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Variational principle of fractional order generalized thermoelasticity

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ABSTRACT

Recently, Youssef constructed a new theory of fractional order generalized thermoelasticity by taking into account the theory of heat conduction in deformable bodies, which depends upon the idea of the Riemann–Liouville fractional integral operator. In this paper, the variational theorem is obtained for the generalized thermoelasticity model for a homogeneous and isotropic body.

1. Introduction

Serious attention has been paid to the generalized thermoelasticity theories in solving thermoelastic problems in place of the classical uncoupled/coupled theory of thermoelasticity. The absence of any elasticity term in the heat conduction equation for uncoupled thermoelasticity appears to be unrealistic, since due to the mechanical loading of an elastic body, the strain so produced causes variation in the temperature field. Moreover, the parabolic type of the heat conduction equation results in an infinite velocity of thermal wave propagation, which also contradicts the actual physical phenomena. Introducing the strain-rate term in the uncoupled heat conduction equation extended the analysis to incorporate coupled thermoelasticity [1]. In this way, although the first shortcoming was over, there remained the parabolic type partial differential equation of heat conduction, which leads to the paradox of infinite velocity of the thermal wave. To eliminate this paradox generalized thermoelasticity theory was developed subsequently. The development of this theory was accelerated by the advent of second sound effects observed experimentally by Ackerman et al. [2] and Ackerman and Overtone [3] in materials at a very low temperature. In heat transfer problems involving very short time intervals and/or very high heat fluxes, it has been revealed that the inclusion of second sound effects in the original theory yields results that are realistic and very much different from those obtained with classical theory of elasticity.

Due to the advancement of pulsed lasers, fast burst nuclear reactors, particle accelerators, etc. which can supply heat pulses with a very fast time-rise [4,5], generalized thermoelasticity theory is receiving serious attention from different researchers. The development of the second sound effect has been nicely reviewed by Chandrasekharaiah [6]. At present mainly two different models of generalized thermoelasticity are being extensively used—one proposed by Lord and Shulman [7] and the other proposed by Green and Lindsay, [8]. The L–S theory suggests one relaxation time and according to this theory, only the Fourier heat conduction equation is modified; while G–L theory suggests two relaxation times and both the energy equation and the equation of motion are modified.

Bahar and Hetnarski [9,10] developed a method for solving coupled thermoelastic problems by using the state-space approach in which the problem is rewritten in terms of the state-space variables, namely the temperature, the displacement

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Nomenclature

λ, μ	Lamé's constants
ρ	Density
C_E	Specific heat at constant strain
α_T	Coefficient of linear thermal expansion
γ	$= (3\lambda + 2\mu)\alpha_T$
t	Time
T	Temperature
T_o	Reference temperature
θ	$= (T - T_o)$ Increment temperature such that $\left\lvert \frac{\theta}{T_o} \right\rvert \ll 1$
σ_{ij}	Components of stress tensor
e_{ij}	Components of strain tensor
u_i	Components of displacement vector
F_i	Body force vector
k	Thermal conductivity
τ_o	Relaxation times

and their gradients. Erbay and Suhubi [11] studied longitudinal wave propagation in an infinite circular cylinder, which is assumed to be made of the generalized thermoelastic material, and thereby obtained the dispersion relation when the surface temperature of the cylinder was kept constant. Generalized thermoelasticity problems for an infinite body with a circular cylindrical hole and for an infinite solid cylinder were solved respectively by Furukawa et al., [12,13]. A problem of generalized thermoelasticity was solved by Sherief [14] by adopting the state-space approach. Chandrasekharaiah and Murthy [15] studied thermoelastic interactions in an isotropic homogeneous unbounded linear thermoelastic body with a spherical cavity, in which the field equations were taken in unified forms covering the coupled, L–S and G–L models of thermoelasticity. The effects of mechanical and thermal relaxations in a heated viscoelastic medium containing a cylindrical hole were studied by Misra et al. [16]. Investigations concerning interactions between magnetic and thermal fields in deformable bodies were carried out by Maugin [17] as well as by Eringen and Maugin, [18]. Subsequently Abd-Alla and Maugin [19] conducted a generalized theoretical study by considering the mechanical, thermal and magnetic field in centro-symmetric magnetizable elastic solids. Among the theoretical contributions to the subject are the proofs of uniqueness theorems under different conditions by Ignaczak [20,21].

This work is dealing with Youssef theory of fractional order generalized thermoelasticity in which the theory of heat conduction in deformable bodies depends upon the idea of the Riemann–Liouville fractional integral operator. Hence, the variational theorem will be obtained for the fractional order generalized thermoelasticity theorem [22].

2. Formulation of the variational theorem

Under the assumption of small deviations of the thermodynamics system from the state of equilibrium, we will consider the statement of virtual external work:

$$\int_v F_i \delta\, u_i \mathrm{d}v + \int_s p_i \delta\, u_i \mathrm{d}s, \tag{1}$$

where v is an arbitrary material volume bounded by a closed and bounded surface s, F_i is the external forces per unit mass and p_i the components of surface traction applied to the surface s.

We have the relation

$$\sigma_{ij}\, n_j = p_i, \tag{2}$$

where σ_{ij} are the stresses components and n_i are the normal components to the surface s.

Using Eq. (2) and Gauss' divergence theorem in the second term of the relation (1), we obtain

$$\int_s p_i\, \delta\, u_i \mathrm{d}s = \int_s \sigma_{ji}\, n_j\, \delta\, u_i \mathrm{d}s = \int_v \sigma_{ji,j}\, \delta\, u_i \mathrm{d}v + \int_v \sigma_{ji}\, \delta\, e_{ij} \mathrm{d}v, \tag{3}$$

where $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$.

The equation of motion takes the form

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i. \tag{4}$$

Using equation of motion (4), Eq. (3) will be reduced to

$$\int_s p_i\,\delta\,u_i \mathrm{d}s + \int_v F_i\,\delta\,u_i \mathrm{d}v = \int_v \rho\,\ddot{u}_i\,\delta\,u_i \mathrm{d}v + \int_v \sigma_{ji}\,\delta\,e_{ij} \mathrm{d}v, \tag{5}$$

Duhamel–Neumann relation takes the form

$$\sigma_{ij} = 2\mu\,e_{ij} + (\lambda\,e_{kk} - \gamma\,\theta)\,\delta_{ij}, \tag{6}$$

where δ_{ij} is the Kronecker delta symbol.

Using Eq. (6) into the second term on the right-hand side of Eq. (5), yields

$$\int_v \sigma_{ji}\,\delta\,e_{ij} \mathrm{d}v = \int_v \left(2\,\mu\,e_{ij} + \lambda\,e_{kk}\,\delta_{ij}\right)\,\delta\,e_{ij}\,\mathrm{d}v - \int_v \gamma\theta\delta e_{ii}\,\mathrm{d}v. \tag{7}$$

We arrive at the theorem of virtual work from Eqs. (5) and (7), we obtain

$$\int_s p_i\,\delta\,u_i \mathrm{d}s + \int_v F_i\,\delta\,u_i \mathrm{d}v - \int_v \rho\,\ddot{u}_i\,\delta\,u_i \mathrm{d}v = \delta\,W - \int_v \gamma\,\theta\delta\,e_{ii} \mathrm{d}v \tag{8}$$

where

$$\delta W = \int_v \left(2\,\mu\,e_{ij}\,\delta\,e_{ij} + \lambda\,e_{kk}\delta\,e_{ii}\right)\,\mathrm{d}v. \tag{9}$$

The function W implies the work of the deformation may be expressed by Naotak et al. [23]:

$$W = \int_v \left(\mu\,e_{ij}\,e_{ij} + \frac{\lambda}{2} e_{kk}\,e_{ii}\right)\,\mathrm{d}v. \tag{10}$$

The three terms on the left-hand side of Eq. (8) express the virtual external work of the body forces, of tractions on the boundary and of inertia forces, respectively, while the right-hand side expresses the virtual internal work.

The entropy balance without internal heat generation is [23]:

$$q_{i,i} = -T\,\dot{\eta} \approx -T_o\dot{\eta}, \tag{11}$$

where q_i are the components of the heat flux and η is the entropy.

We introduce an entropy flux H, which is related to the heat flux through the equation

$$q_i = T_o\dot{H}_i, \tag{12}$$

and

$$\eta = -H_{i,i}. \tag{13}$$

The entropy increment satisfies the following relation for unit mass [23]:

$$T_o\eta = C_E\theta + T_o\gamma\,e_{ij}\delta_{ij}. \tag{14}$$

The modified Fourier law of heat conduction in generalized form of isotropic medium is [22]:

$$q_i + \tau_o\dot{q}_i = -kI^{\alpha-1}\theta_{\,,i}, \tag{15}$$

where

$$I^{\alpha} f(t) = \left\{\begin{array}{ll} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau & \text{for } 0 < \alpha \le 2 \\ f(t) & \text{for } \alpha = 0 \end{array}\right\}, \tag{16}$$

$\Gamma(\alpha)$ is the gamma function and $I^{-\alpha} f(t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t)$.

By eliminating the entropy between Eqs. (13) and (14), we get

$$H_{i,i} = \frac{C_E}{T_o}\theta - \gamma e_{ii}. \tag{17}$$

By eliminating q_i between Eqs. (12) and (15), we obtain

$$T_o\left(\dot{H}_i + \tau_o\,\ddot{H}_i\right) = T_o\left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \tau_o\,\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right) H_i = -k\theta_{\,,i}\,. \tag{18}$$

Without loss of generality, we can put a parameter for the parakeet in the above equation that includes the time derivatives, i.e.

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \tau_o \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} = \beta,$$

hence, the Eq. (18) takes the form

$$\frac{T_o \beta}{k} H_i + \theta_{,i} = 0. \tag{19}$$

Multiplying δH_i by the above equation and integrating over the region v of the body, we find

$$\int_v \left(\frac{T_o \beta}{k} H_i + \theta_{,i} \right) \delta H_i \, \mathrm{d}v = 0. \tag{20}$$

The second term of the Eq. (20) is reduced to

$$\int_v \theta_{,i} \, \delta H_i \, \mathrm{d}v = \int_v (\theta \, \delta H_i)_{,i} \, \mathrm{d}v - \int_v \theta \delta H_{i,i} \, \mathrm{d}v,$$

which gives

$$\int_v \theta_{,i} \, \delta H_i \, \mathrm{d}v = \int_s (\theta \, \delta H_i) \, n_i \, \mathrm{d}s - \int_v \theta \delta H_{i,i} \, \mathrm{d}v. \tag{21}$$

From Eq. (17), we have

$$\delta H_{i,i} = -\frac{C_E}{T_o} \delta\theta - \gamma \delta e_{ii}. \tag{22}$$

Using Eq. (22) in the last term of Eq. (21), we get

$$\int_v \theta_{,i} \, \delta H_i \, \mathrm{d}v = \int_s (\theta \, \delta H_i) \, n_i \, \mathrm{d}s + \frac{C_E}{T_o} \int_v \theta \delta\theta \, \mathrm{d}v + \gamma \int_v \theta \delta e_{ii} \, \mathrm{d}v. \tag{23}$$

Now, Eq. (20) takes the form

$$\int_v \left(\frac{T_o \beta}{k} H_i + \theta_{,i} \right) \delta H_i \, \mathrm{d}v = \frac{T_o \beta}{k} \int_v H_i \delta H_i \, \mathrm{d}v + \int_s (\theta \delta H_i) \, n_i \mathrm{d}s + \frac{C_E}{T_o} \int_v \theta \delta\theta \, \mathrm{d}v + \gamma \int_v \theta \delta e_{ii} \, \mathrm{d}v = 0. \tag{24}$$

We introduced the heat potential P in the form [23]:

$$P = \frac{C_E}{2 \, T_o} \int_v \theta^2 \, \mathrm{d}v, \tag{25}$$

where

$$\delta P = \frac{C_E}{T_o} \int_v \theta \delta\theta \, \mathrm{d}v \tag{26}$$

and the dissipation function D in the form [23]:

$$D = \frac{T_o \beta}{2k} \int_v H_i^2 \, \mathrm{d}v. \tag{27}$$

Hence, we get

$$\delta D = \frac{T_o \beta}{k} \int_v H_i \, \delta H_i \, \mathrm{d}v. \tag{28}$$

Introducing Eqs. (26) and (28) into Eq. (24), we obtain the variational equation for heat conduction

$$\delta P + \delta D + \gamma \int_v \theta \delta e_{ii} \, \mathrm{d}v = -\int_s (\theta \, \delta H_i) \, n_i \, \mathrm{d}s. \tag{29}$$

By elimination the term $\gamma \int_v \theta \delta e_{ii} \, \mathrm{d}v$ in Eqs. (8) and (29), we get

$$\delta W + \delta P + \delta D = \int_s p_i \, \delta \, u_i \mathrm{d}s + \int_v F_i \, \delta \, u_i \mathrm{d}v - \int_v \rho \, \ddot{u}_i \, \delta \, u_i \mathrm{d}v - \int_s (\theta \delta H_i) \, n_i \mathrm{d}s. \tag{30}$$

The terms on the right-hand side of Eq. (30) expresses the virtual external work of the body forces, of tractions on the boundary, of inertia forces, and of heating of the boundary, respectively, while the left-hand side expresses the virtual internal work of deformation, the variation of heat potential, and the variation of the dissipation function, respectively [23].

Introducing the Biot thermoelastic potential ϕ [23]:

$$\phi = W + P = \int_v \left(\mu e_{ij}\, e_{ij} + \frac{\lambda}{2} e_{ii}\, e_{jj} + \frac{C_E}{2\, T_o} \theta^2 \right) \mathrm{d}v. \tag{31}$$

We obtain the variational principal of the fraction order generalized thermoelasticity with one relaxation time in the form

$$\delta\,(\phi + D) = \int_s (p_i\, \delta\, u_i - (\theta \delta H_i)\, n_i)\, \mathrm{d}s\, + \int_v\, (F_i - \rho\, \ddot{u}_i\,)\, \delta\, u_i\, \mathrm{d}v. \tag{32}$$

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