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Toroidal vertex algebras and their modules

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ABSTRACT

We develop a theory of toroidal vertex algebras and their modules, and we give a conceptual construction of toroidal vertex algebras and their modules. As an application, we associate toroidal vertex algebras and their modules to toroidal Lie algebras.

1. Introduction

This is a paper in a series to study extended affine Lie algebras by using vertex algebra language and techniques. In this paper, we introduce and study a theory of $(r+1)$-toroidal vertex algebras with r a positive integer, and we give a general construction. As an example, we associate $(r+1)$-toroidal vertex algebras and their modules to $(r+1)$-toroidal Lie algebras.

Toroidal Lie algebras, which are central extensions of multi-loop Lie algebras, form a special family of what were called extended affine Lie algebras (see [AABGP]), which are a large family of Lie algebras, generalizing affine Kac–Moody Lie algebras in a certain natural way. Affine Kac–Moody Lie algebras are classified as untwisted affine algebras and twisted affine algebras (see [K]), where untwisted affine Lie algebras are the universal central extensions of 1-loop Lie algebras while twisted affine algebras are (or can be realized as) the fixed-point subalgebras of untwisted affine algebras

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with respect to Dynkin diagram automorphisms. It is expected that every extended affine Lie algebra can be realized as the fixed-point subalgebra of a toroidal Lie algebra with respect to a finite abelian group (cf. [A,ABY,ABFP1,ABFP2,ABP1,ABP2]).

It has been well known ([FLM,FZ,DL]; cf. [Li1,Li2]) that vertex algebras can be associated to both untwisted and twisted affine Lie algebras. More specifically, vertex algebras and modules are associated to highest weight modules for untwisted affine Lie algebras, whereas twisted modules for those associated vertex algebras are associated to highest weight modules for the twisted affine Lie algebras. In the literature, a connection of toroidal Lie algebras with vertex algebras has also been known (see [BBS]), which uses one-variable generating functions for toroidal Lie algebras. On the other hand, it is natural and also we need for various purposes to consider multi-variable generating functions (cf. [IKU,IKUX]). Then an important question is what kind of vertex algebra-like structures we can possibly get by using such multi-variable generating functions. This is one of our motivations for introducing a theory of toroidal vertex algebras.

In this paper, for a given positive integer r we introduce a notion of $(r+1)$-toroidal vertex algebra. By definition, an $(r+1)$-toroidal vertex algebra is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot;x_0,\mathbf{x}):\quad & V \to (\operatorname{End} V)\big[\big[x_0^{\pm 1},x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big],\\ & v \mapsto Y(v;x_0,\mathbf{x}), \end{aligned}$$

where $\mathbf{x}=(x_1,\dots,x_r)$, such that for $u,v\in V$,

$$Y(u;x_0,\mathbf{x})v \in V\big[\big[x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big]((x_0)),$$

and

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y(u;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y}) - z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y(v;y_0,\mathbf{y})Y(u;x_0,\mathbf{zy})\\ & = y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big). \end{aligned} \tag{1.1}$$

The existence of a vector $\mathbf{1}$, called the vacuum vector, is also assumed, satisfying

$$Y(\mathbf{1};x_0,\mathbf{x})v=v \quad\text{and}\quad Y(v;x_0,\mathbf{x})\mathbf{1}\in V\big[\big[x_0,x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big] \quad \text{for } v\in V.$$

Note that we here do not have the full creation property for a vertex algebra V:

$$Y(v,x)\mathbf{1}\in V[[x]] \quad\text{and}\quad \big(Y(v,x)\mathbf{1}\big)\big|_{x=0}=v \quad\text{for } v\in V.$$

In this toroidal vertex algebra theory, the vertex operator map $Y(\cdot;x_0,\mathbf{x})$ in general may be *not* injective. That is, the state-field correspondence may be not one-to-one. Furthermore, for a vertex algebra V, one has a canonical operator D, defined by

$$D(v)=\left(\frac{d}{dx}Y(v,x)\mathbf{1}\right)\bigg|_{x=0} \quad\text{for } v\in V,$$

and the following important properties hold:

$$\begin{aligned} \big[D,Y(v,x)\big] &= \frac{d}{dx}Y(v,x),\\ Y(u,x)v &= e^{xD}Y(v,-x)u \quad\text{for } u,v\in V. \end{aligned}$$

For a general toroidal vertex algebra, we *no longer* have such properties. These represent the major differences between the notion of toroidal vertex algebra and that of vertex algebra.

In this paper, we also give a general construction of $(r+1)$-toroidal vertex algebras. Let W be a general vector space. Set

$$\mathcal{E}(W,r)=\operatorname{Hom}\bigl(W,W\bigl[\bigl[x_1^{\pm 1},\dots,x_r^{\pm 1}\bigr]\bigr]((x_0))\bigr).$$

We consider local subsets U of $\mathcal{E}(W,r)$ in the sense that for any $a(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in U$, there exists a nonnegative integer k such that

$$(x_0-z_0)^k\bigl[a(x_0,\mathbf{x}),b(z_0,\mathbf{z})\bigr]=0.$$

It is proved that every local subset generates an $(r+1)$-toroidal vertex algebra in a certain canonical way with W as a module. Roughly speaking, for $a(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in\mathcal{E}(W,r)$, we define $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})\in\mathcal{E}(W,r)$ for $(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r$ in terms of generating function

$$Y_{\mathcal{E}}\bigl(a(y_0,\mathbf{y});z_0,\mathbf{z}\bigr)b(y_0,\mathbf{y})=\sum_{(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r}a(y_0,y)_{m_0,\mathbf{m}}b(y_0,\mathbf{y})z_0^{-m_0-1}\mathbf{z}^{-\mathbf{m}}$$

symbolically by

$$Y_{\mathcal{E}}\bigl(a(y_0,\mathbf{y});z_0,\mathbf{z}\bigr)b(y_0,\mathbf{y})=\bigl(a(x_0,\mathbf{x})b(y_0,\mathbf{y})\bigr)\big|_{x_0=y_0+z_0,\mathbf{x}=\mathbf{yz}}$$

(see Section 3 for the precise definition). Note that for vertex algebras one uses the usual operator product expansion

$$a(x)b(z)\sim\sum_{n\in\mathbb{Z}}(x-z)^{-n-1}C_n(z).$$

For toroidal vertex algebras, we use the following operator product expansion

$$a(x_0,\mathbf{x})b(z_0,\mathbf{z})\sim\sum_{m_0\in\mathbb{Z},\ \mathbf{m}\in\mathbb{Z}^r}(x_0-z_0)^{-m_0-1}\left(\frac{\mathbf{x}}{\mathbf{z}}\right)^{-\mathbf{m}}C_{m_0,\mathbf{m}}(z_0,\mathbf{z}).$$

Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$. Associated to $(\mathfrak{g},\langle\cdot,\cdot\rangle)$, one has a general affine Lie algebra

$$\hat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}\bigl[t,t^{-1}\bigr]\oplus\mathbb{C}\mathbf{k},$$

which is used to construct vertex algebras. Now, consider a central extension of the $(r+1)$-loop Lie algebra

$$\widehat{L_{r+1}(\mathfrak{g})}=\mathfrak{g}\otimes\mathbb{C}\bigl[t_0^{\pm 1},t_1^{\pm 1},\dots,t_r^{\pm 1}\bigr]\oplus\mathbb{C}\mathbf{k},$$

which is referred to as the $(r+1)$-toroidal Lie algebra of $\mathfrak{g}$. We here use this Lie algebra to construct $(r+1)$-toroidal vertex algebras.

Set $B=\mathfrak{g}\otimes\mathbb{C}[t_0,t_1^{\pm 1},\dots,t_r^{\pm 1}]+\mathbb{C}\mathbf{k}$. Let ℓ be a complex number. Define a B-module $(\mathfrak{g}+\mathbb{C})_\ell$ where $(\mathfrak{g}+\mathbb{C})_\ell=\mathfrak{g}\oplus\mathbb{C}$ as a vector space, with $\mathbf{k}$ acting as scalar ℓ, with $\mathfrak{g}\otimes\mathbb{C}[t_0,t_1^{\pm 1},\dots,t_r^{\pm 1}]$ acting trivially on $\mathbb{C}$, and with

$$\left(a\otimes \mathbf{t}^{\mathbf{m}}\right)\cdot b=[a,b],\qquad \left(a\otimes t_0\mathbf{t}^{\mathbf{m}}\right)\cdot b=\langle a,b\rangle\ell,\qquad \left(a\otimes t_0^n\mathbf{t}^{\mathbf{m}}\right)\cdot b=0$$

for $a,b\in\mathfrak{g}$, $\mathbf{m}\in\mathbb{Z}^r$, $n\geqslant 2$. Then form an induced module

$$V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)=U\big(\widehat{L_{r+1}(\mathfrak{g})}\big)\otimes_{U(B)}(\mathfrak{g}+\mathbb{C})_\ell,$$

which naturally contains $\mathfrak{g}$ as a subspace. Set $\mathbf{1}=1\otimes 1\in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$. It is proved that there exists a canonical structure of an $(r+1)$-toroidal vertex algebra on $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$ with $Y(a;x_0,\mathbf{x})=a(x_0,\mathbf{x})$ for $a\in\mathfrak{g}$, where

$$a(x_0,\mathbf{x})=\sum_{(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r}\big(a\otimes t_0^{m_0}\mathbf{t}^{\mathbf{m}}\big)x_0^{-m_0-1}\mathbf{x}^{-\mathbf{m}}.$$

Note that $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$ viewed as an $\widehat{L_{r+1}(\mathfrak{g})}$-module is *not* generated by $\mathbf{1}$, unlike the situation for the vertex algebras associated to affine Lie algebras.

Note that in the literature, certain higher dimension analogues of vertex algebras have been studied before (see [Bor,Li4,Ni,BN]), but toroidal vertex algebras are not vertex algebras in any of those senses. In fact, there is no natural connection between toroidal Lie algebras and any of those higher dimension analogues.

This paper is organized as follows: In Section 2, we define the notion of $(r+1)$-toroidal vertex algebra and the notion of module for an $(r+1)$-toroidal vertex algebra. We also present some basic results. In Section 3, we give a general construction of $(r+1)$-toroidal vertex algebras and their modules. In Section 4, we associate $(r+1)$-toroidal vertex algebras and their modules to $(r+1)$-toroidal Lie algebras.

2. Toroidal vertex algebras and their modules

In this section we define the notion of $(r+1)$-toroidal vertex algebra and the notion of module for an $(r+1)$-toroidal vertex algebra with r a positive integer. We present some basic properties similar to those for ordinary vertex algebras.

For this paper, the scalar field is the field $\mathbb{C}$ of complex numbers, though it works fine with any field of characteristic zero. Letters $x,y,z,x_0,y_0,x_0,x_1,y_1,z_1,\dots$ will be mutually commuting independent formal variables. Let r be a positive integer which is fixed throughout this section. For $\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{Z}^r$, set

$$\mathbf{x}^{\mathbf{m}}=x_1^{m_1}\cdots x_r^{m_r}.$$

As a convention we write

$$\mathbf{x}^{-1}=x_1^{-1}\cdots x_r^{-1},\qquad \mathbf{x}^{\mathbf{m}-1}=x_1^{m_1-1}\cdots x_r^{m_r-1}.$$

We also set

$$\mathrm{Res}_{\mathbf{x}}=\mathrm{Res}_{x_1}\cdots\mathrm{Res}_{x_r}.$$

Definition 2.1. An $(r+1)$-*toroidal vertex algebra* is a vector space V, equipped with a linear map

$$\begin{aligned}Y(\cdot;x_0,\mathbf{x}):\quad & V\to \mathrm{Hom}\big(V,V\big[\big[x_1^{\pm1},\dots,x_r^{\pm1}\big]\big]((x_0))\big)\subset(\mathrm{End}\,V)\big[\big[x_0^{\pm1},x_1^{\pm1},\dots,x_r^{\pm1}\big]\big],\\ & v\mapsto Y(v;x_0,\mathbf{x})=\sum_{(m_0,\mathbf{m})\in\mathbb{Z}^{r+1}}v_{m_0,\mathbf{m}}x_0^{-m_0-1}\mathbf{x}^{-\mathbf{m}}\end{aligned}$$

and a vector $\mathbf{1} \in V$, satisfying the conditions that

$$Y(\mathbf{1}; x_0, \mathbf{x})v = v \quad \text{and} \quad Y(v; x_0, \mathbf{x})\mathbf{1} \in V\big[\big[x_0, x_1^{\pm 1}, \ldots, x_r^{\pm 1}\big]\big] \quad \text{for } v \in V,$$

and that for $u, v \in V$,

$$z_0^{-1}\delta\left(\frac{x_0 - y_0}{z_0}\right)Y(u; x_0, \mathbf{zy})Y(v; y_0, \mathbf{y}) - z_0^{-1}\delta\left(\frac{y_0 - x_0}{-z_0}\right)Y(v; y_0, \mathbf{y})Y(u; x_0, \mathbf{zy})$$
$$= y_0^{-1}\delta\left(\frac{x_0 - z_0}{y_0}\right)Y\big(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}\big), \tag{2.1}$$

where

$$Y(u; x_0, \mathbf{zy}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z}^{r+1}} u_{m_0, \mathbf{m}} x_0^{-m_0 - 1} \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}}.$$

We also define a notion of $(r+1)$-*toroidal vertex algebra without vacuum*, using all the axioms above that do not involve the vacuum vector $\mathbf{1}$. (Note that a notion of vertex algebra without vacuum was introduced and studied by Huang and Lepowsky [HL].)

Remark 2.2. Note that for a vertex algebra U we have the creation property

$$Y(u, x)\mathbf{1} \in U[[x]] \quad \text{and} \quad \big(Y(u, x)\mathbf{1}\big)\big|_{x=0} = u \quad \text{for } u \in U,$$

which particularly implies that the vertex operator map $Y(\cdot, x)$ is injective. For an $(r+1)$-toroidal vertex algebra V, we do not have this full creation property (mainly the second part) and in general the map Y may not be injective.

Let V be an $(r+1)$-toroidal vertex algebra. Just as with ordinary vertex algebras, applying Res_{z_0} and Res_{x_0} to the Jacobi identity (2.1), respectively, we obtain the following *commutator formula* and *iterate formula*:

$$\big[Y(u; x_0, \mathbf{zy}), Y(v; y_0, \mathbf{y})\big] = \mathrm{Res}_{z_0} y_0^{-1}\delta\left(\frac{x_0 - z_0}{y_0}\right)Y\big(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}\big), \tag{2.2}$$

$$Y\big(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}\big) = \mathrm{Res}_{x_0} z_0^{-1}\delta\left(\frac{x_0 - y_0}{z_0}\right)Y(u; x_0, \mathbf{zy})Y(v; y_0, \mathbf{y})$$
$$- \mathrm{Res}_{x_0} z_0^{-1}\delta\left(\frac{y_0 - x_0}{-z_0}\right)Y(v; y_0, \mathbf{y})Y(u; x_0, \mathbf{zy}). \tag{2.3}$$

For $v \in V$, $\mathbf{m} \in \mathbb{Z}^r$, set

$$Y(v; x_0, \mathbf{m}) = \sum_{m_0 \in \mathbb{Z}} v_{m_0, \mathbf{m}} x_0^{-m_0 - 1} = \mathrm{Res}_{\mathbf{x}} \mathbf{x}^{\mathbf{m} - 1} Y(v; x_0, \mathbf{x}) \in \mathrm{Hom}\big(V, V((x_0))\big). \tag{2.4}$$

From (2.2) we have

$$\big[Y(u; x_0, \mathbf{m}), Y(v; y_0, \mathbf{y})\big] = \mathrm{Res}_{z_0} y_0^{-1}\delta\left(\frac{x_0 - z_0}{y_0}\right)\mathbf{y}^{\mathbf{m}} Y\big(Y(u; z_0, \mathbf{m})v; y_0, \mathbf{y}\big). \tag{2.5}$$

Furthermore,

$$\left[Y(u;x_0,\mathbf{m}),Y(v;y_0,\mathbf{n})\right]=\mathrm{Res}_{z_0}\, y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y\left(Y(u;z_0,\mathbf{m})v;y_0,\mathbf{m}+\mathbf{n}\right). \tag{2.6}$$

On the other hand, from (2.3) we have

$$\begin{aligned}Y\left(Y(u;z_0,\mathbf{m})v;y_0,\mathbf{y}\right)&=\mathrm{Res}_{x_0}\, z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)\mathbf{y}^{-\mathbf{m}}Y(u;x_0,\mathbf{m})Y(v;y_0,\mathbf{y})\\&\quad-\mathrm{Res}_{x_0}\, z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)\mathbf{y}^{-\mathbf{m}}Y(v;y_0,\mathbf{y})Y(u;x_0,\mathbf{m}).\end{aligned} \tag{2.7}$$

Just as with ordinary vertex algebras, the Jacobi identity axiom is equivalent to weak commutativity and weak associativity.

Proposition 2.3. *Let V be a vector space equipped with a linear map*

$$Y(\cdot;x_0,\mathbf{x}):V\to \mathrm{Hom}\left(V,V\left[\left[x_1^{\pm 1},\dots,x_r^{\pm 1}\right]\right]((x_0))\right).$$

For $u,v\in V$, the Jacobi identity (2.1) *holds if and only if there exists a nonnegative integer k such that*

$$(x_0-y_0)^kY(u;x_0,\mathbf{x})Y(v;y_0,\mathbf{y})=(x_0-y_0)^kY(v;y_0,\mathbf{y})Y(u;x_0,\mathbf{x}) \tag{2.8}$$

and

$$z_0^kY\left(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\right)=\left((x_0-y_0)^kY(u;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y})\right)\big|_{x_0=y_0+z_0}. \tag{2.9}$$

Note that the commutativity relation (2.8) implies

$$(x_0-y_0)^kY(u;x_0,\mathbf{xy})Y(v;y_0,\mathbf{y})\in \mathrm{Hom}\left(V,V\left[\left[x_1^{\pm 1},y_1^{\pm 1},\dots,x_r^{\pm 1},y_r^{\pm 1}\right]\right]((x_0,y_0))\right),$$

so that the substitution

$$\left((x_0-y_0)^kY(u;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y})\right)\big|_{x_0=y_0+z_0}$$

exists in $\mathrm{Hom}(V,V[[x_1^{\pm 1},y_1^{\pm 1},\dots,x_r^{\pm 1},y_r^{\pm 1}]]((y_0))[[z_0]])$.

Remark 2.4. A familiar version of weak associativity is that for any $u,v,w\in V$, there exists a nonnegative integer l such that

$$(z_0+y_0)^lY\left(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\right)w=(z_0+y_0)^lY(u;z_0+y_0,\mathbf{zy})Y(v;y_0,\mathbf{y})w. \tag{2.10}$$

The equivalence between the two versions of weak associativity essentially follows from [LTW] (Lemma 2.9).

For $(r+1)$-toroidal vertex algebras we have the following skew symmetry:

Proposition 2.5. *Let V be an $(r+1)$-toroidal vertex algebra without vacuum. For $u, v \in V$, we have*

$$Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)=e^{z_0\frac{\partial}{\partial y_0}}\mathbf{z}^{\mathbf{y}\frac{\partial}{\partial \mathbf{y}}}Y\big(Y\big(v;-z_0,\mathbf{z}^{-1}\big)u;y_0,\mathbf{y}\big), \tag{2.11}$$

where

$$\mathbf{z}^{\mathbf{y}\frac{\partial}{\partial \mathbf{y}}}=z_1^{y_1\frac{\partial}{\partial y_1}}\cdots z_r^{y_1\frac{\partial}{\partial y_r}}.$$

Proof. It follows from the standard arguments as in [FHL]. Notice that starting from the left-hand side of the Jacobi identity for the pair (u,v), by changing (u,v) to (v,u), (x_0,y_0) to (y_0,x_0), $(z_0,\mathbf{z})$ to $(-z_0,\mathbf{z}^{-1})$, and then changing $\mathbf{y}$ to $\mathbf{yz}$, we obtain the left-hand side of the Jacobi identity for the pair (v,u). In view of this, we have

$$y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)=x_0^{-1}\delta\left(\frac{y_0+z_0}{x_0}\right)Y\big(Y\big(v;-z_0,\mathbf{z}^{-1}\big)u;x_0,\mathbf{yz}\big).$$

Applying Res_{x_0} to both sides, we obtain

$$\begin{aligned}Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)&=Y\big(Y\big(v;-z_0,\mathbf{z}^{-1}\big)u;y_0+z_0,\mathbf{yz}\big)\\&=e^{z_0\frac{\partial}{\partial y_0}}\mathbf{z}^{\mathbf{y}\frac{\partial}{\partial \mathbf{y}}}Y\big(Y\big(v;-z_0,\mathbf{z}^{-1}\big)u;y_0,\mathbf{y}\big),\end{aligned}$$

as desired. □

Definition 2.6. An *extended* $(r+1)$-*toroidal vertex algebra* is an $(r+1)$-toroidal vertex algebra V, equipped with linear operators $\mathcal{D}_0,\mathcal{D}_1,\dots,\mathcal{D}_r$, satisfying the condition that

$$\begin{aligned}&\mathcal{D}_i(\mathbf{1})=0\quad\text{for } 0\leqslant i\leqslant r,\\&\big[\mathcal{D}_0,Y(v;x_0,\mathbf{x})\big]=Y\big(\mathcal{D}_0(v);x_0,\mathbf{x}\big)=\frac{\partial}{\partial x_0}Y(v;x_0,\mathbf{x}),\\&\big[\mathcal{D}_j,Y(v;x_0,\mathbf{x})\big]=Y\big(\mathcal{D}_j(v);x_0,\mathbf{x}\big)=\left(x_j\frac{\partial}{\partial x_j}\right)Y(v;x_0,\mathbf{x})\end{aligned} \tag{2.12}$$

for $v\in V$, $1\leqslant j\leqslant r$.

For an $(r+1)$-toroidal vertex algebra V, we define a *derivation* of V to be a linear operator D on V satisfying the condition that

$$D(\mathbf{1})=0\quad\text{and}\quad\big[D,Y(v;x_0,\mathbf{x})\big]=Y\big(D(v);x_0,\mathbf{x}\big)\quad\text{for } v\in V. \tag{2.13}$$

Then the linear operators $\mathcal{D}_i$ $(0\leqslant i\leqslant r)$ for an extended $(r+1)$-toroidal vertex algebra are derivations.

From Proposition 2.5 we immediately have:

Corollary 2.7. *Suppose that V is an extended $(r+1)$-toroidal vertex algebra such that $\mathcal{D}_i$ $(i=1,\dots,r)$ are semisimple on V. Then*

$$Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)=Y\big(e^{z_0\mathcal{D}_0}\mathbf{z}^{\mathbf{D}}Y\big(v;-z_0,\mathbf{z}^{-1}\big)u;y_0,\mathbf{y}\big) \tag{2.14}$$

for $u, v \in V$, where

$$\mathbf{z}^{\mathbf{D}} = z_1^{\mathcal{D}_1} z_2^{\mathcal{D}_2} \cdots z_r^{\mathcal{D}_r}.$$

Furthermore, if the vertex operator map Y is injective, we have

$$Y(u; z_0, \mathbf{z})v = e^{z_0\mathcal{D}_0}\mathbf{z}^{\mathbf{D}} Y\big(v; -z_0, \mathbf{z}^{-1}\big)u. \tag{2.15}$$

Lemma 2.8. *Let V be an extended $(r+1)$-toroidal vertex algebra with derivations $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_r$. Suppose that $v \in V$ is an eigenvector of $\mathcal{D}_i$ for $1 \leqslant i \leqslant r$ with eigenvalues $\lambda_1, \ldots, \lambda_r$, respectively. Then $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, r$ and $Y(v; x_0, \mathbf{x}) \in \mathbf{x}^{\lambda}(\operatorname{End} V)[[x_0, x_0^{-1}]]$.*

Proof. From assumption we have

$$\lambda_i Y(v; x_0, \mathbf{x}) = Y\big(\mathcal{D}_i(v); x_0, \mathbf{x}\big) = \left(x_i \frac{\partial}{\partial x_i}\right) Y(v; x_0, \mathbf{x})$$

for $1 \leqslant i \leqslant r$. Notice that

$$Y(v; x_0, \mathbf{x}) \in (\operatorname{End} V)\big[\big[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_r^{\pm 1}\big]\big]$$

and that $(x_i \frac{\partial}{\partial x_i})\mathbf{x}^{\mathbf{m}} = m_i \mathbf{x}^{\mathbf{m}}$ for $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$. Then we have $\lambda_i \in \mathbb{Z}$ and

$$Y(v; x_0, \mathbf{x}) = x_1^{\lambda_1} \cdots x_r^{\lambda_r}(\operatorname{End} V)\big[\big[x_0, x_0^{-1}\big]\big],$$

as desired. □

We define notions of toroidal vertex subalgebra, ideal, and homomorphism, in the obvious ways. The following are straightforward to prove:

Lemma 2.9. *Let V be an $(r+1)$-toroidal vertex algebra and U a subset of V. Denote by $\langle U\rangle$ the linear span of vectors*

$$u^{(1)}_{\mathbf{m}_1} \cdots u^{(k)}_{\mathbf{m}_k} u$$

for $k \geqslant 0$, $u^{(1)}, \ldots, u^{(k)}, u \in U \cup \{\mathbf{1}\}$, $\mathbf{m}_1, \ldots, \mathbf{m}_k \in \mathbb{Z}^{r+1}$. Then $\langle U\rangle$ is a toroidal vertex subalgebra of V.

Lemma 2.10. *Let V and K be $(r+1)$-toroidal vertex algebras. Suppose that ψ is a linear map from V to K, satisfying $\psi(\mathbf{1}) = \mathbf{1}$,*

$$\psi\big(Y(u; x_0, \mathbf{x})v\big) = Y\big(\psi(u); x_0, \mathbf{x}\big)\psi(v) \tag{2.16}$$

for $u \in U$, $v \in V$, where U is a generating subset of V. Then ψ is a homomorphism.

Lemma 2.11. *Let V be an $(r+1)$-toroidal vertex algebra equipped with linear operators $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_r$, such that (2.12) holds for $v \in U$, where U is a generating subset of V. Then V is an extended $(r+1)$-toroidal vertex algebra.*

For any vertex algebra V, the left ideal generated by the vacuum vector is the whole vertex algebra V, but for higher dimension analogs, this is not true in general. We next study the left ideal generated by $\mathbf{1}$ of an $(r+1)$-toroidal vertex algebra.

Definition 2.12. Let V be an $(r+1)$-toroidal vertex algebra. Define V^0 to be the left ideal generated by $\mathbf{1}$, that is, V^0 is the subspace linearly spanned by vectors

$$u^{(1)}_{m_0^{(1)},\mathbf{m}^{(1)}}\cdots u^{(k)}_{m_0^{(k)},\mathbf{m}^{(k)}}\mathbf{1}$$

for $k \geqslant 0$, $u^{(i)} \in V$, $(m_0^{(i)}, \mathbf{m}^{(i)}) \in \mathbb{Z}\times\mathbb{Z}^r$.

First, we have:

Lemma 2.13. *Let V be an $(r+1)$-toroidal vertex algebra. For $u \in V$, $k \in \mathbb{N}$, $\mathbf{m} \in \mathbb{Z}^r$, we have*

$$Y(u_{k,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = 0,$$
$$Y(u_{-k-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = \frac{1}{k!}\left(\frac{\partial}{\partial x_0}\right)^k Y(u; x_0, \mathbf{m})\mathbf{x}^{-\mathbf{m}}. \tag{2.17}$$

Furthermore,

$$Y(u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) \in \mathbf{x}^{-\mathbf{m}}\big(\mathrm{Hom}\big(V, V((x_0))\big)\big) \tag{2.18}$$

for $m_0 \in \mathbb{Z}$ and

$$Y(u; x_0, \mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r} Y(u_{-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}). \tag{2.19}$$

Proof. Taking $v = \mathbf{1}$ in (2.3) we get

$$\begin{aligned} Y\big(Y(u; z_0, \mathbf{z})\mathbf{1}; y_0, \mathbf{y}\big) &= \mathrm{Res}_{x_0} x_0^{-1}\delta\left(\frac{y_0+z_0}{x_0}\right) Y(u; x_0, \mathbf{yz}) \\ &= Y(u; y_0+z_0, \mathbf{yz}) \\ &= e^{z_0\frac{\partial}{\partial y_0}} Y(u; y_0, \mathbf{yz}). \end{aligned} \tag{2.20}$$

From this we obtain

$$Y\big(Y(u; z_0, \mathbf{m})\mathbf{1}; y_0, \mathbf{y}\big) = \mathbf{y}^{-\mathbf{m}} e^{z_0\frac{\partial}{\partial y_0}} Y(u; y_0, \mathbf{m}), \tag{2.21}$$

which implies (2.17). By (2.21), we have

$$Y(u_{-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = \mathbf{x}^{-\mathbf{m}} Y(u; x_0, \mathbf{m}). \tag{2.22}$$

Summing up over $\mathbf{m} \in \mathbb{Z}^r$, we obtain (2.19). □

The following is the main result about V^0:

Proposition 2.14. *Let V be an $(r+1)$-toroidal vertex algebra. Then*

$$V^0 = \operatorname{span}\{v_{m_0,\mathbf{m}}\mathbf{1} \mid v \in V,\ (m_0,\mathbf{m}) \in \mathbb{Z}\times\mathbb{Z}^r\},$$

V^0 is a toroidal vertex subalgebra of V, and for $u \in V^0$,

$$Y(u;x_0,\mathbf{x}) \in \big(\operatorname{Hom}\big(V, V((x_0))\big)\big)\big[x_1^{\pm1},\dots,x_r^{\pm1}\big]. \tag{2.23}$$

Define a linear map $Y^0(\cdot,x_0): V^0 \to (\operatorname{End} V)[[x_0,x_0^{-1}]]$ by

$$Y^0(u,x_0) = Y(u;x_0,\mathbf{x})|_{\mathbf{x}=1} \quad \textit{for } u \in V^0.$$

Then $(V^0, Y^0, \mathbf{1})$ carries the structure of a vertex algebra and (V, Y^0) carries the structure of a V^0-module. Furthermore, we have

$$Y(u;x_0,\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r} Y^0(u_{-1,\mathbf{m}}\mathbf{1},x_0)\mathbf{x}^{-\mathbf{m}} \quad \textit{for } u \in V. \tag{2.24}$$

Proof. Set

$$U = \operatorname{span}\{v_{k,\mathbf{m}}\mathbf{1} \mid v \in V,\ k \in \mathbb{Z},\ \mathbf{m}\in\mathbb{Z}^r\} \subset V.$$

By definition we have $U \subset V^0$. For $u, v \in V$, applying (2.3) to $\mathbf{1}$ and then using the fact that $Y(u;x_0,\mathbf{x})\mathbf{1} \in V[[x_0,x_1^{\pm1},\dots,x_r^{\pm1}]]$ we get

$$Y(u;z_0+y_0,\mathbf{zy})Y(v;y_0,\mathbf{y})\mathbf{1} = Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)\mathbf{1}. \tag{2.25}$$

It follows that U is a left ideal of V and $V^0 \subset U$. Therefore $V^0 = U$. It then follows from (2.18) that for any $v \in V^0$,

$$Y(v;x_0,\mathbf{x}) \in \big(\operatorname{Hom}\big(V, V((x_0))\big)\big)\big[x_1^{\pm1},\dots,x_r^{\pm1}\big].$$

From this, we see that for every $v \in V^0$, $Y^0(v,x_0)$ is a well defined element in $\operatorname{Hom}(V, V((x_0)))$. As V^0 $(=U)$ is a left ideal of V, V^0 is a toroidal vertex subalgebra and we have

$$Y^0(v,x_0)V^0 \subset V^0((x_0)) \quad \text{for } v \in V^0.$$

Let $u, v \in V^0$. It can be readily seen that the Jacobi identity (2.1) for the ordered pair (u,v), after evaluated at $\mathbf{y}=1$, $\mathbf{z}=1$, gives rise to the Jacobi identity that we need for Y^0. Furthermore, we have $\mathbf{1} = \mathbf{1}_{-1,\mathbf{0}}\mathbf{1} \in V^0$ and $Y^0(\mathbf{1},x_0) = Y(\mathbf{1};x_0,\mathbf{x})|_{\mathbf{x}=1} = 1$. To show that $(V^0, Y^0, \mathbf{1})$ carries the structure of a vertex algebra, we need to verify the creation property. By (2.20) we have

$$Y^0\big(Y(u;z_0,\mathbf{z})\mathbf{1},y_0\big)\mathbf{1} = Y(u;y_0+z_0,\mathbf{z})\mathbf{1}.$$

Then

$$\lim_{y_0\to0} Y^0\big(Y(u;z_0,\mathbf{z})\mathbf{1},y_0\big)\mathbf{1} = Y(u;z_0,\mathbf{z})\mathbf{1}.$$

From this it follows that the creation property holds. Therefore, $(V^0, Y^0, \mathbf{1})$ carries the structure of a vertex algebra. As it was mentioned before,

$$Y^0(v, x_0) = Y(v; x_0, \mathbf{x})|_{\mathbf{x}=1} \in \mathrm{Hom}\big(V, V((x_0))\big)$$

for $v \in V^0$, $Y^0(\mathbf{1}, x) = 1$, and Y^0 satisfies the Jacobi identity for modules for a vertex algebra. Thus (V, Y^0) carries the structure of a V^0-module.

Note that for $u \in V$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, by (2.22) we have

$$Y^0(u_{m_0,\mathbf{m}}\mathbf{1}, x_0) = Y(u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x})|_{\mathbf{x}=1} = Y(u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{m}) = \mathbf{x}^{\mathbf{m}} Y(u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}).$$

Using (2.19) we get

$$Y(u; x_0, \mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r} Y(u_{-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r} Y^0(u_{-1,\mathbf{m}}\mathbf{1}, x_0)\mathbf{x}^{-\mathbf{m}},$$

as desired. □

Furthermore, we have:

Proposition 2.15. *Let V be an extended $(r+1)$-toroidal vertex algebra with derivations $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_r$. Then V^0 is an extended $(r+1)$-toroidal vertex subalgebra and $\mathcal{D}_j$ $(1 \leqslant j \leqslant r)$ are semisimple on V^0 with integer eigenvalues. Furthermore, $\mathcal{D}_j$ $(1 \leqslant j \leqslant r)$ are derivations of V^0 viewed as a vertex algebra and $\mathcal{D}_0$ coincides with the D-operator $\mathcal{D}$ of the vertex algebra V^0.*

Proof. It can be readily seen that the toroidal vertex subalgebra V^0 is stable under every derivation of V. In particular, V^0 is stable under $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_r$. For $v \in V$, $m_0 \in \mathbb{Z}$, $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, $1 \leqslant j \leqslant r$, we have

$$\mathcal{D}_j(v_{m_0,\mathbf{m}}\mathbf{1}) = v_{m_0,\mathbf{m}}(\mathcal{D}_j\mathbf{1}) - m_j(v_{m_0,\mathbf{m}}\mathbf{1}) = -m_j(v_{m_0,\mathbf{m}}\mathbf{1}). \tag{2.26}$$

We see that $\mathcal{D}_j$ $(1 \leqslant j \leqslant r)$ act semisimply on V^0 with integer eigenvalues. Furthermore, $\mathcal{D}_j$ $(1 \leqslant j \leqslant r)$ are derivations of V^0 viewed as a vertex algebra because

$$\big[\mathcal{D}_j, Y^0(v, x_0)\big] = \big[\mathcal{D}_j, Y(v; x_0, \mathbf{x})\big]\big|_{\mathbf{x}=1} = Y(\mathcal{D}_j v; x_0, \mathbf{x})|_{\mathbf{x}=1} = Y^0(D_j v, x_0)$$

for $v \in V^0$.

For $u \in V$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, we have

$$\begin{aligned}
\lim_{x_0\to 0} \frac{\partial}{\partial x_0} Y^0(u_{m_0,\mathbf{m}}\mathbf{1}, x_0)\mathbf{1} &= \lim_{x_0\to 0}\left(\frac{\partial}{\partial x_0} Y(u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x})\mathbf{1}\right)\bigg|_{\mathbf{x}=1} \\
&= \lim_{x_0\to 0}\big(Y(\mathcal{D}_0 u_{m_0,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x})\mathbf{1}\big)\big|_{\mathbf{x}=1} \\
&= \lim_{x_0\to 0} Y^0(\mathcal{D}_0 u_{m_0,\mathbf{m}}\mathbf{1}, x_0)\mathbf{1} \\
&= \mathcal{D}_0(u_{m_0,\mathbf{m}}\mathbf{1}).
\end{aligned}$$

Thus $\mathcal{D}_0 = \mathcal{D}$, the D-operator of the vertex algebra V^0. □

On the other hand, we have:

Proposition 2.16. *Let U be an (ordinary) vertex algebra equipped with mutually commuting derivations $\mathcal{D}_1, \dots, \mathcal{D}_r$ which act on U semisimply with integer eigenvalues. For $u \in U$, set*

$$Y(u; x_0, \mathbf{x}) = Y\big(\mathbf{x}^{\mathbf{D}} u, x_0\big),$$

where $\mathbf{x}^{\mathbf{D}} = x_1^{\mathcal{D}_1} \cdots x_r^{\mathcal{D}_r}$. Then U becomes an extended $(r+1)$-toroidal vertex algebra with $\mathcal{D}_0 = \mathcal{D}$, the canonical derivation of vertex algebra U. Furthermore, $U^0 = U$.

Proof. From definition, for $u \in U$ we have

$$Y(u; x_0, \mathbf{x}) \in \big(\mathrm{Hom}\big(U, U((x_0))\big)\big)\big[\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big].$$

We also have

$$\begin{aligned} &Y(\mathbf{1}; x_0, \mathbf{x}) = Y\big(\mathbf{x}^{\mathbf{D}}\mathbf{1}, x_0\big) = Y(\mathbf{1}, x_0) = 1, \\ &Y(u; x_0, \mathbf{x})\mathbf{1} = Y\big(\mathbf{x}^{\mathbf{D}} u, x_0\big)\mathbf{1} = e^{x_0\mathcal{D}_0}\mathbf{x}^{\mathbf{D}} u. \end{aligned} \tag{2.27}$$

Then the vacuum property holds. For $u, v \in U$, noticing that

$$Y\big(Y\big((\mathbf{yz})^{\mathbf{D}} u, z_0\big)\mathbf{y}^{\mathbf{D}} v, y_0\big) = Y\big(\mathbf{y}^{\mathbf{D}} Y\big(\mathbf{z}^{\mathbf{D}} u, z_0\big) v, y_0\big) = Y\big(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}\big),$$

we see that the Jacobi identity for a toroidal vertex algebra holds.

Let d be any derivation of U. It is straightforward to show that $d(\mathbf{1}) = 0$. Furthermore, for $u \in U$ we have

$$d\mathcal{D}_0(u) = \frac{d}{dx}\big(dY(u, x)\mathbf{1}\big)\big|_{x=0} = \frac{d}{dx}\big(Y\big(d(u), x\big)\mathbf{1} + Y(u, x)d(\mathbf{1})\big)\big|_{x=0} = \mathcal{D}_0 d(u).$$

This proves $d\mathcal{D}_0 = \mathcal{D}_0 d$. In particular, we have $\mathcal{D}_0\mathcal{D}_i = \mathcal{D}_i\mathcal{D}_0$ for $1 \leqslant i \leqslant r$. Using this we get

$$\begin{aligned} Y(\mathcal{D}_0 u; x_0, \mathbf{x}) &= Y\big(\mathbf{x}^{\mathbf{D}}\mathcal{D}_0 u, x_0\big) = Y\big(\mathcal{D}_0\mathbf{x}^{\mathbf{D}} u, x_0\big) = \frac{\partial}{\partial x_0} Y\big(\mathbf{x}^{\mathbf{D}} u, x_0\big) \\ &= \frac{\partial}{\partial x_0} Y(u; x_0, \mathbf{x}). \end{aligned}$$

For $1 \leqslant i \leqslant r$, since $\mathcal{D}_i$ acts on U semisimply with integer eigenvalues, we have $(x_i \frac{\partial}{\partial x_i})\mathbf{x}^{\mathbf{D}} u = \mathcal{D}_i \mathbf{x}^{\mathbf{D}} u$ for $u \in U$. Then

$$\begin{aligned} Y(\mathcal{D}_i u; x_0, \mathbf{x}) &= Y\big(\mathbf{x}^{\mathbf{D}}\mathcal{D}_i u, x_0\big) = Y\big(D_i\mathbf{x}^{\mathbf{D}} u, x_0\big) = x_i\frac{\partial}{\partial x_i} Y\big(\mathbf{x}^{\mathbf{D}} u, x_0\big) \\ &= x_i\frac{\partial}{\partial x_i} Y(u; x_0, \mathbf{x}). \end{aligned}$$

This proves that U becomes an extended $(r+1)$-toroidal vertex algebra.

Suppose $u \in U$ such that $\mathcal{D}_j u = m_j u$ for $1 \leqslant j \leqslant r$ with $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$. By (2.27) we have

$$Y(u; x_0, \mathbf{x})\mathbf{1} = e^{x_0\mathcal{D}_0}\mathbf{x}^{\mathbf{D}} u = \mathbf{x}^{\mathbf{m}} e^{x_0\mathcal{D}_0} u,$$

which implies $u = u_{-1,\mathbf{m}}\mathbf{1} \in U^0$. It follows that $U = U^0$. $\square$

Next, we study modules for $(r+1)$-toroidal vertex algebras.

Definition 2.17. Let V be an $(r+1)$-toroidal vertex algebra. A *V-module* is a vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot;x_0,\mathbf{x}):\quad & V\to \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm1},\dots,x_r^{\pm1}\big]\big]((x_0))\big),\\ & v\mapsto Y_W(v;x_0,\mathbf{x}),\end{aligned}$$

satisfying the condition that $Y_W(\mathbf{1};x_0,\mathbf{x})=1_W$ and for $u,v\in V$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y_W(u;x_0,\mathbf{zy})Y_W(v;y_0,\mathbf{y})-z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_W(v;y_0,\mathbf{y})Y_W(u;x_0,\mathbf{zy})\\ &\quad=y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y_W\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big). \end{aligned}\tag{2.28}$$

Just as for an $(r+1)$-toroidal vertex algebra, for a V-module, the same commutator and iterate formulas hold and the Jacobi identity is equivalent to weak commutativity and weak associativity. Furthermore, weak associativity follows from the following *iterate formula*

$$\begin{aligned} Y_W\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)&=\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y_W(u;x_0,\mathbf{zy})Y_W(v;y_0,\mathbf{y})\\ &\quad-\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_W(v;y_0,\mathbf{y})Y_W(u;x_0,\mathbf{zy}). \end{aligned}\tag{2.29}$$

In terms of components, we have

$$\begin{aligned} Y_W(u_{m_0,\mathbf{m}}v;y_0,\mathbf{y})&=\operatorname{Res}_{x_0}\operatorname{Res}_{\mathbf{x}}\mathbf{x}^{\mathbf{m}-1}\mathbf{y}^{-\mathbf{m}}\big((x_0-y_0)^{m_0}Y_W(u;x_0,\mathbf{x})Y_W(v;y_0,\mathbf{y})\\ &\quad-(-y_0+x_0)^{m_0}Y_W(v;y_0,\mathbf{y})Y_W(u;x_0,\mathbf{x})\big) \end{aligned}\tag{2.30}$$

for $u,v\in V$, $(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r$.

Definition 2.18. Let V be an extended $(r+1)$-toroidal vertex algebra. A *V-module* is a module (W,Y_W) for V viewed as an $(r+1)$-toroidal vertex algebra, equipped with linear operators D_i $(i=0,1,\dots,r)$ on W satisfying that for $v\in V$,

$$\begin{aligned} \big[D_0,Y_W(v;x_0,\mathbf{x})\big]&=Y_W(\mathcal{D}_0v;x_0,\mathbf{x})=\frac{\partial}{\partial x_0}Y_W(v;x_0,\mathbf{x}),\\ \big[D_i,Y_W(v;x_0,\mathbf{x})\big]&=Y_W(\mathcal{D}_iv;x_0,\mathbf{x})=x_i\frac{\partial}{\partial x_i}Y_W(v;x_0,\mathbf{x})\quad\text{for } 1\leqslant i\leqslant r. \end{aligned}\tag{2.31}$$

The following is a technical result:

Lemma 2.19. *Let V be an extended $(r+1)$-toroidal vertex algebra and let W be a module for V viewed as an $(r+1)$-toroidal vertex algebra, equipped with linear operators $D_0,D_1,\dots,D_r$, such that* (2.31) *holds for $v\in U$, where U is a generating subset of V. Then W is a V-module.*

Proof. We must prove that (2.31) holds for every $v \in V$. Let A be the collection of $v \in V$ such that (2.31) holds. Let $a, b \in A$ and let $w \in W$. There exists $l \in \mathbb{N}$ such that

$$(x_0 + z_0)^l Y_W\big(Y(a; z_0, \mathbf{z})b; x_0, \mathbf{x}\big)w = (x_0 + z_0)^l Y_W(a; z_0 + x_0, \mathbf{zx}) Y_W(b; x_0, \mathbf{x})w.$$

Using this we get $a_{m,\mathbf{n}}b \in A$ for $(m, \mathbf{n}) \in \mathbb{Z} \times \mathbb{Z}^r$. From assumption, we have $U \subset A$. Since U generates V, we must have $A = V$. Thus W is a V-module. □

Remark 2.20. Note that unlike the case for ordinary vertex algebras, in the definition of a V-module, associativity alone is not enough because of the lack of the skew symmetry for toroidal vertex algebras.

3. Construction of $(r+1)$-toroidal vertex algebras and their modules

In this section, we give a general construction of $(r+1)$-toroidal vertex algebras and their modules from local sets of multi-variable formal vertex operators.

Let W be a vector space and let r be a positive integer, which are both fixed throughout this section. Set

$$\mathcal{E}(W, r) = \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big]((x_0))\big) \subset (\operatorname{End} W)\big[\big[x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big].$$

Let $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in (\operatorname{End} W)[[x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_r^{\pm 1}]]$. We say that $a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are *mutually local* if there exists a nonnegative integer k such that

$$(x_0 - y_0)^k a(x_0, \mathbf{x}) b(y_0, \mathbf{y}) = (x_0 - y_0)^k b(y_0, \mathbf{y}) a(x_0, \mathbf{x}). \tag{3.1}$$

Furthermore, we say that a subset U of $(\operatorname{End} W)[[x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_r^{\pm 1}]]$ is *local* if for any $a(x_0, \mathbf{x})$, $b(x_0, \mathbf{x}) \in U$, $a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are mutually local.

Let $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in \mathcal{E}(W, r)$. Suppose that $a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are mutually local. Let k be a nonnegative integer such that (3.1) holds. Then

$$(x_0 - y_0)^k a(x_0, \mathbf{x}) b(y_0, \mathbf{y}) \in \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big]((x_0, y_0))\big). \tag{3.2}$$

For $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in \mathcal{E}(W, r)$, we say that the ordered pair $(a(x_0, \mathbf{x}), b(x_0, \mathbf{x}))$ is *compatible* if there exists a nonnegative integer k such that (3.2) holds.

Definition 3.1. Let $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in \mathcal{E}(W, r)$. Assume that $(a(x_0, \mathbf{x}), b(x_0, \mathbf{x}))$ is compatible. Define

$$a(x_0, \mathbf{x})_{m_0, \mathbf{m}} b(x_0, \mathbf{x}) \in \mathcal{E}(W, r) \quad \text{for } (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$$

in terms of generating function

$$Y_{\mathcal{E}}\big(a(y_0, \mathbf{y}); z_0, \mathbf{z}\big) b(y_0, \mathbf{y}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} a(y_0, \mathbf{y})_{m_0, \mathbf{m}} b(y_0, \mathbf{y}) z_0^{-m_0 - 1} \mathbf{z}^{-\mathbf{m}} \tag{3.3}$$

by

$$Y_{\mathcal{E}}\big(a(y_0, \mathbf{y}); z_0, \mathbf{z}\big) b(y_0, \mathbf{y}) = z_0^{-k} \big((x_0 - y_0)^k a(x_0, \mathbf{zy}) b(y_0, \mathbf{y})\big)\big|_{x_0 = y_0 + z_0}, \tag{3.4}$$

where k is any nonnegative integer such that (3.2) holds.

To justify the definition, first note that

$$a(x_0,\mathbf{zy})b(y_0,\mathbf{y}) \quad \text{exists in } (\operatorname{End} W)\big[\big[x_0^{\pm 1}, y_0^{\pm 1}, y_1^{\pm 1}, \dots, y_r^{\pm 1}, z_1^{\pm 1}, \dots, z_r^{\pm 1}\big]\big].$$

Furthermore, with (3.2) we have

$$\begin{aligned}
&z_0^{-k}\big((x_0-y_0)^k a(x_0,\mathbf{zy})b(y_0,\mathbf{y})\big)\big|_{x_0=y_0+z_0} \\
&\quad \in z_0^{-k}\operatorname{Hom}\big(W, W\big[\big[y_1^{\pm 1},\dots,y_r^{\pm 1},z_1^{\pm 1},\dots,z_r^{\pm 1}\big]\big]((y_0))\big)[[z_0]] \\
&\quad = \big(\operatorname{Hom}\big(W, W\big[\big[y_1^{\pm 1},\dots,y_r^{\pm 1}\big]\big]((y_0))\big)\big)\big[\big[z_1^{\pm 1},\dots,z_r^{\pm 1}\big]\big]((z_0)).
\end{aligned}$$

Second, it is straightforward to show that the expression on the right-hand side of (3.4) does not depend on the choice of k (cf. [Li3,Li5]).

Set

$$D_0 = \frac{\partial}{\partial x_0} \quad \text{and} \quad D_i = x_i \frac{\partial}{\partial x_i} \quad \text{for } 1 \leqslant i \leqslant r, \tag{3.5}$$

which act on $\mathcal{E}(W,r)$ in the obvious way. We have:

Lemma 3.2. *Suppose that $a(x_0,\mathbf{x})$, $b(x_0,\mathbf{x})$ are compatible (resp. local). Then $D_i a(x_0,\mathbf{x})$ and $b(x_0,\mathbf{x})$ are also compatible (resp. local). Furthermore,*

$$\big[D_i, Y_{\mathcal{E}}\big(a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)\big]b(x_0,\mathbf{x}) = Y_{\mathcal{E}}\big(D_i a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)b(x_0,\mathbf{x}) \tag{3.6}$$

for $0 \leqslant i \leqslant r$, and

$$Y_{\mathcal{E}}\big(D_0 a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)b(x_0,\mathbf{x}) = \frac{\partial}{\partial y_0} Y_{\mathcal{E}}\big(a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)b(x_0,\mathbf{x}), \tag{3.7}$$

$$Y_{\mathcal{E}}\big(D_j a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)b(x_0,\mathbf{x}) = \left(y_j \frac{\partial}{\partial y_j}\right) Y_{\mathcal{E}}\big(a(x_0,\mathbf{x}); y_0,\mathbf{y}\big)b(x_0,\mathbf{x}) \tag{3.8}$$

for $1 \leqslant j \leqslant r$.

Proof. Let k be a nonnegative integer such that (3.2) holds. Then we also have

$$(x_0-y_0)^{k+1} a(x_0,\mathbf{x})b(y_0,\mathbf{y}) \in \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big]((x_0,y_0))\big). \tag{3.9}$$

For $1 \leqslant i \leqslant r$, from (3.2) we immediately have

$$(x_0-y_0)^k \left(x_i \frac{\partial}{\partial x_i} a(x_0,\mathbf{x})\right) b(y_0,\mathbf{y}) \in \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big]((x_0,y_0))\big).$$

For $i=0$, using (3.9) and (3.2) we get

$$\begin{aligned}
&(x_0-y_0)^{k+1}\big(D_0 a(x_0,\mathbf{x})\big)b(y_0,\mathbf{y}) \\
&\quad = \frac{\partial}{\partial x_0}\big((x_0-y_0)^{k+1} a(x_0,\mathbf{x})b(y_0,\mathbf{y})\big) - (k+1)(x_0-y_0)^k a(x_0,\mathbf{x})b(y_0,\mathbf{y}) \\
&\quad \in \operatorname{Hom}\big(W, W\big[\big[x_1^{\pm 1},\dots,x_r^{\pm 1}\big]\big]((x_0,y_0))\big).
\end{aligned}$$

This proves that $D_i a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are compatible. Slightly modifying the arguments, one can show that if $a(x_0, \mathbf{x}), b(x_0, \mathbf{x})$ are local, then $D_i a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are also local.

The other assertions on D_0 follow from the same arguments as in [Li1]. For D_i with $1 \leqslant i \leqslant r$, notice that

$$\left(x\frac{d}{dx}(x^n)\right)\bigg|_{x=yz} = \left(y\frac{\partial}{\partial y}\right)(yz)^n = \left(z\frac{\partial}{\partial z}\right)(yz)^n$$

for $n \in \mathbb{Z}$. Then the assertions on D_i with $1 \leqslant i \leqslant r$ (including (3.8)) follow immediately from definition. $\square$

When $a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are mutually local, $Y_{\mathcal{E}}(a(y_0, \mathbf{y}); z_0, \mathbf{z})b(y_0, \mathbf{y})$ can be defined more explicitly.

Lemma 3.3. *Let $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in \mathcal{E}(W, r)$. Assume that $a(x_0, \mathbf{x})$ and $b(x_0, \mathbf{x})$ are mutually local. Then*

$$\begin{aligned} &Y_{\mathcal{E}}\big(a(y_0, \mathbf{y}); z_0, \mathbf{z}\big)b(y_0, \mathbf{y}) \\ &\quad = \operatorname{Res}_{x_0}\left(z_0^{-1}\delta\left(\frac{x_0 - y_0}{z_0}\right)a(x_0, \mathbf{zy})b(y_0, \mathbf{y}) - z_0^{-1}\delta\left(\frac{y_0 - x_0}{-z_0}\right)b(y_0, \mathbf{y})a(x_0, \mathbf{zy})\right). \end{aligned}$$

In terms of components, we have

$$\begin{aligned} a(y_0, \mathbf{y})_{m_0, \mathbf{m}}b(y_0, \mathbf{y}) = {} & \operatorname{Res}_{x_0}\operatorname{Res}_{\mathbf{x}}\mathbf{x}^{\mathbf{m}-1}\mathbf{y}^{-\mathbf{m}} \\ & \cdot \big((x_0 - y_0)^{m_0}a(x_0, \mathbf{x})b(y_0, \mathbf{y}) - (-y_0 + x_0)^{m_0}b(y_0, \mathbf{y})a(x_0, \mathbf{x})\big) \end{aligned} \tag{3.10}$$

for $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$.

Proof. Let k be a nonnegative integer such that

$$(x_0 - y_0)^k a(x_0, \mathbf{x})b(y_0, \mathbf{y}) = (x_0 - y_0)^k b(y_0, \mathbf{y})a(x_0, \mathbf{x}).$$

Then

$$(x_0 - y_0)^k a(x_0, \mathbf{x})b(y_0, \mathbf{y}) \in \operatorname{Hom}\big(W, W\big[\big[\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}\big]\big]((x_0, y_0))\big).$$

Using basic delta-function properties we obtain

$$\begin{aligned} &z_0^k \operatorname{Res}_{x_0}\left(z_0^{-1}\delta\left(\frac{x_0 - y_0}{z_0}\right)a(x_0, \mathbf{zy})b(y_0, \mathbf{y}) - z_0^{-1}\delta\left(\frac{y_0 - x_0}{-z_0}\right)b(y_0, \mathbf{y})a(x_0, \mathbf{zy})\right) \\ &\quad = \operatorname{Res}_{x_0} z_0^{-1}\delta\left(\frac{x_0 - y_0}{z_0}\right)\big[(x_0 - y_0)^k a(x_0, \mathbf{zy})b\big(y_0, \mathbf{y}\big]\big) \\ &\qquad - \operatorname{Res}_{x_0} z_0^{-1}\delta\left(\frac{y_0 - x_0}{-z_0}\right)\big[(x_0 - y_0)^k b(y_0, \mathbf{y})a(x_0, \mathbf{zy})\big] \\ &\quad = \operatorname{Res}_{x_0} y_0^{-1}\delta\left(\frac{x_0 - z_0}{y_0}\right)\big[(x_0 - y_0)^k a(x_0, \mathbf{zy})b(y_0, \mathbf{y})\big] \\ &\quad = \big[(x_0 - y_0)^k a(x_0, \mathbf{zy})b(y_0, \mathbf{y})\big]\big|_{x_0 = y_0 + z_0} \\ &\quad = z_0^k Y_{\mathcal{E}}\big(a(y_0, \mathbf{y}); z_0, \mathbf{z}\big)b(y_0, \mathbf{y}), \end{aligned}$$

as desired. $\square$

A local subspace U of $\mathcal{E}(W,r)$ is said to be *closed* if

$$a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})\in U$$

for all $a(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in U$, $(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r$. As the first main result of this section we have:

Theorem 3.4. *Let V be a closed local subspace of $\mathcal{E}(W,r)$, containing 1_W. Then $(V,Y_{\mathcal{E}},1_W)$ carries the structure of an $(r+1)$-toroidal vertex algebra and W is a faithful V-module with $Y_W(a(x_0,\mathbf{x});z_0,\mathbf{z})=a(z_0,\mathbf{z})$ for $a(x_0,\mathbf{x})\in V$. Furthermore, if V is also stable under D_i $(i=0,1,\dots,r)$, then V is an extended $(r+1)$-toroidal vertex algebra with $\mathcal{D}_i=D_i$ $(i=0,1,\dots,r)$.*

Proof. First, for $a(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in V$, there exists a nonnegative integer k such that

$$(x_0-y_0)^k a(x_0,\mathbf{x})b(y_0,\mathbf{y})=(x_0-y_0)^k b(y_0,\mathbf{y})a(x_0,\mathbf{x}).$$

Then $a(y_0,\mathbf{y})_{m_0,\mathbf{m}}b(y_0,\mathbf{y})\in V$ for $(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r$ and $a(y_0,\mathbf{y})_{m_0,\mathbf{m}}b(y_0,\mathbf{y})=0$ for $m_0\geqslant k$.

Second, we check the vacuum property for 1_W. In the definition, taking $a(x_0,\mathbf{x})=1_W$, we have

$$1_{m_0,\mathbf{m}}b(x_0,\mathbf{x})=\operatorname{Res}_{y_0}\operatorname{Res}_{\mathbf{y}}\mathbf{y}^{\mathbf{m}-1}\mathbf{x}^{-\mathbf{m}}\big((y_0-x_0)^{m_0}-(-x_0+y_0)^{m_0}\big)b(x_0,\mathbf{x}),$$

which equals $b(x_0,\mathbf{x})$ when $m_0=-1$ and $\mathbf{m}=\mathbf{0}$ and equals 0 otherwise. That is,

$$Y_{\mathcal{E}}(1_W;z_0,\mathbf{z})b(x_0,\mathbf{x})=b(x_0,\mathbf{x}). \tag{3.11}$$

On the other hand, taking $b(x_0,\mathbf{x})=1_W$, we have

$$a(x_0,\mathbf{x})_{m_0,\mathbf{m}}1_W=\operatorname{Res}_{y_0}\operatorname{Res}_{\mathbf{y}}\mathbf{y}^{\mathbf{m}-1}\mathbf{x}^{-\mathbf{m}}\big((y_0-x_0)^{m_0}-(-x_0+y_0)^{m_0}\big)a(y_0,\mathbf{y}),$$

which equals 0 whenever $m_0\geqslant 0$. For $m_0<0$, we have

$$a(x_0,\mathbf{x})_{m_0,\mathbf{m}}1_W=\frac{1}{(-m_0-1)!}\left(\frac{\partial}{\partial x_0}\right)^{-m_0-1}\mathbf{x}^{-\mathbf{m}}a(x_0,\mathbf{m}). \tag{3.12}$$

Thus

$$\sum_{\mathbf{m}\in\mathbb{Z}^r}a(x_0,\mathbf{x})_{-1,\mathbf{m}}1_W=a(x_0,\mathbf{x}).$$

Third, we establish the Jacobi identity. Let $a(x_0,\mathbf{x}),b(x_0,\mathbf{x}),c(x_0,\mathbf{x})\in V$. Using definition we have

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{zy}\big)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})\\
&=z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)\operatorname{Res}_{s_0}x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)a(s_0,\mathbf{zyt})Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})\\
&\quad-z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)\operatorname{Res}_{s_0}x_0^{-1}\delta\left(\frac{t_0-s_0}{-x_0}\right)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})a(s_0,\mathbf{zyt})\\
&=\operatorname{Res}_{s_0}\operatorname{Res}_{u_0}z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)T,
\end{aligned}$$

where

$$\begin{aligned}T={}&x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\\&-x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)a(s_0,\mathbf{zyt})c(t_0,\mathbf{t})b(u_0,\mathbf{yt})\\&-x_0^{-1}\delta\left(\frac{t_0-s_0}{-x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)b(u_0,\mathbf{yt})c(t_0,\mathbf{t})a(s_0,\mathbf{zyt})\\&+x_0^{-1}\delta\left(\frac{t_0-s_0}{-x_0}\right)y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)c(t_0,\mathbf{t})b(u_0,\mathbf{yt})a(s_0,\mathbf{zyt}).\end{aligned}$$

Let k be a positive integer such that

$$\begin{aligned}(x_0-y_0)^k a(x_0,\mathbf{x})b(y_0,\mathbf{y})&=(x_0-y_0)^k b(y_0,\mathbf{y})a(x_0,\mathbf{x}),\\(x_0-y_0)^k a(x_0,\mathbf{x})c(y_0,\mathbf{y})&=(x_0-y_0)^k c(y_0,\mathbf{y})a(x_0,\mathbf{x}),\\(x_0-y_0)^k b(x_0,\mathbf{x})c(y_0,\mathbf{y})&=(x_0-y_0)^k c(y_0,\mathbf{y})b(x_0,\mathbf{x}).\end{aligned}$$

Using delta-function substitutions we get

$$\begin{aligned}x_0^k y_0^k z_0^k z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)T&=z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)(s_0-t_0)^k(u_0-t_0)^k(x_0-y_0)^k T\\&=z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)(s_0-t_0)^k(u_0-t_0)^k(s_0-u_0)^k T\\&=z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)t_0^{-1}\delta\left(\frac{s_0-x_0}{t_0}\right)t_0^{-1}\delta\left(\frac{u_0-y_0}{t_0}\right)\\&\quad\cdot\left[(s_0-t_0)^k(u_0-t_0)^k(s_0-u_0)^k a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});\,y_0,\mathbf{y}\big)Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});\,x_0,\mathbf{zy}\big)c(t_0,\mathbf{t})\\&=\mathrm{Res}_{s_0}\,\mathrm{Res}_{u_0}\,z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)T',\end{aligned}$$

where

$$\begin{aligned}T'={}&x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)b(u_0,\mathbf{yt})a(s_0,\mathbf{zyt})c(t_0,\mathbf{t})\\&-x_0^{-1}\delta\left(\frac{t_0-s_0}{-x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)b(u_0,\mathbf{yt})c(t_0,\mathbf{t})a(s_0,\mathbf{zyt})\\&-x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)a(s_0,\mathbf{zyt})c(t_0,\mathbf{t})b(u_0,\mathbf{yt})\\&+x_0^{-1}\delta\left(\frac{t_0-s_0}{-x_0}\right)y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)c(t_0,\mathbf{t})a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt}),\end{aligned}$$

and furthermore we have

$$
\begin{aligned}
&x_0^k y_0^k z_0^k z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{zy}\big)c(t_0,\mathbf{t})\\
&\quad=\operatorname{Res}_{s_0}\operatorname{Res}_{u_0}z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)t_0^{-1}\delta\left(\frac{s_0-x_0}{t_0}\right)t_0^{-1}\delta\left(\frac{u_0-y_0}{t_0}\right)\\
&\qquad\cdot\big[(s_0-t_0)^k(u_0-t_0)^k(s_0-u_0)^k a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\big].
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y_{\mathcal{E}}\big(Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});z_0,\mathbf{z}\big)b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})\\
&=y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)\Big[\operatorname{Res}_{u_0}y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)Y_{\mathcal{E}}\big(a(u_0,\mathbf{yt});z_0,\mathbf{z}\big)b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\\
&\quad-y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)c(t_0,\mathbf{t})Y_{\mathcal{E}}\big(a(u_0,\mathbf{yt});z_0,\mathbf{z}\big)b(u_0,\mathbf{yt})\Big]\\
&=y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)\operatorname{Res}_{u_0}\operatorname{Res}_{s_0}T'',
\end{aligned}
$$

where

$$
\begin{aligned}
T''={}&y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)z_0^{-1}\delta\left(\frac{s_0-u_0}{z_0}\right)a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\\
&-y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)z_0^{-1}\delta\left(\frac{u_0-s_0}{-z_0}\right)b(u_0,\mathbf{yt})a(s_0,\mathbf{zyt})c(t_0,\mathbf{t})\\
&-y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)z_0^{-1}\delta\left(\frac{s_0-u_0}{z_0}\right)c(t_0,\mathbf{t})a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})\\
&+y_0^{-1}\delta\left(\frac{t_0-u_0}{-y_0}\right)z_0^{-1}\delta\left(\frac{u_0-s_0}{-z_0}\right)c(t_0,\mathbf{t})b(u_0,\mathbf{yt})a(s_0,\mathbf{zyt}),
\end{aligned}
$$

and furthermore we have

$$
\begin{aligned}
&x_0^k y_0^k z_0^k y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y_{\mathcal{E}}\big(Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});z_0,\mathbf{z}\big)b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})\\
&\quad=\operatorname{Res}_{s_0}\operatorname{Res}_{u_0}y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)t_0^{-1}\delta\left(\frac{s_0-x_0}{t_0}\right)t_0^{-1}\delta\left(\frac{u_0-y_0}{t_0}\right)\\
&\qquad\cdot\big[(s_0-t_0)^k(u_0-t_0)^k(s_0-u_0)^k a(s_0,\mathbf{zyt})b(u_0,\mathbf{yt})c(t_0,\mathbf{t})\big].
\end{aligned}
$$

Note that

$$
(x_0-y_0)^k x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right)=(s_0-u_0)^k x_0^{-1}\delta\left(\frac{s_0-t_0}{x_0}\right)y_0^{-1}\delta\left(\frac{u_0-t_0}{y_0}\right).
$$

Then

$$\begin{aligned}
&x_0^k y_0^k z_0^k z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{zy}\big)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t})\\
&\quad - z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{zy}\big)c(t_0,\mathbf{t})\\
&= x_0^k y_0^k z_0^k y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y_{\mathcal{E}}\big(Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});z_0,\mathbf{z}\big)b(t_0,\mathbf{t});y_0,\mathbf{y}\big)c(t_0,\mathbf{t}),
\end{aligned}$$

from which we immediately get

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y_{\mathcal{E}}(a;x_0,\mathbf{zy})Y_{\mathcal{E}}(b;y_0,\mathbf{y})c - z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y_{\mathcal{E}}(b;y_0,\mathbf{y})Y_{\mathcal{E}}(a;x_0,\mathbf{zy})c\\
&= y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y_{\mathcal{E}}\big(Y_{\mathcal{E}}(a;z_0,\mathbf{z})b;y_0,\mathbf{y}\big)c,
\end{aligned}$$

as desired. This establishes Jacobi identity. Therefore, $(V, Y_{\mathcal{E}}, 1_W)$ carries the structure of an $(r+1)$-toroidal vertex algebra.

For $a(x_0,\mathbf{x}) \in V$, set $Y_W(a(x_0,\mathbf{x}); z_0,\mathbf{z}) = a(z_0,\mathbf{z})$. Then, for $a(x_0,\mathbf{x}), b(x_0,\mathbf{x}) \in V$, $w \in W$, we have

$$\begin{aligned}
&Y_W\big(Y_{\mathcal{E}}\big(a(x_0,\mathbf{x});z_0,\mathbf{z}\big)b(x_0,\mathbf{x});y_0,\mathbf{y}\big)w\\
&= \big(Y_{\mathcal{E}}\big(a(x_0,\mathbf{x});z_0,\mathbf{z}\big)b(x_0,\mathbf{x})\big)\big|_{x_0=y_0,\mathbf{x}=\mathbf{y}}w\\
&= \big(Y_{\mathcal{E}}\big(a(y_0,\mathbf{y});z_0,\mathbf{z}\big)b(y_0,\mathbf{y})\big)w\\
&= \mathrm{Res}_{t_0} z_0^{-1}\delta\left(\frac{t_0-y_0}{z_0}\right)a(t_0,\mathbf{zy})b(y_0,\mathbf{y})w - \mathrm{Res}_{t_0} z_0^{-1}\delta\left(\frac{y_0-t_0}{-z_0}\right)b(y_0,\mathbf{y})a(t_0,\mathbf{zy})w\\
&= \mathrm{Res}_{t_0} z_0^{-1}\delta\left(\frac{t_0-y_0}{z_0}\right)Y_W\big(a(x_0,\mathbf{x});t_0,\mathbf{zy}\big)Y_W\big(b(x_0,\mathbf{x});y_0,\mathbf{y}\big)w\\
&\quad - \mathrm{Res}_{t_0} z_0^{-1}\delta\left(\frac{y_0-t_0}{-z_0}\right)Y_W\big(b(x_0,\mathbf{x});y_0,\mathbf{y}\big)Y_W\big(a(x_0,\mathbf{x});t_0,\mathbf{zy}\big)w.
\end{aligned}$$

Therefore, (W, Y_W) carries the structure of a V-module which is faithful as $V \subset \mathcal{E}(W,r)$. For the last assertion, it is clear from Lemma 3.2. □

Next, we shall show that each local subset of $\mathcal{E}(W,r)$ gives rise to a closed local subspace. The following technical result follows from the same arguments as in the single variable case (see [Li1]):

Lemma 3.5. *Assume that elements $a(x_0,\mathbf{x}), b(x_0,\mathbf{x}), c(x_0,\mathbf{x})$ of $\mathcal{E}(W,r)$ are pairwise local. Then for any $(m_0,\mathbf{m}) \in \mathbb{Z}\times\mathbb{Z}^r$, $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})$ and $c(x_0,\mathbf{x})$ are local.*

We have:

Proposition 3.6. *Let V be a maximal local subspace of $\mathcal{E}(W,r)$. Then V contains 1_W, is closed and stable under $D_0, D_1, \ldots, D_r$. Furthermore, $(V, Y_{\mathcal{E}}, 1_W)$ carries the structure of an extended $(r+1)$-toroidal vertex algebra.*

Proof. Notice that $V+\mathbb{C}1_W$ is local and contains V. With V maximal we have $V+\mathbb{C}1_W=V$. Thus $1_W\in V$. Let $a(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in V$, $(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r$. By Lemma 3.5, for any $c(x_0,\mathbf{x})\in V$, $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})$ and $c(x_0,\mathbf{x})$ are local. In particular, $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})$ is local with $a(x_0,\mathbf{x})$ and $b(x_0,\mathbf{x})$. Furthermore, using Lemma 3.5 again, we see that $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})$ is local with itself. Thus $V+\mathbb{C}a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})$ is local. With V maximal we must have $a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})\in V$. This proves that V is closed. Using a similar argument and using Lemma 3.2 (the first part) we see that V is stable under $D_0,D_1,\dots,D_r$. Then it follows from Theorem 3.4 that V is an extended $(r+1)$-toroidal vertex algebra. □

Furthermore, we have:

Theorem 3.7. *Let U be a local subset of $\mathcal{E}(W,r)$. Denote by $\langle U\rangle$ the linear span of the elements of the form*

$$a^{(1)}(x_0,\mathbf{x})_{m_0^{(1)},\mathbf{m}^{(1)}}\cdots a^{(k)}(x_0,\mathbf{x})_{m_0^{(k)},\mathbf{m}^{(k)}}b(x_0,\mathbf{x})$$

with $k\in\mathbb{N}$, $a^{(i)}(x_0,\mathbf{x}),b(x_0,\mathbf{x})\in U\cup\{1_W\}$, $(m_0^{(i)},\mathbf{m}^{(i)})\in\mathbb{Z}\times\mathbb{Z}^r$. Then $\langle U\rangle$ is the (unique) smallest closed local subspace containing $U\cup\{1_W\}$, and $(\langle U\rangle,Y_{\mathcal{E}},1_W)$ carries the structure of an $(r+1)$-toroidal vertex algebra with W as a faithful module. Furthermore, $\mathbb{C}[D_0,D_1,\dots,D_r]\langle U\rangle$ is an extended $(r+1)$-toroidal vertex algebra and W is a module for $\mathbb{C}[D_0,D_1,\dots,D_r]\langle U\rangle$ viewed as an $(r+1)$-toroidal vertex algebra.

Proof. By Zorn's Lemma, there exists a maximal local subspace V of $\mathcal{E}(W,r)$, containing U. By Proposition 3.6, V contains 1_W and is closed, and V is an extended $(r+1)$-toroidal vertex algebra with $\mathcal{D}_i=D_i$ $(i=0,1,\dots,r)$. As $U\subset V$, we have $\langle U\rangle\subset V$, so that $\langle U\rangle$ is local. It follows from induction and (2.3) that $\langle U\rangle$ is closed. Then $\langle U\rangle$ is an $(r+1)$-toroidal vertex algebra with W as a faithful module. On the other hand, we have

$$\mathbb{C}[D_0,D_1,\dots,D_r]\langle U\rangle\subset V.$$

It follows from Lemma 3.2 and induction that $\mathbb{C}[D_0,D_1,\dots,D_r]\langle U\rangle$ is a subalgebra of V. Consequently, $\mathbb{C}[D_0,D_1,\dots,D_r]\langle U\rangle$ is an extended $(r+1)$-toroidal vertex algebra. □

We shall also need the following result:

Lemma 3.8. *Let V be a closed local subspace of $\mathcal{E}(W,r)$, let*

$$a(x_0,\mathbf{x}),b(x_0,\mathbf{x}),c_0(x_0,\mathbf{x}),c_1(x_0,\mathbf{x}),\dots,c_k(x_0,\mathbf{x})\in V,$$

and let $\mathbf{m}\in\mathbb{Z}^r$. If

$$\big[a(x_0,\mathbf{m}),b(y_0,\mathbf{y})\big]=\mathbf{y}^{\mathbf{m}}\sum_{j=0}^{k}c_j(y_0,\mathbf{y})\frac{1}{j!}\left(\frac{\partial}{\partial y_0}\right)^j x_0^{-1}\delta\left(\frac{y_0}{x_0}\right),\tag{3.13}$$

where $a(x_0,\mathbf{x})=\sum_{\mathbf{m}\in\mathbb{Z}^r}a(x_0,\mathbf{m})\mathbf{x}^{-\mathbf{m}}$, then

$$a(x_0,\mathbf{x})_{j,\mathbf{m}}b(x_0,\mathbf{x})=c_j(x_0,\mathbf{x})\tag{3.14}$$

for $0\leqslant j\leqslant k$ and $a(x_0,\mathbf{x})_{j,\mathbf{m}}b(x_0,\mathbf{x})=0$ for $j>k$.

Proof. For $j \geqslant 0$, $\mathbf{m} \in \mathbb{Z}^r$, from Lemma 3.3 we have

$$\begin{aligned} a(y_0,\mathbf{y})_{j,\mathbf{m}}b(y_0,\mathbf{y}) &= \operatorname{Res}_{x_0}\operatorname{Res}_{\mathbf{x}}\mathbf{x}^{\mathbf{m}-1}\mathbf{y}^{-\mathbf{m}}(x_0-y_0)^j\big[a(x_0,\mathbf{x}),b(y_0,\mathbf{y})\big] \\ &= \operatorname{Res}_{x_0}(x_0-y_0)^j\mathbf{y}^{-\mathbf{m}}\big[a(x_0,\mathbf{m}),b(y_0,\mathbf{y})\big] \\ &= \operatorname{Res}_{x_0}(x_0-y_0)^j\sum_{i=0}^{k}c_i(y_0,\mathbf{y})\frac{1}{i!}\left(\frac{\partial}{\partial y_0}\right)^i y_0^{-1}\delta\left(\frac{x_0}{y_0}\right) \\ &= \operatorname{Res}_{x_0}(x_0-y_0)^j\sum_{i=0}^{k}c_i(y_0,\mathbf{y})\big((x_0-y_0)^{-i-1}-(-y_0+x_0)^{-i-1}\big). \end{aligned}$$

If $j>k$, we have $(x_0-y_0)^j((x_0-y_0)^{-i-1}-(-y_0+x_0)^{-i-1})=0$ for $0\leqslant i\leqslant k$, so that $a(y_0,\mathbf{y})_{j,\mathbf{m}}b(y_0,\mathbf{y})=0$. If $0\leqslant j\leqslant i$, we have

$$\begin{aligned} &\operatorname{Res}_{x_0}(x_0-y_0)^j\big((x_0-y_0)^{-i-1}-(-y_0+x_0)^{-i-1}\big) \\ &\quad= \operatorname{Res}_{x_0}\big((x_0-y_0)^{-(i-j)-1}-(-y_0+x_0)^{-(i-j)-1}\big) \\ &\quad= \delta_{ij}. \end{aligned}$$

In this case, we get

$$a(y_0,\mathbf{y})_{j,\mathbf{m}}b(y_0,\mathbf{y}) = c_j(y_0,\mathbf{y}),$$

as desired. □

Furthermore we have:

Proposition 3.9. *Suppose that V is a closed local subspace of $\mathcal{E}(W,r)$ and let*

$$a(x_0,\mathbf{x}),b(x_0,\mathbf{x}),c_0(x_0,\mathbf{x}),c_1(x_0,\mathbf{x}),\dots,c_k(x_0,\mathbf{x})\in V,\qquad \mathbf{m}\in\mathbb{Z}^r.$$

If

$$\big[a(x_0,\mathbf{m}),b(y_0,\mathbf{y})\big] = \mathbf{y}^{\mathbf{m}}\sum_{j=0}^{k}c_j(y_0,\mathbf{y})\frac{1}{j!}\left(\frac{\partial}{\partial y_0}\right)^j x_0^{-1}\delta\left(\frac{y_0}{x_0}\right), \tag{3.15}$$

where $a(x_0,\mathbf{x})=\sum_{\mathbf{m}\in\mathbb{Z}^r}a(x_0,\mathbf{m})\mathbf{x}^{-\mathbf{m}}$, then

$$\begin{aligned} &\big[Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{m}\big),Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{y}\big)\big] \\ &\quad= \mathbf{y}^{\mathbf{m}}\sum_{j=0}^{k}Y_{\mathcal{E}}\big(c_j(t_0,\mathbf{t});y_0,\mathbf{y}\big)\frac{1}{j!}\left(\frac{\partial}{\partial y_0}\right)^j x_0^{-1}\delta\left(\frac{y_0}{x_0}\right). \end{aligned} \tag{3.16}$$

In terms of components,

$$\begin{aligned}&\big[Y_{\mathcal{E}}\big(a(t_0,\mathbf{t});x_0,\mathbf{m}\big),Y_{\mathcal{E}}\big(b(t_0,\mathbf{t});y_0,\mathbf{n}\big)\big]\\&\quad=\sum_{j=0}^{k}Y_{\mathcal{E}}\big(c_j(t_0,\mathbf{t});y_0,\mathbf{m}+\mathbf{n}\big)\frac{1}{j!}\left(\frac{\partial}{\partial y_0}\right)^j x_0^{-1}\delta\left(\frac{y_0}{x_0}\right).\end{aligned}\tag{3.17}$$

Proof. We may assume that V contains 1_W, so that V is an $(r+1)$-toroidal vertex algebra. By Lemma 3.8 we have

$$a(x_0,\mathbf{x})_{j,\mathbf{m}}b(x_0,\mathbf{x})=c_j(x_0,\mathbf{x})$$

for $0\leqslant j\leqslant k$ and $a(x_0,\mathbf{x})_{j,\mathbf{m}}b(x_0,\mathbf{x})=0$ for $j>k$. Then it follows from (2.5) immediately. □

As we need, next we establish an analog of a theorem of Xu (see [X1,X2]; cf. [LL]).

Theorem 3.10. *Let V be a vector space equipped with a linear map*

$$\begin{aligned}Y(\cdot;x_0,\mathbf{x}):\quad &V\to \operatorname{Hom}\big(V,V\big[\big[x_1^{\pm1},\dots,x_r^{\pm1}\big]\big]((x_0))\big),\\ &v\mapsto Y(v;x_0,\mathbf{x})=\sum_{m\in\mathbb{Z},\ \mathbf{n}\in\mathbb{Z}^r}v_{m,\mathbf{n}}x_0^{-m-1}\mathbf{x}^{-n}\end{aligned}$$

and let U be a subset of V, satisfying the conditions that $\{Y(u;x_0,\mathbf{x})\mid u\in U\}$ is a local subset of $\mathcal{E}(V,r)$, V is linearly spanned by vectors

$$u^{(1)}_{\mathbf{m}_1}\cdots u^{(k)}_{\mathbf{m}_k}u$$

for $u^{(1)},\dots,u^{(k)},u\in U$, $\mathbf{m}_1,\dots,\mathbf{m}_k\in\mathbb{Z}^{r+1}$ with $k\in\mathbb{N}$, and that

$$\begin{aligned}Y\big(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)&=\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y(u;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y})\\&\quad-\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y(v;y_0,\mathbf{y})Y(u;x_0,\mathbf{zy})\end{aligned}\tag{3.18}$$

for $u\in U$, $v\in V$. Then (V,Y) carries the structure of an $(r+1)$-toroidal vertex algebra without vacuum.

Proof. Set

$$\tilde{U}=\big\{Y(u;x_0,\mathbf{x})\bigm| u\in U\big\}\subset\mathcal{E}(V,r).$$

By assumption, $\tilde{U}$ is a local subset. Then by Theorem 3.7, $\tilde{U}$ generates an $(r+1)$-toroidal vertex algebra $\langle\tilde{U}\rangle$, which is also a local subspace. For $a\in U$, $v\in V$, combining (3.18) with Lemma 3.3 we get

$$\begin{aligned}Y\big(Y(a;z_0,\mathbf{z})v;y_0,\mathbf{y}\big)&=\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y(a;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y})\\&\quad-\operatorname{Res}_{x_0}z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y(v;y_0,\mathbf{y})Y(a;x_0,\mathbf{zy})\\&=Y_{\mathcal{E}}\big(Y(a;y_0,\mathbf{y});z_0,\mathbf{z}\big)Y(v;y_0,\mathbf{y}).\end{aligned}\tag{3.19}$$

Then it follows from the span assumption that Y maps V into $\langle\tilde{U}\rangle$. Since $\langle\tilde{U}\rangle$ is a local subspace of $\mathcal{E}(W,r)$, $\{Y(v;x_0,\mathbf{x}) \mid v\in V\}$ as a subset of $\langle\tilde{U}\rangle$ is local. Now it remains to establish weak associativity, which will be achieved in the following by using iterate formula and induction.

Let K consist of each $u\in V$ such that (3.18) holds for every $v\in V$. Now, let $a,b\in K$, $v\in V$. From (3.18) with (a,b) in the places of (u,v), there exists a nonnegative integer l such that

$$(x_0+y_0)^l Y\big(Y(a;x_0,\mathbf{x})b;y_0,\mathbf{y}\big)v=(x_0+y_0)^l Y(a;x_0+y_0,\mathbf{xy})Y(b;y_0,\mathbf{y})v.$$

Using this, (3.19), and the weak associativity for $\langle\tilde{U}\rangle$, replacing l with a large one if necessary, we have

$$\begin{aligned}
&(x_0+y_0)^l Y\big(Y\big(Y(a;x_0,\mathbf{x})b;y_0,\mathbf{y}\big)v;z_0,\mathbf{z}\big)\\
&\quad=(x_0+y_0)^l Y\big(Y(a;x_0+y_0,\mathbf{xy})Y(b;y_0,\mathbf{y})v;z_0,\mathbf{z}\big)\\
&\quad=(x_0+y_0)^l Y_{\mathcal{E}}\big(\overline{Y(a;z_0,\mathbf{z})};x_0+y_0,\mathbf{xy}\big)\overline{Y\big(Y(b;y_0,\mathbf{y})v;z_0,\mathbf{z}\big)}\\
&\quad=(x_0+y_0)^l Y_{\mathcal{E}}\big(\overline{Y(a;z_0,\mathbf{z})};x_0+y_0,\mathbf{xy}\big)Y_{\mathcal{E}}\big(\overline{Y(b;z_0,\mathbf{z})};y_0,\mathbf{y}\big)\overline{Y(v;z_0,\mathbf{z})}\\
&\quad=(x_0+y_0)^l Y_{\mathcal{E}}\big(Y_{\mathcal{E}}\big(\overline{Y(a;z_0,\mathbf{z})};x_0,\mathbf{x}\big)\overline{Y(b;z_0,\mathbf{z})};y_0,\mathbf{y}\big)\overline{Y(v;z_0,\mathbf{z})}\\
&\quad=(x_0+y_0)^l Y_{\mathcal{E}}\big(\overline{Y\big(Y(a;x_0,\mathbf{x})b;z_0,\mathbf{z}\big)};y_0,\mathbf{y}\big)\overline{Y(v;z_0,\mathbf{z})},
\end{aligned}$$

where $\overline{X}=X$ for all the bar objects; the only purpose is to make the equation easier to read. Multiplying both sides by formal series $(y_0+x_0)^{-l}$ we get

$$Y\big(Y\big(Y(a;x_0,\mathbf{x})b;y_0,\mathbf{y}\big)v;z_0,\mathbf{z}\big)=Y_{\mathcal{E}}\big(\overline{Y\big(Y(a;x_0,\mathbf{x})b;z_0,\mathbf{z}\big)};y_0,\mathbf{y}\big)\overline{Y(v;z_0,\mathbf{z})}.$$

Furthermore, using Lemma 3.3, we get

$$\begin{aligned}
&Y\big(Y\big(Y(a,x_0,\mathbf{x})b;y_0,\mathbf{y}\big)v;z_0,\mathbf{z}\big)\\
&\quad=Y_{\mathcal{E}}\big(Y\big(Y(a,x_0,\mathbf{x})b;z_0,\mathbf{z}\big);y_0,\mathbf{y}\big)Y(v;z_0,\mathbf{z})\\
&\quad=\operatorname{Res}_{x_1} y_0^{-1}\delta\left(\frac{x_1-z_0}{y_0}\right)Y\big(Y(a;x_0,\mathbf{x})b;x_1,\mathbf{yz}\big)Y(v;z_0,\mathbf{z})\\
&\qquad-\operatorname{Res}_{x_1} y_0^{-1}\delta\left(\frac{z_0-x_1}{-y_0}\right)Y(v;z_0,\mathbf{z})Y\big(Y(a;x_0,\mathbf{x})b;x_1,\mathbf{yz}\big).
\end{aligned}$$

This shows that $a_{m_0,\mathbf{m}}b\in K$ for $m_0\in\mathbb{Z}$, $\mathbf{m}\in\mathbb{Z}^r$. It follows from induction and the span assumption that $V=K$. This proves that (3.18) holds for *all* $u,v\in V$. Then weak associativity follows. Therefore, (V,Y) carries the structure of an $(r+1)$-toroidal vertex algebra without vacuum. □

Remark 3.11. Note that in the proof of Theorem 3.10, if we can show that the map Y is injective as for the one-variable case, the proof can be simplified by using the toroidal vertex algebra structure transported from $\langle\tilde{U}\rangle$. Unfortunately, Y may be not injective.

4. $(r+1)$-Toroidal vertex algebras and modules associated to toroidal Lie algebras

In this section, by using the general construction established in Section 3 we associate $(r+1)$-toroidal vertex algebras and their modules to toroidal Lie algebras.

Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra, equipped with a (possibly degenerate) symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$. Form the $(r+1)$-loop Lie algebra

$$L_{r+1}(\mathfrak{g}) = \mathfrak{g}\otimes\mathbb{C}\big[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_r^{\pm 1}\big],$$

where $\mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is the algebra of Laurent polynomials in mutually commuting variables $t_0, t_1, \ldots, t_r$. Form a 1-dimensional central extension

$$\widehat{L_{r+1}(\mathfrak{g})} = L_{r+1}(\mathfrak{g})\oplus\mathbb{C}\mathbf{k} = \big(\mathfrak{g}\otimes\mathbb{C}\big[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_r^{\pm 1}\big]\big)\oplus\mathbb{C}\mathbf{k}, \tag{4.1}$$

where $\mathbf{k}$ is central and

$$\big[a\otimes t_0^{m_0}\mathbf{t}^{\mathbf{m}}, b\otimes t_0^{n_0}\mathbf{t}^{\mathbf{n}}\big] = [a,b]\otimes t_0^{m_0+n_0}\mathbf{t}^{\mathbf{m}+\mathbf{n}} + m_0\langle a,b\rangle\delta_{m_0+n_0,0}\delta_{\mathbf{m}+\mathbf{n},0}\mathbf{k} \tag{4.2}$$

for $a, b\in\mathfrak{g}$, $m_0, n_0\in\mathbb{Z}$, $\mathbf{m},\mathbf{n}\in\mathbb{Z}^r$.

Notice that Lie algebra $\widehat{L_{r+1}(\mathfrak{g})}$ can also be considered as the affine Lie algebra of the r-loop Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ with respect to the symmetric invariant bilinear form defined by

$$\big\langle a\otimes\mathbf{t}^{\mathbf{m}}, b\otimes\mathbf{t}^{\mathbf{n}}\big\rangle = \langle a,b\rangle\delta_{\mathbf{m}+\mathbf{n},0}\quad\text{for } a,b\in\mathfrak{g},\ \mathbf{m},\mathbf{n}\in\mathbb{Z}^r.$$

For $a\in\mathfrak{g}$, $\mathbf{m}\in\mathbb{Z}^r$, set

$$a(\mathbf{m},z) = \sum_{k\in\mathbb{Z}} a(k,\mathbf{m})z^{-k-1}, \tag{4.3}$$

where $a(k,\mathbf{m}) = a\otimes t_0^k\mathbf{t}^{\mathbf{m}}$. The Lie bracket relations (4.2) amount to

$$\big[a(\mathbf{m},z_1), b(\mathbf{n},z_2)\big] = [a,b](\mathbf{m}+\mathbf{n},z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + \langle a,b\rangle\delta_{\mathbf{m}+\mathbf{n},0}\mathbf{k}\frac{\partial}{\partial z_2}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) \tag{4.4}$$

for $a,b\in\mathfrak{g}$, $\mathbf{m},\mathbf{n}\in\mathbb{Z}^r$. Furthermore, for $a\in\mathfrak{g}$, set

$$a(x_0,\mathbf{x}) = \sum_{n_0\in\mathbb{Z}}\sum_{\mathbf{n}\in\mathbb{Z}^r}\big(a\otimes t_0^{n_0}\mathbf{t}^{\mathbf{n}}\big)x_0^{-n_0-1}\mathbf{x}^{-\mathbf{n}} = \sum_{\mathbf{n}\in\mathbb{Z}^r} a(\mathbf{n},x_0)\mathbf{x}^{-\mathbf{n}}. \tag{4.5}$$

Then (4.4) amounts to

$$\big[a(\mathbf{m},x_0), b(z_0,\mathbf{z})\big] = \mathbf{z}^{\mathbf{m}}\left([a,b](z_0,\mathbf{z})x_0^{-1}\delta\left(\frac{z_0}{x_0}\right) + \langle a,b\rangle\mathbf{k}\frac{\partial}{\partial z_0}x_0^{-1}\delta\left(\frac{z_0}{x_0}\right)\right). \tag{4.6}$$

Define $r+1$ derivations d_i $(0\leqslant i\leqslant r)$ on $\widehat{L_{r+1}(\mathfrak{g})}$ by

$$d_0 = -1\otimes\frac{\partial}{\partial t_0},\qquad d_i = -1\otimes t_i\frac{\partial}{\partial t_i}\quad\text{for } 1\leqslant i\leqslant r. \tag{4.7}$$

We have

$$\big[d_0, a(x_0,\mathbf{x})\big] = \frac{\partial}{\partial x_0} a(x_0,\mathbf{x}), \qquad \big[d_i, a(x_0,\mathbf{x})\big] = x_i \frac{\partial}{\partial x_i} a(x_0,\mathbf{x}) \tag{4.8}$$

for $1 \leqslant i \leqslant r$, $a \in \mathfrak{g}$. Set

$$\begin{aligned} \widehat{L_{r+1}(\mathfrak{g})}_+ &= \mathfrak{g} \otimes \mathbb{C}\big[t_0, t_1^{\pm 1}, \dots, t_r^{\pm 1}\big], \\ \widehat{L_{r+1}(\mathfrak{g})}_- &= \mathfrak{g} \otimes t_0^{-1}\mathbb{C}\big[t_0^{-1}, t_1^{\pm 1}, \dots, t_r^{\pm 1}\big], \end{aligned} \tag{4.9}$$

which are Lie subalgebras of $\widehat{L_{r+1}(\mathfrak{g})}$. Clearly, both are stable under the actions of derivations $d_0, d_1, \dots, d_r$.

We view $\widehat{L_{r+1}(\mathfrak{g})}$ as a $\mathbb{Z}$-graded Lie algebra with

$$\widehat{L_{r+1}(\mathfrak{g})}_{(n)} = \begin{cases} \mathfrak{g} \otimes t_0^{-n}\mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] & \text{if } n \neq 0, \\ \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] + \mathbb{C}\mathbf{k} & \text{if } n = 0 \end{cases} \tag{4.10}$$

for $n \in \mathbb{Z}$. We have

$$d_0 \cdot \widehat{L_{r+1}(\mathfrak{g})}_{(n)} \subset \widehat{L_{r+1}(\mathfrak{g})}_{(n+1)} \quad \text{and} \quad d_i \cdot \widehat{L_{r+1}(\mathfrak{g})}_{(n)} = \widehat{L_{r+1}(\mathfrak{g})}_{(n)} \tag{4.11}$$

for $1 \leqslant i \leqslant r$.

Lemma 4.1. *Let ℓ be any complex number. Then there exists an $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module structure on $\mathfrak{g} \oplus \mathbb{C}$, which is uniquely determined by $\mathbf{k} = \ell$ (a scalar),*

$$\begin{aligned} \big(a \otimes t_0^k \mathbf{t}^{\mathbf{m}}\big) \cdot 1 &= 0, \\ \big(a \otimes t_0^k \mathbf{t}^{\mathbf{m}}\big) \cdot b &= \begin{cases} [a,b] & \text{if } k = 0, \\ \langle a,b\rangle \ell & \text{if } k = 1, \\ 0 & \text{if } k \geqslant 2 \end{cases} \end{aligned} \tag{4.12}$$

for $a, b \in \mathfrak{g}$, $k \in \mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^r$ with $k \geqslant 0$. Furthermore, $\mathfrak{g} \oplus \mathbb{C}$ is an $\mathbb{N}$-graded $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module with $\deg \mathbb{C} = 0$ *and* $\deg \mathfrak{g} = 1$.

Proof. First, we show that $\mathfrak{g} \oplus \mathbb{C}$ is a module for Lie algebra $\mathfrak{g}[t_0]$ under the action given as a special case with $\mathbf{m} = 0$. This can be proved straightforwardly. Here we give a proof using the vertex algebra associated to affine Lie algebra $\hat{\mathfrak{g}}$. Recall (see [FZ], cf. [LL]) that associated to $\hat{\mathfrak{g}}$ with level ℓ, one has a vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ whose underlying space is the universal level-ℓ vacuum $\hat{\mathfrak{g}}$-module. View $\mathfrak{g} \oplus \mathbb{C}$ as a subspace of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ by identifying $a + \alpha \in \mathfrak{g} + \mathbb{C}$ with $a(-1)\mathbf{1} + \alpha\mathbf{1} \in V_{\hat{\mathfrak{g}}}(\ell, 0)$, where $\mathbf{1}$ denotes the canonical highest weight vector. For $a, b \in \mathfrak{g}$, $n \geqslant 0$, we have

$$a(n)\mathbf{1} = 0, \qquad a(0)b = [a,b], \qquad a(1)b = \ell\langle a,b\rangle\mathbf{1}, \qquad a(m)b = 0 \quad \text{for } m \geqslant 2.$$

It follows that $\mathfrak{g} + \mathbb{C}$ is a $\mathfrak{g}[t_0]$-submodule of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ as desired. Second, equip $\mathfrak{g} \oplus \mathbb{C}$ with the evaluation module structure for the r-loop Lie algebra $(\mathfrak{g}[t_0]) \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ with $\mathbf{t} = 1$ (through the Lie algebra homomorphism sending $a \otimes t_0^k t^m$ to $a \otimes t_0^k$). As $\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k}$ is a direct sum of Lie algebras, by letting $\mathbf{k}$ act as scalar ℓ we obtain an $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module structure on $\mathfrak{g} \oplus \mathbb{C}$, as desired. It is clear that $\mathfrak{g} \oplus \mathbb{C}$ is an $\mathbb{N}$-graded module. $\square$

Let $\ell \in \mathbb{C}$. Denote by $(\mathfrak{g} + \mathbb{C})_\ell$ the $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module obtained in Lemma 4.1. We form an induced module

$$V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0) = U\big(\widehat{L_{r+1}(\mathfrak{g})}\big) \otimes_{U(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})} (\mathfrak{g} + \mathbb{C})_\ell, \tag{4.13}$$

which is an $\mathbb{N}$-graded $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ. By the P–B–W theorem, we have

$$V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0) = U\big(\widehat{L_{r+1}(\mathfrak{g})}_-\big) \otimes (\mathfrak{g} \oplus \mathbb{C})$$

as a vector space. It follows from the $\mathbb{N}$-grading that $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ is a restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module.

Set

$$\mathbf{1} = 1 \otimes 1 \in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0).$$

Identify $\mathfrak{g}$ with the subspace $1 \otimes \mathfrak{g}$, through the map $a \mapsto 1 \otimes a$.

We have:

Theorem 4.2. *Let ℓ be any complex number. Then there exists an $(r+1)$-toroidal vertex algebra structure on $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, which is uniquely determined by the conditions that $\mathbf{1}$ is the vacuum vector and that*

$$Y(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g}.$$

Proof. As $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0) = U(\widehat{L_{r+1}(\mathfrak{g})})(\mathfrak{g} + \mathbb{C}\mathbf{1})$, the uniqueness follows immediately. We now establish the existence by applying Theorem 3.10. Let W be any restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ. In particular, we can (and we shall) take W to be $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$. Set

$$U_W = \big\{a(x_0, \mathbf{x}) \bigm| a \in \mathfrak{g}\big\} \cup \{1_W\}.$$

For $a, b \in \mathfrak{g}$, from (4.4) we have

$$(x_0 - z_0)^2\big[a(\mathbf{m}, x_0), b(\mathbf{n}, z_0)\big] = 0$$

for $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$, which implies

$$(x_0 - z_0)^2\big[a(x_0, \mathbf{x}), b(z_0, \mathbf{z})\big] = 0.$$

Thus U_W is a local subset of $\mathcal{E}(W, r)$. By Theorem 3.7, U_W generates an $(r+1)$-toroidal vertex algebra $\langle U_W \rangle$ inside $\mathcal{E}(W, r)$ and W is a $\langle U_W \rangle$-module with

$$Y_W\big(\beta(x_0, \mathbf{x}); z_0, \mathbf{z}\big) = \beta(z_0, \mathbf{z}) \quad \text{for } \beta(x_0, \mathbf{x}) \in \langle U_W \rangle.$$

For $a, b \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$, with (4.6) by Proposition 3.9, we have

$$\begin{aligned}
&\big[Y_{\mathcal{E}}\big(a(t_0, \mathbf{t}); x_0, \mathbf{m}\big), Y_{\mathcal{E}}\big(b(t_0, \mathbf{t}); z_0, \mathbf{z}\big)\big] \\
&\quad = \mathbf{z}^{\mathbf{m}}\left(Y_{\mathcal{E}}\big([a, b](t_0, \mathbf{t}); z_0, \mathbf{z}\big)x_0^{-1}\delta\left(\frac{z_0}{x_0}\right) + \langle a, b\rangle \mathbf{k} \frac{\partial}{\partial z_0} x_0^{-1}\delta\left(\frac{z_0}{x_0}\right)\right).
\end{aligned}$$

Thus, $\langle U_W\rangle$ becomes an $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ with $a(z_0,\mathbf{z})$ acting as $Y_{\mathcal{E}}(a(x_0,\mathbf{x});z_0,\mathbf{z})$ for $a\in\mathfrak{g}$. Let $a,b\in\mathfrak{g}$, $m_0\in\mathbb{Z}$, $\mathbf{m}\in\mathbb{Z}^r$ with $m_0\geqslant 0$. With relation (4.6), in view of Lemma 3.8 we have

$$a(x_0,\mathbf{x})_{m_0,\mathbf{m}}1_W=0,$$

and

$$a(x_0,\mathbf{x})_{m_0,\mathbf{m}}b(x_0,\mathbf{x})=\begin{cases}[a,b](x_0,\mathbf{x}) & \text{for } m_0=0,\\ \ell\langle a,b\rangle 1_W & \text{for } m_0=1,\\ 0 & \text{for } m_0\geqslant 2.\end{cases}$$

This implies that $U_W+\mathbb{C}1_W$ is an $(\widehat{L_{r+1}(\mathfrak{g})}_+ +\mathbb{C}\mathbf{k})$-submodule of $\langle U_W\rangle$ and that the obvious map ψ_W from $\mathfrak{g}+\mathbb{C}$ to $U_W+\mathbb{C}1_W$ is an $(\widehat{L_{r+1}(\mathfrak{g})}_+ +\mathbb{C}\mathbf{k})$-module homomorphism. It then follows from the construction of $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$ that there exists an $\widehat{L_{r+1}(\mathfrak{g})}$-module homomorphism $\tilde{\psi}_W$ from $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$ to $\langle U_W\rangle$, extending ψ_W.

Now, take $W=V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$. Denote $\tilde{\psi}_W$ by $\tilde{\psi}_{x_0,\mathbf{x}}$ to indicate the dependence on those variables and also denote $1_{V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)}$ simply by 1. For $v\in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$, set

$$Y(v;x_0,\mathbf{x})=\tilde{\psi}_{x_0,\mathbf{x}}(v).$$

For $a\in\mathfrak{g}$, we have

$$Y(a;x_0,\mathbf{x})=\tilde{\psi}_{x_0,\mathbf{x}}(a)=a(x_0,\mathbf{x}).$$

Then $\{Y(a;x_0,\mathbf{x})\mid a\in\mathfrak{g}\}\cup\{1\}$ is local. For $a\in\mathfrak{g}$, $v\in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$, we have

$$\begin{aligned}
&Y\big(Y(a;z_0,\mathbf{z})v;\,y_0,\mathbf{y}\big)\\
&\quad=\tilde{\psi}_{y_0,\mathbf{y}}\big(a(z_0,\mathbf{z})v\big)\\
&\quad=Y_{\mathcal{E}}\big(a(y_0,\mathbf{y});z_0,\mathbf{z}\big)\tilde{\psi}_{y_0,\mathbf{y}}(v)\\
&\quad=\mathrm{Res}_{x_0}\,z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)a(x_0,\mathbf{zy})\tilde{\psi}_{y_0,\mathbf{y}}(v)-z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)\tilde{\psi}_{y_0,\mathbf{y}}(v)a(x_0,\mathbf{zy})\\
&\quad=\mathrm{Res}_{x_0}\,z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y(a;x_0,\mathbf{zy})Y(v;\,y_0,\mathbf{y})\\
&\qquad-\mathrm{Res}_{x_0}\,z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y(v;\,y_0,\mathbf{y})Y(a;x_0,\mathbf{zy}).
\end{aligned}$$

Now it follows immediately from Theorem 3.10 that $(V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0),Y)$ carries the structure of an $(r+1)$-toroidal vertex algebra as desired. □

Furthermore, we have:

Theorem 4.3. *Let W be any restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ. Then there exists a $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell,0)$-module structure on W, which is uniquely determined by*

$$Y_W(a;x_0,\mathbf{x})=a(x_0,\mathbf{x})\quad \textit{for } a\in\mathfrak{g}.$$

On the other hand, suppose that (W, Y_W) is a $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$-module. Then W becomes a restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ with

$$a(x_0, \mathbf{x}) = Y_W(a; x_0, \mathbf{x}) \quad \textit{for } a \in \mathfrak{g}.$$

Proof. Recall the first part of the proof of Theorem 4.2: We have an $(r+1)$-toroidal vertex algebra $\langle U_W\rangle$ with W as a canonical module and we showed that $\langle U_W\rangle$ is an $\widehat{L_{r+1}(\mathfrak{g})}$-module with $a(z_0, \mathbf{z})$ acting as $Y_{\mathcal{E}}(a(x_0, \mathbf{x}); z_0, \mathbf{z})$ for $a \in \mathfrak{g}$ and that there exists an $\widehat{L_{r+1}(\mathfrak{g})}$-module homomorphism $\tilde{\psi}_W$ from $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ to $\langle U_W\rangle$, sending $\mathbf{1}$ to 1_W and $a \in \mathfrak{g}$ to $a(x_0, \mathbf{x})$. For $a \in \mathfrak{g}$, $v \in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, we have

$$\tilde{\psi}_W\big(Y(a; y_0, \mathbf{y})v\big) = \tilde{\psi}_W\big(a(y_0, \mathbf{y})v\big) = Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); y_0, \mathbf{y}\big)\tilde{\psi}_W(v) = Y_{\mathcal{E}}\big(\tilde{\psi}_W(a); y_0, \mathbf{y}\big)\tilde{\psi}_W(v).$$

Since $\mathfrak{g}$ generates $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, it follows that $\tilde{\psi}_W$ is a homomorphism of $(r+1)$-toroidal vertex algebras. With W as a canonical $\langle U_W\rangle$-module, consequently, W becomes a $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$-module with

$$Y_W(v; z_0, \mathbf{z}) = Y_W\big(\tilde{\psi}_W(v); z_0, \mathbf{z}\big) = \tilde{\psi}_W(v)(z_0, \mathbf{z})$$

for $v \in V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$. In particular, we have

$$Y_W(a; z_0, \mathbf{z}) = Y_W\big(\tilde{\psi}_W(a); z_0, \mathbf{z}\big) = Y_W\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big) = a(z_0, \mathbf{z})$$

for $a \in \mathfrak{g}$. This proves the first assertion.

On the other hand, let (W, Y_W) be a $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$-module. From (4.12) we have

$$\begin{aligned} &a_{k,\mathbf{m}} \cdot 1 = 0, \\ &a_{k,\mathbf{m}} \cdot b = \begin{cases} [a, b] & \text{if } k = 0, \\ \langle a, b\rangle\ell\mathbf{1} & \text{if } k = 1, \\ 0 & \text{if } k \geqslant 2 \end{cases} \end{aligned} \tag{4.14}$$

for $a, b \in \mathfrak{g}$, $k \geqslant 0$, $\mathbf{m} \in \mathbb{Z}^r$. Combining this with the commutator formula (2.5), we get

$$\begin{aligned} &\operatorname{Res}_{\mathbf{x}} \mathbf{x}^{\mathbf{m}-1}\big[Y_W(a; x_0, \mathbf{x}), Y_W(b; z_0, \mathbf{z})\big] \\ &= \mathbf{z}^{\mathbf{m}}\left(Y_W\big([a, b]; z_0, \mathbf{z}\big)x_0^{-1}\delta\left(\frac{z_0}{x_0}\right) + \ell\langle a, b\rangle Y_W(\mathbf{1}; z_0, \mathbf{z})\frac{\partial}{\partial z_0}x_0^{-1}\delta\left(\frac{z_0}{x_0}\right)\right). \end{aligned}$$

It follows that W is a restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ with $a(x_0, \mathbf{x}) = Y_W(a; x_0, \mathbf{x})$ for $a \in \mathfrak{g}$. □

Remark 4.4. Recall from Proposition 2.14 that for any $(r+1)$-toroidal vertex algebra V, the left ideal V^0 generated by $\mathbf{1}$ is an ordinary vertex algebra, where

$$V^0 = \big\{u_{m_0,\mathbf{m}}\mathbf{1} \bigm| u \in V,\ m_0 \in \mathbb{Z},\ \mathbf{m} \in \mathbb{Z}^r\big\},$$

and

$$Y^0(v, x_0) = Y(v; x_0, \mathbf{x})|_{\mathbf{x}=1} \quad \text{for } v \in V^0.$$

Assume $V = V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$. As $\mathfrak{g} + \mathbb{C}\mathbf{1}$ generates $V_{\widehat{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, it follows that

$$V^0 = U\big(\widehat{L_{r+1}(\mathfrak{g})}\big)\mathbf{1} = V_{\widehat{L_r(\mathfrak{g})}}(\ell, 0)$$

as a vector space, where $V_{\widehat{L_r(\mathfrak{g})}}(\ell, 0)$ is the vertex algebra associated to the affine Lie algebra $\widehat{L_r(\mathfrak{g})}$ of level ℓ. For $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$, using (2.22) we have

$$Y^0\big(\big(a \otimes t_0^{-1} \otimes \mathbf{t}^{\mathbf{m}}\big)\mathbf{1}, x_0\big) = Y\big(\big(a \otimes t_0^{-1} \otimes \mathbf{t}^{\mathbf{m}}\big)\mathbf{1}; x_0, \mathbf{x}\big)\big|_{\mathbf{x}=1} = Y(a; x_0, \mathbf{m}) = a(x_0, \mathbf{m}).$$

It follows (from induction) that V^0 coincides with the vertex algebra $V_{\widehat{L_r(\mathfrak{g})}}(\ell, 0)$.

In what follows, by modifying the previous construction we shall construct an extended $(r+1)$-toroidal vertex algebra. First, we extend the toroidal Lie algebra $\widehat{L_{r+1}(\mathfrak{g})}$ by derivations $d_0, d_1, \ldots, d_r$ as defined in (4.7). Set

$$D = \mathbb{C}d_0 + \mathbb{C}d_1 + \cdots + \mathbb{C}d_r, \tag{4.15}$$

an abelian Lie algebra acting on $\widehat{L_{r+1}(\mathfrak{g})}$ by derivations. We have a Lie algebra

$$\widetilde{L_{r+1}(\mathfrak{g})} = \widehat{L_{r+1}(\mathfrak{g})} \oplus D, \tag{4.16}$$

which is the semi-direct product Lie algebra. Recall the $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module $(\mathfrak{g} + \mathbb{C})_\ell$ with $\ell \in \mathbb{C}$ (see Lemma 4.1). Form an induced $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k} + D)$-module

$$\tilde{K}_\ell = U\big(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k} + D\big) \otimes_{U(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})} (\mathfrak{g} + \mathbb{C})_\ell.$$

It can be readily seen that $U(D)D \cdot 1$ is a submodule of $\tilde{K}_\ell$. Define K_ℓ to be the quotient module of $\tilde{K}_\ell$ modulo relation $D \cdot 1 = 0$. As a D-module,

$$K_\ell = \big(S(D) \otimes \mathfrak{g}\big) \oplus \mathbb{C}, \tag{4.17}$$

the direct sum of the free D-module on $\mathfrak{g}$ and the trivial module $\mathbb{C}$.

Then, we form an induced $\widetilde{L_{r+1}(\mathfrak{g})}$-module

$$V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0) = U\big(\widetilde{L_{r+1}(\mathfrak{g})}\big) \otimes_{U(\widehat{L_{r+1}(\mathfrak{g})}_+ + D + \mathbb{C}\mathbf{k})} K_\ell, \tag{4.18}$$

which is an $\mathbb{N}$-graded $\widetilde{L_{r+1}(\mathfrak{g})}$-module of level ℓ. We have:

Theorem 4.5. *On $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, there exists a structure of an extended $(r+1)$-toroidal vertex algebra, which is uniquely determined by the condition*

$$\mathcal{D}_i = d_i \quad \textit{for } 0 \leqslant i \leqslant r \quad \textit{and} \quad Y(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x}) \quad \textit{for } a \in \mathfrak{g}.$$

Proof. First we shall use Theorem 3.10 to obtain an $(r+1)$-toroidal vertex algebra structure. Recall

$$K_\ell = \mathbb{C}[d_0, d_1, \dots, d_r]\mathfrak{g} + \mathbb{C} \subset V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0).$$

Let W be any restricted $\widetilde{L_{r+1}(\mathfrak{g})}$-module of level ℓ. Set

$$U_W = \left\{ d_i^k a(x_0, \mathbf{x}) \;\middle|\; a \in \mathfrak{g},\ 0 \leqslant i \leqslant r,\ k \geqslant 0 \right\}.$$

Then U_W is a local subset of $\mathcal{E}(W, r)$. By Theorem 3.7, U_W generates an extended $(r+1)$-toroidal vertex algebra $\mathbb{C}[D_0, D_1, \dots, D_r]\langle U_W\rangle$ in $\mathcal{E}(W, r)$, denoted locally by $\overline{\langle U_W\rangle}$. It follows from the same argument as in the proof of Theorem 4.2 that $\overline{\langle U_W\rangle}$ is an $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ with

$$a(z_0, \mathbf{z}) = Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big) \quad \text{for } a \in \mathfrak{g}.$$

As $\overline{\langle U_W\rangle}$ is an extended $(r+1)$-toroidal vertex algebra with $\mathcal{D}_i = D_i$ $(i = 0, 1, \dots, r)$, we have

$$\big[D_0, Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big)\big] = \frac{\partial}{\partial z_0} Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big),$$
$$\big[D_i, Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big)\big] = z_i \frac{\partial}{\partial z_i} Y_{\mathcal{E}}\big(a(x_0, \mathbf{x}); z_0, \mathbf{z}\big)$$

for $a \in \mathfrak{g}$, $1 \leqslant i \leqslant r$. Thus $\overline{\langle U_W\rangle}$ is an $\widetilde{L_{r+1}(\mathfrak{g})}$-module of level ℓ with $d_i = D_i$ for $0 \leqslant i \leqslant r$. From the proof of Theorem 4.2 we see that the linear map $\psi_W : \mathfrak{g} + \mathbb{C} \to \overline{\langle U_W\rangle}$, defined by

$$\psi_W(a + \lambda) = a(x_0, \mathbf{x}) + \lambda 1_W \quad \text{for } a \in \mathfrak{g},\ \lambda \in \mathbb{C},$$

is an $(\widehat{L_{r+1}(\mathfrak{g})}_+ + \mathbb{C}\mathbf{k})$-module homomorphism. We also have

$$d_i \cdot 1_W = D_i(1_W) = 0 \quad \text{for } 0 \leqslant i \leqslant r.$$

It follows from the construction of $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ that there exists an $\widetilde{L_{r+1}(\mathfrak{g})}$-module homomorphism $\tilde{\psi}_W$ from $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ to $\overline{\langle U_W\rangle}$, extending ψ_W.

Take $W = V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ and let $\tilde{\psi}(x_0, \mathbf{x})$ denote the corresponding map $\tilde{\psi}_W$. Define

$$Y(v; x_0, \mathbf{x}) = \tilde{\psi}(x_0, \mathbf{x})(v) \quad \text{for } v \in V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0).$$

We have

$$Y\big(d_0^k a; x_0, \mathbf{x}\big) = \tilde{\psi}(x_0, \mathbf{x})\big(d_0^k a\big) = D_0^k\big(\tilde{\psi}(x_0, \mathbf{x})(a)\big) = \left(\frac{\partial}{\partial x_0}\right)^k Y(a; x_0, \mathbf{x}),$$
$$Y\big(d_i^k a; x_0, \mathbf{x}\big) = \tilde{\psi}(x_0, \mathbf{x})\big(d_i^k a\big) = D_i^k\big(\tilde{\psi}(x_0, \mathbf{x})(a)\big) = \left(x_i \frac{\partial}{\partial x_i}\right)^k Y(a; x_0, \mathbf{x})$$

for $a \in \mathfrak{g}$, $k \geqslant 0$, $1 \leqslant i \leqslant r$. As in the proof of Theorem 4.2, it follows from Theorem 3.10 with $U = K_\ell$ that $(V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0), Y)$ carries the structure of an $(r+1)$-toroidal vertex algebra. We also have

$$[d_0, Y(a; x_0, \mathbf{x})] = [d_0, a(x_0, \mathbf{x})] = \frac{\partial}{\partial x_0} a(x_0, \mathbf{x}) = \frac{\partial}{\partial x_0} Y(a; x_0, \mathbf{x}),$$
$$[d_i, Y(a; x_0, \mathbf{x})] = [d_i, a(x_0, \mathbf{x})] = x_i \frac{\partial}{\partial x_i} a(x_0, \mathbf{x}) = x_i \frac{\partial}{\partial x_i} Y(a; x_0, \mathbf{x}).$$

Then it follows that

$$[d_0, Y(d_j^k a; x_0, \mathbf{x})] = \frac{\partial}{\partial x_0} Y(d_j^k a; x_0, \mathbf{x}),$$
$$[d_i, Y(d_j^k a; x_0, \mathbf{x})] = x_i \frac{\partial}{\partial x_i} Y(d_j^k a; x_0, \mathbf{x}).$$

As K_ℓ generates $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$, by Lemma 2.11, $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ is an extended $(r+1)$-toroidal vertex algebra with $\mathcal{D}_i = d_i$ $(i = 0, 1, \ldots, r)$. □

We also have:

Theorem 4.6. *For any restricted $\widetilde{L_{r+1}(\mathfrak{g})}$-module W of level ℓ, there exists a structure of a $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$-module Y_W on W, uniquely determined by*

$$Y_W(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g}.$$

On the other hand, for any $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$-module (W, Y_W), W is a restricted $\widetilde{L_{r+1}(\mathfrak{g})}$-module of level ℓ with

$$a(x_0, \mathbf{x}) = Y_W(a; x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g} \quad \text{and} \quad d_i = D_i \quad \text{for } 0 \leqslant i \leqslant r.$$

Remark 4.7. Let W be a restricted $\widehat{L_{r+1}(\mathfrak{g})}$-module of level ℓ. Then using Theorem 3.7 one can show that W is a module for $V_{\widetilde{L_{r+1}(\mathfrak{g})}}(\ell, 0)$ viewed as an $(r+1)$-toroidal vertex algebra, satisfying

$$Y_W(\mathcal{D}_0(v); x_0, \mathbf{x}) = \frac{\partial}{\partial x_0} Y_W(v; x_0, \mathbf{x}),$$
$$Y_W(\mathcal{D}_i(v); x_0, \mathbf{x}) = x_i \frac{\partial}{\partial x_i} Y_W(v; x_0, \mathbf{x})$$

for $v \in V$, $1 \leqslant i \leqslant r$.

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