let $P''$ be a $v^1_p$–$v^1_q$ path in $G_2$ of every length $l'', l'' \geq m - k - 1$ for $m \geq 6$ and $l'' \geq 3$ for $m = 5$. The length $l_1$ of an $x$–y path $P_1$ obtained from merging $P'$ and $P''$ is any integer in the range $2m - 2k + 3 \leq l_1 \leq 3 \cdot 2^{m-2} - f^0_v - f^1_v - f^3_v - 1$ for $m \geq 6$ and in the range $2^{m-2} + 4 \leq l_1 \leq 2^{m-2} - f^0_v - f^1_v - f^3_v - 1$ for $m = 5$. It is straightforward to check that $2^{m-2} + 4 \leq m - k \leq 2m - 3 \leq f^1_v - f^0_v$ for $m \geq 6$ and $2m - 2k + 3 \leq f^3_v$ for $m = 5$.

Let $x$ and $y$ be vertices in $G_1$ and $G_3$, respectively, and let $x = v^1_i$ and $y = v^3_j$. When $j \neq 1$, we assume w.l.o.g. that path $(x, v^2_i, v^3_j)$ and $G_3$ are fault-free. Letting $P'$ be a $v^1_i$–$y$ path in $G_3$ of every length $l'$, $3 \leq l' \leq 2^{m-2} - 1$, by Lemma 4. Then, we have an $x$–y path $P_0 = (x, v^1_i, v^2_i, P')$ of every length $l_0, 6 \leq l_0 \leq 2^{m-2} + 2$. Note that $6 \leq m - k + 1$ unless $m = 5$ and $k = 1$. To construct a longer path $P_1$, let
Claim 5 is considered later in Lemma 8. It will be proved that $G(2^2, 4) \setminus F$ with $|F| = f_0 = 1$ is 5-panconnected. We also exclude the exceptional case in our discussion. Now, we will construct an $s \rightarrow t$ path of every length $m - k + 1$ or more. An $s \rightarrow t$ path of every length between $m - k + 1$ and $3 \cdot 2^{m-2} - 2$ is constructed in Subcases 2.1 through 2.4, and a path of every length $3 \cdot 2^{m-2} - 1$ or more is constructed in Subcases 2.5 through 2.7.

Subcase 2.1. Both $s$ and $t$ are contained in $G_0$.
Assume w.l.o.g. that $F_{0,1} \cup F_1 = \emptyset$. Let $s = v^0_0$ and $t = v^0_j$. Letting $P'$ be a $v^1_0 - v^1_j$ path in $G_1$ of every length $l'$, $m - k - 1 \leq l' \leq 2^{m-2} - 1$, we have an $s \rightarrow t$ path $P_0 = (s, P', t)$ of every length $l_0$, $m - k + 1 \leq l_0 \leq 2^{m-2} + 1$. Letting $P''$ be a $v^1_0 - v^1_j$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $l'' \geq m - k + 1$, we have a longer path $P_1 = (s, P''', t)$ of every length $l_1$, $m - k + 3 \leq l_1 \leq 3 \cdot 2^{m-2} - f_1^1 - f^2_v - f^3_v + 1$.

Subcase 2.2. $s$ and $t$ are contained in $G_0$ and $G_1$, respectively.
Let $s = v^0_0$ and $t = v^0_j$. Let $v^0_0$ be a vertex in $G_0$ which is either $s$ or at least adjacent to it such that $i \neq j$ and path $(s, v^0_i, v^1_j)$ is fault-free. Letting $P'$ be a $v^1_0 - v^1_j$ path in $G_1$ of every length $l'$, $m - k - 1 \leq l' \leq 2^{m-2} - 1$, we have an $s \rightarrow t$ path $P_0 = (s, P', t)$ of every length $l_0$, $m - k + 1 \leq l_0 \leq 2^{m-2} - 1$. Letting $P''$ be a $v^1_0 - v^1_j$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $l'' \geq m - k + 1$, we have an $s \rightarrow t$ path $P_1 = (s, P''', t)$ of every length $l_1$, $m - k + 3 \leq l_1 \leq 3 \cdot 2^{m-2} - f_1^1 - f^2_v - f^3_v + 1$.

Subcase 2.3. $s$ and $t$ are contained in $G_0$ and $G_2$, respectively.
Let $s = v^0_0$ and $t = v^0_j$, and assume w.l.o.g. that $j \neq 1$. When $F_{0,1} \cup F_1 \cup F_{1,2} = \emptyset$, letting $P'$ be a $v^1_0 - v^1_j$ path in $G_1$ of every length $l' \geq m - k - 1$, we have an $s \rightarrow t$ path $P_0 = (s, v^0_0, P', v^1_j)$ of every length $l_0$, $m - k + 1 \leq l_0 \leq 2^{m-2} + 1$. To construct a longer path, let $v^0_0$ be a vertex in $G_0$ adjacent to $s$ such that $(s, v^0_0, v^1_j)$ is a fault-free path. Letting $P''$ be a $v^1_0 - v^1_j$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s \rightarrow t$ path $P_1 = (s, v^0_0, P'', v^1_j)$ of every length $l_1$, $m - k + 3 \leq l_1 \leq 3 \cdot 2^{m-2} - f_1^1 - f^2_v - f^3_v + 1$.

Subcase 2.4. Both $s$ and $t$ are contained in $G_1 \oplus G_2 \oplus G_3$. By Claim 5, we have an $s \rightarrow t$ path $P_0$ in $G_1 \oplus G_2 \oplus G_3$ of every length $l_0$, $m - k + 1 \leq l_0 \leq 3 \cdot 2^{m-2} - f_1^1 - f^2_v - f^3_v - 1$.

Subcase 2.5. Both $s$ and $t$ are contained in $G_0 \oplus G_1$. For a vertex $x$ in $G_0 \oplus G_1$, we denote by $\bar{x}$ the vertex in $G_2 \oplus G_3$ adjacent to $x$. If $f_0 + f_{0,1} + f_1 \leq m - 4$, then there exists an $s \rightarrow t$ hamiltonian path $P'$ in $G_0 \oplus G_1$ by Lemma 3(b). Let $(x, y)$ be an edge on $P'$ such that $\bar{x}, (x, \bar{x}, \bar{y})$, and $(y, \bar{y})$ are fault-free, so that $P' = (s, Q_1, x, y, Q_2, t)$. Letting $P''$ be an $\bar{x} \rightarrow \bar{y}$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s \rightarrow t$ path $P_0 = (s, Q_1, x, P'', y, Q_2, t)$ of every length $l_0$, $2^{m-1} - f_0^1 - f_0^1 + m - k + 1 \leq l_0 \leq 2^{m-1} - f_0^1 - f_0^1$. If $f_0 + f_{0,1} + f_1 = m - 3$, there exists a hamiltonian cycle $(s, Q_1, x, t, Q_2, y)$ in $G_0 \oplus G_1$. Then, letting $P''$ be an $\bar{x} \rightarrow \bar{y}$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s \rightarrow t$ path $P_2 = (s, Q_1, x, P'', y, Q_2^R, t)$ of every length $2^{m-1} - f_0^1 - f_0^1 + m - k + 1$ or more. Here, $Q_2^R$ denotes the reverse path of $Q_2$, that is, $Q_2^R = (z_l, z_{l-1}, \ldots, z_1)$ for $Q_2 = (z_1, z_2, \ldots, z_j)$. Obviously, $2^{m-1} - f_0^1 - f_0^1 + m - k + 1 \leq 3 \cdot 2^{m-2} - 1$ for any $m \geq 5$.

Subcase 2.6. $s$ is contained in $G_0 \oplus G_1$ and $t$ is contained in $G_2 \oplus G_3$. If $f_0 + f_{0,1} + f_1 \leq m - 4$, we let $s$ be a vertex in $G_0 \oplus G_1$ such that $x \neq s, \bar{x} \neq t$, and all of $x, \bar{x}, (x, \bar{x})$ are fault-free. Then, letting $P'$ be an $s \rightarrow x$ hamiltonian path in $G_0 \oplus G_1$ and $P''$ be an $\bar{x} \rightarrow t$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s \rightarrow t$ path $P_2 = (P', P'')$ of every length $l_2$, $2^{m-1} - f_0^1 - f_0^1 + m - k + 1 \leq l_2 \leq 2^{m-1} - f_0^1 - f_0^1$. If $f_0 + f_{0,1} + f_1 = m - 3$, there exists a hamiltonian cycle $(s, x, Q, y)$ in $G_0 \oplus G_1$. Assume w.l.o.g. that $\bar{y} \neq t$. Letting $P''$ be a $\bar{y} \rightarrow t$ path of every length $m - k + 1$ or more, we have an $s \rightarrow t$ path $P_2 = (s, x, Q, y, P'')$ of every length $2^{m-1} - f_0^1 - f_0^1 + m - k + 1$ or more.

Subcase 2.7. Both $s$ and $t$ are contained in $G_2$.
There exists a hamiltonian path in $G_0 \setminus F_0$ by Lemma 3(a) and let the hamiltonian path be $(v^0_0, Q, v^0_1)$. Assume
Let \( G \) and \( G' \) be graphs. Let \( \gamma \) be a function that assigns a label to each edge of \( G \). The function \( \gamma \) is called \( k \)-connected if for any two vertices \( u, v \) in \( G \), there exists a \( k \)-connected path from \( u \) to \( v \). A graph \( G \) is \( k \)-connected if for any two vertices \( u, v \) in \( G \), there exists a \( k \)-connected path from \( u \) to \( v \).

**Lemma 7.** \( G(2^3, 4) \backslash F \) with \( |F| = 2 \) and \( f_0 = f_j = 1 \) for some \( j = 1, 2, 3 \) is 6-panconnected.

**Proof.** We can see that \( G(8, 4) \times K_2 \) is 1-fault 5-panconnected since \( G(8, 4) \times K_2 \) is a four-dimensional restricted HL-graph and every four-dimensional restricted HL-graph was shown to be 1-fault 5-panconnected in [20]. Due to vertex symmetry, we assume \( f_1 = 0 \) (either \( f_2 = 1 \) or \( f_3 = 1 \)). When \( s \) and \( t \) are contained in \( G_0 \oplus G_1 \), there exists an \( s-t \) path \( P_0 \) of every length \( l_0 \), \( 5 \leq l_0 \leq 2^4 - f_0^0 - 1 \). For some edge \( (x, y) \) on \( P_0 \) such that the vertices \( x_0 \) and \( y_0 \) in \( G_0 \oplus G_1 \) adjacent to \( x \) and \( y \), respectively, are fault-free, letting \( P' \) be an \( x \)-\( y \)-\( x_0 \) path in \( G_0 \oplus G_1 \) of every length \( l' \geq 5 \), we can obtain an \( s-t \) path \( P_1 \) from \( P_0 \) and \( P' \). The length \( l_1 \) of \( P_1 \) is any integer in the range \( 11 \leq l_1 \leq 2^5 - f_0 - 1 \).

When \( s \) is contained in \( G_0 \oplus G_1 \) and \( t \) is contained in \( G_2 \oplus G_3 \), we first construct an \( s-t \) path of every length \( 7 \) or more. There exists an edge \( (x, \bar{x}) \) joining a vertex \( x \) in \( G_0 \oplus G_1 \) and a vertex \( \bar{x} \) in \( G_2 \oplus G_3 \) such that (i) \( x \) is adjacent to \( t \), (ii) \( x \neq \bar{x} \), and (iii) \( \bar{x} \) is fault-free. Letting \( P' \) be an \( s \)-\( x \)-\( t \) path in \( G_0 \oplus G_1 \) of every length \( l' \geq 5 \), we have an \( s-t \) path \( P_0 = (P', \bar{x}, t) \) of every length \( l_0 \), \( 7 \leq l_0 \leq 2^4 - f_0^0 + 1 \). Replacing the edge \( (x, \bar{x}) \) on \( P_0 \) with an \( x \)-\( \bar{x} \)-\( x_0 \) path \( P'' \) in \( G_2 \oplus G_3 \) of every length \( l'' \geq 5 \) results in an \( s-t \) path \( P_1 = (P', P'') \) of every length \( l_1 \), \( 11 \leq l_1 \leq 2^5 - f_0 - 1 \). It remains to construct an \( s-t \) path of length \( 7 \). If the vertex \( x \) in \( G_0 \oplus G_1 \) adjacent to \( t \) is fault-free and different from \( s \), then the above construction with \( \bar{x} = x \) and \( t = \bar{x} \) will be sufficient. Symmetrically, if \( \bar{x} \neq t \) and \( \bar{x} \) is fault-free, we are done. Thus, we assume that \( s \) is adjacent to \( t \), or both \( s \) and \( \bar{x} \) are the faulty elements.

For the subcase where \( s \) is adjacent to \( t \), let \( s = v_1^0 \) and \( t = v_3^0 \). If \( f_3 = 0 \), we are done since \( G_3 \oplus G_0 \) is 1-fault 5-panconnected. Otherwise \((f_1 = f_2 = 0)\), letting \( P' \) be a \( v_1^0 \)-\( v_3^0 \) path in \( G_1 \) of length \( 3 \) by Lemma 6(a), we have an \( s-t \) path \((s, P', v_3^0, t)\) of length \( 6 \). Finally, let us consider the subcase where both \( s \) and \( \bar{x} \) are faulty vertices. Since \( \bar{x} \) is faulty and \( f_1 = 0 \), \( t \) is contained in \( G_3 \). Let \( t = v_3^0 \). If \( s \) is contained in \( G_0 \), regarding \( \bar{x} \) as a virtual fault-free vertex, we find an \( s-\bar{x} \) path in \( G_0 \) of length \( 5 \). Letting the path found be \((s, Q, v_3^0, \bar{x})\), we have an \( s-t \) path \((s, Q, v_3^0, v_{j-1}^3, t)\) of length \( 6 \). If \( s \) is contained in \( G_1 \), let \( v_j^3 \) be a vertex adjacent to \( t \) such that \( v_j^3 \neq s \). Observe that path \((v_j^0, v_j^2, v_j^3, t)\) is fault-free. Letting \( P' \) be an \( s-v_j^3 \) path in \( G_1 \) of length \( 3 \), by Lemma 6(a), there exists an \( s-t \) path \((P', v_j^3, v_j^3, t)\) of length \( 6 \). □

**Lemma 8.** \( G(2^3, 4) \backslash F \) with \( |F| = f_0 = 1 \) for some \( i = 0, 1, 2, 3 \) is 5-panconnected.

**Proof.** By Lemma 7, it suffices to construct an \( s-t \) path of length \( 5 \). If \( s \) and \( t \) are contained in \( G_i \oplus G_{i+1 \mod 4} \) for some \( i = 0, 1, 2, 3 \), we are done since \( G(8, 4) \times K_2 \) is 1-fault 5-panconnected. It is assumed w.l.o.g. that \( s = v_1^0 \) and \( t = v_j^3 \) for some \( j \neq 1 \). We can see that (i) \( f_2 = 0 \) and \((s, v_1^3, v_j^3)\) is a fault-free path, or (ii) \( f_0 = 0 \) and \((t, v_1^0, v_j^0)\) is a fault-free path. If condition (i) is satisfied, we have an \( s-t \) path \((s, v_1^3, P'')\), where \( P'' \) is a \( v_1^3 \)-\( t \) path in \( G_2 \) of length \( 3 \); otherwise, an \( s-t \) path can be constructed symmetrically. □

**Remark 1.** Let \( l_m^* \) be the minimum \( l_m^* \) such that \( G(2^m, 4) \) is \( m-3 \)-fault \( l_m^* \)-panconnected. Theorem 1 suggests an upper bound \( m + 1 \) on \( l_m^* \). Of course, \( l_m^* \) cannot be smaller than \( D_m \), and thus we have \( \lceil \frac{3m-1}{4} \rceil \leq l_m^* \leq m + 1 \).

4. Edge-pancyclicity of faulty \( G(2^m, 4) \)

In this section, we will show that \( G(2^m, 4), m \geq 3, \) is \( m-3 \)-fault nearly edge-pancyclic. Since an \( f \)-fault \( l \)-panconnected graph is always \( f \)-fault \( l+1 \)-edge-pancyclic, by Theorem 1, we have the following lemma.

**Lemma 9.** \( G(2^m, 4), m \geq 3, \) is \( m-3 \)-fault \( m+2 \)-edge-pancyclic.
We are to show that $G(m, 4)$, $m \geq 3$, with at most $m - 3$ faulty elements, has a cycle of every length $l$, $l = 4, 6, 7, 8, \ldots, m + 1$, passing through an arbitrary fault-free edge.

**Lemma 10.** (a) $G(2^3, 4)$ is 0-fault almost edge-pancyclic.
(b) $G(2^4, 4)$ is 1-fault nearly edge-pancyclic.

**Proof.** The statement (a) is obvious from Lemma 6(a). To prove (b), it suffices to construct a cycle of length 4 that passes through an arbitrary edge $e$ by Lemma 9. There are two cases up to symmetry. If $e = (v_0^0, v_0^1)$, then at least one of the two cycles $(v_0^0, v_1^0, v_2^0, v_3^0)$ and $(v_0^0, v_1^1, v_2^1, v_3^0)$ is fault-free. If $e = (v_0^0, v_0^1)$, then cycles $(v_0^0, v_0^1, v_1^1, v_1^0)$ or $(v_0^0, v_1^0, v_2^0, v_2^1)$ are fault-free. Thus, we have the lemma. □

**Theorem 2.** $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault nearly edge-pancyclic.

**Proof.** For $m = 3, 4$, the theorem holds by Lemma 10. Assume $m \geq 5$. Let $e$ be an arbitrary fault-free edge whose two end-vertices are also fault-free. By Lemma 9, it suffices to construct a cycle of every even length $l = 4, 6, 7, 8, \ldots, m + 1$, that passes through $e$. There are two cases up to symmetry.

**Case 1.** $e = (v_i^0, v_i^1)$.
If $f_0 \leq (m - 2) - 3$, we have a cycle of every even length $l$, $l = 4, 6, 7, 8, \ldots, 2m - f_0^0$, and we are done since $m + 1 \leq 2m - 2 - (m - 5) \leq 2m - 2 - f_0^0$ for any $m \geq 5$. Let $f_0 \geq m - 4$. Then, there exists at most one faulty element outside $G_0$, and thus $F_{0,1} \cup F_1 = \emptyset$ or $F_{0,3} \cup F_3 = \emptyset$. Assume w.l.o.g. that $F_{0,1} \cup F_1 = \emptyset$. There exists a $v_i^1 - v_{i+1}^0$ path $P'$ in $G_1$ of length $l'$, $l' = 1, 3, 4, 5, \ldots, 2m - 1$, by Lemma 4. Thus, we have a cycle $(v_i^0, P', v_i^0)$ of every length $l$, $l = 4, 6, 7, 8, \ldots, 2m - 2 + 2$.

**Case 2.** $e = (v_i^0, v_i^1)$.
We first construct a cycle of every even length $l$, $4 \leq l \leq m + 1$. Let $F' = \{ (v_a^0, v_b^1) \in F \text{ or } (v_a^0, v_b^0) \in F \} \cup \{ (v_a^0, v_b^0) \in E \}$. Obviously, $|F' \cup F_0| \leq m - 3$. By Lemma 3(a), there exists a hamiltonian path in $G_0 \setminus F_0 \cup F'$ of length at least $2m - 2 - (m - 3) - 1$. The hamiltonian path passes through $v_i^0$, and thus we can construct a fault-free $v_i^0$-path of every length $k$, $1 \leq k \leq \left\lfloor \frac{2m - 2 - (m - 3)}{2} \right\rfloor$. Let the $v_i^0$-path in $G_0 \setminus F_0 \cup F'$ be $(v_i^0, v_i^{0,1}, v_i^{1,0}, \ldots, v_i^{k,0})$. Then, by the construction, $v_i^0$-path $(v_i^1, v_i^0, v_i^1, \ldots, v_i^0)$ is also fault-free. Furthermore, the edge $(v_i^0, v_i^1)$ is fault-free. Thus, we have a cycle $(v_i^0, v_i^1, \ldots, v_i^0, v_i^1, \ldots, v_i^1, v_i^0)$ of length $2k + 2$ for every $k$, $1 \leq k \leq \left\lfloor \frac{2m - 2 - (m - 3)}{2} \right\rfloor$. The construction of every even cycle passing through $e$ is completed since $2\left[ \frac{2m - 2 - (m - 3)}{2} \right] + 2 \geq 2m - 2 - (m - 3) + 1 \geq m + 1$ for any $m \geq 5$.

Now, it remains to construct a cycle of every odd length $l$, $7 \leq l \leq m + 1$. We first claim that for some vertex $v_i^0$ in $G_0$ adjacent to $v_i^0$, a cycle $C_2 = (v_i^0, v_i^0, v_i^1, v_i^1, v_i^0, v_i^0)$ associated with $v_i^0$ is fault-free. There are in total $m - 2$ cycles associated with vertices in $G_0$ adjacent to $v_i^0$, and any two cycles among them are disjoint excluding $v_i^0$, $v_i^1$, and $v_i^1$. Note that it is impossible for both $v_i^1$ and $v_i^{0,1}$ to be adjacent to $v_i^0$ since $G(2^m, 4)$ has no cycle of length 3. Since there are at most $m - 3$ faulty elements, at least one of the cycles is fault-free. Thus, the claim is proved. Observe that $C_2$ has a single edge in $G_0$, in $G_1$, and in $G_3$, respectively. It is straightforward to see that at least one of $G_0$, $G_1$, and $G_3$ has at most $m - 3$ faulty elements. Assume w.l.o.g. that $f_0 \leq m - 5$. Remember that the cycle $C_2$ is of length 7. Since $G_0$ is $m - 5$-faulty nearly edge-pancyclic, $G_0 \setminus F_0$ has a cycle $C$ passing through $(v_i^0, v_i^1)$ of even every even length $l'$, $4 \leq l' \leq 2m - 2 - f_0^0$. Thus, there exists a $v_i^0 - v_{i+1}^0$ path $P = C \setminus (v_i^0, v_i^0)$ in $G_0$ of every odd length $l''$, $3 \leq l'' \leq 2m - 2 - f_0^0 - 1$. If we replace the edge $(v_i^0, v_i^0)$ of $C_2$ with $v_i^0 - v_{i+1}^0$ path $P$, we have a cycle of every odd length $l$, $9 \leq l \leq 2m - 2 - f_0^0 + 5$. Obviously, $m + 1 \leq 2m - 2 - (m - 5) + 5 \leq 2m - 2 - f_0^0 + 5$ for any $m \geq 5$. This completes the proof. □

**Remark 2.** $G(2^4, 4)$ has a unique cycle $(v_i^0, v_i^0, v_i^0, v_i^0)$ of length 5 passing through edge $(v_i^0, v_i^0)$. Thus, we cannot say that every $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault almost edge-pancyclic.

5. Concluding remarks

In this paper, we have proven that every recursive circulant $G(2^m, 4)$ with $m \geq 3$ is $m - 3$-fault $m + 1$-panconnected. Here, the upper bound $m - 3$ on the number of faulty elements is the maximum possible in a sense that, for any $f$
with \( f \geq m - 2 \), there exists a fault set \( F \) with \( |F| = f \) such that \( G(2^m, 4) \setminus F \) is not \( l_m \)-panconnected for any \( l_m, l_m \leq |V(G \setminus F)| - 1 \). We have also shown that the result on fault-panconnectivity of \( G(2^m, 4) \) leads to the fact that \( G(2^m, 4), m \geq 3 \), is \( m - 3 \)-fault nearly edge-pancyclic. There remain a number of interesting issues for future research. Finding the minimum \( l^*_m \) such that \( G(2^m, 4) \) is \( m - 3 \)-fault \( l^*_m \)-panconnected will be one of them.

References