Appl. Math. Lett. Vol. 4, No. 6, pp. 63–67, 1991
Printed in Great Britain.
0893-9659/91 $3.00 + 0.00

A Trace Theorem in Kinetic Theory

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(Received May 1991)

Abstract. A mathematical assumption made in a recent paper by Cercignani on the traces of the solutions of initial-boundary value problems for the Boltzmann Equation is shown here to be true in the case of a domain whose boundary is a Lyapunoff surface with purely diffusing boundary conditions.

1. INTRODUCTION

The global existence of a weak solution for the initial-boundary value problem for the Boltzmann Equation was obtained by Hamdache [1] by means of the renormalization method of DiPerna and Lions [2]. His proof, however, introduces rather specific restrictions in the case of no net mass flow through the boundary. Recently, Cercignani [3] examined more general conditions, proving a theorem on the traces of the solutions that is needed in order to extend the results of Hamdache. In his proof, Cercignani assumed that a linear operator had a bounded inverse, since this is "physically meaningful."

The aim of this note is to show that this assumption is, in fact, mathematically correct in the case of a domain whose boundary is a Lyapunoff surface with purely diffusing boundary conditions.

2. PROBLEM FORMULATION

If Ω is an open set of R^3 with boundary $\partial\Omega$ we denote by $\mathbf{n}(\mathbf{x})$ the unit internal normal at $\mathbf{x} \in \partial\Omega$ and by $d\sigma$ the Lebesgue measure on $\partial\Omega$; $f(\mathbf{x},\boldsymbol{\xi})$ denotes, as usual, the molecular density of a rarefied gas in the vessel Ω, density of molecules at the point $\mathbf{x} \in \Omega$ with velocity $\boldsymbol{\xi} \in R^3$. Let us now define $D = \Omega \times R^3$ and $\partial D^{\pm} = \{(\mathbf{x},\boldsymbol{\xi}) \in \partial\Omega \times R^3 : \ \pm\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x}) > 0\}$ and put, in the sense of distribution,

$$(\Lambda f)(\mathbf{x},\boldsymbol{\xi}) = \boldsymbol{\xi}\cdot\frac{\partial f}{\partial \mathbf{x}}. \tag{2.1}$$

We also put $d\sigma^{\pm} = |\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x})|\,d\sigma\,d\boldsymbol{\xi}$ (on $\partial D^{\pm}$) and we introduce the spaces

$$W^p = \{f \in L^p(D, d\mathbf{x}\,d\boldsymbol{\xi}) \mid \Lambda f \in L^p(D, d\mathbf{x}\,d\boldsymbol{\xi})\} \tag{2.2}$$

$$L^{p,\pm} = L^p(\partial D^{\pm}, d\sigma^{\pm}). \tag{2.3}$$

Now the trace operator $\gamma_D^{\pm}$ are first defined on $C_0^1(\bar{D})$ by [4]

$$\gamma_D^{\pm} f = f_{|\partial D^{\pm}} \quad f \in C_0^1(\bar{D}) \tag{2.4}$$

and then [3,4] extended on W^1, but in general not with values in $L^{1,\pm}$ as is needed in order to solve the initial-boundary value problem for the Boltzmann Equation in L^1 [1]. If, however, we impose suitable boundary conditions, then we can make some progress in the direction

Supported by the Italian National Project "Equazioni di Evoluzione e Applicazioni Fisico-Matematiche"

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of proving that $\gamma_D^{\pm} f \in L^{1,\pm}$. To this end, let f satisfy a linear boundary condition of the form [1]:

$$[\gamma_D^{+} f] = (1-\alpha)\, K[\gamma_D^{-} f] + \alpha h, \tag{2.5}$$

where $h(\mathbf{x},\boldsymbol{\xi})$ is a given $L^{1,+}$-function, $\alpha \in [0,1]$ and $K : L^{1,-} \to L^{1,+}$ is a positive mass-preserving scattering operator which also transforms the restriction M_w^- of the wall Maxwellian M_w to $\boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x}) < 0$ into the restriction M_w^+ of M_w to $\boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x}) > 0$.

Hamdache proved that $\gamma_D^{\pm} f \in L^{1,\pm}$ when $\alpha \neq 0$ and subsequently Cercignani was able to show this regularity property even when $\alpha = 0$, which is the case of no net mass flow through the wall. In order to review Cercignani's result let us introduce the following operators.

First, we define the operator P which reflects $\boldsymbol{\xi}$, i.e.,

$$[P\psi](\mathbf{x},\boldsymbol{\xi}) = \psi(\mathbf{x},-\boldsymbol{\xi}) \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}). \tag{2.6}$$

Then we denote by λ_D^+ the operator that carries $\gamma_D^-\psi$ into $\gamma_D^+\psi$ defined by

$$[\lambda_D^+\psi](\mathbf{x},\boldsymbol{\xi}) = \psi(\tilde{\mathbf{x}},\boldsymbol{\xi}) \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}) \in \partial D^+, \tag{2.7}$$

here $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x},\boldsymbol{\xi})$ denotes the point of intersection of the ray $\ell(\mathbf{x},\boldsymbol{\xi}) = \{\mathbf{y} \;:\, \mathbf{y} = \mathbf{x} + \tau\boldsymbol{\xi},\ \tau < 0\}$ with $\partial\Omega$, which is closest to $\mathbf{x}$. Finally, we define the operator

$$F = I - (PK)' P \lambda_D^+ :\ L^{\infty,-} \to L^{\infty,-}, \tag{2.8}$$

where I is the identity operator and $(PK)' :\ L^{\infty,-} \to L^{\infty,-}$ is the dual operator of $PK : L^{1,-} \to L^{1,-}$ defined by means of the duality product

$$\langle (PK)'\psi, f\rangle_- = \langle \psi, (PK) f\rangle_- \quad \forall\, \psi \in L^{\infty,-},\ \forall\, f \in L^{1,-}, \tag{2.9}$$

where for the real value functions $\phi(\mathbf{x},\boldsymbol{\xi}) \in L^{\infty,-}$ and $g(\mathbf{x},\boldsymbol{\xi}) \in L^{1,-}$ we put

$$\langle \phi, g\rangle_- = \int_{\partial\Omega}\int_{\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x})<0} \phi(\mathbf{x},\boldsymbol{\xi})\, g(\mathbf{x},\boldsymbol{\xi})\, |\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x})|\, d\sigma\, d\boldsymbol{\xi} = \int\int_{\partial D^-} \phi(\mathbf{x},\boldsymbol{\xi})\, g(\mathbf{x},\boldsymbol{\xi})\, d\sigma^-. \tag{2.10}$$

Let us now consider for any functions $\psi \in L^{\infty,-}$ the decomposition into a constant part

$$P_{\mathcal{M}}\psi = \frac{\langle \psi, M_w^-\rangle_-}{\langle 1, M_w^-\rangle_-} \tag{2.11}$$

plus the remainder

$$[P_{\mathcal{O}}\psi]\,(\mathbf{x},\boldsymbol{\xi}) = \psi(\mathbf{x},\boldsymbol{\xi}) - P_{\mathcal{M}}\psi. \tag{2.12}$$

All this makes sense if $\partial\Omega$ has finite measure, in fact, if this holds, then $M_w^- \in L^{1,-}$. Now $L^{\infty,-}$ is decomposed into the direct sum of the subspaces $\mathcal{O}$ and $\mathcal{M}$ of the functions having the form $P_{\mathcal{O}}\psi$ and $P_{\mathcal{M}}\psi$. Then Cercignani proved [3] the following:

THEOREM 1. *Let $f \in W^1$, $|\boldsymbol{\xi}|^2 f \in L^1(D, d\mathbf{x}\, d\boldsymbol{\xi})$, $|\boldsymbol{\xi}|^2 \Lambda f \in L^1(D, d\mathbf{x}\, d\boldsymbol{\xi})$. If the boundary condition (2.5) applies with $\alpha = 0$ and if the linear operator F has a bounded inverse in the subspace $\mathcal{O}$ of $L^{\infty,-}$, then $\gamma_D^{\pm} f \in L^{1,\pm}$.*

3. MAIN RESULTS

Here we prove that F has a bounded inverse in $\mathcal{O}$, provided that $\partial\Omega$ is a Lyapunoff surface and that K is purely diffusing. Let us first recall the following definitions:

DEFINITION 1. *A surface S in R^3 is called a Lyapunoff surface if:*

i) *at every point of S there is a well-defined tangent plane, and consequently a well-defined normal;*
ii) *there is a number $r > 0$, the same for all points of S such that if one takes the part Σ of S lying inside the Lyapunoff sphere $B(y_0, r)$ with center at an arbitrary point $y_0 \in S$ and radius r, then the lines parallel to the normal to S at y_0 meet Σ at most once; and*
iii) *there are two numbers $\Lambda > 0$ and $\lambda, 0 < \lambda \leq 1$, the same for the whole of S, such that for any two points $y_1, y_2 \in S$,*

$$|\Theta| < \Lambda \, |y_1 - y_2|^\lambda, \tag{3.1}$$

where Θ is the angle between the normal to S at y_1 and y_2.

DEFINITION 2. *A scattering operator K is purely diffusing if:*

$$[Kf](\mathbf{x},\boldsymbol{\xi}) = M_w^+(\boldsymbol{\xi}) \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} [f](\mathbf{x},\boldsymbol{\xi}') \, |\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x})| \, d\boldsymbol{\xi}' \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}) \in \partial D^+ \tag{3.2}$$

with the normalization condition

$$\int_{\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x})>0} M_w^+(\boldsymbol{\xi}) |\boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x})| \, d\boldsymbol{\xi} = 1 \qquad \text{a.e. } \mathbf{x} \in \partial\Omega. \tag{3.3}$$

A simple calculation shows that the linear operator F is now given by

$$[F\psi](\mathbf{x},\boldsymbol{\xi}) = \psi(\mathbf{x},\boldsymbol{\xi}) - \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}') \, \psi(\tilde{\mathbf{x}}, -\boldsymbol{\xi}') \, |\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x})| \, d\boldsymbol{\xi}' \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}) \in \partial D^- \tag{3.4}$$

Now we prove that $rk \; F = \mathcal{O}$ and that $F : \; \mathcal{O} \to \mathcal{O}$ has a bounded inverse if $\partial\Omega$ is a Lyapunoff surface. Let us first prove that $rk \; F \subseteq \mathcal{O}$. To this end we have only to check that

$$P_{\mathcal{M}}[F\psi] = 0 \quad \forall \, \psi \in L^{\infty,-} \tag{3.5}$$

but Equation (3.3) implies that $\langle 1, M_w^- \rangle_- = \mu(\partial\Omega)$ so that Equation (3.5) is written

$$\frac{1}{\mu(\partial\Omega)} \int_{\partial\Omega} \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}') \, \psi(\mathbf{x},\boldsymbol{\xi}') \, |\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x})| \, d\sigma \, d\boldsymbol{\xi}' =$$

$$\frac{1}{\mu(\partial\Omega)} \int_{\partial\Omega} \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}') \, \psi(\tilde{\mathbf{x}}, -\boldsymbol{\xi}') \, |\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x})| \, d\sigma \, d\boldsymbol{\xi}', \tag{3.6}$$

which is true if we let in the l.h.s. $\boldsymbol{\xi}' \to -\boldsymbol{\xi}', \mathbf{x} \to \tilde{\mathbf{x}}$ and observe that $\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x}) < 0 \iff -\boldsymbol{\xi}' \cdot \mathbf{n}(\tilde{\mathbf{x}}) < 0$. In order to show that $rk \; F = \mathcal{O}$, let $\sigma(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{O} \subset L^{\infty,-}$ be given and observe that if $\psi \in L^{\infty,-}$ is a solution of

$$[F\psi](\mathbf{x},\boldsymbol{\xi}) = \sigma(\mathbf{x},\boldsymbol{\xi}) \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}) \in \partial D^-, \tag{3.7}$$

then it is written in the following form

$$\psi(\mathbf{x},\boldsymbol{\xi}) = A(\mathbf{x}) + \sigma(\mathbf{x},\boldsymbol{\xi}) \qquad \text{a.e. } (\mathbf{x},\boldsymbol{\xi}) \in \partial D^-, \tag{3.8}$$

where $A(\mathbf{x})$ is a function in $L^{\infty,-}$, independent of $\boldsymbol{\xi}$, such that

$$A(\mathbf{x}) = s(\mathbf{x}) + \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}') \, A(\tilde{\mathbf{x}}) \, |\boldsymbol{\xi}' \cdot \mathbf{n}(\mathbf{x})| \, d\boldsymbol{\xi}' \qquad \text{a.e. } \mathbf{x} \in \partial\Omega \tag{3.9}$$

with $s(\mathbf{x})$ given by

$$s(\mathbf{x}) = \int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}')\,\sigma(\tilde{\mathbf{x}},-\boldsymbol{\xi}')\,|\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})|\,d\boldsymbol{\xi}' \qquad \text{a.e. } \mathbf{x}\in\partial\Omega. \tag{3.10}$$

Please note that $\sigma(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{O} \subset L^{\infty,-} \Longrightarrow s(\mathbf{x}) \in \mathcal{O} \subset L^{\infty,-}$, *i.e.*, $s(\mathbf{x}) \in L^{\infty,-}$ and satisfies

$$\int_{\partial\Omega} s(\mathbf{x})\,d\sigma = 0. \tag{3.11}$$

In order to solve Equation (3.9) let us put [5, pp. 137–138]

$$\mathbf{u}' = \frac{\boldsymbol{\xi}'}{|\boldsymbol{\xi}'|} = \frac{\mathbf{x}-\tilde{\mathbf{x}}}{|\mathbf{x}-\tilde{\mathbf{x}}|}, \tag{3.12}$$

where of course $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x},\boldsymbol{\xi}')$ and observe that

$$\int_{\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})<0} M_w^-(\boldsymbol{\xi}')\,A(\tilde{\mathbf{x}})\,|\boldsymbol{\xi}'\cdot\mathbf{n}(\mathbf{x})|\,d\boldsymbol{\xi}' = \frac{1}{\pi}\int_{\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})<0} A(\tilde{\mathbf{x}})\,|\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})|\,d\mathbf{u}' \qquad \text{a.e. } \mathbf{x}\in\partial\Omega \tag{3.13}$$

which follows from Equation (3.3) and the fact that

$$\int_{\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})<0} |\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})|\,d\mathbf{u}' = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta = \pi. \tag{3.14}$$

Now Equation (3.9) is written

$$A(\mathbf{x}) = s(\mathbf{x}) + \int_{\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})<0} A(\tilde{\mathbf{x}})\,|\mathbf{u}'\cdot\mathbf{n}(\mathbf{x})|\,d\mathbf{u}' \qquad \text{a.e. } \mathbf{x}\in\partial\Omega. \tag{3.15}$$

If we observe that $(\mathbf{x}-\tilde{\mathbf{x}}) = \mathbf{u}'\,|\mathbf{x}-\tilde{\mathbf{x}}|$ and that $|\mathbf{x}-\tilde{\mathbf{x}}|^2\,d\mathbf{u}' = |\mathbf{u}'\cdot\mathbf{n}(\tilde{\mathbf{x}})|\,d\tilde{\sigma}$, [$d\tilde{\sigma}$ being the surface element at $\tilde{\mathbf{x}}$] and put $\mathbf{x} = \mathbf{P}$, $\tilde{\mathbf{x}} = \mathbf{Q}$ we finally obtain

$$A(\mathbf{P}) = s(\mathbf{P}) + \frac{1}{\pi}\int_{\partial\Omega(\mathbf{P})} A(\mathbf{Q})\,b(\mathbf{P},\mathbf{Q})\,d\sigma(\mathbf{Q}) \qquad \text{a.e. } \mathbf{P}\in\partial\Omega, \tag{3.16}$$

where $b(\mathbf{P},\mathbf{Q})$ is given by

$$b(\mathbf{P},\mathbf{Q}) = \frac{|(\mathbf{P}-\mathbf{Q})\cdot\mathbf{n}(\mathbf{P})|\,|(\mathbf{P}-\mathbf{Q})\cdot\mathbf{n}(\mathbf{Q})|}{|\mathbf{P}-\mathbf{Q}|^4} \tag{3.17}$$

and $\partial\Omega(\mathbf{P})$ is the part of the surface $\partial\Omega$ which is seen from the point $\mathbf{P}$.

Equation (3.16) is a linear integral equation in R^3 with a singular kernel [6–8]; $A(\mathbf{P})=$ const is the only solution in L^2 (and therefore in $L^\infty \subset L^2$) of the corresponding homogeneous equation [$s(\mathbf{P}) = 0$] (this follows from Equation (3.14) and Schwarz's inequality).

Equation (3.16) can now be solved in an L^2-framework, by means of Fredholm theorems, provided that $b(\mathbf{P},\mathbf{Q})$ has a *weak* singularity, *i.e.*, [7, p. 6, Equation (1.11)] has the form

$$b(\mathbf{P},\mathbf{Q}) = \frac{B(\mathbf{P},\mathbf{Q})}{|\mathbf{P}-\mathbf{Q}|^\alpha} \tag{3.18}$$

with $B(\mathbf{P},\mathbf{Q})$ a bounded function and $\alpha =$ const. such that $0<\alpha<2$. If Equation (3.18) holds and if $s(\mathbf{P}) \in L^2$ fulfills the integral condition (3.11), then all the solutions of Equation (3.16) in L^2 are written [6, Chapter 1, Section 15]

$$A(\mathbf{P}) = C + A_{\mathcal{O}}(\mathbf{P}), \tag{3.19}$$

where C is an arbitrary constant and $A_{\mathcal{O}}(\mathbf{P})$ a particular solution such that [9, p. 199]

$$\int_{\partial\Omega} A_{\mathcal{O}}(\mathbf{P})\, d\sigma(\mathbf{P}) = 0. \tag{3.20}$$

Moreover, if Equation (3.18) is satisfied, and if $s(\mathbf{P}) \in L^\infty$, then Equation (3.19) holds in L^∞, *i.e.*, $A_{\mathcal{O}} \in L^\infty$ [6, p. 94, Theorem 3], and finally Equation (3.19) shows that Equation (3.9) [and consequently Equation (3.7)] has a unique solution in $\mathcal{O} \cong L^\infty/\mathcal{M}$. Thus the linear operator F has a bounded inverse in $\mathcal{O}$, provided that condition (3.18) is satisfied. To this end, we have only to recall the *Definition 1* or, more precisely, Equation (3.1) written for the points $y_1 = \mathbf{P}$ and $y_2 = \mathbf{Q}$ (this is possible because both lie on the boundary $\partial\Omega$); now if we denote by Λ and λ the Lyapunoff constants of $\partial\Omega$ we have:

$$\frac{|(\mathbf{P}-\mathbf{Q})\cdot\mathbf{n}(\mathbf{P})|\,|(\mathbf{P}-\mathbf{Q})\cdot\mathbf{n}(\mathbf{Q})|}{|\mathbf{P}-\mathbf{Q}|^4} \le \frac{\Lambda^2}{|\mathbf{P}-\mathbf{Q}|^{2(1-\lambda)}}, \tag{3.21}$$

thus Equation (3.18) is valid if we let $\alpha = 2(1-\lambda)$ and $B(\mathbf{P},\mathbf{Q}) = b(\mathbf{P},\mathbf{Q})\,|\mathbf{P}-\mathbf{Q}|^{2(1-\lambda)}$.

4. Concluding Remarks

The existence of a bounded inverse in $\mathcal{O}$ of F is the result that is needed in order to prove Cercignani's theorem. In the present paper we have shown that F has a bounded inverse in $\mathcal{O}$, provided that $\partial\Omega$ is a Lyapunoff surface with purely diffusing boundary conditions; the same result should apply for a more general scattering operator K. A detailed examination will be presented in a forthcoming paper.

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