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Monophonic numbers of the join and composition of connected graphs

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Abstract

In this paper, we describe the monophonic sets in the join and composition of two connected graphs. We also determine the monophonic numbers of the join of any two connected graphs and the composition $G[K_n]$ of a connected graph G and the complete graph K_n. Lower and upper bounds are obtained for the monophonic number of the composition $G[H]$, where G is connected and H is a connected non-complete graph.

Keywords: Monophonic set; Monophonic number; Join; Composition

1. Introduction

Given a connected graph $G = (V(G), E(G))$ and vertices u and v of G, we call any u–v path of length $d_G(u, v)$ (length of the shortest path connecting u and v) as u–v *geodesic*. Any chordless path connecting u and v is called a u–v *m-path*. The *monophonic closure* of a subset S of $V(G)$ is $J_G[S] = \bigcup_{u,v \in S} J_G[u, v]$, where $J_G[u, v]$ is the set containing u and v and all vertices lying on some u–v m-path. If $J_G[S] = V(G)$, then we call S a *monophonic set* in G. A monophonic set in G of minimum order is called a *minimum monophonic set* in G. The order of a minimum monophonic set in G is called the *monophonic number* of G and is denoted by $\mathrm{mn}(G)$.

It is easily verified that every monophonic set in G contains every extreme vertex (a vertex in G whose neighbors or vertices adjacent to it induce a complete subgraph of G). For example, for any positive integer $n > 1$, the endvertices of the path P_n are elements of every monophonic set in P_n. In fact, the set containing the endvertices of P_n is the unique minimum monophonic set in P_n; hence, $\mathrm{mn}(P_n) = 2$. For $n \geqslant 4$, it can be shown that $\mathrm{mn}(C_n) = 2$, where C_n is the cycle of order n. For any connected graph G of order $n \geqslant 1$, it is routine to show that $\mathrm{mn}(G) = n$ if and only if G is the complete graph K_n of order n.

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The geodetic counterparts of the concepts of monophonic set and monophonic number of a graph had been considered by Chartrand, Harary, and Zhang in [4] and [5]. Recently, Canoy and Cagaanan also studied those concepts for the join and composition of graphs (see [2,3]).

The concepts of monophonic set and monophonic number of a graph have not yet been fully explored and investigated. These concepts appeared in [1] and were studied by Pelayo et al. in [7] and [9].

In this paper, the authors consider these two monophonicity concepts of a graph. In particular, the monophonic sets in the join and composition of two connected graphs will be characterized. Furthermore, lower and upper bounds of the monophonic number of some of these graphs will be determined. The authors of this present paper, however, neither characterized the monophonic sets in the cartesian product of graphs nor determined their monophonic numbers.It is still unknown if the monophonic number of the cartesian product of connected graphs has an upper bound involving the monophonic numbers of each of the graphs. Hence, determination of upper bounds (or exact values) of the monophonic number of the cartesian product of graphs remains an open problem.

A subset S of $V(G)$ is said to be *m-convex* if, for every pair of vertices $x, y \in S$, the vertex set of every x–y m-path is contained in S. It is easy to verify that S is m-convex if and only if $J_G[S] = S$.

Recently, Paluga and Canoy [8] characterized the m-convex sets in the join, composition, and cartesian product of graphs.

Throughout this paper, $G = (V(G), E(G))$ is a simple undirected connected graph. For other graph theoretic terms which are assumed here, readers are advised to refer to [6].

2. Monophonic number of the join of graphs

Definition 1. The *join* of graphs G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Our first result characterizes all monophonic sets in the join $G + K_n$.

Theorem 2. *Let G be a non-complete graph. A subset $S \subseteq V(G + K_n)$ is a monophonic set in $G + K_n$ if and only if $S \cap V(G)$ is a monophonic set in G.*

Proof. Suppose $S \subseteq V(G + K_n)$ is a monophonic set in $G + K_n$. Let $S_1 = S \cap V(G)$. Since S is a monophonic set in $G + K_n$, $S_1 \neq \emptyset$. If $S_1 = V(G)$, then we are done. So suppose that $S_1 \neq V(G)$. Let $x \in V(G)\backslash S_1$. Then there exists $y, z \in S$ such that $x \in J_{G+K_n}[y, z]$. Since $x \neq y$ and $x \neq z$, $d_{G+K_n}(y, z) \neq 1$; hence, $d_{G+K_n}(y, z) = 2$. This implies that $y, z \in S_1$. In this case, any y–z m-path containing x cannot contain a vertex of $V(K_n)$. Since every m-path in $G + K_n$ that does not contain a vertex of K_n is an m-path in G and $x \in J_{G+K_n}[y, z]$, it follows that $x \in J_G[y, z]$. Therefore, $J_G[S_1] = V(G)$. This shows that S_1 is a monophonic set in G.

Conversely, suppose that $S_1 = S \cap V(G)$ is a monophonic set in G. Since G is non-complete, the subgraph $\langle S_1 \rangle$ induced by S_1 is not complete. This means that there exist vertices $a, b \in S_1$ such that $d_G(a, b) \neq 1$. Hence, $d_{G+K_n}(a, b) = 2$ and (a, v, b) is an a–b m-path for every $v \in V(K_n)$. Therefore, $V(K_n) \subseteq J_{G+K_n}[a, b]$; hence, $V(G + K_n) = V(G) \cup V(K_n) = J_{G+K_n}[S_1]$. This shows that S_1 is a monophonic set in $G + K_n$. Since $S_1 \subseteq S$, it follows that S is a monophonic set in $G + K_n$. $\square$

Theorem 3. *If G is a graph of order n, then $\mathrm{mn}(G) = n$ if and only if $G = K_n$.*

Proof. Suppose that $\mathrm{mn}(G) = n$. Suppose further that G is not complete. Then there exist $a, b \in V(G)$ such that $d_G(a, b) \neq 1$. Let $c \in J_G[a, b]\backslash\{a, b\}$ and consider $S = V(G)\backslash\{c\}$. Then S is not m-convex in G; hence $J_G[S] \neq S$. Since $c \in J_G[a, b]$, where $a, b \in S$, it follows that $c \in J_G[S]$. This implies that $V(G) \subseteq J_G[S]$. Therefore, $J_G[S] = V(G)$, that is, S is a monophonic set in G. Hence, $\mathrm{mn}(G) \leqslant |S| = n - 1$, contrary to our assumption. Therefore, G must be the complete graph K_n.

Conversely, suppose that G is a complete graph. Then every subset C of $V(G)$ is m-convex in G and hence, $J_G[C] = C$. Thus, a subset C of $V(G)$ satisfies $J_G[C] = V(G)$ if and only if $C = V(G)$. This implies that $C = V(G)$ is the only monophonic set in G. Therefore, $\mathrm{mn}(G) = n$. $\square$

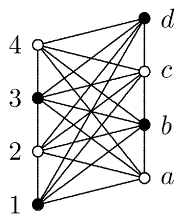

Fig. 1.

An immediate consequence of Theorems 2 and 3 is the following corollary.

Corollary 4. *Let G be a graph of order p. Then*

$$\mathrm{mn}(G+K_n)=\begin{cases} p+n & \text{if } G=K_p,\\ \mathrm{mn}(G) & \text{if } G\neq K_p.\end{cases}$$

Proof. The first part of the corollary follows from Theorem 3. By Theorem 2, it can be shown that $S\subseteq V(G+K_n)$ is a minimum monophonic set in $G+K_n$ if and only if $S\subseteq V(G)$ and is a minimum monophonic set in G. Thus, the result follows. $\square$

Lemma 5. *Let G and H be non-complete graphs and $S\subseteq V(G+H)$. If $S\cap V(G)$ is a monophonic set in G or $S\cap V(H)$ is a monophonic set in H, then S is a monophonic set in $G+H$.*

Proof. Assume that $S_1=S\cap V(G)$ is a monophonic set in G. If $S_1=V(G)$, then S_1 is a monophonic set in $G+H$; hence, S is a monophonic set in $G+H$. Suppose $S_1\neq V(G)$. Let $x\in V(G)\backslash S_1$. Then there exist $y,z\in S_1$ such that $x\in J_G[y,z]$. Since y and z are different from x, $d_G(y,z)\neq 1$; hence, $d_{G+H}(y,z)=2$. It follows that $V(H)\subseteq J_{G+H}[y,z]\subseteq J_{G+H}[S_1]$. Further, since $x\in J_G[y,z]$ and every m-path in G is an m-path in $G+H$, it follows that $x\in J_{G+H}[S_1]$. Since x was arbitrary, $V(G)=S_1\cup(V(G)\backslash S_1)\subseteq J_{G+H}[S_1]$. Thus, $V(G+H)\subseteq J_{G+H}[S_1]$. If $V(G)=S_1$, then we can choose $y',z'\in V(G)$ with $d_G(y',z')\neq 1$. Then, $V(H)\subseteq J_{G+H}[y',z']$. Hence, $V(G+H)\subseteq J_{G+H}[S_1]$. In both cases, we have $J_{G+H}[S_1]=V(G+H)$. This shows that $S_1$is a monophonic set in $G+H$. Since $S_1\subseteq S$, it follows that S is a monophonic set in $G+H$.

A similar argument can be used to prove the other case. $\square$

The following example will show that the converse of Lemma 5 is not true.

Example 6. Consider the graphs $G=P_4=(1,2,3,4)$ and $H=P_4=(a,b,c,d)$ and the graph $G+H=P_4+P_4$ illustrated in Fig. 1. The set $S=\{1,3,b,d\}$ is clearly a monophonic set in P_4+P_4. However, $S_1=S\cap V(G)=\{1,3\}$ and $S_2=S\cap V(H)=\{b,d\}$ are not monophonic sets in G and H, respectively. One can easily verify that $\mathrm{mn}(P_4+P_4)=2$.

The following result characterizes the monophonic sets in $G+H$ for non-complete graphs G and H.

Theorem 7. *Let G and H be non-complete graphs. A subset S of $V(G+H)$ is a monophonic set in $G+H$ if and only if at least one of the following conditions holds*:

(a) *$S_1=S\cap V(G)$ is monophonic in G*;
(b) *$S_2=S\cap V(H)$ is monophonic in H; or*
(c) *There exist $x,y\in S_1=S\cap V(G)$ and $u,v\in S_2=S\cap V(H)$ such that $xy\notin E(G)$ and $uv\notin E(H)$.*

Proof. Suppose $S\subseteq V(G+H)$ is a monophonic set in $G+H$. If (a) or (b) holds, then we are done. So, suppose (a) and (b) are not satisfied by S. Then $J_G[S_1]\neq V(G)$ and $J_H[S_2]\neq V(H)$. Let $h\in V(H)\backslash J_H[S_2]$. Since S is a monophonic set in $G+H$, there exist $x,y\in S$ such that $h\in J_{G+H}[x,y]$. Since $h\neq x$ and $h\neq y$, it follows that $d_{G+H}(x,y)=2$. This means that either $x,y\in S_1$ or $x,y\in S_2$. Now, since $h\notin J_H[S_2]$, $x,y\notin S_2$; hence,

$x, y \in S_1$. Moreover, $xy \notin E(G)$. Similarly, there exist $u, v \in S_2$ such that $uv \notin E(H)$. This shows that condition (c) holds.

For the converse, suppose first that (c) holds. Then, clearly, $V(H) \subseteq J_{G+H}[x, y]$ and $V(G) \subseteq J_{G+H}[u, v]$. These inclusions imply that $V(G+H) \subseteq J_{G+H}[S]$; hence, S is a monophonic set in $G+H$. If (a) or (b) holds, then S is monophonic in $G+H$ by Lemma 5. $\square$

Theorem 8. *Let G and H be non-complete graphs. Then* $\mathrm{mn}(G+H) = \min\{4,\ \mathrm{mn}(G),\ \mathrm{mn}(H)\}$.

Proof. Consider the following cases:

Case 1: Assume that $\mathrm{mn}(G)=\min\{4,\ \mathrm{mn}(G),\ \mathrm{mn}(H)\}$ and therefore, $\mathrm{mn}(G)\leqslant 4$. Assume also that S is a minimum monophonic set in G. Then $\mathrm{mn}(G) = |S|$. By Lemma 5, S is a monophonic set in $G+H$. Suppose that there exists a monophonic set S' of $G+H$ such that $|S'| < |S|$. Because S is a minimum monophonic set in G, $S' \cap V(G)$ is not a monophonic set in G (otherwise, we get a contradiction). Furthermore, since $|S'| < |S| \leqslant 4$, S' does not satisfy the condition (c) in Theorem 7. Thus, by Theorem 7, $S^* = S' \cap V(H)$ has to be a monophonic set in H. This implies that $\mathrm{mn}(H) \leqslant |S^*| \leqslant |S'| < |S| \leqslant 4$. This contradicts the assumption that $\mathrm{mn}(G) \leqslant \mathrm{mn}(H)$. Therefore, S is a minimum monophonic set in $G+H$ and $\mathrm{mn}(G+H) = \mathrm{mn}(G)$.

Case 2: Assume that $\mathrm{mn}(H)=\min\{4,\ \mathrm{mn}(G),\ \mathrm{mn}(H)\}$. Arguing as in *Case* 1, we obtain that $\mathrm{mn}(G+H)=\mathrm{mn}(H)$.

Case 3: Assume that $4=\min\{4,\ \mathrm{mn}(G),\ \mathrm{mn}(H)\}$. Choose $x, y \in V(G)$ and $u, v \in V(H)$ such that $xy \notin E(G)$ and $uv \notin E(H)$. By Theorem 7(c), the set $S=\{x, y, u, v\}$ is a monophonic set in $G+H$. Therefore, $\mathrm{mn}(G+H) \leqslant |S| = 4$. Suppose that there exists a monophonic set S' in $G+H$ such that $|S'| \leqslant 3$. Since S' does not satisfy the condition (c) in Theorem 7 and because of this theorem, either $S_1 = S' \cap V(G)$ is a monophonic set in G or $S_2 = S' \cap V(H)$ is a monophonic set in H. As a consequence, we get either $\mathrm{mn}(G) \leqslant |S_1| \leqslant 3$ or $\mathrm{mn}(H) \leqslant |S_2| \leqslant 3$, contradicting our assumption that $4 = \min\{4,\ \mathrm{mn}(G),\ \mathrm{mn}(H)\}$. Thus, $\mathrm{mn}(G+H) = 4$. $\square$

3. Monophonic number of the composition of graphs

Definition 9. The *composition* of two graphs G_1 and G_2, denoted by $G = G_1[G_2]$, is the graph with $V(G) = V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1v_1 \in E(G_1)$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$.

Lemma 10. *Let G be a graph. If* $P = ((u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_r, v_r))$, *where* $r \geqslant 2$, *is an m-path in* $G[K_n]$, *then we have the following possibilities*:

(i) *If the* u_i*'s are distinct*, *then* $(u_1, u_2, \ldots, u_r)$ *is an m-path in G*.
(ii) *If the* u_i*'s are not distinct*, *then* $r = 2$.

Proof. (i) Suppose the u_i's are distinct. Since $d_{G[K_n]}((u_i, v_i), (u_{i+1}, v_{i+1}))=1$ for all $i=1, 2, \ldots, r-1$, $d_G(u_i, u_{i+1})=1$ for all $i = 1, 2, \ldots, r-1$. Since $d_{G[K_n]}((u_i, v_i), (u_j, v_j)) \geqslant 2$ whenever $1 \leqslant i < j-1 \leqslant r$, $d_G(u_i, u_j) \geqslant 2$ whenever $1 \leqslant i < j-1 \leqslant r$. Thus, $(u_1, u_2, \ldots, u_r)$ is an m-path in G.

(ii) Suppose $u_i = u_j$, for some i, j with $i < j$ and $r \geqslant 3$. Then $d_{G[K_n]}((u_i, v_i), (u_j, v_j)) = 1$. Consequently, $i \in \{1, \ldots, r-1\}$ and $j = i+1$. If $1 \leqslant i \leqslant r-2$, then $d_{G[K_n]}((u_i, v_i), (u_{i+2}, v_{i+2})) = 1$, a contradiction. If $i = r-1$, then $d_{G[K_n]}((u_{r-2}, v_{r-2}), (u_r, v_r)) = 1$, a contradiction. Hence, if the u_i's are not distinct, then $r = 2$. $\square$

Lemma 11. *Let it G be a graph and H a non-complete graph. If* $P = ((u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r))$, *where* $r \geqslant 2$ *is an m-path in* $G[H]$, *then we have the following possibilities*:

(i) *If the* u_i*'s are distinct*, *then* $(u_1, u_2, \ldots, u_r)$ *is an m-path in G*.
(ii) *If the* u_i*'s are all equal*, *then* $(v_1, v_2, \ldots, v_r)$ *is an m-path in H*.
(iii) *If the* u_i*'s are not distinct and not all equal*, *then* $r = 3$.

Proof. (i) This part is proved as in Lemma 10.

(ii) Suppose $u_1 = u_2 = \ldots = u_r$. Since $d_{G[H]}((u_i, v_i), (u_{i+1}, v_{i+1})) = 1$ for all $i = 1, \ldots, r-1$, $d_H(v_i, v_{i+1}) = 1$ for all $i = 1, 2, \ldots, r-1$. Since $d_{G[H]}((u_i, v_i), (u_j, v_j)) \geqslant 2$ whenever $1 \leqslant i < j+1 \leqslant r$, $d_H(v_i, v_j) \geqslant 2$ whenever $1 \leqslant i < j+1 \leqslant r$. Thus, $(v_1, v_2, \ldots, v_r)$ is an m-path in H.

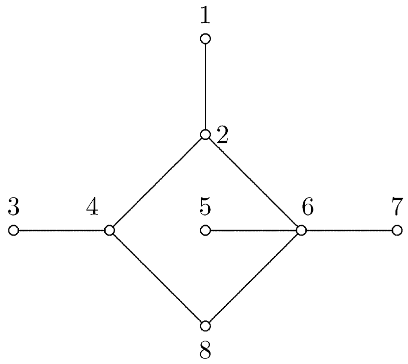

Fig. 2.

(iii) Suppose the u_i's are not distinct and not all are equal (This is possible since H is non-complete). Let $i, j \in \{1, 2, \ldots, r\}$, $i < j$, such that $u_i = u_j$ and $k \in \{1, 2, \ldots, r\}$ such that $d_G(u_i, u_k) = 1$. If $k < i$, then $\langle\{(u_k, v_k), (u_i, v_i), (u_j, v_j)\}\rangle$ is a cycle. If $k > j$, then $\langle\{(u_k, v_k), (u_i, v_i), (u_j, v_j)\}\rangle$ is a cycle in $G[H]$. If $k > i + 1$, $\langle\{(u_i, v_i), (u_{i+1}, v_{i+1}), (u_k, v_k)\}\rangle$ is a cycle in $G[H]$. If $k < j - 1$, then $\langle\{(u_{j-1}, v_{j-1}), (u_j, v_j), (u_k, v_k)\}\rangle$ is a cycle in $G[H]$. Hence, in any of these cases we have a contradiction. Thus, $k = i + 1$ and $k = j - 1$. Now, if $i > 1$ and $u_{i-1} = u_i$, then $d_{G[H]}((u_{i-1}, v_{i-1}), (u_k, v_k)) = 1$ and $\langle\{(u_{i-1}, v_{i-1}), (u_i, v_i), (u_k, v_k)\}\rangle$is a cycle. If $i > 1$ and $d_G(u_{i-1}, u_i) = 1$, then $d_{G[H]}((u_{i-1}, v_{i-1}), (u_j, v_j))$=1 and $\langle\{(u_{i-1}, v_{i-1}), (u_i, v_i), (u_k, v_k), (u_j, v_j)\}\rangle$ is a cycle. Again, in both cases we have a contradiction. Thus, $i = 1$. If $j < r$ and $u_{j+1} = u_j$, then $\langle\{(u_j, v_j), (u_{j+1}, v_{j+1}), (u_k, v_k)\}\rangle$ is a cycle, a contradiction. If $j < r$ and $d_G(u_j, u_{j+1}) = 1$, then $\langle\{(u_i, v_i), (u_k, v_k), (u_k, v_k), (u_{j+1}, v_{j+1})\}\rangle$ is a cycle, a contradiction. Thus, $j = r$.

Therefore, $r = 3$, $u_1 = u_3$ and $d_G(u_1, u_2) = 1$. □

Definition 12. Let G be a graph and $A \subseteq V(G)$. A point a in A is called a *monophonic interior point* of A if $a \in J_G[A\backslash\{a\}]$. The set of all monophonic interior points of A is denoted by A°.

Example 13. Let G be the graph illustrated in Fig. 2. Let $A = \{2, 4, 5, 6, 7, 8\}$, $B = \{1, 4, 3\}$, and $C = \{1, 7\}$. Since $2 \in J_G[A\backslash\{2\}]$, 2 is a monophonic interior point of A. Notice that the vertices 4, 6 and 8 are also monophonic interior points of A. Thus, $A^\circ = \{2, 4, 6, 8\}$ and no others. Also, it can be seen from Fig. 2 that $B^\circ = \{4\}$ and $C^\circ = \emptyset$.

Theorem 14. *Let G be a graph. Then* $C = \bigcup_{a \in S}(\{a\} \times T_a)$ *is a monophonic set in* $G[K_n]$, *where* $S \subseteq V(G)$ *and* $T_a \subseteq V(H)$ *for every* $a \in S$, *if and only if S is a monophonic set in G and* $T_a = V(K_n)$ *for every* $a \in S\backslash S^\circ$.

Proof. Suppose $C = \bigcup_{a \in S}(\{a\} \times T_a)$ is monophonic in $G[K_n]$. Let $u \in V(G)$ and $v \in V(K_n)$. If $u \in S$, then $u \in J_G[S]$. Suppose $u \in V(G)\backslash S$. Since C is monophonic in $G[K_n]$, there exists an m-path

$$((u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)),$$

where $(u_1, v_1), (u_r, v_r) \in C$ and $(u_k, v_k) = (u, v)$ for some k, $1 < k < r$. Since $u \notin S$, then $u \neq u_1$ and $u \neq u_r$. If the u_i's are not distinct, then by Lemma 10, $r = 2$, a contradiction. Thus, u_i's are distinct and by Lemma 10, $(u_1, u_2, \ldots, u_r)$ is an m-path in G. Consequently, $u \in J_G[S]$. Thus, S is a monophonic set in G.

Suppose there exists $a \in S\backslash S^\circ$ such that $T_a \neq V(K_n)$. Let $v \in V(K_n)\backslash T_a$. Then there exists an m-path

$$((u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)),$$

where $(u_1, v_1), (u_r, v_r) \in C$ and $(u_k, v_k) = (a, v)$ for some k, $1 \leqslant k \leqslant r$. By Lemma 10, $(u_1, u_2, \ldots, u_r)$ is an m-path in G or $r = 2$ and $u_1 = u_2$. If $(u_1, u_2, \ldots, u_r)$ is an m-path in G, $a = u_k \in J_G[S\backslash\{u_k\}]$, a contradiction. If $r = 2$ and $u_1 = u_2$, $v = v_1$ or $v = v_2$, a contradiction. Thus, $T_a = V(K_n)$ whenever $a \in S\backslash S^\circ$.

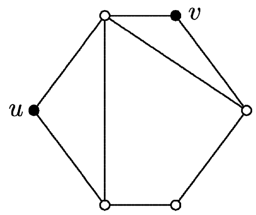

Fig. 3.

For the converse, suppose S is monophonic in G and $T_a = V(K_n)$ whenever $a \in S\backslash S^{\circ}$. Let $(u, v) \in V(G[K_n])$. Suppose $u \in V(G)\backslash S$ or $u \in S^{\circ}$. Since S is monophonic in G, there exists an m-path $(u_1, u_2, \ldots, u_r)$ such that $u_1, u_r \in S$ and $u_k = u$ for some k, $1 < k < r$. Let $v_1 \in T_{u_1}$ and $v_r \in T_{u_r}$. Then

$$((u_1, v_1), \ldots, (u_{k-1}, v_1), (u_k, v), (u_{k+1}, v_r), \ldots, (u_r, v_r))$$

is an m-path in $G[K_n]$. Since $(u_1, v_1), (u_r, v_r) \in C$, $(u, v) \in J_{G[K_n]}[C]$. Suppose $u \in S\backslash S^{\circ}$. Then $T_u = V(K_n)$; hence, $(u, v) \in C$.

Thus, $(u, v) \in J_{G[K_n]}[C]$. Accordingly, C is a monophonic set in $G[K_n]$. $\square$

The next result follows directly from Theorem 14.

Corollary 15. *Let G be a graph. Then*

$$\mathrm{mn}(G[K_n]) = \min\{n|S| - (n-1)|S^{\circ}| : S \textit{ is monophonic in } G\}.$$

Proof. Let S be a monophonic set in G and let $x \in V(K_n)$. Set $T_a = \{x\}$ for each $a \in S^{\circ}$. By Theorem 14,

$$C_S = \bigcup_{a \in S}(\{a\} \times T_a) = \left[\bigcup_{a \in S^{\circ}} \{(a, x)\}\right] \cup \left[\bigcup_{a \in S\backslash S^{\circ}} (\{a\} \times V(K_n))\right]$$

is a monophonic set in $G[K_n]$. Clearly, $|C_S| = n|S| - (n-1)|S^{\circ}|$. Therefore, the result follows. $\square$

Definition 16. A vertex of a graph G is an *extreme vertex* if its neighbors induce a complete subgraph of G. A graph G is called an *extreme monophonic graph* if $\mathrm{Ext}(G)$ (set containing all the extreme vertices of G) is a monophonic set in G.

Example 17. Consider the graph G illustrated in Fig. 3. Clearly, the vertices u and v are the only extreme vertices of G. Also, it is easy to verify that $S = \mathrm{Ext}(G) = \{u, v\}$ is the unique minimum monophonic set in G. Therefore, G is an extreme monophonic graph.

The following corollary is a consequence of Corollary 15.

Corollary 18. *Let G be a graph. If G is an extreme monophonic graph, then* $\mathrm{mn}(G[K_n]) = n|\mathrm{Ext}(G)|$.

Proof. Let $S = \mathrm{Ext}(G)$. By assumption, S is a monophonic set in G. It is easy to show that $S^{\circ} = \emptyset$. Now let S_1 be a monophonic set in G. Then $S \subseteq S_1$ and $S_1^{\circ} \subseteq S_1\backslash S$. Hence,

$$n|S_1| - (n-1)|S_1^{\circ}| \geqslant n|S_1| - (n-1)|S_1\backslash S| = n|S| + |S_1| - |S| \geqslant n|S|.$$

By Corollary 15, $\mathrm{mn}(G[K_n]) = n|S| = n|\mathrm{Ext}(G)|$. $\square$

Definition 19. A subset S of $V(G)$ is a *monophonic closed set* in G if for every $a \in V(G)\backslash J_G[S]$ there exists $s \in S$ such that $d_G(a, s) = 1$. A monophonic closed set with minimum cardinality is called a *minimum monophonic closed set*. Its cardinality is denoted by $\mathrm{m}_c(G)$.

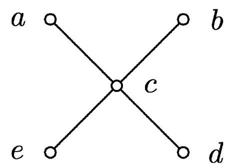

Fig. 4.

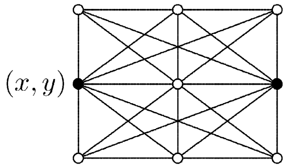

Fig. 5.

Example 20. Consider the graph G illustrated in Fig. 4. The set $A=\{a,b,d,e\}$ is the minimum monophonic set in G, i.e, $\mathrm{mn}(G)=4$. Set $B=\{c\}$ is not a monophonic set but is a monophonic closed set since each vertex in the set $\{a,b,d,e\} = V(G)\backslash J_G[\{c\}]$ is of distance one from vertex c. Since B is a minimum monophonic closed set, $\mathrm{m}_c(G)=1$.

Remark 21. Every monophonic set in G is a monophonic closed set. Thus, $\mathrm{m}_c(G) \leqslant \mathrm{mn}(G)$.

Theorem 22. *Let G be a graph and H a non-complete graph. If $C=\bigcup_{a\in S}(\{a\}\times T_a)$ is a monophonic set in $G[H]$, where $S\subseteq G$ and $T_a \subseteq V(H)$ for every $a\in S$, then S is a monophonic closed set in G.*

Proof. Suppose $C=\bigcup_{a\in S}(\{a\}\times T_a)$ is a monophonic set in $G[H]$. Suppose further that S is not a monophonic closed set in G. Then there exists $a\in V(G)\backslash J_G[S]$ such that $d_G(a,s)\geqslant 2$ for every $s\in S$. Let $b\in V(H)$. Since C is monophonic in $G[H]$, there exists an m-path $((u_1,v_1),(u_2,v_2),\ldots,(u_r,v_r))$ such that $(u_1,v_1),(u_r,v_r)\in C$ and $(a,b)=(u_k,v_k)$ for some k, $1<k<r$. Suppose the u_i's are distinct. Then by Lemma 11, $(u_1,u_2,\ldots,u_r)$ is an m-path in G. Consequently, $a=u_k\in J_G[S]$, a contradiction. Suppose the u_i's are all equal. Then $a=u_k=u_1\in S$, a contradiction. Suppose the u_i's are not distinct and not all equal. Then by Lemma 11, $r=3$, $u_1=u_3$ and $d_G(u_1,u_2)=1$. Consequently, $k=2$ and $d_G(a,u_1)=1$, a contradiction. Thus, S is a monophonic closed set in G. $\square$

The graph $P_3[P_3]$ illustrated in Fig. 5 shows that the converse of Theorem 22 is not true. This is shown in the next example.

Example 23. Consider the graph $P_3[P_3]$ illustrated in Fig. 5. Clearly, $\{x\}$ is a is a monophonic closed set in P_3. However, $\{(x,y)\}$ is not a monophonic set in $P_3[P_3]$.

The next result gives a lower bound of the monophonic number of the composition of G and H, where H is non-complete.

Corollary 24. *Let G be a graph and H a non-complete graph. Then $\mathrm{m}_c(G)\leqslant\mathrm{mn}(G[H])$. If G has a minimum monophonic closed set S such that S is monophonic and $S^\circ=S$, then $\mathrm{m}_c(G)=\mathrm{mn}(G[H])$.*

Proof. The inequality follows directly from Theorem 22.

Next, suppose that S be a minimum monophonic closed set such that S is monophonic in G and $S^\circ=S$. For each $s\in S$, choose $b_s\in V(H)$. We will show that $C=\bigcup_{s\in S}\{(s,b_s)\}$ is a monophonic set in $G[H]$. To this end, let $(u,v)\in V(G[H])$. Since S is monophonic, there exists an m-path $(u_1,\ldots,u_k,\ldots,u_r)$ such that $u_1,u_r\in S$ and $u_k=u$

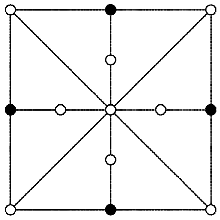

Fig. 6.

for some k. Since $S^{\circ} = S$, then we can further assume that $1 < k < r$. Note that

$$((u_1, b_{u_1}), \ldots, (u_{k-1}, b_{u_1}), (u_k, v), (u_{k+1}, b_{u_r}), \ldots, (u_r, b_{u_r}))$$

is an m-path in $G[H]$ and $(u_1, b_{u_1}), (u_r, b_{u_r}) \in C$. Consequently, $(u, v) \in J_{G[H]}[C]$. Accordingly, C is a monophonic set in $G[H]$. Thus, $\mathrm{mn}(G[H]) \leqslant \mathrm{m}_c(G)$. Combining this with the first part of this corollary, we obtain that $\mathrm{m}_c(G) = \mathrm{mn}(G[H])$. $\square$

Example 25. The graph illustrated in Fig. 6 shows that the conditions in the second part of Corollary 24 are, indeed, possible.

It is easy to verify that the shaded vertices of G in Fig. 6 form a minimum monophonic closed set in G. Moreover, this set of vertices is monophonic and its elements are its interior points.

Theorem 26. *Let G be a graph and H a non-complete graph. Suppose $S \subseteq V(G)$ is a monophonic closed set in G and T_a is a monophonic set in H for all $a \in S$. Then $C = \bigcup_{a \in S}(\{a\} \times T_a)$ is a monophonic set in $G[H]$.*

Proof. Suppose $S \subseteq V(G)$ is a monophonic closed set in G and T_a is a monophonic set in H for all $a \in S$. Let $x = (u, v) \in V(G[H])$. If $x \in C$, then we are done. Assume $x \notin C$. Consider the following cases:

Case 1: Suppose $u \in S$ and $v \notin T_u$.

Then T_u is monophonic in H. Thus there exists an m-path $(v_1, v_2, \ldots, v_n)$ in H such that $v_1, v_n \in T_u$ and $v = v_s$ for some s, $1 < s < n$. Now $((u, v_1), (u, v_2), \ldots, (u, v_n))$ is an m-path in $G[H]$ where $(u, v_1), (u, v_n) \in C$. Thus, $x = (u, v) = (u, v_s) \in J_{G[H]}[C]$.

Case 2: Suppose $u \in J_G[S] \backslash S$.

Then there exists an m-path $(u_1, u_2, \ldots, u_m)$ in G such that $u_1, u_m \in S$ and $u = u_t$ for some t with $1 < t < m$. Let $v_1 \in T_{u_1}$ and $v_m \in T_{u_m}$. Note that

$$((u_1, v_1), \ldots, (u_{t-1}, v_1), (u_t, v), (u_{t+}, v_m), \ldots, (u_m, v_n))$$

is an m-path in $G[H]$. Moreover, $(u_1, v_1), (u_m, v_m) \in C$ and $u_t = u$.

Thus, $x = (u, v) \in J_{G[H]}[C]$.

Case 3: Suppose $u \in V(G) \backslash J_G[S]$.

Since S is a monophonic closed set, there exists $s \in S$ such that $d_G(u, s) = 1$. Since T_s is monophonic and H is non-complete, the induced subgraph $\langle T_s \rangle_H$ of T_s in H is non-complete. Let $v_1, v_2 \in T_s$ such that $d_H(v_1, v_2) \geqslant 2$. Note that $((s, v_1), (u, v), (s, v_2))$ is an m-path in $G[H]$ and $(s, v_1), (s, v_2) \in C$. Thus, $x = (u, v) \in J_{G[H]}[C]$.

Hence, C is monophonic in $G[H]$. $\square$

Example 27. The shaded vertices illustrated in Fig. 7 form a monophonic set in $C_6[P_3]$. But the second component of these vertices do not form a monophonic set in P_3. Therefore, the converse of Theorem 26 is not true.

Corollary 28. *Let G be a graph and H a non-complete graph. Then*

$$\mathrm{mn}(G[H]) \leqslant \mathrm{m}_c(G)\mathrm{mn}(H).$$

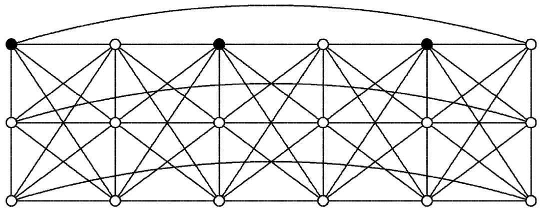

Fig. 7.

The graph in Fig. 5 shows that the upper bound in Corollary 28 is sharp. Note that the shaded vertices of the graph in Fig. 5 form a minimum monophonic set in $P_3[P_3]$, that is, $\mathrm{mn}(P_3[P_3]) = 2$. On the other hand, $\mathrm{m}_c(P_3) = 1$ and $\mathrm{mn}(P_3) = 2$. Thus, $\mathrm{mn}(P_3[P_3]) = 2 = \mathrm{m}_c(P_3)\mathrm{mn}(P_3)$.

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