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J. Differential Equations 191 (2003) 206–233

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient

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Received June 19, 2002

Abstract

In this paper we consider an initial boundary value problem for a parabolic inclusion whose multivalued nonlinearity is characterized by Clarke's generalized gradient of some locally Lipschitz function, and whose elliptic operator may be a general quasilinear operator of Leray–Lions type. Recently, extremality results have been obtained in case that the governing multivalued term is of special structure such as, multifunctions given by the usual subdifferential of convex functions or subgradients of so-called dc-functions. The main goal of this paper is to prove the existence of extremal solutions within a sector of appropriately defined upper and lower solutions for quasilinear parabolic inclusions with general Clarke's gradient. The main tools used in the proof are abstract results on nonlinear evolution equations, regularization, comparison, truncation, and special test function techniques as well as tools from nonsmooth analysis.

MSC: 35A15; 35B05; 35R70; 49J52

Keywords: Quasilinear parabolic inclusions; Initial boundary value problem; Extremal solutions; Upper and lower solutions; Pseudo-monotone operators; Generalized gradient; Nonsmooth analysis; Regularization; Comparison

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0022-0396/03/$ - see front matter

doi:10.1016/S0022-0396(03)00022-6

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $Q = \Omega \times (0, \tau)$ and $\Gamma = \partial\Omega \times (0, \tau)$, with $\tau > 0$. Consider the problem with the unknown $u = u(x, t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + \partial g(\cdot, \cdot, u) \ni Fu + h & \text{in } Q, \\ u(\cdot, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Here A is a second-order quasilinear differential operator in divergence form of Leray–Lions type

$$Au(x, t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, t, u(x, t), \nabla u(x, t)),$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$, and F is the Nemytski operator associated with a Carathéodory function f: $Q \times \mathbb{R} \to \mathbb{R}$. The function g: $Q \times \mathbb{R} \to \mathbb{R}$ is such that $g(\cdot, \cdot, s)$: $Q \to \mathbb{R}$ is measurable and $g(x, t, \cdot)$: $\mathbb{R} \to \mathbb{R}$ is locally Lipschitz. The notation ∂g stands for the generalized gradient in the sense of Clarke (cf. [9]) with respect to the third variable. Homogeneous initial and boundary conditions have been taken into account only for the sake of simplifying our presentation. Problem (1.1) may also be considered as the multivalued version of a parabolic hemivariational inequality. Hemivariational inequalities arise, e.g., in mechanical problems governed by nonconvex, possibly nonsmooth energy functionals, called superpotentials, which appear if nonmonotone, multivalued constitutive laws are taken into account, cf. [10,11].

Recently in [3,4,7] the existence of extremal solutions has been proved under the assumption that the multifunction of the inclusion is either given by the subdifferential of some convex function or Clarke's gradient of so-called dc-functions, see also [5] for a quasilinear elliptic problem with multivalued flux boundary conditions.

The main goal of this paper is to extend the extremality results to parabolic inclusions with a general Clarke's gradient. More precisely, we are going to show the existence of extremal solutions within a sector of appropriately defined upper and lower solutions, and prove some compactness of the solution set within this sector. The main tools used in the proofs are abstract results on nonlinear evolution equations, regularization, comparison, truncation, and special test function techniques as well as tools from nonsmooth analysis.

Our result extends previous works of the authors where the multivalued gradient term were in a more particular form. Specifically, the existence of extremal solutions has been proved in [7] under the assumption that $s \mapsto g(\cdot, \cdot, s)$ is convex and $\partial g(\cdot, \cdot, s)$ is the usual subdifferential, and in [3,4] the function $s \mapsto g(\cdot, \cdot, s)$ was supposed to be a dc-function, which means that g is of the form $g(\cdot, \cdot, s) = j_1(\cdot, \cdot, s) - j_2(\cdot, \cdot, s)$, where

$s \mapsto j_k(\cdot,\cdot,s)$ are convex, $k = 1, 2$. The crucial point in the extremality proof is to show that the solution set is upward (downward) directed. One of the arguments to prove this in the above-cited papers was the maximal monotonicity of the subdifferential of a convex function. In the present paper we were able to deal with a general Clarke's gradient, which only is restricted by a kind of one-sided growth condition, cf. (H1)(ii). This strongly extends the class of multifunctions that can be taken into account. The proof of our result is more involved and requires different tools.

2. Hypotheses

Let $2 \leqslant p < \infty$ and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. We set

$$V = L^p(0,\tau; W^{1,p}(\Omega))$$

with the dual $V^* = L^q(0,\tau; (W^{1,p}(\Omega))^*)$ and

$$W = \left\{ w \in V \colon \frac{\partial w}{\partial t} \in V^* \right\},$$

where the derivative $\partial/\partial t$ is understood in the sense of vector-valued distributions, see [12]. It is known that W is a reflexive, separable Banach space endowed with the norm

$$||w||_W = ||w||_V + \left|\left| \frac{\partial w}{\partial t} \right|\right|_{V^*}$$

and the embedding $W \subset L^p(Q)$ is compact, see [12]. We introduce

$$V_0 = L^p(0,\tau; W_0^{1,p}(\Omega))$$

with the dual $V_0^* = L^q(0,\tau; W^{-1,q}(\Omega))$ and

$$W_0 = \left\{ w \in V_0 \colon \frac{\partial w}{\partial t} \in V_0^* \right\}.$$

The function h is supposed to satisfy $h \in V_0^*$.

We assume the following hypotheses on the coefficient functions a_i, $i = 1, \ldots, N$, entering the definition of the operator A.

(A1) a_i: $Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, i.e. $a_i(\cdot,\cdot,s,\xi)$: $Q \to \mathbb{R}$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $a_i(x,t,\cdot,\cdot)$: $\mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous for a.e. $(x,t) \in Q$. In addition, one has

$$|a_i(x,t,s,\xi)| \leqslant k_0(x,t) + c_0(|s|^{p-1} + |\xi|^{p-1})$$

for a.e. $(x,t)\in Q$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, for some constant $c_0>0$ and some function $k_0\in L^q(Q)$.

(A2) $\sum_{i=1}^N (a_i(x,t,s,\xi)-a_i(x,t,s,\xi'))(\xi_i-\xi_i')>0$
for a.e. $(x,t)\in Q$, for all $s\in\mathbb{R}$ and all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$.

(A3) $\sum_{i=1}^N a_i(x,t,s,\xi)\xi_i\geqslant \nu|\xi|^p-k_1(x,t)$
for a.e. $(x,t)\in Q$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, for some constant $\nu>0$ and some function $k_1\in L^1(Q)$.

(A4) $|a_i(x,t,s,\xi)-a_i(x,t,s',\xi)|\leqslant[k_2(x,t)+|s|^{p-1}+|s'|^{p-1}+|\xi|^{p-1}]\omega(|s-s'|)$
for a.e. $(x,t)\in Q$, for all $s,s'\in\mathbb{R}$ and all $\xi\in\mathbb{R}^N$, for some function $k_2\in L^q(Q)$ and a continuous function ω: $[0,+\infty)\to[0,+\infty)$ satisfying

$$\int_{0^+}\frac{1}{\omega(r)}\,dr=+\infty.$$

For example, we can take $\omega(r)=cr$, with $c>0$, in (A4).

Let $\langle\cdot,\cdot\rangle$ be the duality pairing between V_0^* and V_0. By (A1) the semilinear form a associated with the operator A by

$$\langle Au,\varphi\rangle := a(u,\varphi)=\sum_{i=1}^N\int_Q a_i(x,t,u,\nabla u)\frac{\partial\varphi}{\partial x_i}\,dx\,dt,\quad \forall u,\varphi\in V_0, \tag{2.1}$$

is well-defined, and the operator $A: V_0\to V_0^*$ is continuous and bounded.

We denote by $L_+^p(Q)$ the positive cone of nonnegative elements of $L^p(Q)$. A partial ordering in $L^p(Q)$ is defined by $u\leqslant v$ if and only if $v-u\in L_+^p(Q)$. If $\underline{u},\bar{u}\in W_0$ with $\underline{u}\leqslant\bar{u}$, we denote by

$$[\underline{u},\bar{u}]=\{u\in W_0:\ \underline{u}\leqslant u\leqslant\bar{u}\}$$

the order interval formed by $\underline{u}$ and $\bar{u}$.

We define the notion of weak solution of problem (1.1).

Definition 2.1. A function $u\in W_0$ is called a *solution* of problem (1.1) if $Fu\in L^q(Q)$ and if there is a function $v\in L^q(Q)$ such that

(i) $u(\cdot,0)=0$ in Ω,
(ii) $v(x,t)\in\partial g(x,t,u(x,t))$ for a.e. $(x,t)\in Q$,
(iii) $\langle\frac{\partial u}{\partial t},\varphi\rangle+\langle Au,\varphi\rangle+\int_Q v(x,t)\varphi(x,t)\,dx\,dt=\int_Q(Fu)(x,t)\varphi(x,t)\,dx\,dt+\langle h,\varphi\rangle$, for all $\varphi\in V_0$.

We now give an extension of the upper and lower solutions for single-valued equations to the multivalued problem (1.1).

Definition 2.2. A function $\bar{u} \in W$ is called an *upper solution* of problem (1.1) if $F\bar{u} \in L^q(Q)$ and if there is a function $\bar{v} \in L^q(Q)$ such that

(i) $\bar{u}(x,0) \geqslant 0$ in Ω and $\bar{u} \geqslant 0$ on Γ,
(ii) $\bar{v}(x,t) \in \partial g(x,t,\bar{u}(x,t))$ for a.e. $(x,t) \in Q$,
(iii) $\langle \frac{\partial \bar{u}}{\partial t}, \varphi \rangle + \langle A\bar{u}, \varphi \rangle + \int_Q \bar{v}(x,t)\varphi(x,t)\, dx\, dt \geqslant \int_Q (F\bar{u})(x,t)\, \varphi(x,t)\, dx\, dt + \langle h, \varphi \rangle$, for all $\varphi \in V_0 \cap L^p_+(Q)$.

Similarly, a function $\underline{u} \in W$ is called a *lower solution* of problem (1.1) if the reversed inequalities hold in Definition 2.2 with $\bar{u}, \bar{v}$ replaced by $\underline{u}, \underline{v}$.

We additionally impose the following hypotheses on problem (1.1).

(H1) There exist an upper solution $\bar{u}$ and lower solution $\underline{u}$ of problem (1.1) such that $\underline{u} \leqslant \bar{u}$.
(H2) The function $g : Q \times \mathbb{R} \to \mathbb{R}$ satisfies
  (i) $g(\cdot,\cdot,s) : Q \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}$.
  (ii) $g(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and there exist constants $\alpha > 0$ and $c_1 \geqslant 0$ such that

$$\xi_1 \leqslant \xi_2 + c_1(s_2 - s_1)^{p-1}$$

  for a.e. $(x,t) \in Q$, for all $\xi_i \in \partial g(x,t,s_i)$, $i = 1,2$, and for all s_1, s_2 with $\underline{u}(x,t) - \alpha \leqslant s_1 < s_2 \leqslant \bar{u}(x,t) + \alpha$.
  (iii) There is a function $k_3 \in L^q_+(Q)$ such that

$$|z| \leqslant k_3(x,t)$$

  for a.e. $(x,t) \in Q$, for all $s \in [\underline{u}(x,t) - 2\alpha, \bar{u}(x,t) + 2\alpha]$ and all $z \in \partial g(x,t,s)$, where α is the one entering (ii).
(H3) The function f: $Q \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and there exists $k_4 \in L^q_+(Q)$ such that

$$|f(x,t,s)| \leqslant k_4(x,t)$$

for a.e. $(x,t) \in Q$, for all $s \in [\underline{u}(x,t), \bar{u}(x,t)]$.

In the appendix we shall prove that the following result holds.

Lemma 2.1. *Assume* (A1)–(A4) *and* (H1)–(H3) *be satisfied. Then problem* (1.1) *admits at least one solution u within the order interval* $[\underline{u}, \bar{u}]$ *formed by the given lower and upper solutions* $\underline{u}$ *and* $\bar{u}$, *respectively.*

Definition 2.3. A solution u^* is the *greatest solution* within $[\underline{u}, \bar{u}]$ if for any solution $u \in [\underline{u}, \bar{u}]$ we have $u \leqslant u^*$. Similarly, u_* is the *least solution* within $[\underline{u}, \bar{u}]$ if for any solution $u \in [\underline{u}, \bar{u}]$ we have $u_* \leqslant u$. The least and greatest solutions are the *extremal* ones.

Remark 2.1. One possibility to determine upper and lower solutions of the multivalued problem (1.1) is to replace the problem by the following single-valued one

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + \hat{g}(\cdot,\cdot,u) = Fu + h & \text{in } Q, \\ u(\cdot,0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.2}$$

where $\hat{g} : Q \times \mathbb{R} \to \mathbb{R}$ may be any single-valued measurable selection of ∂g. Then obviously any upper (lower) solution $\bar{u}$ ($\underline{u}$) of the single-valued problem (2.2) is an upper (lower solution) of the multivalued one with $\bar{v} := \hat{g}(\cdot,\cdot,\bar{u})$ ($\underline{v} := \hat{g}(\cdot,\cdot,\underline{u})$). We illustrate this technique with two examples.

Example 2.1. Let $p = q = 2$ and, for some $h \in V_0^*$, consider the initial-Dirichlet boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,t,\nabla u(x,t)) + \partial g(\cdot,\cdot,u) \ni Fu + h & \text{in } Q, \\ u(\cdot,0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{E}$$

Here $g : Q \times \mathbb{R} \to \mathbb{R}$ verifies condition (H2)(i), $g(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and the generalized gradient ∂g satisfies

(i) $\xi_1 \leqslant \xi_2 + c_1(s_2 - s_1)$ for a.e. $(x,t) \in Q$ and for all $\xi_i \in \partial g(x,t,s_i)$, $i = 1,2$, with $s_1 < s_2$, and c_1 some positive constant.
(ii) There is some function $k_5 \in L_+^2(Q)$ such that $|\xi| \leqslant k_5(x,t) + c_2|s|$ for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$ and $\xi \in \partial g(x,t,s)$.

Further we assume conditions (A1)–(A3) for a_i (note that (A4) is trivially satisfied), and suppose $f : Q \times \mathbb{R} \to \mathbb{R}$ to be a Carathéodory function having the following growth:

(iii) $|f(x,t,s)| \leqslant k_6(x,t) + c_2|s|$, for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$, and with some function $k_6 \in L_+^2(Q)$ and a positive constant c_2.

Now we consider the following uniquely solvable single-valued problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,t,\nabla u(x,t)) - (k_5(x,t) + c_2|u|) & \\ \quad = k_6(x,t) + c_2|u| + h & \text{in } Q, \\ u(\cdot,0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{U}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,t,\nabla u(x,t)) + (k_5(x,t) + c_2|u|) \\ \quad = -(k_6(x,t) + c_2|u|) + h & \text{in } Q, \\ u(\cdot,0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{L}$$

Denote the unique solutions of (U) and (L) by $\bar{u}$ and $\underline{u}$, respectively. Then by comparison we get $\underline{u} \leqslant \bar{u}$. Furthermore, $\bar{u}$ and $\underline{u}$ are upper and lower solutions for problem (E). To verify this for the case of the upper solution, let $\hat{g}$ be any single-valued measurable selection of ∂g, then the conditions of Definition 2.2 are satisfied with $\bar{v} = \hat{g}(\cdot,\cdot,\bar{u})$. Similarly one verifies that $\underline{u}$ is a lower solution. One easily sees also that all the hypotheses (H1) and (H2) are fulfilled. For instance, the function $k_3(x,t)$ in (H2) (iii) is

$$k_3(x,t) = k_5(x,t) + c_2 \max\{|\bar{u}(x,t) + c_0|, |\underline{u}(x,t) - c_0|\},$$

with a constant $c_0(=2\alpha) > 0$, while $k_4(x,t)$ required in (H3) is

$$k_4(x,t) = k_6(x,t) + c_2 \max\{|\bar{u}(x,t)|, |\underline{u}(x,t)|\}.$$

Thus our main result (i.e., Theorem 3.1 below) can be applied.

Example 2.2. We give an example where Theorem 3.1 below provides nonnegative bounded solutions of initial-Dirichlet boundary value problem (1.1). Assume

(i) $a_i(x,t,0,0) = a_i(x,t,1,0) = 0$ for a.e. $(x,t) \in Q$, $i = 1,\dots,N$.
(ii) $h = 0$.
(iii) $g : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $g(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in Q$.
(iv) There exist constants $\alpha > 0$ and $c_1 \geqslant 0$ such that

$$\xi_1 \leqslant \xi_2 + c_1(s_2 - s_1)^{p-1}$$

for a.e. $(x,t) \in Q$, for all $\xi_i \in \partial g(x,t,s_i)$, $i = 1,2$, and for all s_1, s_2 with $-\alpha \leqslant s_1 < s_2 \leqslant 1 + \alpha$.
(v) There is some function $k_3 \in L_+^q(Q)$ such that $|z| \leqslant k_3(x,t)$ for a.e. $(x,t) \in Q$, for all $s \in [-2\alpha, 1 + 2\alpha]$ and $z \in \partial g(x,t,s)$.
(vi) $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for which there exists $k_4 \in L_+^q(Q)$ such that

$$|f(x,t,s)| \leqslant k_4(x,t) \quad \text{for a.e. } (x,t) \in Q, \ \forall s \in [0,1].$$

(vii) One has for a.e. $(x,t) \in Q$ that

$$\max\{z\colon\ z \in \partial g(x,t,1)\} \geqslant f(x,t,1)$$

and

$$\min\{z:\ z\in\partial g(x,t,0)\}\leqslant f(x,t,0).$$

By taking $\bar{v}(x,t)=\max\{z:\ z\in\partial g(x,t,1)\}$ in Definition 2.2, it is clear that $\bar{u}=1$ is an upper solution of problem (1.1). Similarly, setting $\underline{v}(x,t)=\min\{z:\ z\in\partial g(x,t,0)\}$, we find that $\underline{u}=0$ is a lower solution of problem (1.1). Assumptions (H1)–(H3) are readily verified from the imposed conditions. Theorem 3.1 can be applied yielding a solution of problem (1.1) which belongs to the interval [0,1].

3. Main result

Let us denote by $\mathscr{S}$ the set of the solutions of problem (1.1) enclosed by the lower and upper solutions, i.e.

$$\mathscr{S}=\{u\in W_0:\ u\in[\underline{u},\bar{u}] \text{ and } u \text{ is a solution of } (1.1)\}.$$

In view of Lemma 2.1 it follows that $\mathscr{S}\neq\emptyset$. Next, we state our main result.

Theorem 3.1. *Assume* (A1)–(A4) *and* (H1)–(H3) *be satisfied. Then problem* (1.1) *admits extremal solutions u within the order interval* $[\underline{u},\bar{u}]$ *formed by the given lower and upper solutions* $\underline{u}$ *and* $\bar{u}$, *respectively.*

In the following we proceed to prove Theorem 3.1. Throughout this section we assume that the hypotheses of Theorem 3.1 are fulfilled. We shall prove the existence of the greatest solution of problem (1.1), while a similar reasoning will lead to the existence of the least solution of problem (1.1).

The next lemma expresses that the set $\mathscr{S}$ is *upward directed*, i.e. whenever $u_1,u_2\in\mathscr{S}$ there is an $u\in\mathscr{S}$ such that $u_1\leqslant u$ and $u_2\leqslant u$.

Lemma 3.1. *Let* $u_1,u_2\in\mathscr{S}$. *Then there exists a function* $u\in\mathscr{S}$ *satisfying*

$$\max\{u_1,u_2\}\leqslant u.$$

Proof. The proof of Lemma 3.1 will be done in several steps.

Step 1: *Preliminaries.* Let $u_0:=\max\{u_1,u_2\}$. For $k=0,1,2$ we define the truncation mapping T_k as follows:

$$(T_k u)(x,t)=\begin{cases}\bar{u}(x,t) & \text{if } u(x,t)>\bar{u}(x,t),\\ u(x,t) & \text{if } u_k(x,t)\leqslant u(x,t)\leqslant\bar{u}(x,t),\\ u_k(x,t) & \text{if } u(x,t)<u_k(x,t).\end{cases}$$

With α given in (H2)(ii) we introduce the truncation operator T^α by

$$(T^\alpha u)(x,t) = \begin{cases} \bar{u}(x,t)+\alpha & \text{if } u(x,t) > \bar{u}(x,t)+\alpha, \\ u(x,t) & \text{if } \underline{u}(x,t)-\alpha \leqslant u(x,t) \leqslant \bar{u}(x,t)+\alpha, \\ \underline{u}(x,t)-\alpha & \text{if } u(x,t) < \underline{u}(x,t)-\alpha. \end{cases}$$

It is known that the truncation operators T_k, $k=0,1,2$, and T^α are continuous and bounded from V into V (see, e.g., [6]).

Let ρ: $\mathbb{R}\to\mathbb{R}$ be a mollifier function, that is $\rho \in C_0^\infty((-1,1))$, $\rho \geqslant 0$ and

$$\int_{-\infty}^{+\infty} \rho(s)\, ds = 1.$$

For any $\varepsilon > 0$ we define the regularization g^ε of g with respect to the third variable by convolution, i.e.

$$g^\varepsilon(x,t,s) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} g(x,t,s-\zeta)\rho\left(\frac{\zeta}{\varepsilon}\right) d\zeta.$$

Let us define $G_\alpha^\varepsilon : L^p(Q) \to L^q(Q)$ by

$$G_\alpha^\varepsilon u := (g^\varepsilon)'(\cdot,\cdot,(T^\alpha u)(\cdot,\cdot)). \tag{3.1}$$

The definition makes sense since, by (H2)(iii), $k_3 \in L^q(Q)$ and we have that

$$|(G_\alpha^\varepsilon u)(x,t)| = |(g^\varepsilon)'(x,t,(T^\alpha u)(x,t))| \leqslant k_3(x,t) \tag{3.2}$$

for a.e. $(x,t)\in Q$, for all $u \in L^p(Q)$ and for all ε with $0<\varepsilon<\alpha$. In order to show that (3.2) is true, we see from (H2)(iii) that

$$(g^\varepsilon)'(x,t,(T^\alpha u)(x,t)) \in \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} \partial g(x,t,(T^\alpha u)(x,t)-\zeta)\rho\left(\frac{\zeta}{\varepsilon}\right) d\zeta. \tag{3.3}$$

Here we used Aubin–Clarke Theorem (cf. [9]) whose application is possible due to the inequalities

$$\underline{u}(x,t)-2\alpha \leqslant \underline{u}(x,t)-\alpha-\zeta \leqslant (T^\alpha u)(x,t)-\zeta \leqslant \bar{u}(x,t)+\alpha-\zeta \leqslant \bar{u}(x,t)+2\alpha.$$

Using again (H2)(iii) it results that

$$|(g^\varepsilon)'(x,t,(T^\alpha u)(x,t))| \leqslant \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} k_3(x,t)\rho\left(\frac{\zeta}{\varepsilon}\right) d\zeta = k_3(x,t),$$

i.e. (3.2) is true.

Next we introduce the cut-off function $b : Q \times \mathbb{R} \to \mathbb{R}$ given by

$$b(x,t,s) = \begin{cases} (s - \bar{u}(x,t))^{p-1} & \text{if } s > \bar{u}(x,t), \\ 0 & \text{if } u_0(x,t) \leqslant s \leqslant \bar{u}(x,t), \\ -(u_0(x,t) - s)^{p-1} & \text{if } s < u_0(x,t). \end{cases} \tag{3.4}$$

We have that b is a Carathéodory function satisfying the growth condition

$$|b(x,t,s)| \leqslant k_5(x,t) + c_2|s|^{p-1} \tag{3.5}$$

for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, where $c_2 > 0$ is a constant and $k_5 \in L^q(Q)$. Moreover, one has the following estimate:

$$\int_Q b(x,t,u(x,t))\, u(x,t)\, dx\, dt \geqslant c_3 \|u\|^p_{L^p(Q)} - c_4, \quad \forall u \in L^p(Q), \tag{3.6}$$

for some constants $c_3 > 0$ and $c_4 > 0$.

By (3.5), the Nemytski operator $B : L^p(Q) \to L^q(Q)$ defined by

$$Bu(x,t) = b(x,t,u(x,t)) \tag{3.7}$$

is continuous and bounded.

We introduce the following regularized truncated problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + G^{\varepsilon}_{\alpha} u + \lambda Bu & \\ \quad = F \circ T_0 u + \displaystyle\sum_{i=1}^{2} |F \circ T_i u - F \circ T_0 u| + h & \text{in } Q, \\ u(\cdot, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{P$_\varepsilon$}$$

where $\lambda > 0$ is any constant satisfying $\lambda > c_1$. In the next steps we shall study existence, convergence and comparison properties on problem (P$_\varepsilon$).

Step 2: *Existence of solutions of* (P$_\varepsilon$) $(0 < \varepsilon < \alpha)$. Let the operator $L = \partial/\partial t : D(L) \subset V_0 \to V_0^*$, with the domain

$$D(L) = \{u \in W_0 : \ u(\cdot, 0) = 0 \ \text{in} \ \Omega\}$$

defined by

$$\langle Lu, \varphi \rangle = \int_0^\tau \left\langle \frac{\partial u}{\partial t}(t), \varphi(t) \right\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} dt, \quad \forall u \in D(L), \varphi \in V_0.$$

The linear operator L is closed, densely defined and maximal monotone (cf. [12]).

Fix $0<\varepsilon<\alpha$. Problem (P_ε) can be reformulated as follows:

$$u\in D(L), \quad (L+A+G_\alpha^\varepsilon+\lambda B)u = Eu+h \quad \text{in } V_0^*, \tag{3.8}$$

where the operator $E: L^p(Q)\to L^q(Q)$ is defined by

$$Eu := F\circ T_0 u + \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u|.$$

Using (H3) and the continuity of the truncation operators T_k, $k=0,1,2$, we have that the operator $E: L^p(Q)\to L^q(Q)$ is continuous and uniformly bounded. In addition, since the embedding $W_0\subset L^p(Q)$ is compact, endowing $D(L)\subset W_0$ with the graph norm

$$||u||_{D(L)} = ||u||_{V_0} + ||Lu||_{V_0^*} = ||u||_{W_0},$$

we obtain that $E: D(L)\to L^q(Q)\subset V_0^*$ is completely continuous.

Similarly, using now (3.2) and the continuity of the truncation operator T^α, we derive that the operator $G_\alpha^\varepsilon: L^p(Q)\to L^q(Q)$ is continuous and uniformly bounded. Using the compactness of the embedding $W_0\subset L^p(Q)$ yields that the continuous, bounded operators $G_\alpha^\varepsilon, B: D(L)\to L^q(Q)\subset V_0^*$ are completely continuous on $D(L)$ endowed with the graph norm topology.

The Leray–Lions conditions (A1)–(A3) and the properties of the operators G_α^ε, B, E imply that $A+G_\alpha^\varepsilon+\lambda B-E: D(L)\subset V_0\to V_0^*$ is continuous, bounded and pseudomonotone with respect to the graph norm topology of $D(L)$ (see, [6, Theorem E.3.2]). Thus the mapping $L+A+G_\alpha^\varepsilon+\lambda B-E: D(L)\to V_0^*$ is surjective provided that $A+G_\alpha^\varepsilon+B-E: V_0\to V_0^*$ is coercive. This means that there exists at least a solution of problem (3.8), which solves problem (P_ε).

We show that the coerciveness property of $A+G_\alpha^\varepsilon+\lambda B-E: V_0\to V_0^*$ is satisfied. Using (2.1), (3.7), (A3), (3.6) as well as the uniform boundedness of the operators G_α^ε and E, one has

$$\begin{aligned}
&\langle (A+G_\alpha^\varepsilon+\lambda B-E)u,u\rangle \\
&\quad = \langle Au,u\rangle + \lambda\langle Bu,u\rangle_{L^q(Q),L^p(Q)} + \langle (G_\alpha^\varepsilon - E)u,u\rangle \\
&\quad \geqslant \sum_{i=1}^{N} \int_Q a_i(x,t,u,\nabla u)\frac{\partial u}{\partial x_i}\,dx\,dt + \lambda\int_Q b(x,t,u(x,t))\,u(x,t)\,dx\,dt \\
&\qquad - ||(G_\alpha^\varepsilon - E)u||_{V_0^*}\,||u||_{V_0} \\
&\quad \geqslant \nu\int_Q |\nabla u|^p\,dx\,dt - \int_Q k_1(x,t)\,dx\,dt + \lambda c_3||u||^p_{L^p(Q)} - \lambda c_4 - \tilde{c}||u||_{V_0} \\
&\quad \geqslant \bar{c}||u||^p_{V_0} - c, \quad \forall u\in V_0,
\end{aligned} \tag{3.9}$$

where $\tilde{c}, \bar{c}, c$ are positive constants. Thus the use of hypothesis $p \geqslant 2$ in (3.9) ensures that $A + G_\alpha^\varepsilon + \lambda B - E : V_0 \to V_0^*$ is coercive. The existence of a solution of problem (P_ε) is proved.

In the next step the notation "$\rightharpoonup$" stands for the weak convergence, while "$\to$" represents the strong convergence with respect to different topologies.

Step 3: *Convergence of solutions of* $(\mathrm{P}_{\varepsilon_n})$. Let (ε_n) be a sequence such that $\varepsilon_n \in (0, \alpha)$ and $\varepsilon_n \to 0$ as $n \to \infty$. We know from the previous step that for all n problem $(\mathrm{P}_{\varepsilon_n})$ has at least a solution denoted by u_n.

Let us show that the sequence (u_n) is bounded in W_0. First, we remark that

$$\begin{aligned} \langle Lu, u \rangle &= \int_0^\tau \left\langle \frac{\partial u}{\partial t}(t), u(t) \right\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} dt \\ &= \int_0^\tau \frac{\partial}{\partial t}\left(\frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 \right) dt = \frac{1}{2} \|u(\tau)\|_{L^2(\Omega)}^2 \geqslant 0, \quad \forall u \in D(L). \end{aligned} \tag{3.10}$$

Using that u_n is a solution of $(\mathrm{P}_{\varepsilon_n})$, (3.10) and (3.9) with u_n in place of u we can write

$$\|h\|_{V_0^*} \|u_n\|_{V_0} \geqslant \langle h, u_n \rangle = \langle Lu_n, u_n \rangle + \langle (A + G_\alpha^{\varepsilon_n} + \lambda B - E) u_n, u_n \rangle \geqslant \bar{c} \|u_n\|_{V_0}^p - c.$$

Since $p \geqslant 2$, the previous inequality implies that (u_n) is bounded in V_0. Using again that u_n is a solution of $(\mathrm{P}_{\varepsilon_n})$, we have

$$\frac{\partial u_n}{\partial t} = (-A - G_\alpha^{\varepsilon_n} - \lambda B + E) u_n + h \quad \text{in } V_0^*.$$

The boundedness of the sequence (u_n) in V_0 ensures that the right-hand side in the previous equality is bounded in V_0^*. This implies that $(\frac{\partial u_n}{\partial t})$ is bounded in V_0^*, so the sequence (u_n) is bounded in W_0.

In the following we prove that there is a subsequence of (u_n) having the properties below.

(i) $u_n \rightharpoonup u$ in W_0, i.e. $u_n \rightharpoonup u$ in V_0 and $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in V_0^* as $n \to \infty$,
(ii) $u_n \to u$ in $L^p(Q)$ as $n \to \infty$,
(iii) $G_\alpha^{\varepsilon_n} u_n \rightharpoonup v$ in $L^q(Q)$ as $n \to \infty$, where $v(x,t) \in \partial g(x, t, (T^\alpha u)(x,t))$ for a.e. $(x,t) \in Q$.

Property (i) is a consequence of the boundedness of (u_n) in the reflexive Banach space W_0, while condition (ii) results from property (i) and the compactness of the embedding $W_0 \subset L^p(Q)$.

By (3.2) and (H2)(iii), $G_\alpha^{\varepsilon_n} u_n$ is bounded in $L^q(Q)$, thus along a subsequence $G_\alpha^{\varepsilon_n} u_n \rightharpoonup v$ in $L^q(Q)$, for some $v \in L^q(Q)$. In order to obtain (iii) we have to prove that $v(x,t) \in \partial g(x, t, (T^\alpha u)(x,t))$ for a.e. $(x,t) \in Q$, with $u \in W_0$ entering (i), (ii) and $v \in L^q(Q)$ entering (iii).

To this end, let us first establish the following inequality:

$$\int_Q \limsup_{n\to\infty}\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt$$
$$\geqslant \langle v,w\rangle_{L^q(Q),L^p(Q)},\quad \forall w\in L^p(Q), \tag{3.11}$$

where the notation g^0 stands for the generalized directional derivative in the sense of Clarke [9] of g with respect to the third variable. For any $w\in L^p(Q)$, using (3.1), (3.3) and Proposition 2.1.2 in [9], we have

$$\begin{aligned}
&\langle G_\alpha^{\varepsilon_n}u_n,w\rangle_{L^q(Q),L^p(Q)}\\
&\quad=\langle (g^{\varepsilon_n})'(T^\alpha u_n),w\rangle_{L^q(Q),L^p(Q)}\\
&\quad=\int_Q (g^{\varepsilon_n})'(x,t,(T^\alpha u_n)(x,t))w(x,t)\,dx\,dt\\
&\quad=\int_Q\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} z_n(x,t,\zeta)\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)w(x,t)\,dx\,dt\\
&\quad\leqslant\int_Q\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt,
\end{aligned}$$

with $z_n(x,t,\zeta)\in\partial g(x,t,(T^\alpha u_n)(x,t)-\zeta)$. Passing to the upper limit in the previous inequality and using $G_\alpha^{\varepsilon_n}u_n\rightharpoonup v$ in $L^q(Q)$ as well as Fatou's lemma (see, e.g., [2, p. 54]) we obtain

$$\begin{aligned}
\langle v,w\rangle_{L^q(Q),L^p(Q)}&=\lim_{n\to\infty}\langle G_\alpha^{\varepsilon_n}u_n,w\rangle_{L^q(Q),L^p(Q)}\\
&\leqslant\limsup_{n\to\infty}\int_Q\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)\right.\\
&\qquad\left.-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt\\
&\leqslant\int_Q\limsup_{n\to\infty}\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)\right.\\
&\qquad\left.-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt,
\end{aligned}$$

i.e. (3.11). The application of Fatou's lemma was possible due to the inequalities

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\\
&\quad=\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} z_n(x,t,\zeta)w(x,t)\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\\
&\quad\leqslant\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} k_3(x,t)w(x,t)\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta=k_3(x,t)w(x,t),
\end{aligned}$$

with $k_3 w\in L^1(Q)$, and

$$\begin{aligned}
&\int_Q\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt\\
&\quad\geqslant-\int_Q\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} |z_n(x,t,\zeta)|\,|w(x,t)|\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)dx\,dt\\
&\quad\geqslant-\int_Q k_3(x,t)|w(x,t)|\,dx\,dt,
\end{aligned}$$

where $z_n(x,t,\zeta)\in\partial g(x,t,(T^\alpha u_n)(x,t)-\zeta)$ is fixed such that

$$g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))=z_n(x,t,\zeta)w(x,t).$$

Next we show that

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)\\
&\quad\leqslant g^0(x,t,(T^\alpha u)(x,t);w(x,t))\quad\text{for a.e. }(x,t)\in Q,\ \forall w\in L^p(Q). \qquad (3.12)
\end{aligned}$$

Towards the proof of (3.12) we note that, by (ii) and the continuity of T^α, we get $T^\alpha u_n\to T^\alpha u$ in $L^p(Q)$ as $n\to\infty$. Then passing eventually to a subsequence it results

$$(T^\alpha u_n)(x,t)\to(T^\alpha u)(x,t)\quad\text{for a.e. }(x,t)\in Q\text{ as }n\to\infty. \qquad (3.13)$$

Thus to prove (3.12) it is sufficient to show that (3.12) holds for every $w\in L^p(Q)$ and every point $(x,t)\in Q$ satisfying (3.13) (because (3.13) is valid for a.e. $(x,t)\in Q$). Fix $w\in L^p(Q)$ and any point $(x,t)\in Q$ satisfying (3.13). Let an arbitrary number $\varepsilon>0$. The upper semicontinuity of $g^0(x,t,\cdot;w(x,t))$ yields a number $\delta>0$ such that for all ξ with $|\xi-(T^\alpha u)(x,t)|<\delta$ one has

$$g^0(x,t,\xi;w(x,t))<g^0(x,t,(T^\alpha u)(x,t);w(x,t))+\varepsilon. \qquad (3.14)$$

On the other hand, the convergence in (3.13) gives a positive integer n_ε (depending on (x,t)) such that

$$\begin{aligned}|(T^\alpha u_n)(x,t)-\zeta-(T^\alpha u)(x,t)| &\leqslant |(T^\alpha u_n)(x,t)-(T^\alpha u)(x,t)|+|\zeta|\\ &\leqslant |(T^\alpha u_n)(x,t)-(T^\alpha u)(x,t)|+\varepsilon_n<\delta,\quad \forall n\geqslant n_\varepsilon,\ \forall \zeta\in(-\varepsilon_n,\varepsilon_n).\end{aligned}$$

This allows us to apply (3.14) with $\xi=(T^\alpha u_n)(x,t)-\zeta$ to get

$$g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))<g^0(x,t,(T^\alpha u)(x,t);w(x,t))+\varepsilon$$

for all $n\geqslant n_\varepsilon$ and all $\zeta\in(-\varepsilon_n,\varepsilon_n)$. Consequently, we may write

$$\begin{aligned}&\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\\ &\quad=\frac{1}{\varepsilon_n}\int_{-\varepsilon_n}^{\varepsilon_n} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\\ &\quad<g^0(x,t,(T^\alpha u)(x,t);w(x,t))+\varepsilon.\end{aligned}$$

Passing to the upper limit as $n\to\infty$ we derive that

$$\begin{aligned}&\limsup_{n\to\infty}\left(\frac{1}{\varepsilon_n}\int_{-\infty}^{+\infty} g^0(x,t,(T^\alpha u_n)(x,t)-\zeta;w(x,t))\rho\left(\frac{\zeta}{\varepsilon_n}\right)d\zeta\right)\\ &\quad\leqslant g^0(x,t,(T^\alpha u)(x,t);w(x,t))+\varepsilon.\end{aligned}$$

As $\varepsilon>0$ was arbitrary, we conclude that (3.12) holds true.

Combining (3.11) and (3.12) it results that

$$\int_Q v(x,t)w(x,t)\,dx\,dt\leqslant\int_Q g^0(x,t,(T^\alpha u)(x,t);w(x,t))\,dx\,dt \tag{3.15}$$

for all $w\in L^p(Q)$. We use Lebesgue's point argument in (3.15).

Let an arbitrarily fixed $r\in\mathbb{R}$ and the open ball $B((\bar{x},\bar{t}),\eta)$ in Q centered at some fixed point $(\bar{x},\bar{t})$ and of radius $\eta>0$. Denote by $\chi_{B((\bar{x},\bar{t}),\eta)}$ the characteristic function of $B((\bar{x},\bar{t}),\eta)$. Setting $w=\chi_{B((\bar{x},\bar{t}),\eta)}r$ in (3.15), we have

$$\int_Q v(x,t)\chi_{B((\bar{x},\bar{t}),\eta)}(x,t)r\,dx\,dt\leqslant\int_Q g^0(x,t,(T^\alpha u)(x,t);\chi_{B((\bar{x},\bar{t}),\eta)}(x,t)r)\,dx\,dt.$$

This inequality can be equivalently written as

$$\begin{aligned}\frac{1}{m(B((\bar{x},\bar{t}\,),\eta))}\int_{B((\bar{x},\bar{t}\,),\eta)} v(x,t)r\,dx\,dt\\ \leqslant \frac{1}{m(B((\bar{x},\bar{t}\,),\eta))}\int_{B((\bar{x},\bar{t}\,),\eta)} g^0(x,t,(T^{\alpha}u)(x,t);r)\,dx\,dt,\end{aligned}$$

where $m(B((\bar{x},\bar{t}\,),\eta))$ denotes the measure of $B((\bar{x},\bar{t}\,),\eta)$. Since the functions v and $g^0(\cdot,\cdot,(T^{\alpha}u)(\cdot,\cdot);r)$ belong to $L^q(Q)$, letting $\eta\to 0$ in the previous inequality, we arrive at

$$v(\bar{x},\bar{t}\,)r\leqslant g^0(\bar{x},\bar{t},(T^{\alpha}u)(\bar{x},\bar{t}\,);r),\quad \forall r\in\mathbb{R}.$$

The definition of the generalized gradient of Clarke gives

$$v(\bar{x},\bar{t}\,)\in\partial g(\bar{x},\bar{t},(T^{\alpha}u)(\bar{x},\bar{t}\,)),$$

which completes the proof of assertion (iii).

Our aim is to pass to the weak limit as $n\to\infty$ in $(\mathrm{P}_{\varepsilon_n})$. First we show that

$$Au_n \rightharpoonup Au \quad \text{in } V_0^* \text{ as } n\to\infty. \tag{3.16}$$

To this end we shall use the pseudo-monotonicity of $A: V_0\to V_0^*$ with respect to the graph norm topology of $D(L)$. Let us show that

$$\limsup_{n\to\infty}\ \langle Au_n,u_n-u\rangle\leqslant 0. \tag{3.17}$$

By (3.10) we have

$$\left\langle\frac{\partial u_n}{\partial t},u_n-u\right\rangle=\left\langle\frac{\partial(u_n-u)}{\partial t},u_n-u\right\rangle+\left\langle\frac{\partial u}{\partial t},u_n-u\right\rangle\geqslant\left\langle\frac{\partial u}{\partial t},u_n-u\right\rangle.$$

This inequality combined with $(\mathrm{P}_{\varepsilon_n})$ implies

$$\begin{aligned}\left\langle\frac{\partial u}{\partial t},u_n-u\right\rangle+\langle Au_n,u_n-u\rangle+\langle G_{\alpha}^{\varepsilon}u_n,u_n-u\rangle_{L^q(Q),L^p(Q)}\\ +\langle(\lambda B-E)u_n,u_n-u\rangle\leqslant\langle h,u_n-u\rangle.\end{aligned}$$

Passing to the upper limit in the relation above and using properties (i)–(iii) as well as the fact that $\lambda B-E: D(L)\subset V_0\to V_0^*$ is completely continuous with respect to graph norm, we arrive at (3.17).

The pseudo-monotonicity of $A: V_0\to V_0^*$ with respect to the graph norm opology of $D(L)$ in conjunction with $u_n\rightharpoonup u$ in W_0 (see (i)) and (3.17) imply (3.16) (cf., e.g., [1]).

Passing to the weak limit in V_0^* as $n\to\infty$ in problem $(\mathrm{P}_{\varepsilon_n})$ and making use of the convergences (i), (3.16), (iii) as well as of the complete continuity of $\lambda B-E$

from $D(L) \subset W_0$ into V_0^* we conclude that u is a solution of the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + v + \lambda Bu = Eu + h & \text{in } V_0^*, \\ v \in \partial g(\cdot, \cdot, (T^\alpha u)(\cdot, \cdot)) & \text{a.e. in } Q. \end{cases} \tag{P$_0$}$$

Additionally, the operator L being closed, we have that its graph is closed and convex, thus weakly closed. This leads to $u \in D(L)$. In the next step we show that the solution u of problem (P_0) satisfies $u_0 \leqslant u \leqslant \bar{u}$.

Step 4: *Comparison* $u_0 \leqslant u \leqslant \bar{u}$. In order to prove $u_0 \leqslant u$ we show that $u_k \leqslant u$, $k = 1, 2$. Since $u_k \in \mathscr{S}$ it follows that for $k = 1, 2$, $u_k \in W_0$ and verifies (1.1), thus

$$\begin{cases} \dfrac{\partial u_k}{\partial t} + Au_k + v_k = Fu_k + h & \text{in } V_0^*, \\ v_k \in \partial g(\cdot, \cdot, u_k(\cdot, \cdot)) & \text{a.e. in } Q. \end{cases} \tag{3.18}$$

Substracting the equality in (P_0) from the one in (3.18) it results that

$$\begin{aligned} &\frac{\partial (u_k - u)}{\partial t} + Au_k - Au + v_k - v - \lambda Bu \\ &\quad = Fu_k - F \circ T_0 u - \sum_{i=1}^{2} |F \circ T_i u - F \circ T_0 u| \quad \text{in } V_0^*. \end{aligned} \tag{3.19}$$

By (A4), for any fixed $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \varepsilon)$ such that

$$\int_{\delta(\varepsilon)}^{\varepsilon} \frac{1}{\omega(r)}\, dr = 1.$$

We define the function $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ by

$$\theta_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \delta(\varepsilon), \\ \int_{\delta(\varepsilon)}^{s} \dfrac{1}{\omega(r)}\, dr & \text{if } \delta(\varepsilon) \leqslant s \leqslant \varepsilon, \\ 1 & \text{if } s > \varepsilon. \end{cases}$$

It is clear that, for each $\varepsilon > 0$, the function θ_ε is continuous, piecewise differentiable and the derivative is nonnegative and bounded. Therefore the function θ_ε is Lipschitz continuous and nondecreasing. In addition, it satisfies

$$\theta_\varepsilon \to \chi_{\{s > 0\}} \quad \text{as } \varepsilon \to 0, \tag{3.20}$$

where $\chi_{\{s>0\}}$ is the characteristic function of the set $\{s>0\}$. Moreover, one has

$$\theta_\varepsilon'(s) = \begin{cases} \dfrac{1}{\omega(s)} & \text{if } \delta(\varepsilon) < s < \varepsilon, \\ 0 & \text{if } s \notin [\delta(\varepsilon), \varepsilon]. \end{cases}$$

Taking in the weak formulation of (3.19) the test function $\theta_\varepsilon(u_k - u) \in V_0 \cap L^p_+(Q)$ it follows

$$\begin{aligned} &\left\langle \frac{\partial(u_k - u)}{\partial t}, \theta_\varepsilon(u_k - u) \right\rangle + \langle Au_k - Au, \theta_\varepsilon(u_k - u) \rangle \\ &\qquad + \int_Q (v_k - v)\theta_\varepsilon(u_k - u)\, dx\, dt - \lambda \int_Q (Bu)\theta_\varepsilon(u_k - u)\, dx\, dt \\ &\quad = \int_Q \left(Fu_k - F\circ T_0 u - \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right) \theta_\varepsilon(u_k - u)\, dx\, dt. \end{aligned} \tag{3.21}$$

Let Θ_ε be the primitive of the function θ_ε defined by

$$\Theta_\varepsilon(s) = \int_0^s \theta_\varepsilon(r)\, dr.$$

We obtain for the first term on the left-hand side of (3.21) (cf., e.g., [8]) that

$$\left\langle \frac{\partial(u_k - u)}{\partial t}, \theta_\varepsilon(u_k - u) \right\rangle = \int_\Omega \Theta_\varepsilon(u_k - u)(x, \tau)\, dx \geqslant 0. \tag{3.22}$$

Using (A4) and (A2), the second term on the left-hand side of (3.21) can be estimated as follows:

$$\begin{aligned} &\langle Au_k - Au, \theta_\varepsilon(u_k - u) \rangle \\ &\quad = \sum_{i=1}^{N} \int_Q (a_i(x,t,u_k,\nabla u_k) - a_i(x,t,u,\nabla u)) \frac{\partial}{\partial x_i} \theta_\varepsilon(u_k - u)\, dx\, dt \\ &\quad \geqslant \sum_{i=1}^{N} \int_Q (a_i(x,t,u_k,\nabla u_k) - a_i(x,t,u_k,\nabla u)) \frac{\partial(u_k - u)}{\partial x_i} \theta_\varepsilon'(u_k - u)\, dx\, dt \\ &\qquad - N \int_Q (k_2 + |u_k|^{p-1} + |u|^{p-1} + |\nabla u|^{p-1})\, \omega(|u_k - u|)\theta_\varepsilon'(u_k - u)|\nabla(u_k - u)|\, dx\, dt \\ &\quad \geqslant - N \int_{\{\delta(\varepsilon) < u_k - u < \varepsilon\}} \gamma |\nabla(u_k - u)|\, dx\, dt, \end{aligned} \tag{3.23}$$

where $\gamma = k_2 + |u_k|^{p-1} + |u|^{p-1} + |\nabla u|^{p-1} \in L^q(Q)$. The term on the right-hand side of (3.23) tends to zero as $\varepsilon \to 0$.

By (3.20), the application of Lebesgue's dominated convergence theorem implies

$$\lim_{\varepsilon\to 0} \int_Q \left(v_k - v - \lambda Bu - Fu_k + F\circ T_0 u + \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right) \theta_\varepsilon(u_k - u)\, dx\, dt$$
$$= \int_Q \left(v_k - v - \lambda Bu - Fu_k + F\circ T_0 u + \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right) \chi_{\{u_k > u\}}\, dx\, dt. \quad (3.24)$$

Using (3.22), (3.23) and passing to the limit as $\varepsilon \to 0$ in (3.21), the convergence in (3.24) and the definitions of the truncation operators $T_0,\ T_1,\ T_2$ allow us to deduce

$$\begin{aligned}
&-\lambda \int_Q Bu\chi_{\{u_k > u\}}\, dx\, dt \\
&\leqslant \int_Q \left(v - v_k + Fu_k - F\circ T_0 u - \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right) \chi_{\{u_k > u\}}\, dx\, dt \\
&= \int_{\{u_k > u\}} \left(v - v_k + Fu_k - F\circ T_0 u - \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right) dx\, dt \\
&\leqslant \int_{\{u_k > u\}} (v - v_k)\, dx\, dt. \qquad (3.25)
\end{aligned}$$

If (x,t) is such that $u(x,t) < u_k(x,t)$, from the definition of T^α, we see that $\underline{u}(x,t) - \alpha \leqslant (T^\alpha u)(x,t) < u_k(x,t) \leqslant \bar{u}(x,t) + \alpha$. Applying (H2)(ii) we derive

$$v(x,t) - v_k(x,t) \leqslant c_1 (u_k(x,t) - (T^\alpha u)(x,t))^{p-1},$$

with v in (iii) and v_k in (3.18). Combining the previous inequality with (3.25) and making use of (3.4), (3.7) we obtain

$$\begin{aligned}
&\lambda \int_{\{u_k > u\}} (u_0 - u)^{p-1}\, dx\, dt \\
&= -\lambda \int_{\{u_k > u\}} Bu\, dx\, dt \\
&\leqslant c_1 \int_{\{u_k > u\}} (u_k - T^\alpha u)^{p-1}\, dx\, dt.
\end{aligned}$$

For (x,t) such that $u(x,t) < u_k(x,t)$, by the definition of T^α, we have $(u_k - T^\alpha u)(x,t) \leqslant (u_0 - u)(x,t)$, which ensures that

$$(\lambda - c_1) \int_{\{u_k > u\}} (u_0 - u)^{p-1}\, dx\, dt \leqslant 0.$$

Since $c_1 < \lambda$ (see (H2)(ii)) and $(u_0 - u)(x,t) > 0$ whenever $(u_k - u)(x,t) > 0$, we infer from the previous inequality that the Lebesgue measure of the set $\{u_k > u\}$ is equal to 0. This implies that $u_k \leqslant u$ a.e. in Q, for $k = 1, 2$, thus $u_0 \leqslant u$.

In order to prove $u \leqslant \bar{u}$, we use Definition 2.2 and (P_0) to deduce

$$\begin{aligned}\left\langle \frac{\partial(u-\bar{u})}{\partial t}, \theta_\varepsilon(u-\bar{u})\right\rangle &+ \langle Au - A\bar{u}, \theta_\varepsilon(u-\bar{u})\rangle \\ &+ \int_Q (v-\bar{v})\theta_\varepsilon(u-\bar{u})\,dx\,dt + \lambda \int_Q (Bu)\theta_\varepsilon(u-\bar{u})\,dx\,dt \\ &\leqslant - \int_Q \left(F\bar{u} - F\circ T_0 u - \sum_{i=1}^{2} |F\circ T_i u - F\circ T_0 u| \right)\theta_\varepsilon(u-\bar{u})\,dx\,dt.\end{aligned}$$

Using similar arguments as in proving (3.25), on the basis of (3.20) we obtain

$$\lambda \int_Q Bu\chi_{\{u>\bar{u}\}}\,dx\,dt \leqslant \int_{\{u>\bar{u}\}} (\bar{v} - v)\,dx\,dt.$$

If (x,t) is such that $u(x,t) > \bar{u}(x,t)$, we have that $\underline{u}(x,t) - \alpha \leqslant \bar{u}(x,t) < T^\alpha u(x,t) \leqslant \bar{u}(x,t) + \alpha$. Applying (H2)(ii) we get

$$\bar{v}(x,t) - v(x,t) \leqslant c_1 (T^\alpha u(x,t) - \bar{u}(x,t))^{p-1},$$

with v in (iii) and $\bar{v}$ in Definition 2.2, (ii). Consequently, in view of (3.4), (3.7) we deduce that

$$\lambda \int_{\{u>\bar{u}\}} (u-\bar{u})^{p-1}\,dx\,dt \leqslant c_1 \int_{\{u>\bar{u}\}} (T^\alpha u - \bar{u})^{p-1}\,dx\,dt.$$

Since $T^\alpha u(x,t) \leqslant u(x,t)$ whenever $u(x,t) > \bar{u}(x,t)$ it follows

$$(\lambda - c_1) \int_{\{u>\bar{u}\}} (u-\bar{u})^{p-1}\,dx\,dt \leqslant 0.$$

In view of $c_1 < \lambda$ (see (H2)(ii)) we obtain that $u \leqslant \bar{u}$ a.e. in Q.

Step 5: *Completion of the proof.* By the previous step any solution u of problem (P_0) satisfies $u_0 \leqslant u \leqslant \bar{u}$. Thus $Bu = 0$ and, since $T_i u = u$ for $i = 0, 1, 2$, one has $Eu = Fu$. In addition, we see that $v(x,t) \in \partial g(x,t,u(x,t))$ a.e. $(x,t) \in Q$ because $T^\alpha u = u$. Hence u is a solution of problem (1.1) satisfying $u_0 \leqslant u \leqslant \bar{u}$. The proof is complete. $\square$

Lemma 3.2. *The solution set $\mathscr{S}$ is bounded in W_0. Any sequence in $\mathscr{S}$ contains a weakly convergent subsequence in W_0 and its limit belongs to $\mathscr{S}$.*

Proof. Since $\mathscr{S} \subset [\underline{u}, \bar{u}]$, from (H2)(iii) we see that ∂g is bounded in $L^q(Q)$ on $[\underline{u}, \bar{u}]$, while (H3) implies that F is bounded in $L^q(Q)$ on $[\underline{u}, \bar{u}]$.

We claim that $\mathscr{S}$ is bounded in W_0. The coerciveness condition (A3) for $A : V_0 \to V_0^*$ yields

$$||u||_{V_0} \leqslant c', \quad \forall u \in \mathscr{S}, \tag{3.26}$$

for some constant $c' > 0$. Indeed, for any $u \in \mathscr{S}$ one has

$$\frac{\partial u}{\partial t} = -Au - v + Fu + h \quad \text{in } V_0^*, \tag{3.27}$$

with $v(x,t) \in \partial g(x,t,u(x,t))$ a.e. $(x,t) \in Q$. Then one obtains

$$\left\langle \frac{\partial u}{\partial t}, u \right\rangle + \langle Au, u \rangle = \langle Fu - v, u \rangle_{L^q(Q), L^p(Q)} + \langle h, u \rangle.$$

Using (3.10), the boundedness of F, ∂g in $L^q(Q)$ on $[\underline{u}, \bar{u}]$ and (A3) we arrive at

$$\begin{aligned} M||u||_{V_0} \geqslant \langle Au, u \rangle &= \sum_{i=1}^{N} \int_Q a_i(x,t,u,\nabla u) \frac{\partial u}{\partial x_i} \, dx \, dt \\ &\geqslant \nu ||\nabla u||^p_{L^p(Q)} - ||k_1||_{L^p(Q)} = \nu ||u||^p_{V_0} - ||k_1||_{L^p(Q)}, \end{aligned}$$

for some constant $M > 0$, which proves (3.26). By (3.26) and the boundedness of $A : V_0 \to V_0^*$ and of F, ∂g in $L^q(Q) \subset V_0^*$ we deduce from (3.27) that

$$\left\| \frac{\partial u}{\partial t} \right\|_{V_0^*} \leqslant c'', \quad \forall u \in \mathscr{S}, \tag{3.28}$$

for some constant $c'' > 0$. From (3.26) and (3.28) we obtain the boundedness of $\mathscr{S}$ in W_0, which is the first part in Lemma 3.2.

Let a sequence (u_n) in $\mathscr{S}$. By the reflexivity of W_0 we find a subsequence of (u_n), denoted again by (u_n), such that

$$u_n \rightharpoonup u \text{ in } W_0, \quad u_n \to u \text{ in } L^p(Q) \text{ and a.e. in } Q \text{ as } n \to \infty, \tag{3.29}$$

for some $u \in W_0$, where the compactness of the embedding $W_0 \subset L^p(Q)$ has been used.

Since L is a closed linear operator, its graph is weakly closed, so $u_n \rightharpoonup u$ in W_0 implies $u \in D(L)$.

From the fact that $(u_n) \subset \mathscr{S}$ we have that $u_n \in W_0$ and

$$\frac{\partial u_n}{\partial t} + Au_n + v_n = Fu_n + h, \tag{3.30}$$

with $v_n \in \partial g(\cdot, \cdot, u_n(\cdot, \cdot))$. Hypothesis (H2)(iii) ensures that (v_n) is bounded in $L^q(Q)$. Then there exists a subsequence of (v_n), denoted by (v_n), such that

$$v_n \rightharpoonup v \quad \text{in } L^q(Q) \quad \text{as } n \to \infty, \tag{3.31}$$

for some $v \in L^q(Q)$.

Next we show that

$$v(\cdot,\cdot)\in\partial g(\cdot,\cdot,u(\cdot,\cdot)). \tag{3.32}$$

Using $v_n \rightharpoonup v$ in $L^q(Q)$, $v_n\in\partial g(\cdot,\cdot,u_n(\cdot,\cdot))$, (3.29), Fatou's lemma and the upper semicontinuity of $g^0(x,t,\cdot;w(x,t)):\mathbb{R}\to\mathbb{R}$, we deduce that

$$\begin{aligned}\int_Q v(x,t)w(x,t)\,dx\,dt &= \lim_{n\to\infty}\int_Q v_n(x,t)w(x,t)\,dx\,dt\\ &\leqslant \limsup_{n\to\infty}\int_Q g^0(x,t,u_n(x,t);w(x,t))\,dx\,dt\\ &\leqslant \int_Q \limsup_{n\to\infty} g^0(x,t,u_n(x,t);w(x,t))\,dx\,dt\\ &\leqslant \int_Q g^0(x,t,u(x,t);w(x,t))\,dx\,dt.\end{aligned}$$

In order to use Lebesgue's point argument, fix $r\in\mathbb{R}$, $(\bar{x},\bar{t})\in Q$, $\eta>0$ and let $w=\chi_{B((\bar{x},\bar{t}),\eta)}r$ in the previous inequality, with $\chi_{B((\bar{x},\bar{t}),\eta)}$ the characteristic function of the open ball $B((\bar{x},\bar{t}),\eta)$. We obtain

$$\begin{aligned}&\frac{1}{m(B((\bar{x},\bar{t}),\eta))}\int_{B((\bar{x},\bar{t}),\eta)} v(x,t)r\,dx\,dt\\ &\quad\leqslant\frac{1}{m(B((\bar{x},\bar{t}),\eta))}\int_{B((\bar{x},\bar{t}),\eta)} g^0(x,t,u(x,t);r)\,dx\,dt,\end{aligned}$$

where $m(B((\bar{x},\bar{t}),\eta))$ is the measure of $B((\bar{x},\bar{t}),\eta)$. Letting $\eta\to 0$ in the previous inequality we infer

$$v(\bar{x},\bar{t})r\leqslant g^0(\bar{x},\bar{t},u(\bar{x},\bar{t});r),\quad \forall r\in\mathbb{R}.$$

Using the definition of the generalized gradient of Clarke, we deduce that (3.32) is satisfied.

From (3.30) it results

$$\begin{aligned}&\left\langle\frac{\partial u_n}{\partial t},u_n-u\right\rangle+\langle Au_n,u_n-u\rangle\\ &\quad=\langle Fu_n,u_n-u\rangle_{L^q(Q),L^p(Q)}-\langle v_n,u_n-u\rangle_{L^q(Q),L^p(Q)}+\langle h,u_n-u\rangle.\end{aligned} \tag{3.33}$$

By (3.10) we derive

$$\left\langle\frac{\partial u_n}{\partial t},u_n-u\right\rangle=\left\langle\frac{\partial(u_n-u)}{\partial t},u_n-u\right\rangle+\left\langle\frac{\partial u}{\partial t},u_n-u\right\rangle\geqslant\left\langle\frac{\partial u}{\partial t},u_n-u\right\rangle.$$

Using this inequality in (3.33) and passing to the upper limit as $n \to \infty$, on the basis of (3.29), (3.31) and the boundedness of $F(u_n)$ (see (H3)), we arrive at

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leqslant 0.$$

By the pseudo-monotonicity of A with respect to the graph norm topology of $D(L)$, this inequality and $u_n \rightharpoonup u$ in W_0 imply that $Au_n \rightharpoonup Au$ in V_0^* (cf., e.g., [1]). This allows us to pass to the limit as $n \to \infty$ in (3.30), obtaining

$$\frac{\partial u}{\partial t} + Au + v = Fu + h \quad \text{in } V_0^*.$$

As v satisfies (3.32) it follows that u is a solution of (1.1).

Combining $u_n \to u$ a.e. in Q (see (3.29)) with $u_n \in [\underline{u}, \bar{u}]$ leads to $u \in [\underline{u}, \bar{u}]$. Therefore $u \in \mathscr{S}$ and the proof is complete. □

Proof of Theorem 3.1. We show the existence of the greatest solution of (1.1). Since W_0 is separable we have that $\mathscr{S} \subset W_0$ is separable, so there exists a countable, dense subset $Z = \{z_n\colon n \in \mathbb{N}\}$ of $\mathscr{S}$. By Lemma 3.1, $\mathscr{S}$ is upward directed, so we can construct an increasing sequence $(u_n) \subset \mathscr{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathscr{S}$ such that

$$\max\{z_n, u_n\} \leqslant u_{n+1} \leqslant \bar{u}.$$

The existence of u_{n+1} is due to Lemma 3.1. By Lemma 3.2 we find a subsequence of (u_n), denoted again (u_n), and an element $u \in \mathscr{S}$ such that $u_n \rightharpoonup u$ in W_0, $u_n \to u$ in $L^p(Q)$ and $u_n(x,t) \to u(x,t)$ a.e. $(x,t) \in Q$. This last property of (u_n) combined with its increasing monotonicity imply that $u = \sup_n u_n$. By construction, we see that

$$\max\{z_1, z_2, \dots, z_n\} \leqslant u_{n+1} \leqslant u, \quad \forall n,$$

thus $Z \subset [\underline{u}, u]$. Since the interval $[\underline{u}, u]$ is closed in W_0, we infer

$$\mathscr{S} \subset \bar{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u],$$

which in conjunction with $u \in \mathscr{S}$ ensures that u is the greatest solution of (1.1).

The existence of the least solution of (1.1) can be proved in a similar way using Lemma 3.2 and a corresponding dual formulation of Lemma 3.1. This completes the proof. □

Corollary 3.1. *The solution set $\mathscr{S}$ is weakly compact in W_0, and compact in V_0.*

Proof. The weak compactness in W_0 is the contents of Lemma 3.2. We only need to show that $\mathscr{S}$ is compact in V_0. Let us be given any sequence $(u_n) \subset \mathscr{S}$. Then we have to prove that there is a subsequence of (u_n) which is strongly convergent in V_0 to some $u \in \mathscr{S}$. The weak compactness of $\mathscr{S}$ in W_0 implies the existence of a subsequence denoted by (u_k) which is weakly convergent in W_0 to some $u \in \mathcal{S}$. Hypotheses (A1)–(A3) ensure that the operator A satisfies the (S_+)-property with respect to the graph norm topology of L (see [6, Theorem E.3.2]), which means that whenever (u_k) is weakly convergent to u in W_0 and satisfies $\limsup_{k\to\infty} \langle Au_k, u_k - u\rangle \leqslant 0$, then (u_k) is strongly convergent in V_0 to u. Since $\limsup_{k\to\infty} \langle Au_k, u_k - u\rangle \leqslant 0$ has already been shown in the proof of Lemma 3.2, the (S_+)-property of A immediately implies that the weak limit $u \in \mathcal{S}$ in W_0 of the sequence (u_k) is its strong limit in V_0, and thus the compactness of $\mathcal{S}$ in V_0. $\square$

Acknowledgments

The authors are very grateful to the referee for the careful reading and useful remarks.

Appendix

Proof of Lemma 2.1. The proof of Lemma 2.1 follows the steps of the proof of Lemma 3.1.

Step 1: *Preliminaries*.

We consider the following regularized truncated problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + G_\alpha^\varepsilon u + \lambda Bu = F \circ Tu + h & \text{in } Q, \\ u(\cdot, 0) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{$\tilde{\mathrm{P}}_\varepsilon$}$$

where λ is some constant satisfying $\lambda > c_1$. Here the operator $G_\alpha^\varepsilon : L^p(Q) \to L^q(Q)$ is the one in (3.1) and verifies (3.2) for $0 < \varepsilon < \alpha$. The operator $B : L^p(Q) \to L^q(Q)$ is defined analogously as in (3.7) for $b : Q \times \mathbb{R} \to \mathbb{R}$ given by

$$b(x,t,s) = \begin{cases} (s - \bar{u}(x,t))^{p-1} & \text{if } s > \bar{u}(x,t), \\ 0 & \text{if } \underline{u}(x,t) \leqslant s \leqslant \bar{u}(x,t), \\ -(\underline{u}(x,t) - s)^{p-1} & \text{if } s < \underline{u}(x,t). \end{cases}$$

Then b is a Carathéodory function satisfying (3.5) and (3.6), while B is continuous and bounded. The truncation operator T: $V_0 \to V_0$ is defined by

$$(Tu)(x,t) = \begin{cases} \bar{u}(x,t) & \text{if } u(x,t) > \bar{u}(x,t), \\ u(x,t) & \text{if } \underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t), \\ \underline{u}(x,t) & \text{if } u(x,t) < \underline{u}(x,t) \end{cases}$$

and is continuous and bounded.

Step 2: *Existence of solution of* $(\tilde{\mathrm{P}}_\varepsilon)$ $(0<\varepsilon<\alpha)$. For a fixed ε with $0<\varepsilon<\alpha$, problem $(\tilde{\mathrm{P}}_\varepsilon)$ can be reformulated

$$u \in D(L), \quad (L + A + G_\alpha^\varepsilon + \lambda B - F \circ T)u = h \quad \text{in } V_0^*,$$

where $L = \frac{\partial}{\partial t}$ is as in the proof of Lemma 3.1. The same arguments as the ones in the proof of Lemma 3.1 ensure that $A + G_\alpha^\varepsilon + \lambda B - F \circ T$ is continuous, bounded, pseudo-monotone with respect to the graph norm of $D(L)$, and coercive, while L is maximal monotone (see [12]). Thus $L + A + G_\alpha^\varepsilon + \lambda B - F \circ T : D(L) \to V_0^*$ is surjective, so problem $(\tilde{\mathrm{P}}_\varepsilon)$ has at least a solution.

Step 3: *Convergence of solutions of* $(\tilde{\mathrm{P}}_{\varepsilon_n})$. Let a sequence (ε_n) satisfying $\varepsilon_n \in (0,\alpha)$ and $\varepsilon_n \to 0$ as $n \to \infty$. For each n let u_n be a solution of problem $\tilde{\mathrm{P}}_{\varepsilon_n}$ given by the previous step.

Using that u_n is a solution of $(\tilde{\mathrm{P}}_{\varepsilon_n})$, (3.10) and (3.9) we obtain that (u_n) is bounded in V_0. This combined with $(\tilde{\mathrm{P}}_{\varepsilon_n})$ implies that $(\frac{\partial u_n}{\partial t})$ is bounded in V_0^*. Hence the sequence (u_n) is bounded in W_0.

In the same way as in the proof of Lemma 3.1 we can show that the following properties hold:

(i) $u_n \rightharpoonup u$ in W_0, i.e. $u_n \rightharpoonup u$ in V_0 and $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in V_0^* as $n \to \infty$,
(ii) $u_n \to u$ in $L^p(Q)$ as $n \to \infty$,
(iii) $G_\alpha^{\varepsilon_n} u_n \rightharpoonup v$ in $L^q(Q)$ as $n \to \infty$, where $v(x,t) \in \partial g(x,t,(T^\alpha u)(x,t))$ for a.e. $(x,t) \in Q$.

On the basis of $(\tilde{\mathrm{P}}_{\varepsilon_n})$ and (3.10) we have

$$\left\langle \frac{\partial u}{\partial t}, u_n - u \right\rangle + \langle Au_n, u_n - u \rangle + \langle G_\alpha^\varepsilon u_n, u_n - u \rangle_{L^q(Q),L^p(Q)} + \langle (\lambda B - F \circ T)u_n, u_n - u \rangle \leqslant \langle h, u_n - u \rangle.$$

Passing here to the upper limit as $n \to \infty$ in $(\tilde{\mathrm{P}}_{\varepsilon_n})$ and using properties (i)–(iii) as well as the fact that $\lambda B - F \circ T : D(L) \subset V_0 \to V_0^*$ is completely continuous with respect to graph norm topology, we obtain

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leqslant 0.$$

Taking into account that $u_n \rightharpoonup u$ in W_0, the pseudo-monotonicity of A: $V_0 \to V_0^*$ with respect to the graph norm of $D(L)$ yields

$$Au_n \rightharpoonup Au \text{ in } V_0^* \text{ as } n \to \infty,$$

(see [1]). Letting now $n \to \infty$ in problem $(\tilde{\mathrm{P}}_{\varepsilon_n})$ and making use of the above convergence as well as (i), (iii) and the complete continuity of $\lambda B - F \circ T$

from $D(L) \subset W_0$ into V_0^* we conclude that u is a solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + v + \lambda Bu = F \circ Tu + h & \text{in } V_0^*, \\ v \in \partial g(\cdot, \cdot, (T^\alpha u)(\cdot, \cdot)) & \text{a.e. in } Q. \end{cases} \tag{$\tilde{\mathrm{P}}_0$}$$

In addition, the closedness of L yields $u \in D(L)$.

Step 4: *Comparison* $\underline{u} \leqslant u \leqslant \bar{u}$. Let us first check that $\underline{u} \leqslant u$. Using the definition of the lower solution and the fact that u is a solution of problem ($\tilde{\mathrm{P}}_0$) it results that

$$\begin{aligned} &\left\langle \frac{\partial(\underline{u} - u)}{\partial t}, \theta_\varepsilon(\underline{u} - u) \right\rangle + \langle A\underline{u} - Au, \theta_\varepsilon(\underline{u} - u) \rangle \\ &\qquad + \int_Q (\underline{v} - v)\theta_\varepsilon(\underline{u} - u)\, dx\, dt - \lambda \int_Q (Bu)\theta_\varepsilon(\underline{u} - u)\, dx\, dt \\ &\quad \leqslant \int_Q (F\underline{u} - F \circ Tu)\theta_\varepsilon(\underline{u} - u)\, dx\, dt, \end{aligned}$$

with θ_ε as in Step 4 of Lemma 3.1. Proceeding in the same way as in proving (3.25), on the basis of the previous inequality and (3.20) we obtain

$$\begin{aligned} -\lambda \int_Q Bu\chi_{\{\underline{u} > u\}}\, dx\, dt &\leqslant \int_Q (v - \underline{v} + F\underline{u} - F \circ Tu)\chi_{\{\underline{u} > u\}}\, dx\, dt \\ &= \int_{\{\underline{u} > u\}} (v - \underline{v} + F\underline{u} - F \circ Tu)\, dx\, dt \\ &\leqslant \int_{\{\underline{u} > u\}} (v - \underline{v})\, dx\, dt. \end{aligned} \tag{4.1}$$

If (x,t) is such that $u(x,t) < \underline{u}(x,t)$, we see that $\underline{u}(x,t) - \alpha \leqslant T^\alpha u(x,t) < \underline{u}(x,t) \leqslant \bar{u}(x,t) + \alpha$. Hypothesis (H2)(ii) implies

$$v(x,t) - \underline{v}(x,t) \leqslant c_1(\underline{u}(x,t) - T^\alpha u(x,t))^{p-1},$$

with v in (iii) and $\underline{v} \in \partial g(\cdot, \cdot, \underline{u}(\cdot, \cdot))$. Using (3.4), (3.7), the previous inequality and (4.1) we deduce

$$\lambda \int_{\{\underline{u} > u\}} (\underline{u} - u)^{p-1}\, dx\, dt = -\lambda \int_{\{\underline{u} > u\}} Bu\, dx\, dt \leqslant c_1 \int_{\{\underline{u} > u\}} (\underline{u} - T^\alpha u)^{p-1}\, dx\, dt.$$

For (x,t) such that $u(x,t) < \underline{u}(x,t)$, by the definition of T^α, we have $u(x,t) \leqslant (T^\alpha u)(x,t)$, thus

$$(\lambda - c_1) \int_{\{\underline{u} > u\}} (\underline{u} - u)^{p-1}\, dx\, dt \leqslant 0.$$

Since $c_1 < \lambda$ (see (H2)(ii)) it results that the Lebesgue measure of the set $\{\underline{u} > u\}$ is equal to 0. This implies that $\underline{u} \leqslant u$ a.e. in Q.

In order to prove $u \leqslant \bar{u}$, we use Definition 2.2 and $(\tilde{\mathrm{P}}_0)$ to deduce

$$\left\langle \frac{\partial(u-\bar{u})}{\partial t}, \theta_\varepsilon(u-\bar{u}) \right\rangle + \langle Au - A\bar{u}, \theta_\varepsilon(u-\bar{u}) \rangle + \int_Q (v-\bar{v})\theta_\varepsilon(u-\bar{u})\,dx\,dt + \lambda \int_Q (Bu)\theta_\varepsilon(u-\bar{u})\,dx\,dt \leqslant \int_Q (F\circ Tu - F\bar{u})\theta_\varepsilon(u-\bar{u})\,dx\,dt.$$

Similar arguments as in proving (3.25) based on (3.20) yield

$$\lambda \int_Q Bu\chi_{\{u>\bar{u}\}}\,dx\,dt \leqslant \int_{\{u>\bar{u}\}} (\bar{v} - v + F\circ Tu - F\bar{u})\,dx\,dt \leqslant \int_{\{u>\bar{u}\}} (\bar{v}-v)\,dx\,dt.$$

If (x,t) is such that $u(x,t) > \bar{u}(x,t)$, we have that $\underline{u}(x,t) - \alpha \leqslant \bar{u}(x,t) < T^\alpha u(x,t) \leqslant \bar{u}(x,t) + \alpha$. Applying (H2)(ii) we get

$$\bar{v}(x,t) - v(x,t) \leqslant c_1(T^\alpha u(x,t) - \bar{u}(x,t))^{p-1},$$

with v in (iii) and $\bar{v}$ in Definition 2.2, (ii). Thus in view of (3.4), (3.7) we obtain

$$\lambda \int_{\{u>\bar{u}\}} (u-\bar{u})^{p-1}\,dx\,dt \leqslant c_1 \int_{\{u>\bar{u}\}} (T^\alpha u - \bar{u})^{p-1}\,dx\,dt.$$

Since $T^\alpha u(x,t) \leqslant u(x,t)$ whenever $u(x,t) > \bar{u}(x,t)$ it results that

$$(\lambda - c_1) \int_{\{u>\bar{u}\}} (u-\bar{u})^{p-1}\,dx\,dt \leqslant 0.$$

In view of $c_1 < \lambda$ (see (H2)(ii)) it follows that $u \leqslant \bar{u}$ a.e. in Q.

Step 5: *Completion of the proof*. From the previous step any solution u of problem $(\tilde{\mathrm{P}}_0)$ satisfies $\underline{u} \leqslant u \leqslant \bar{u}$. It follows that $Bu = 0$ and $Tu = u$. In addition, one has that $v(x,t) \in \partial g(x,t,u(x,t))$ a.e. $(x,t) \in Q$ since $T^\alpha u = u$. We conclude that u is a solution of problem (1.1) satisfying $\underline{u} \leqslant u \leqslant \bar{u}$. The proof of Lemma 2.1 is complete. □

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