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# Panconnectivity and edge-pancyclicity of faulty recursive circulant $G(2^m, 4)$ ☆

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**Abstract**

In this paper, we investigate a problem on embedding paths into recursive circulant $G(2^m, 4)$ with faulty elements (vertices and/or edges) and show that each pair of vertices in recursive circulant $G(2^m, 4)$, $m \geq 3$, are joined by a fault-free path of every length from $m + 1$ to $|V(G(2^m, 4) \setminus F)| - 1$ inclusive for any fault set $F$ with $|F| \leq m - 3$. The bound $m - 3$ on the number of acceptable faulty elements is the maximum possible. Moreover, recursive circulant $G(2^m, 4)$ has a fault-free cycle of every length from 4 to $|V(G(2^m, 4) \setminus F)|$ inclusive excluding 5 passing through an arbitrary fault-free edge for any fault set $F$ with $|F| \leq m-3$.


*Keywords:* Panconnected; Edge-pancyclic; Embedding; Linear arrays; Rings; Recursive circulants; Fault tolerance; Interconnection networks

## 1. Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnection network is an important issue in parallel processing [5,11,19,21–24]. An interconnection network is often modelled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modelled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges.

In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. A graph $G$ is called *$f$-fault hamiltonian* (resp. *$f$-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \setminus F$ for any set $F$ of faulty elements with $|F| \leq f$. On the other hand, if the paths joining each pair of vertices of every length shorter than or equal to a hamiltonian path are required the problem is concerned with panconnectivity of the graph. If the cycles of arbitrary size (up to a hamiltonian cycle) are required the problem is concerned with pancyclicity of the graph.

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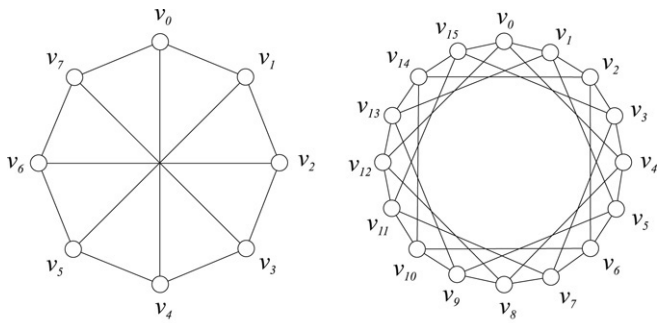


(a) $G(8,4)$ (b) $G(16,4)$

Fig. 1. Examples of $G(N, d)$.

**Definition 1.** A graph $G$ is called *$f$-fault $l$-panconnected* if each pair of fault-free vertices are joined by a path in $G \backslash F$ of every length from $l$ to $|V(G \backslash F)| - 1$ inclusive for any set $F$ of faulty elements with $|F| \leq f$.

**Definition 2.** A graph $G$ is called *$f$-fault almost edge-pancyclic* (resp. *$f$-fault nearly edge-pancyclic*) if for any set $F$ of faulty elements with $|F| \leq f$, there exists a cycle of every length from 4 to $|V(G \backslash F)|$ inclusive (resp. from 4 to $|V(G \backslash F)|$ inclusive excluding 5) that passes through an arbitrary fault-free edge.

Panconnectivity of some interconnection networks without faulty elements was reported in the literature. A graph $G$ is said to be *panconnected* (resp. *almost panconnected*) if each pair of vertices $s$ and $t$ in $G$ are joined by an $s$–$t$ path of every length from $d(s, t)$ to $V(G) - 1$ (resp. from $d(s, t) + 2$ to $V(G) - 1$) inclusive. Here, $d(s, t)$ denotes the distance between $s$ and $t$. Recursive circulant $G(2^m, 2)$ [16], alternating group graphs [5], and augmented cubes [13] are panconnected, and recursive circulant $G(2^m, 4)$ [16], locally twisted cubes [14], and twisted cubes [7] are almost panconnected. Recently, fault-panconnectivity of a family of hypercube-like interconnection networks called *restricted HL-graphs* was investigated in [20]. It was shown that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault $2m - 3$-panconnected. The family includes many interconnection networks proposed in the literature such as twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes.

Edge-pancyclicity of some fault-free interconnection networks such as recursive circulants, crossed cubes, twisted cubes was studied in [1], [9], and [8]. A graph $G$ is called *$f$-fault $l$-edge-pancyclic* if for any fault set $F$ with $|F| \leq f$, there exists a cycle of every length from $l$ to $|V(G \backslash F)|$ inclusive that passes through an arbitrary fault-free edge. An $f$-fault $l$-panconnected graph is obviously $f$-fault $l + 1$-edge-pancyclic. In the presence of faulty elements, the fault-panconnectivity result in [20] implies that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault $2m - 2$-edge-pancyclic.

Pancyclicity and fault-pancyclicity of various interconnection networks were investigated. A graph $G$ is called *$f$-fault pancyclic* (resp. *$f$-fault almost pancyclic*) if $G \backslash F$ contains a cycle of every length from 3 to $|V(G \backslash F)|$ inclusive (resp. 4 to $|V(G \backslash F)|$ inclusive) for any fault set $F$ with $|F| \leq f$. The works on fault-pancyclicity can be summarized as saying that many interconnection networks of degree $\delta$ are $\delta - 2$-fault pancyclic or $\delta - 2$-fault almost pancyclic depending on the existence of length 3 cycles in the network; for example, augmented cubes [13], recursive circulants [2,17], Möbius cubes [11], crossed cubes [23], twisted cubes [24], and restricted HL-graphs [20].

A recursive circulant is an interconnection network proposed in [18]. Recursive circulant $G(N, d)$, $d \geq 2$, is defined as follows: the vertex set $V = \{v_0, v_1, v_2, \ldots, v_{N-1}\}$, and the edge set $E = \{(v_i, v_j) \mid$ there exists $k$, $0 \leq k \leq \lceil \log_d N \rceil - 1$, such that $i + d^k \equiv j \pmod N\}$. $G(N, d)$ is a circulant graph with $N$ vertices and jumps of powers of $d$, $d^0, d^1, \ldots, d^{\lceil \log_d N \rceil - 1}$. Examples of $G(N, d)$ are shown in Fig. 1.

In this work, our attention is restricted to $G(N, d)$ with $N = 2^m$ and $d = 4$. $G(2^m, 4)$, whose degree is $m$, compares favorably to the hypercube $Q_m$. While retaining attractive properties of hypercube $Q_m$ such as node symmetry, recursive structure, the maximum connectivity, etc., it achieves noticeable improvements in diameter [18] and possesses a complete binary tree with $2^m - 1$ vertices as a subgraph [12]. A recursive circulant has a cycle-based construction, and thus it is expected to have nice properties concerned with cycles. $G(N, d)$ with degree 3 or higher is hamiltonian-connected [6]. $G(N, d)$ with $N = cd^m$ and $1 \leq c < d$ is hamiltonian decomposable [3,10,15], that is, the set of edges can be partitioned into edge-disjoint hamiltonian cycles (and a 1-factor when the degree is odd). In [10], the edge forwarding index and bisection width for recursive circulants were also analyzed.

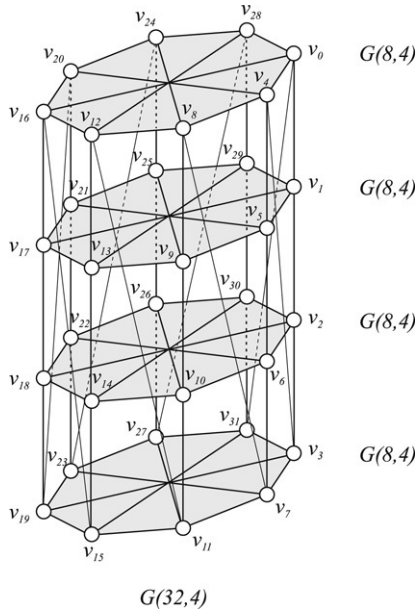


Fig. 2. Recursive structure of $G(32, 4)$.

In this paper, we investigate panconnectivity and edge-pancyclicity of recursive circulant $G(2^m, 4)$ with faulty elements. It will be shown that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected and $m-3$-fault nearly edge-pancyclic. The bound $m-3$ on the number of acceptable faulty elements for $G(2^m, 4)$ to be $l$-panconnected for any fixed $l$ (less than the number of fault-free vertices) is the maximum possible in a sense that no graph of degree $m$ is $m-2$-fault $l$-panconnected as well as hamiltonian-connected.

In the rest of this paper, we will use standard terminology for graphs (see Ref. [4]). This paper is organized as follows. In the next section, we will present some basic properties of recursive circulant $G(2^m, 4)$. Panconnectivity and edge-pancyclicity of faulty recursive circulant $G(2^m, 4)$ will be proved in Sections 3 and 4, respectively. Finally in Section 5, the concluding remarks of this paper will be given.

## 2. Recursive circulant $G(2^m, 4)$

Recursive circulant $G(N, d)$ can also be defined as the Cayley graph of the cyclic group $\mathbb{Z}_N$ with the generating set $\{d^0, d^1, \ldots, d^{\lceil \log_d N \rceil - 1}\}$. Every Cayley graph over a general group is vertex symmetric, and thus regular. Recursive circulant $G(N, d)$ has a recursive structure when $N = cd^m$, $1 \leq c < d$ [18]. In other words, $G(cd^m, d)$ can be defined recursively by utilizing the following property.

**Property 1** (*[18]*). *Let $V_i$ be a subset of vertices in $G(cd^m, d)$ such that $V_i = \{v_j \mid j \equiv i \pmod d\}$, $m \geq 1$. For $0 \leq i \leq d-1$, the subgraph of $G(cd^m, d)$ induced by $V_i$ is isomorphic to $G(cd^{m-1}, d)$.*

$G(cd^m, d)$, $m \geq 1$, can be constructed recursively on $d$ copies of $G(cd^{m-1}, d)$ as follows. Let $G_i(V_i, E_i)$, $0 \leq i \leq d-1$, be a copy of $G(cd^{m-1}, d)$. We assume that $V_i = \{v_0^i, v_1^i, \ldots, v^i_{cd^{m-1}-1}\}$, and $G_i$ is isomorphic to $G(cd^{m-1}, d)$ with the isomorphism mapping $v_j^i$ to $v_j$. We relabel $v_j^i$ by $v_{jd+i}$. The vertex set $V$ of $G(cd^m, d)$ is $\bigcup_{0 \leq i \leq d-1} V_i$, and the edge set $E$ is $\bigcup_{0 \leq i \leq d-1} E_i \cup X$, where $X = \{(v_j, v_{j'}) \mid j+1 \equiv j' \pmod{cd^m}\}$. The construction of $G(32, 4)$ on four copies of $G(8, 4)$ is illustrated in Fig. 2. Note that recursive circulant $G(2^m, 4)$ has a recursive structure when $m \geq 2$. In the recursive structure, $G(2^m, 4)$ consists of four *components* $G_0$, $G_1$, $G_2$, and $G_3$; each of them is isomorphic to $G(2^{m-2}, 4)$. A vertex in $G_i$ is represented by $v_j^i$, $0 \leq j < 2^{m-2}$, $0 \leq i \leq 3$, as well as $v_{j'}$, $0 \leq j' < 2^m$, without saying in which $G_i$ the vertex is contained.

Hereafter in this paper, we denote by $G_i \oplus G_j$ and $G_i \oplus G_j \oplus G_k$ for some $0 \leq i, j, k \leq 3$ the subgraphs of $G(2^m, 4)$ induced by $V_i \cup V_j$ and $V_i \cup V_j \cup V_k$, respectively. Let $F$ be the set of faulty elements in $G(2^m, 4)$. $F_i$ denotes the set of faulty elements in $G_i$, $i = 0, 1, 2, 3$, and $F_{i,i+1 \bmod 4}$ denotes the set of faulty edges joining vertices in $G_i$ and vertices

in $G_{i+1 \bmod 4}$, so that $F = \bigcup_{0 \le i \le 3}(F_i \cup F_{i,i+1 \bmod 4})$. Let $f_i = |F_i|$ and $f_{i,i+1 \bmod 4} = |F_{i,i+1 \bmod 4}|$. We denote by $f_v^i$ the number of faulty vertices in $G_i$, and by $f_v$ the total number of faulty vertices, so that $f_v = \sum_{0 \le i \le 3} f_v^i$.

From now on, all arithmetic on the indices of vertices will be assumed to be done modulo $2^m$. Some properties of recursive circulant $(2^m, 4)$ explored to establish our main results are listed below, where the *diameter* $D_m$ of $G(2^m, 4)$ is defined as the maximum distance between any two vertices in the graph.

**Lemma 1** (*Shortest Path [18]*). *Let $G_0$, $G_1$, $G_2$, and $G_3$ be the components of* $(2^m, 4)$. (a) *Every shortest path joining a pair of vertices $v_0^i$ and $v_j^i$ passes through only vertices in $G_i$.* (b) *There exists a shortest path between $v_0^0$ and $v_j^i$ passing through $v_j^0$ when $i = 1$, and passing through $v_{j+1}^0$ when $i = 3$. In the case $i = 2$, there exists a shortest path between $v_0^0$ and $v_j^i$ passing through $v_j^0$ when $d(v_0^0, v_j^0) \le d(v_0^0, v_{j+1}^0)$, and passing through $v_{j+1}^0$ when $d(v_0^0, v_{j+1}^0) \le d(v_0^0, v_j^0)$.*

**Lemma 2** (*Diameter [18]*). (a) $D_{m-2} + 1 \le D_m \le D_{m-2} + 2$ *for* $m \ge 2$. (b) $D_m = \lceil \frac{3m-1}{4} \rceil$.

**Lemma 3** (*Fault-Hamiltonicity [22,19]*). (a) $G(2^m, 4)$, $m \ge 3$, *is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.* (b) *The product $G(2^m, 4) \times K_2$ of $G(2^m, 4)$ and $K_2$, $m \ge 3$, is $m-2$-fault hamiltonian-connected and $m-1$-fault hamiltonian.*

Lemma 3(a) implies that $G(2^m, 4)$, $m \ge 3$, with at most $m-1$ faulty elements has a hamiltonian path joining some pair of fault-free vertices.

## 3. Panconnectivity of faulty $G(2^m, 4)$

In this section, we will show that $G(2^m, 4)$, $m \ge 3$, is $m-3$-fault $m+1$-panconnected. Throughout this paper, a path in a graph is represented as a sequence of vertices. A path joining a pair of vertices $s$ and $t$ is called an *$s$–$t$ path*.

Panconnectivity of fault-free recursive circulants $G(2^m, 2^k)$ was investigated in [16]. It was shown that between any pair of vertices $s$ and $t$, there exists a path of every length $d(s, t) + \Delta$ or longer for some $\Delta$. One of the results is given in the following, which will be utilized for our purpose.

**Lemma 4** (*[16]*). $G(2^m, 4)$ *is almost panconnected. That is, between any pair of vertices $s$ and $t$ in $G(2^m, 4)$, there exists a path of every length $l$, $d(s, t) + 2 \le l \le 2^m - 1$.*

A *concatenation* of two paths $(x_1, x_2, \ldots, x_p)$ and $(y_1, y_2 \ldots, y_q)$ is defined to be the path $(x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$.

**Lemma 5.** (a) $G(2^m, 4)$, $m \ge 3$, *is 0-fault $D_m + 1$-panconnected.*
(b) $G(2^m, 4)$, $m \ge 5$, *is 0-fault $D_m$-panconnected.*

**Proof.** We prove (a) by induction on $m$. Due to Lemma 4, it suffices to show that for any pair of vertices $s$ and $t$ with $d(s, t) = D_m$, there exists a path of length $D_m + 1$ between them. For $m = 3, 4$, the construction is immediate by inspection. Let $m \ge 5$. We assume $s = v_1^0$ without loss of generality. There are two cases up to symmetry. If $t = v_j^1$ for some $j$ ($\neq 1$), we first find a $v_1^1$–$t$ path $P'$ in $G_1$ of length $D_m$. The path $P'$ exists since $D_{m-2} + 1 \le D_m$. Then, $(s, P')$ is a desired path of length $D_m + 1$. Now, let $t = v_j^2$ for some $j$ ($\neq 1$). By Lemma 1, $D_m$ is equal to $d(v_1^2, t) + 2$ or $d(v_0^2, t) + 2$. We assume w.l.o.g. that $D_m = d(v_1^2, t) + 2$. Letting $P''$ be a shortest $v_1^2$–$t$ path in $G_2$, we have a path $(s, v_0^3, v_1^3, P'')$ of length $d(v_1^2, t) + 3 = D_m + 1$.

To prove (b), we assume that each $G_i$ is $D_{m-2}+1$-panconnected and furthermore, whenever $m-2 \ge 5$, it is $D_{m-2}$-panconnected. It suffices to construct a path of length $D_m$ joining every pair of vertices $s$ and $t$ with $d(s, t) = D_m - 1$. Let $s = v_1^0$. If $t = v_j^0$ for some $j$ ($\neq 1$), there exists an $s$–$t$ path in $G_0$ of every length $D_{m-2} + 1$ or longer, and thus we are done. When $t = v_j^1$ for some $j$ ($\neq 1$), there exists a $v_1^1$–$t$ path $P'$ of length $D_{m-2} + 1$, and $(s, P')$ is an $s$–$t$ path of length $D_{m-2} + 2$. If $D_m = D_{m-2} + 2$, we are done. Suppose otherwise ($D_m = D_{m-2} + 1$); observe $m \ge 7$. Note that $D_3$, $D_4$, $D_5$, and $D_6$ are 2, 3, 4, 5, respectively. Employing the assumption that $G_1$ is $D_{m-2}$-panconnected, we have an $s$–$t$ path $(s, P'')$ of length $D_m$, where $P''$ is a $v_1^1$–$t$ path in $G_1$ of length $D_{m-2}$. Finally when $t = v_j^2$ for some $j$ ($\neq 1$), assuming w.l.o.g. that $d(s, t) = d(v_1^2, t) + 2$, a concatenation of $(s, v_0^3, v_1^3)$ and a shortest $v_1^2$–$t$ path in $G_2$ results in an $s$–$t$ path of length $d(s, t) + 1 = D_m$. Thus, the proof is completed. □

Now, we are to investigate panconnectivity of faulty recursive circulants. We will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected. For $m = 3, 4$, we have the following lemma.

**Lemma 6.** (a) $G(8, 4)$ *is* 0-*fault* 3-*panconnected.*
(b) $G(16, 4)$ *is* 0-*fault* 4-*panconnected and* 1-*fault* 5-*panconnected.*

**Proof.** Lemma 5(a) says that $G(8, 4)$ is 0-fault 3-panconnected and $G(16, 4)$ is 0-fault 4-panconnected. To show that $G(16, 4)$ is 1-fault 5-panconnected, we need to construct an $s$–$t$ path of every length 5 or longer for any pair of fault-free vertices $s$ and $t$ in $G(16, 4)$ with one faulty element. When the faulty element is a vertex $v_f$, the construction of an $s$–$t$ path in $G(16, 4)\backslash v_f$ is by a case analysis and omitted here. Suppose there exists a faulty edge $(x, y)$. If $\{x, y\} = \{s, t\}$, letting $(x, y)$ be a *virtual* fault-free edge, Lemma 5(a) is applied. Otherwise, letting $x \notin \{s, t\}$ be a *virtual* faulty vertex, an $s$–$t$ path of every length up to 14 is constructed. An $s$–$t$ hamiltonian path of length 15 also exists due to Lemma 3(a). □

To prove the main result for $m \geq 5$, we exploit the recursive structure of $G(2^m, 4)$ and a technique: so called "strong induction". In other words, assuming that each component $G_i$ which is isomorphic to $G(2^{m-2}, 4)$ is not only $m-5$-fault $m-1$-panconnected but also $\frac{m-5}{2}$-fault $m-2$-panconnected and $\frac{m-5}{2^2}$-fault $m-3$-panconnected and so on, we show that $G(2^m, 4)$ is $m-3$-fault $m+1$-panconnected and $\frac{m-3}{2}$-fault $m$-panconnected and so on.

**Theorem 1.** $G(2^m, 4)$, $m \geq 3$, *is* $\lfloor \frac{m-3}{2^k} \rfloor$-*fault* $m-k+1$-*panconnected for any integer* $k$, $0 \leq k \leq L(m-3)+1$, *where* $L(n) = \lfloor \log_2 n \rfloor$ *for* $n \geq 1$ *and* $L(0) = 0$.

**Proof.** By Lemma 6, the theorem holds for $m = 3, 4$. Hereafter, we assume $m \geq 5$. Observe that $\lfloor \frac{m-3}{2^k} \rfloor = 0$ if $k = L(m-3)+1$, and that $\lfloor \frac{m-3}{2^k} \rfloor = 1$ if $k = L(m-3)$. When $k = L(m-3)+1$, due to Lemma 5(b), the theorem holds. We claim that $D_m = \lceil \frac{3m-1}{4} \rceil \leq m - L(m-3)$ for any $m \geq 5$. The inequality can be checked for small $m$ in the following table. For $m \geq 19$, it suffices to show that $\frac{3m-1}{4} + 1 \leq m - \lfloor \log_2(m-3) \rfloor$ or equivalently $4\lfloor \log_2(m-3) \rfloor + 3 \leq m$. Let $k \geq 4$ be an integer such that $2^k \leq m-3 < 2^{k+1}$. Then, we have $4\lfloor \log_2(m-3) \rfloor + 3 = 4k+3$ and $2^k + 3 \leq m$. Obviously $4k+3 \leq 2^k + 3$ for any $k \geq 4$, and thus the claim is proved.

| $m$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lceil \frac{3m-1}{4} \rceil$ | 4 | 5 | 5 | 6 | 7 | 8 | 8 | 9 | 10 | 11 | 11 | 12 | 13 | 14 |
| $m - L(m-3)$ | 4 | 5 | 5 | 6 | 7 | 8 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |

From now on, we assume $0 \leq k \leq L(m-3)$. It suffices to consider the case $|F| = \lfloor \frac{m-3}{2^k} \rfloor$ since otherwise, we can choose $\lfloor \frac{m-3}{2^k} \rfloor - |F|$ fault-free edges and regard them as *virtual* faults when we construct an $s$–$t$ path of every length $m-k+1$ or more. Observe that $|F| - 1 = \lfloor \frac{(m-2^k)-3}{2^k} \rfloor \leq \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for any $k \geq 1$. Thus, assuming $f_0 \geq f_j$ for any $j = 1, 2, 3$, there are two cases.

*Case* 1. $f_i \leq \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for every $i = 0, 1, 2, 3$.
It is straightforward to see that $k \leq L((m-2)-3)+1$. Thus, each $G_i \backslash F_i$ is $m-k-1$-panconnected. Furthermore when $m = 5$, $G_i$ is fault-free and, by Lemma 6(a), it is 3-panconnected. We first consider panconnectivity of $G_0 \oplus G_1$.

**Claim 1.** *Each pair of vertices* $x$ *and* $y$ *in* $G_0 \oplus G_1$ *are joined by an* $x$–$y$ *path of every length* $l$, $m-k+1 \leq l \leq 2^{m-1} - f_v^0 - f_v^1 - 1$.

To prove the claim, let $x$ and $y$ be vertices in $G_0$ first. There exists an $x$–$y$ path $P_0$ in $G_0$ of every length $l_0$, $m-k-1 \leq l_0 \leq 2^{m-2} - f_v^0 - 1$. To construct a longer path $P_1$ that passes through vertices in $G_1$ as well as vertices in $G_0$, let $P'$ be an $x$–$y$ path in $G_0$ of every length $l'$, $2m-5 \leq l' \leq 2^{m-2} - f_v^0 - 1$. Then, there is an edge $(v_i^0, v_j^0)$ on $P'$ such that all of $v_i^1$, $(v_i^0, v_i^1)$, $v_j^1$, and $(v_j^0, v_j^1)$ are fault-free since each faulty element can "block" at most two such candidate edges and the number of faulty elements is at most $m-3$. The path $P_1$ can be obtained from merging $P'$ and a $v_i^1$–$v_j^1$ path $P''$ in $G_1$ with the edges $(v_i^0, v_i^1)$ and $(v_j^0, v_j^1)$. When $m \geq 6$, the length $l''$ of $P''$ is any integer in the range $m-k-1 \leq l'' \leq 2^{m-2} - f_v^1 - 1$ and thus the length $l_1$ of $P_1$ is in the range $(2m-5) + (m-k-1) + 1 \leq l_1 \leq 2^{m-1} - f_v^0 - f_v^1 - 1$. It is straightforward to see that $(2m-5) + (m-k-1) + 1 \leq (2^{m-2} - f_v^0 - 1) + 1$ since $3m-5 \leq 2^{m-2} - (m-5)$ for every $m \geq 6$. When

$m = 5$, observing $f_i = 0$ for each $i$, we have $5 \le l' \le 7$. Furthermore, by Lemma 6(a), we have $3 \le l'' \le 7$. Thus, $9 \le l_1 \le 15$. It remains to construct an $x$–$y$ path of length 8. Let $v_p^0$ (resp. $v_q^0$) be a vertex in $G_0$ which is either $x$ (resp. $y$) or at least adjacent to it such that (i) $v_p^0 \ne y$ and $v_q^0 \ne x$, (ii) $(v_p^0, v_p^1)$ and $(v_q^0, v_q^1)$ are fault-free, and (iii) $v_p^0 \ne v_q^0$. Since there exists a $v_p^1$–$v_q^1$ path $P''$ of every length $l''$, $3 \le l'' \le 7$, we have an $x$–$y$ path $P_1 = (s, v_p^0, v_p^1, P'', v_q^1, v_q^0, y)$ of length 8. Therefore, we have an $x$–$y$ path of every length $l$, $m - k - 1 \le l \le 2^{m-1} - f_v^0 - f_v^1 - 1$.

Now, let $x$ be a vertex in $G_0$ and $y$ be a vertex in $G_1$. Let $v_p^1$ be a vertex in $G_1$ which is either $y$ or at least adjacent to it such that (i) $v_p^0 \ne x$ and (ii) $v_p^0$ ,$v_p^1$, and $(v_p^0, v_p^1)$ are fault-free. The existence of such a vertex $v_p^1$ is due to there being $m - 1$ candidates and at most $m - 2$ blocking elements (the source $x$ and at most $m - 3$ faulty elements). Letting $P'$ be an $x$–$v_p^0$ path in $G_0$ of every length $l'$, $m - k - 1 \le l' \le 2^{m-2} - f_v^0 - 1$, we have an $x$–$y$ path $P_0 = (P', v_p^1, y)$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} - f_v^0$. To construct a longer path, we let $(v_q^0, v_q^1)$ be an edge such that (i) $v_q^0 \ne x$ and $v_q^1 \ne y$, and (ii) $v_q^0$, $v_q^1$, and $(v_q^0, v_q^1)$ are fault-free. Letting $P'$ be an $x$–$v_q^0$ path in $G_0$ of every length $l'$, $m - k - 1 \le l' \le 2^{m-2} - f_v^0 - 1$, and letting $P''$ be a $v_q^1$–$y$ path in $G_1$ of every length $l''$, $m - k - 1 \le l'' \le 2^{m-2} - f_v^1 - 1$, we have an $x$–$y$ path $P_1 = (P', P'')$ of every length $l_1$, $2m - 2k - 1 \le l_1 \le 2^{m-1} - f_v^0 - f_v^1 - 1$. We have $2m - 2k - 1 \le 2^{m-2} - f_v^0 + 1$ since $2m - 1 \le 2^{m-2} - (m - 5) + 1$ for every $m \ge 5$. Therefore, we have an $x$–$y$ path of every length $m - k + 1$ or more. This completes the proof of Claim 1.

Note that for each of $G_1 \oplus G_2$, $G_2 \oplus G_3$, and $G_3 \oplus G_0$, we can establish the same statement as Claim 1 since we do not use the assumption of $f_0 \ge f_1, f_2, f_3$ in the proof. From now on, we will construct an $s$–$t$ path of every length $l$, $m - k + 1 \le l \le 2^m - f_v - 1$. We assume w.l.o.g. that $s$ is contained in $G_0$.

*Subcase* 1.1. $t$ is a vertex in $G_0$, $G_1$, or $G_3$.
We assume w.l.o.g. that $t$ is contained in $G_0 \oplus G_1$. By Claim 1, there exists an $s$–$t$ path $P_0$ in $G_0 \oplus G_1$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-1} - f_v^0 - f_v^1 - 1$. Let $P'$ be an $s$–$t$ path in $G_0 \oplus G_1$ of every length $l'$, $2m - 5 \le l' \le 2^{m-1} - f_v^0 - f_v^1 - 1$. There is an edge $(x, y)$ on $P'$ such that $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free, where $\bar{x}$ and $\bar{y}$ are the vertices in $G_2 \oplus G_3$ adjacent to $x$ and $y$, respectively. Letting $P''$ be an $\bar{x}$–$\bar{y}$ path in $G_2 \oplus G_3$ of every length $l''$, $m - k + 1 \le l'' \le 2^{m-1} - f_v^2 - f_v^3 - 1$, an $s$–$t$ path $P_1$ can be obtained from merging $P'$ and $P''$ with edges $(x, \bar{x})$ and $(y, \bar{y})$. The length $l_1$ of $P_1$ is any integer in the range $3m - k - 3 \le l_1 \le 2^m - f_v - 1$. It holds true that $3m - k - 3 \le 2^{m-1} - f_v^0 - f_v^1 - 1 + 1$ since $3m - 3 \le 2^{m-1} - (m - 3)$ for any $m \ge 5$. Thus, we have an $s$–$t$ path of every length $m - k + 1$ or more.

*Subcase* 1.2. $t$ is a vertex in $G_2$.
We let $s = v_1^0$ and $t = v_j^2$ for some $j$. First, we will construct an $s$–$t$ path $P_0$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor$. Let us consider the subcase when $|F| - 1 \le f_0 + f_2 \le |F|$. In this subcase, we assume w.l.o.g. that $j \ne 1$. (Suppose otherwise; we can construct an $s$–$t$ path $P_0$ with the roles of $G_1$ and $G_3$ being interchanged in a symmetric way.) If all of $v_1^1$, $(s, v_1^1)$, $v_j^1$, and $(t, v_j^1)$ are fault-free, letting $P'$ be a $v_1^1$–$v_j^1$ path in $G_1$ of every length $l'$, $m - k - 1 \le l' \le 2^{m-2} - f_v^1 - 1$, we have an $s$–$t$ path $P_0 = (s, P', t)$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} - f_v^1 + 1$. Obviously, $2^{m-2} - \lfloor \frac{m-5}{2} \rfloor \le 2^{m-2} - f_v^1 + 1$. Suppose otherwise; exactly one among the four elements $v_1^1$, $(s, v_1^1)$, $v_j^1$, and $(t, v_j^1)$ is faulty. If $j \ne 0$, an $s$–$t$ path $P_0$ passing through vertices in $G_3$ can be constructed symmetrically. Let $j = 0$ and let $v_i^2$ be a vertex in $G_2$ adjacent to $t$ such that $v_i^2$ and $(t, v_i^2)$ are fault-free. There is a $v_0^3$–$v_i^3$ path $P'$ in $G_3$ of every length $l'$, $3 \le l' \le 2^{m-2} - 1$, by Lemma 4. Note that $G_3$ is fault-free and $v_0^3$ is adjacent to $v_i^3$. Thus, the length $l_0$ of $P_0 = (s, P', v_i^2, t)$ is any integer in the range $6 \le l_0 \le 2^{m-2} + 2$. Observe that $6 \le m - k + 1$ for any $m$ and $k$ with $m \ge 5$ and $0 \le k \le L(m - 3)$ except only when $m = 5$ and $k = 1$ ($|F| = 1$). For the exceptional case, regarding the faulty element as a *virtual* fault-free one, we will construct two vertex-disjoint $s$–$t$ paths of length 5. Letting $P'$ be an $s$–$v_0^0$ path of length 3 in $G_0$ and $P''$ be a $v_1^2$–$t$ path of length 3 in $G_2$, we have two paths $(P', v_0^1, t)$ and $(s, v_1^1, P'')$. At least one of the two is a fault-free path since $|F| = 1$.

Now we will construct an $s$–$t$ path $P_0$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor$, when $f_0 + f_2 \le |F| - 2$ ($|F| \ge 2$). Remember that $f_2 \le f_0$. Then, in the following claim, we can obtain a result stronger than that $G_2 \backslash F_2$ is $m - k - 1$-panconnected.

**Claim 2.** *$G_2 \backslash F_2$ is $m - k - 2$-panconnected.*

To prove the claim, it suffices to show that $f_2 \le \lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor$ and $k+1 \le L((m-2)-3)+1$. Suppose $f_2 \ge \lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor + 1$; we have $f_0 + f_2 \ge 2\lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor + 2 \ge \lfloor \frac{(m-2)-3}{2^k} \rfloor + 1 \ge \lfloor \frac{m-3}{2^k} + \frac{2 \cdot 2^k - 2}{2^k} \rfloor - 1 \ge \lfloor \frac{m-3}{2^k} \rfloor - 1 = |F| - 1$, which is a contradiction. Suppose $k \ge L((m-2)-3)+1$; we have $|F| = \lfloor \frac{m-3}{2^k} \rfloor \le \lfloor \frac{m-3}{2^{L((m-2)-3)+1}} \rfloor \le 1$ since $m-3 < 2 \cdot 2^{L((m-2)-3)+1}$ for any $m \ge 5$. This is a contradiction to $|F| \ge 2$. Thus, we have the claim.

In the subcase of $f_0 + f_2 \le |F| - 2$, we assume w.l.o.g. that $p \ne j$ for each vertex $v_p^0$ adjacent to $s$. (Suppose otherwise; we can construct an $s$–$t$ path $P_0$ passing through a vertex in $G_3$ instead of a vertex in $G_1$ in a symmetric way. Note that for any pair of vertices $v_i$ and $v_{i+1}$, there exists no vertex adjacent to both $v_i$ and $v_{i+1}$ since $G(2^m, 4)$ does not have a cycle of length 3.) There exists a vertex $v_p^0$ adjacent to $s$ such that $(s, v_p^0, v_p^1, v_p^2)$ is a fault-free path (and $v_p^2 \ne t$). Letting $P'$ be a $v_p^2$–$t$ path in $G_2$ of every length $l'$, $m-k-2 \le l' \le 2^{m-2} - f_v^2 - 1$, we have an $s$–$t$ path $P_0 = (s, v_p^0, v_p^1, P')$ of every length $l_0$, $m-k+1 \le l_0 \le 2^{m-2} - f_v^2 + 2$. Obviously, $2^{m-2} - \lfloor \frac{m-5}{2} \rfloor \le 2^{m-2} - f_v^2 + 2$.

We are to construct a longer path $P_1$ that passes through vertices in $G_0$, $G_1$, and $G_2$. There exists a fault-free vertex $v_i^2$ in $G_2$ adjacent to $t$ such that all of $v_i^1$, $(v_i^2, t)$, and $(v_i^2, v_i^1)$ are fault-free. Letting $P'$ be an $s$–$v_i^1$ path in $G_0 \oplus G_1$ of every length $l'$, $m-k+1 \le l' \le 2^{m-1} - f_v^0 - f_v^1 - 1$, we have an $s$–$t$ path $P_1 = (P', v_i^2, t)$ of every length $l_1$, $m-k+3 \le l_1 \le 2^{m-1} - f_v^0 - f_v^1 + 1$. Observe that $m-k+3 \le 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor + 1$ since $m+3 \le 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor + 1$ for any $m \ge 5$. Finally, it remains to construct a path $P_2$ longer than $P_1$. $P_2$ is constructed from $P_1$ by replacing the edge $(v_i^2, t)$ with a $v_i^2$–$t$ path in $G_2 \oplus G_3$ of every length $l''$, $m-k+1 \le l'' \le 2^{m-1} - f_v^2 - f_v^3 - 1$. Then, the length $l_2$ of $P_2$ is any integer in the range $2m-2k+3 \le l_2 \le 2^m - f_v - 1$. Observe that $2m-2k+3 \le 2^{m-1} - f_v^0 - f_v^1 + 2$ since $2m+3 \le 2^{m-1} - (m-3) + 2 \le 2^{m-1} - f_v^0 - f_v^1 + 2$ for any $m \ge 5$.

*Case* 2. Either $k \ge 1$ and $F_0 = F$ or $k = 0$ and $|F_0| \ge |F| - 1$.
In this case, we have $f_0 \ge 1$. Let us consider panconnectivity of $G_1$, $G_1 \oplus G_2$, $G_2 \oplus G_3$, and $G_1 \oplus G_2 \oplus G_3$ first in the following Claim 3 through 5.

**Claim 3.** *$G_i \backslash F_i$ is $m-k-1$-panconnected for every $i = 1, 2, 3$, except only when $m = 5$, $k = 0$, $f_0 = 1$, and $f_j = 1$ for some $j = 1, 2, 3$.*

Recall that $G_i$ is $\lfloor \frac{(m-2)-3}{2^k} \rfloor$-fault $m-k-1$-panconnected. The claim holds for $k \ge 1$ or $m \ge 6$ and $k = 0$ since $|F_i| = 0$ for $k \ge 1$ and $\lfloor \frac{(m-2)-3}{2^k} \rfloor = m-5 \ge |F_i|$ for $m \ge 6$ and $k = 0$. If $m = 5$ and $k = 0$, we have $|F| = 2$, and thus the claim holds only when $f_1 = f_2 = f_3 = 0$. This completes the proof of the claim.

For the exceptional case of Claim 3, it will be proved later in Lemma 7 that $G(2^5, 4) \backslash F$ with $|F| = 2$ and $f_0 = f_j = 1$ for some $j = 1, 2, 3$ is 6-panconnected. Hereafter in this proof, we will exclude the exceptional case. Then, we have $|F_i| \le \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for every $i = 1, 2, 3$. By virtue of Claim 1, we have Claim 4.

**Claim 4.** *$G_1 \oplus G_2 \backslash F$ and $G_2 \oplus G_3 \backslash F$ are $m-k+1$-panconnected.*

**Claim 5.** *$G_1 \oplus G_2 \oplus G_3 \backslash F$ is $m-k+1$-panconnected with an exception of $m = 5$ and $k = 1$.*

To prove the claim, between any pair of vertices $x$ and $y$, an $x$–$y$ path of every length $m-k+1$ or more will be constructed. First, we consider the case where $x$ and $y$ are contained in $G_1 \oplus G_2$. There exists an $x$–$y$ path $P_0$ in $G_0 \oplus G_1$ of every length $l_0$, $m-k+1 \le l_0 \le 2^{m-1} - f_v^1 - f_v^2 - 1$, by Claim 4. To construct a longer path, we assume w.l.o.g. that $F_3 = F_{2,3} = \emptyset$ if both $x$ and $y$ are contained in $G_2$. Let $P'$ be an $x$–$y$ path in $G_1 \oplus G_2$ of every length $l' \ge 2^{m-2} + 4$. Then, there exists an edge $(v_p^2, v_q^2)$ on $P'$ such that $v_p^3$, $(v_p^2, v_p^3)$, $v_q^3$, and $(v_q^2, v_q^3)$ are fault-free. Let $P''$ be a $v_p^3$–$v_q^3$ path in $G_3$ of every length $l''$, $l'' \ge m-k-1$ for $m \ge 6$ and $l'' \ge 3$ for $m = 5$. The length $l_1$ of an $x$–$y$ path $P_1$ obtained from merging $P'$ and $P''$ is any integer in the range $2^{m-2} + 4 + m - k \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 - 1$ for $m \ge 6$ and in the range $2^{m-2} + 4 + 4 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 - 1$ for $m = 5$. It is straightforward to check that $2^{m-2} + 4 + m - k \le 2^{m-1} - f_v^1 - f_v^2$ for $m \ge 6$ and $2^{m-2} + 4 + 4 \le 2^{m-1} - f_v^1 - f_v^2 = 2^{m-1}$ for $m = 5$.

Let $x$ and $y$ be vertices in $G_1$ and $G_3$, respectively, and let $x = v_1^1$ and $y = v_j^3$. When $j \ne 1$, we assume w.l.o.g. that path $(x, v_1^2, v_1^3)$ and $G_3$ are fault-free. Letting $P'$ be a $v_1^3$–$y$ path in $G_3$ of every length $l' \ge m-k-1$, we have an $x$–$y$ path $P_0 = (x, v_1^2, P')$ of every length $l_0$, $m-k+1 \le l_0 \le 2^{m-2} + 1$. When $j = 1$ ($y = v_1^3$), there exists a vertex $v_i^1$ adjacent to $x$ such that path $(x, v_i^1, v_i^2, v_i^3, y)$ is fault-free. Assuming w.l.o.g. that $G_3$ is fault-free, we find a $v_i^3$–$y$ path $P'$ in $G_3$ of every length $l'$, $3 \le l' \le 2^{m-2} - 1$, by Lemma 4. Then, we have an $x$–$y$ path $P_0 = (x, v_i^1, v_i^2, P')$ of every length $l_0$, $6 \le l_0 \le 2^{m-2} + 2$. Note that $6 \le m-k+1$ unless $m = 5$ and $k = 1$. To construct a longer path $P_1$, let

$(v_p^2, v_p^3)$ be a fault-free edge with $v_p^2$ and $v_p^3$ being fault-free. Letting $P'$ be an $x$–$v_p^2$ path in $G_1 \oplus G_2$ of every length $l'$, $m-k+1 \le l' \le 2^{m-1} - f_v^1 - f_v^2 - 1$, and $P''$ be a $v_p^3$–$y$ path in $G_3$ of every length $l''$, $D_{m-2}+1 \le l'' \le 2^{m-2}-1$, we have an $x$–$y$ path $P_1 = (P', P'')$ of every length $l_1$, $m - k + D_{m-2} + 3 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - 1$. Since $m + D_{m-2} + 3 \le 2^{m-2} + 2$ for any $m \ge 5$, an $x$–$y$ path of every length $m - k + 1$ or more is constructed. Thus, the proof of the claim is completed.

The exceptional case of Claim 5 is considered later in Lemma 8. It will be proved that $G(2^5, 4)\backslash F$ with $|F| = f_0 = 1$ is 5-panconnected. We also exclude the exceptional case in our discussion. Now, we will construct an $s$–$t$ path of every length $m - k + 1$ or more. An $s$–$t$ path of every length between $m - k + 1$ and $3 \cdot 2^{m-2} - 2$ is constructed in Subcases 2.1 through 2.4, and a path of every length $3 \cdot 2^{m-2} - 1$ or more is constructed in Subcases 2.5 through 2.7.

*Subcase* 2.1. Both $s$ and $t$ are contained in $G_0$.
Assume w.l.o.g. that $F_{0,1} \cup F_1 = \emptyset$. Let $s = v_1^0$ and $t = v_j^0$. Letting $P'$ be a $v_1^1$–$v_j^1$ path in $G_1$ of every length $l'$, $m - k - 1 \le l' \le 2^{m-2} - 1$, we have an $s$–$t$ path $P_0 = (s, P', t)$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} + 1$. Letting $P''$ be a $v_1^1$–$v_j^1$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $l'' \ge m - k + 1$, we have a longer path $P_1 = (s, P'', t)$ of every length $l_1$, $m - k + 3 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 + 1$.

*Subcase* 2.2. $s$ and $t$ are contained in $G_0$ and $G_1$, respectively.
Let $s = v_1^0$ and $t = v_j^1$. Let $v_i^0$ be a vertex in $G_0$ which is either $s$ or at least adjacent to it such that $i \ne j$ and path $(s, v_i^0, v_i^1)$ is fault-free. Letting $P'$ be a $v_i^1$–$t$ path in $G_1$ of every length $l' \ge m - k - 1$, we have an $s$–$t$ path $P_0 = (s, v_i^0, P')$ of every length $l_0$, $m-k+1 \le l_0 \le 2^{m-2} - f_v^1$. Letting $P''$ be a $v_i^1$–$t$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $l'' \ge m-k+1$, we have an $s$–$t$ path $P_1 = (s, v_i^0, P'')$ of every length $l_1$, $m-k+3 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3$.

*Subcase* 2.3. $s$ and $t$ are contained in $G_0$ and $G_2$, respectively.
Let $s = v_1^0$ and $t = v_j^2$, and assume w.l.o.g. that $j \ne 1$. When $F_{0,1} \cup F_1 \cup F_{1,2} = \emptyset$, letting $P'$ be a $v_1^1$–$v_j^1$ path in $G_1$ of every length $l' \ge m - k - 1$, we have an $s$–$t$ path $P_0$ of every length $l_0$, $m - k + 1 \le l_0 \le 2^{m-2} + 1$. If $F_{0,1} \cup F_1 \cup F_{1,2} \ne \emptyset$, then $F_{0,3} \cup F_3 \cup F_{3,2} = \emptyset$. When $j \ne 0$, an $s$–$t$ path $P_0$ passing through vertices in $G_3$ can be constructed symmetrically. When $j = 0$, letting $P'$ be a $v_0^3$–$v_1^3$ path in $G_3$ of length 3 or more, we have an $s$–$t$ path $P_0 = (s, P', v_1^2, t)$ of every length $l_0$, $6\ (\le m - k + 1) \le l_0 \le 2^{m-2} + 2$. To construct a longer path, let $v_i^0$ be a vertex in $G_0$ adjacent to $s$ such that $(s, v_i^0, v_i^1)$ is a fault-free path. Letting $P''$ be a $v_i^1$–$t$ path in $G_1 \oplus G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s$–$t$ path $P_1 = (s, v_i^0, P'')$ of every length $l_1$, $m - k + 3 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 + 1$.

*Subcase* 2.4. Both $s$ and $t$ are contained in $G_1 \oplus G_2 \oplus G_3$.
By Claim 5, we have an $s$–$t$ path $P_0$ in $G_1 \oplus G_2 \oplus G_3$ of every length $l_0$, $m-k+1 \le l_0 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 - 1$.

*Subcase* 2.5. Both $s$ and $t$ are contained in $G_0 \oplus G_1$.
For a vertex $x$ in $G_0 \oplus G_1$, we denote by $\bar{x}$ the vertex in $G_2 \oplus G_3$ adjacent to $x$. If $f_0 + f_{0,1} + f_1 \le m-4$, then there exists an $s$–$t$ hamiltonian path $P'$ in $G_0 \oplus G_1$ by Lemma 3(b). Let $(x, y)$ be an edge on $P'$ such that $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free, so that $P' = (s, Q_1, x, y, Q_2, t)$. Letting $P''$ be an $\bar{x}$–$\bar{y}$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s$–$t$ path $P_2 = (s, Q_1, x, P'', y, Q_2, t)$ of every length $l_2$, $2^{m-1} - f_v^0 - f_v^1 + m - k + 1 \le l_2 \le 2^m - f_v - 1$. If $f_0 + f_{0,1} + f_1 = m - 3$, there exists a hamiltonian cycle $(s, Q_1, x, t, Q_2, y)$ in $G_0 \oplus G_1$. Then, letting $P''$ be an $\bar{x}$–$\bar{y}$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s$–$t$ path $P_2 = (s, Q_1, x, P'', y, Q_2^R, t)$ of every length $2^{m-1} - f_v^0 - f_v^1 + m - k + 1$ or more. Here, $Q_2^R$ denotes the reverse of path $Q_2$, that is, $Q_2^R = (z_l, z_{l-1}, \ldots, z_1)$ for $Q_2 = (z_1, z_2, \ldots, z_l)$. Obviously, $2^{m-1} - f_v^0 - f_v^1 + m - k + 1 \le 3 \cdot 2^{m-2} - 1$ for any $m \ge 5$.

*Subcase* 2.6. $s$ is contained in $G_0 \oplus G_1$ and $t$ is contained in $G_2 \oplus G_3$.
If $f_0 + f_{0,1} + f_1 \le m - 4$, we let $x$ be a vertex in $G_0 \oplus G_1$ such that $x \ne s$, $\bar{x} \ne t$, and all of $x$, $\bar{x}$, $(x, \bar{x})$ are fault-free. Then, letting $P'$ be an $s$–$x$ hamiltonian path in $G_0 \oplus G_1$ and $P''$ be an $\bar{x}$–$t$ path in $G_2 \oplus G_3$ of every length $m - k + 1$ or more, we have an $s$–$t$ path $P_2 = (P', P'')$ of every length $l_2$, $2^{m-1} - f_v^0 - f_v^1 + m - k + 1 \le l_2 \le 2^m - f_v - 1$. If $f_0 + f_{0,1} + f_1 = m - 3$, there exists a hamiltonian cycle $(s, x, Q, y)$ in $G_0 \oplus G_1$. Assume w.l.o.g. that $\bar{y} \ne t$. Letting $P''$ be a $\bar{y}$–$t$ path of every length $m - k + 1$ or more, we have an $s$–$t$ path $P_2 = (s, x, Q, y, P'')$ of every length $2^{m-1} - f_v^0 - f_v^1 + m - k + 1$ or more.

*Subcase* 2.7. Both $s$ and $t$ are contained in $G_2$.
There exists a hamiltonian path in $G_0 \backslash F_0$ by Lemma 3(a) and let the hamiltonian path be $(v_a^0, Q, v_b^0)$. Assume

w.l.o.g. that $v_a^1$, $(v_a^0, v_a^1)$, $v_{b-1}^3$, and $(v_b^0, v_{b-1}^3)$ are all fault-free. We first find an $s$–$t$ path $P'$ in $G_2$ of every length $m-k-1$ or more by Claim 3. Since $m-k-1 \geq 4$, there exists an edge $(v_x^2, v_y^2)$ on $P'$ such that $x \neq a$, $y \neq b-1$, and all of $v_x^1$, $(v_x^2, v_x^1)$, $v_y^3$, $(v_y^2, v_y^3)$ are fault-free. Let $P' = (s, Q_1, v_x^2, v_y^2, Q_2, t)$. Then, we have an $s$–$t$ path $P_2 = (s, Q_1, v_x^2, P'', v_a^0, Q, v_b^0, P''', v_y^2, Q_2, t)$, where $P''$ is a $v_x^1$–$v_a^1$ path in $G_1$ of length $m-k-1$ or more and $P'''$ is a $v_{b-1}^3$–$v_y^3$ path in $G_3$ of length $m-k-1$ or more. The length $l_2$ of $P_2$ is any integer in the range $2^{m-2} - f_v^0 + 3(m-k) - 1 \leq l_2 \leq 2^m - f_v - 1$. It is straightforward to check that $2^{m-2} - f_v^0 + 3(m-k) - 1 \leq 3 \cdot 2^{m-2} - 1$ for any $m \geq 5$. □

**Lemma 7.** *$G(2^5, 4) \backslash F$ with $|F| = 2$ and $f_0 = f_j = 1$ for some $j = 1, 2, 3$ is 6-panconnected.*

**Proof.** We can see that $G(8,4) \times K_2$ is 1-fault 5-panconnected since $G(8,4) \times K_2$ is a four-dimensional restricted HL-graph and every four-dimensional restricted HL-graph was shown to be 1-fault 5-panconnected in [20]. Due to vertex symmetry, we assume $f_1 = 0$ (either $f_2 = 1$ or $f_3 = 1$). When $s$ and $t$ are contained in $G_0 \oplus G_1$, there exists an $s$–$t$ path $P_0$ of every length $l_0$, $5 \leq l_0 \leq 2^4 - f_v^0 - 1$. For some edge $(x, y)$ on $P_0$ such that the vertices $\bar{x}$ and $\bar{y}$ in $G_2 \oplus G_3$ adjacent to $x$ and $y$, respectively, are fault-free, letting $P'$ be an $\bar{x}$–$\bar{y}$ path in $G_2 \oplus G_3$ of every length $l' \geq 5$, we can obtain an $s$–$t$ path $P_1$ from merging $P_0$ and $P'$. The length $l_1$ of $P_1$ is any integer in the range $11 \leq l_1 \leq 2^5 - f_v - 1$.

When $s$ is contained in $G_0 \oplus G_1$ and $t$ is contained in $G_2 \oplus G_3$, we first construct an $s$–$t$ path of every length 7 or more. There exists an edge $(x, \bar{x})$ joining a vertex $x$ in $G_0 \oplus G_1$ and a vertex $\bar{x}$ in $G_2 \oplus G_3$ such that (i) $\bar{x}$ is adjacent to $t$, (ii) $x \neq s$, and (iii) path $(t, \bar{x}, x)$ is fault-free. Letting $P'$ be an $s$–$x$ path in $G_0 \oplus G_1$ of every length $l' \geq 5$, we have an $s$–$t$ path $P_0 = (P', \bar{x}, t)$ of every length $l_0$, $7 \leq l_0 \leq 2^4 - f_v^0 + 1$. Replacing the edge $(\bar{x}, t)$ on $P_0$ with an $\bar{x}$–$t$ path $P''$ in $G_2 \oplus G_3$ of every length $l'' \geq 5$ results in an $s$–$t$ path $P_1 = (P', P'')$ of every length $l_1$, $11 \leq l_1 \leq 2^5 - f_v - 1$. It remains to construct an $s$–$t$ path of length 6. If the vertex $\bar{t}$ in $G_0 \oplus G_1$ adjacent to $t$ is fault-free and different from $s$, then the above construction with $\bar{t} = x$ and $t = \bar{x}$ will be sufficient. Symmetrically, if $\bar{s} \neq t$ and $\bar{s}$ is fault-free, we are done. Thus, we assume that $s$ is adjacent to $t$, or both $\bar{s}$ and $\bar{t}$ are the faulty elements.

For the subcase where $s$ is adjacent to $t$, let $s = v_1^0$ and $t = v_0^3$. If $f_3 = 0$, we are done since $G_3 \oplus G_0$ is 1-fault 5-panconnected. Otherwise ($f_1 = f_2 = 0$), letting $P'$ be a $v_1^1$–$v_0^1$ path in $G_1$ of length 3 by Lemma 6(a), we have an $s$–$t$ path $(s, P', v_0^2, t)$ of length 6. Finally, let us consider the subcase where both $\bar{s}$ and $\bar{t}$ are faulty vertices. Since $\bar{t}$ is faulty and $f_1 = 0$, $t$ is contained in $G_3$. Let $t = v_i^3$. If $s$ is contained in $G_0$, regarding $\bar{t}$ as a *virtual* fault-free vertex, we find an $s$–$\bar{t}$ path in $G_0$ of length 5. Letting the path found be $(s, Q, v_j^0, \bar{t})$, we have an $s$–$t$ path $(s, Q, v_j^0, v_{j-1}^3, t)$ of length 6. If $s$ is contained in $G_1$, let $v_j^3$ be a vertex adjacent to $t$ such that $v_j^1 \neq s$. Observe that path $(v_j^1, v_j^2, v_j^3, t)$ is fault-free. Letting $P'$ be an $s$–$v_j^1$ path in $G_1$ of length 3, by Lemma 6(a), there exists an $s$–$t$ path $(P', v_j^2, v_j^3, t)$ of length 6. □

**Lemma 8.** *$G(2^5, 4) \backslash F$ with $|F| = f_i = 1$ for some $i = 0, 1, 2, 3$ is 5-panconnected.*

**Proof.** By Lemma 7, it suffices to construct an $s$–$t$ path of length 5. If $s$ and $t$ are contained in $G_i \oplus G_{i+1 \bmod 4}$ for some $i = 0, 1, 2, 3$, we are done since $G(8,4) \times K_2$ is 1-fault 5-panconnected. It is assumed w.l.o.g. that $s = v_1^0$ and $t = v_j^2$ for some $j \neq 1$. We can see that (i) $f_2 = 0$ and $(s, v_1^1, v_1^2)$ is a fault-free path, or (ii) $f_0 = 0$ and $(t, v_j^1, v_j^0)$ is a fault-free path. If condition (i) is satisfied, we have an $s$–$t$ path $(s, v_1^1, P')$, where $P'$ is a $v_1^2$–$t$ path in $G_2$ of length 3; otherwise, an $s$–$t$ path can be constructed symmetrically. □

**Remark 1.** Let $l_m^*$ be the minimum $l_m$ such that $G(2^m, 4)$ is $m-3$-fault $l_m$-panconnected. Theorem 1 suggests an upper bound $m+1$ on $l_m^*$. Of course, $l_m^*$ cannot be smaller than $D_m$, and thus we have $\lceil \frac{3m-1}{4} \rceil \leq l_m^* \leq m+1$.

## 4. Edge-pancyclicity of faulty $G(2^m, 4)$

In this section, we will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault nearly edge-pancyclic. Since an $f$-fault $l$-panconnected graph is always $f$-fault $l+1$-edge-pancyclic, by Theorem 1, we have the following lemma.

**Lemma 9.** *$G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+2$-edge-pancyclic.*

We are to show that $G(2^m, 4)$, $m \geq 3$, with at most $m-3$ faulty elements, has a cycle of every length $l$, $l = 4, 6, 7, 8, \ldots, m+1$, passing through an arbitrary fault-free edge.

**Lemma 10.** (a) $G(2^3, 4)$ *is* 0*-fault almost edge-pancyclic.*
(b) $G(2^4, 4)$ *is* 1*-fault nearly edge-pancyclic.*

**Proof.** The statement (a) is obvious from Lemma 6(a). To prove (b), it suffices to construct a cycle of length 4 that passes through an arbitrary edge $e$ by Lemma 9. There are two cases up to symmetry. If $e = (v_0^0, v_1^0)$, then at least one of the two cycles $(v_0^0, v_1^0, v_2^0, v_3^0)$ and $(v_0^0, v_1^0, v_1^1, v_0^1)$ is fault-free. If $e = (v_0^0, v_0^1)$, then cycles $(v_0^0, v_0^1, v_1^1, v_1^0)$ or $(v_0^0, v_0^1, v_3^1, v_3^0)$ are fault-free. Thus, we have the lemma. □

**Theorem 2.** $G(2^m, 4)$*,* $m \geq 3$*, is* $m-3$*-fault nearly edge-pancyclic.*

**Proof.** For $m = 3, 4$, the theorem holds by Lemma 10. Assume $m \geq 5$. Let $e$ be an arbitrary fault-free edge whose two end-vertices are also fault-free. By Lemma 9, it suffices to construct a cycle of every length $l$, $l = 4, 6, 7, 8, \ldots, m+1$, that passes through $e$. There are two cases up to symmetry.

*Case* 1. $e = (v_i^0, v_j^0)$.

If $f_0 \leq (m-2)-3$, we have a cycle of every length $l$, $l = 4, 6, 7, 8, \ldots, 2^{m-2} - f_v^0$, and we are done since $m+1 \leq 2^{m-2} - (m-5) \leq 2^{m-2} - f_v^0$ for any $m \geq 5$. Let $f_0 \geq m-4$. Then, there exists at most one faulty element outside $G_0$, and thus $F_{0,1} \cup F_1 = \emptyset$ or $F_{0,3} \cup F_3 = \emptyset$. Assume w.l.o.g. that $F_{0,1} \cup F_1 = \emptyset$. There exists a $v_i^1$–$v_j^1$ path $P'$ in $G_1$ of length $l'$, $l' = 1, 3, 4, 5, \ldots, 2^{m-1} - 1$, by Lemma 4. Thus, we have a cycle $(v_i^0, P', v_j^0)$ of every length $l$, $l = 4, 6, 7, 8, \ldots, 2^{m-2} + 2$.

*Case* 2. $e = (v_i^0, v_i^1)$.

We first construct a cycle of every even length $l$, $4 \leq l \leq m+1$. Let $F' = \{v_a^0 | v_a^1 \in F \text{ or } (v_a^0, v_a^1) \in F\} \cup \{(v_a^0, v_b^0) | (v_a^1, v_b^1) \in F\}$. Obviously, $|F' \cup F_0| \leq m-3$. By Lemma 3(a), there exists a hamiltonian path in $G_0 \backslash F_0 \cup F'$ of length at least $2^{m-2} - (m-3) - 1$. The hamiltonian path passes through $v_i^0$, and thus we can construct a fault-free $v_i^0$-path of every length $k$, $1 \leq k \leq \lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor$. Let the $v_i^0$-path in $G_0 \backslash F_0 \cup F'$ be $(v_i^0, v_{i_1}^0, v_{i_2}^0, \ldots, v_{i_k}^0)$. Then, by the construction, $v_i^1$-path $(v_i^1, v_{i_1}^1, v_{i_2}^1, \ldots, v_{i_k}^1)$ is also fault-free. Furthermore, the edge $(v_{i_k}^0, v_{i_k}^1)$ is fault-free. Thus, we have a cycle $(v_i^0, v_{i_1}^0, \ldots, v_{i_k}^0, v_{i_k}^1, \ldots, v_{i_1}^1, v_i^1)$ of length $2k+2$ for every $k$, $1 \leq k \leq \lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor$. The construction of every even cycle passing through $e$ is completed since $2\lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor + 2 \geq 2^{m-2} - (m-3) + 1 \geq m+1$ for any $m \geq 5$.

Now, it remains to construct a cycle of every odd length $l$, $7 \leq l \leq m+1$. We first claim that for some vertex $v_z^0$ in $G_0$ adjacent to $v_i^0$, a cycle $C_z = (v_i^0, v_z^0, v_{z-1}^3, v_z^3, v_z^2, v_z^1, v_i^1)$ associated with $v_z^0$ is fault-free. There are in total $m-2$ cycles associated with vertices in $G_0$ adjacent to $v_i^0$, and any two cycles among them are disjoint excluding $v_i^0$, $v_i^1$, and $(v_i^0, v_i^1)$. Note that it is impossible for both $v_z^0$ and $v_{z-1}^0$ to be adjacent to $v_i^0$ since $G(2^m, 4)$ has no cycle of length 3. Since there are at most $m-3$ faulty elements, at least one of the cycles is fault-free. Thus, the claim is proved. Observe that $C_z$ has a single edge in $G_0$, in $G_1$, and in $G_3$, respectively. It is straightforward to see that at least one of $G_0$, $G_1$, and $G_3$ has at most $m-5$ faulty elements. Assume w.l.o.g. that $f_0 \leq m-5$. Remember that the cycle $C_z$ is of length 7. Since $G_0$ is $m-5$-fault nearly edge-pancyclic, $G_0 \backslash F_0$ has a cycle $C$ passing through $(v_i^0, v_z^0)$ of every even length $l'$, $4 \leq l' \leq 2^{m-2} - f_v^0$. Thus, there exists a $v_i^0$–$v_z^0$ path $P = C \backslash (v_i^0, v_z^0)$ in $G_0$ of every odd length $l''$, $3 \leq l'' \leq 2^{m-2} - f_v^0 - 1$. If we replace the edge $(v_i^0, v_z^0)$ of $C_z$ with $v_i^0$–$v_z^0$ path $P$, we have a cycle of every odd length $l$, $9 \leq l \leq 2^{m-2} - f_v^0 + 5$. Obviously, $m+1 \leq 2^{m-2} - (m-5) + 5 \leq 2^{m-2} - f_v^0 + 5$ for any $m \geq 5$. This completes the proof. □

**Remark 2.** $G(2^4, 4)$ has a unique cycle $(v_0^0, v_0^1, v_0^2, v_0^3, v_1^0)$ of length 5 passing through edge $(v_0^0, v_1^0)$. Thus, we cannot say that every $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault almost edge-pancyclic.

## 5. Concluding remarks

In this paper, we have proven that every recursive circulant $G(2^m, 4)$ with $m \geq 3$ is $m-3$-fault $m+1$-panconnected. Here, the upper bound $m-3$ on the number of faulty elements is the maximum possible in a sense that, for any $f$

with $f \geq m-2$, there exists a fault set $F$ with $|F| = f$ such that $G(2^m, 4)\backslash F$ is not $l_m$-panconnected for any $l_m$, $l_m \leq |V(G\backslash F)| - 1$. We have also shown that the result on fault-panconnectivity of $G(2^m, 4)$ leads to the fact that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault nearly edge-pancyclic. There remain a number of interesting issues for future research. Finding the minimum $l_m^*$ such that $G(2^m, 4)$ is $m-3$-fault $l_m^*$-panconnected will be one of them.

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