The Singularly Perturbed Domain and the Characterization for the Eigenfunctions with Neumann Boundary Condition

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1. Introduction

The eigenvalue problem of the Laplace operator and its relation to the geometric structure of the domain have been considered by many authors. It is usually the case that the eigenvalues and their eigenfunctions vary continuously under the smooth deformation of the domain. On the other hand, if the convergence of the domain is weak (i.e., the topological type is not preserved or some part of the domain degenerates), many characteristic phenomena occur and they are not easy to analyze in general. But it is important to deal with these cases to get some insight into the geometrical dependence of the eigenvalue problem. We are concerned with a moving domain \( \Omega(\zeta) \) \((\zeta > 0: \text{parameter})\) which partially degenerates as \( \zeta \to 0 \), and we give an elaborate characterization of the behaviors of the eigenfunctions of the Laplace operator with the Neumann boundary condition. The continuous dependence of the eigenvalues of the elliptic operator with the boundary condition under a regular variation of the domain is shown in R. Courant and D. Hilbert [5], I. Babuska and R. Výborný [2], and some other literature in various situations. For a singular variation of the domain, J. Rauch and M. Taylor [21], S. Ozawa [18, 19], I. Chavel, and E. A. Feldman [4] have dealt with the domain with a small hole or a domain where a very thin tubular neighborhood of a submanifold is removed, and they have shown the convergence of the eigenvalues to those of the original domain. Especially, S. Ozawa has obtained some very elaborate asymptotic behaviors of the eigenvalues with some boundary conditions. On the other hand, the domain \( \Omega(\zeta) \) in this paper belongs to another type of singular variation because \( \Omega(\zeta) \) decreases as \( \zeta \to 0 \) and thus is quite different from the above. This type of domain was first studied by J. T. Beale [3], who characterized the set of the scattering frequencies (the
square root of some spectrum) in the process of degenerating the domain. M. Mobo-Hidalgo and E. Sanchez-Palencia [15] have dealt with a more general domain than that of [3] and have proved a result corresponding to [3; Theorem 1 – (a)]. For the case of a manifold, C. Anne [1] has dealt with a manifold with a thin handle and obtained the limit of the set of the eigenvalues. K. Fukaya [7] has dealt with a collapse of the Riemannian manifold under the boundedness condition on the Ricci and the sectional curvatures and he has characterized the eigenvalues and the limit operator of the Laplacian. (See also E. Sanchez-Palencia [22] for other topics.)

The moving domain which we deal with in this paper is expressed as

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta) \quad (\zeta > 0; \text{small}) \quad \text{(Fig. 1)},$$

where $D_1$ and $D_2$ are mutually disjoint bounded domains in $\mathbb{R}^n$ and $Q(\zeta)$ is "cylindrical" and monotonously approaches a 1-dim line segment as $\zeta \to 0$. We remark that if $D_1$ is an exterior domain, $\Omega(\zeta)$ is equal to that in J. T. Beale [3]. Let $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ be the set of the eigenvalues of $-A$ on $\Omega(\zeta)$ for the Neumann boundary condition. Applying similar arguments to those in J. T. Beale [3], we can separate $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ as follows, under some assumption (cf. (A.3) in Section 2.),

$$\{\mu_k(\zeta)\}_{k=1}^{\infty} = \{\omega_k(\zeta)\}_{k=1}^{\infty} \cup \{\lambda_k(\zeta)\}_{k=1}^{\infty}, \quad \text{(1.1)}$$

where $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ approaches the set of the eigenvalues $-A$ in $D_1 \cup D_2$ with the Neumann boundary condition and $\{\lambda_k(\zeta)\}_{k=1}^{\infty}$ approaches the set of the eigenvalues of the operator $-d^2/dz^2$ on the line segment $(L = \cap_{\zeta > 0} Q(\zeta))$ with the Dirichlet boundary condition on the endpoints of $L$ as $\zeta \to 0$ where $z$ is the canonical parameter on $L$. Let $\{\Phi_{k, \zeta}\}_{k=1}^{\infty}$ be the
orthonormalized eigenfunctions corresponding to \( \{ \mu_k(\zeta) \}_{k=1}^{\infty} \) which are separated according to the decomposition (1.1)

\[
\{ \Phi_{k, \zeta} \}_{k=1}^{\infty} = \{ \phi_{k, \zeta} \}_{k=1}^{\infty} \cup \{ \psi_{k, \zeta} \}_{k=1}^{\infty} \quad (\zeta > 0),
\]

where \( \phi_{k, \zeta} \) and \( \psi_{k, \zeta} \) correspond to \( \omega_k(\zeta) \) and \( \lambda_k(\zeta) \), respectively. Our main purpose is to characterize the asymptotic behavior of \( \phi_{k, \zeta} \) and \( \psi_{k, \zeta} \), respectively, when \( \zeta \to 0 \). We will prove that \( \phi_{k, \zeta} \) for small \( \zeta > 0 \) is approximated by \( \phi_k \) in \( D_1 \cup D_2 \) and by \( V_k \) in \( \Omega(\zeta) \), where \( \omega_k \) and \( \phi_k \) are the \( k \)-th eigenvalues and one of the corresponding eigenfunctions of the following eigenvalue problem (1.3) and \( V_k \) is the solution (uniquely determined under our assumption (A.3)) of the following two point boundary value problem (1.4) whose boundary condition is given by \( \phi_k \):

\[
\begin{align*}
\Delta \phi + \omega \phi &= 0 \text{ in } D_1 \cup D_2, \\
\frac{\partial \phi}{\partial v} &= 0 \text{ on } \partial D_1 \cup \partial D_2, \\
\frac{d^2 V}{dz^2} + \omega_k V &= 0, \quad z \in L, \\
V(p_i) &= \phi_k(p_i) \quad (i = 1, 2),
\end{align*}
\]

where \( p_1 \) and \( p_2 \) are the endpoints of \( L \). We will also prove that \( d_{n-1}^{1/2} \xi^{(n-1)/2} \psi_{k, \zeta} \) converges to 0 uniformly in \( D_1 \cup D_2 \) and that \( d_{n-1}^{1/2} \xi^{(n-1)/2} \psi_{k, \zeta} \mid_{\Omega(\zeta)} \) approaches \( \sin(k\pi/2)(1-z) \) or \( \sin(k\pi/2)(1-z) \) when \( \zeta \to 0 \) where \( d_{n-1} \) is the \( (n-1) \)-dimensional Lebesgue measure of the unit ball in \( \mathbb{R}^{n-1} \) (Theorem 2). Second, we will investigate the behavior of \( \psi_{k, \zeta} \) itself, especially the exact decay rate of \( \psi_{k, \zeta} \) in \( D_1 \cup D_2 \); i.e., we will prove that

\[
\psi_{k, \zeta}(x) \sim O(\xi^{(n-1)/2})
\]

uniformly on any compact subset of \( \bigcup_{i=1}^{2} (\bar{D}_i \setminus \{ p_i \}) \),

\[
\| \psi_{k, \zeta} \|_{L^2(\Omega(\zeta))} \sim O(\xi^{(n-1)/2})
\]

while \( \| \psi_{k, \zeta} \|_{L^2(\Omega(\zeta))} = 1 \) (Theorem 3).

By (1.5) and (1.6), we see the rate of the degeneration of the function space \( L^2(\Omega(\zeta)) \), which is associated with the partial degeneration of the domain \( \Omega(\zeta) \) as \( \zeta \to 0 \). 

Our results are applicable to the investigation of some delicate behaviors and the structure of the solutions of reaction-diffusion equations or systems (such as those in [8, 9, 11, 12, 13, 16, 17, 23]) in \( \Omega(\zeta) \) with the Neumann boundary condition on \( \partial \Omega(\zeta) \). More precisely, when we reduce these equations to a finite dimensional problem in the Lyapunov–Schmidt procedure in the neighborhood of a solution or an approximate solution by using the eigenfunctions of the linearized problem, the singular behaviors of the eigenfunctions associated with the partial degeneration of \( \Omega(\zeta) \) may give rise to a difficulty in the reduced equation and thus we need to obtain some elaborate estimates of the behaviors of the eigenfunctions such as in.
Theorems 3 and 4 and by this we can get a "well-" reduced equation. The application of the results of this paper is given in [13]. In the proofs of Theorems 1 and 2, the revised version of the results obtained in [11, 12], (which is given in Proposition 3.1) will be essentially applied. In the proof of Theorem 3, some comparison functions will be used and they will be constructed in Section 4. All the functions which appear in this paper are real valued.

2. Formulation and Main Results

We specify the singularly perturbed domain $\Omega(\zeta)$ in $\mathbb{R}^n$ in the form

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta),$$

where $D_i$ ($i = 1, 2$) and $Q(\zeta)$ are defined in the following conditions where $x' = (x_2, x_3, ..., x_n) \in \mathbb{R}^{n-1}$.

(A.1) $D_1$ and $D_2$ are bounded domains with $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ in $\mathbb{R}^n$ with smooth boundaries which satisfy the following conditions for some positive constant $\zeta_* > 0$. $\overline{D}_1 \cap \{x = (x_1, x') \in \mathbb{R}^n | x_1 \leq 1, |x'| < 3\zeta_*\} = \{(1, x') \in \mathbb{R}^n | |x'| < 3\zeta_*\}$, $\overline{D}_2 \cap \{x = (x_1, x') \in \mathbb{R}^n | x_1 \geq -1, |x'| < 3\zeta_*\} = \{(-1, x') \in \mathbb{R}^n | |x'| < 3\zeta_*\}$.

(A.2) $Q(\zeta) = R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$, $R_1(\zeta) = \{(x_1, x') \in \mathbb{R}^n | 1 - 2\zeta < x_1 \leq 1, |x'| < \zeta \rho((x_1 - 1)/\zeta)\}$, $R_2(\zeta) = \{(x_1, x') \in \mathbb{R}^n | -1 \leq x_1 < -1 + 2\zeta, |x'| < \zeta \rho((-1 - x_1)/\zeta)\}$, $\Gamma(\zeta) = \{(x_1, x') \in \mathbb{R}^n | -1 + 2\zeta \leq x_1 \leq -1 - 2\zeta, \rho'\zeta < \zeta, |x'| < \zeta\}$, where $\rho \in C^0((-2, 0]) \cap C^\infty((-2, 0))$ is a positive valued monotone increasing function such that $\rho(0) = 2$, $\rho(s) = 1$ for $s \in (-2, -1]$, $\rho'(s) > 0$ for $s \in (-1, 0)$, and the inverse function of $\rho_{[(-1, 0]}$ satisfies $\lim_{\zeta \to 0} (\rho_{[(-1, 0]} / \zeta^k) = 0$ for any nonnegative integer $k$.

By these conditions $\Omega(\zeta)$ is a bounded domain in $\mathbb{R}^n$ with a $C^\infty$-boundary ($0 < \zeta < \zeta_*$). We prepare some notations for the later arguments,

$$p_1 = (1, 0, ..., 0), \quad p_2 = (-1, 0, ..., 0),$$

$$L = \bigcap_{0 < \zeta < \zeta_*} \overline{Q}(\zeta) = \{(z, 0, ..., 0) \in \mathbb{R}^n | -1 \leq z \leq 1\},$$

$$\Sigma_1(\eta) = \{(x_1, x') \in \mathbb{R}^n | x_1 > 1, |x - p_1| < \eta\},$$

$$\Sigma_2(\eta) = \{(x_1, x') \in \mathbb{R}^n | x_1 < -1, |x - p_2| < \eta\}.$$

Definition 1. Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ and $\{\Phi_k(\zeta)\}_{k=1}^\infty$ be respectively the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of the corresponding eigenfunctions which are
orthonormalized as \((\Phi_k, \phi_m, \zeta)_{L^2(\Omega(\zeta))} = \delta_{k,m} \) of the following eigenvalue problem of the Laplacian with the Neumann boundary condition
\[
\Delta \Phi + \mu \Phi = 0 \quad \text{in } \Omega(\zeta),
\]
\[
\frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \partial \Omega(\zeta),
\]
where \(\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \) (Laplacian) and \(\nu\) denotes the unit outward normal vector on \(\partial \Omega(\zeta)\).

The continuity theorem of the eigenvalue, under the smooth deformation of the domain (cf. Courant and Hilbert [5]), asserts that each \(\mu_k(\zeta) (k = 1, 2, 3, \ldots)\) varies continuously in \(\zeta \) \((0 < \zeta < \zeta_0)\), but it says nothing about \(\lim_{\zeta \to 0} \mu_k(\zeta) (k \geq 1)\). We begin with the behavior of the set of the eigenvalues \(\{\mu_k(\zeta)\}_{k=1}^\infty\) when \(\zeta \to 0\). By the arguments in J. T. Beale [3], one can characterize the set \(\{\lim_{\zeta \to 0} \mu_k(\zeta)\}_{k=1}^\infty\) by using the spectral information of \(D_1, D_2,\) and \(L\). We pose a condition (A.3) on \(D_1\) and \(D_2\) (without loss of essence of the problem), by which we can avoid some inessential and complicated arguments, and we can state the asymptotic behavior of \(\{\mu_k(\zeta)\}_{k=1}^\infty\) explicitly (Theorem 1). For completeness, we carry the proof of Theorem 1 within our formulation together with the proof of Theorem 2 in Section 5.

**Definition 2.** Let \(\{\omega_k\}_{k=1}^\infty\) be the sequence of the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problem in \(D_1 \cup D_2\).
\[
\Delta \phi + \omega \phi = 0 \quad \text{in } D_1 \cup D_2,
\]
\[
\frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial D_1 \cup \partial D_2,
\]
\((0 = \omega_1 = \omega_2 < \omega_3 \leq \cdots \to \infty)\).

**Definition 3.** We put \(\lambda_k = (k\pi/2)^2\) and \(S_k(z) = (\sin(k\pi/2))(1 - z)\) \((k \geq 1)\) which are respectively the eigenvalues and the complete system of eigenfunctions of the eigenvalue problem
\[
d^2S/dz^2 + \lambda S = 0, \quad -1 < z < 1
\]
\[
S(1) = S(-1) = 0.
\]

We assume the following condition.
\[(A.3) \quad \{\lambda_k\}_{k=1}^\infty \cap \{\omega_k\}_{k=1}^\infty = \emptyset.\]
In the following theorems in this section, we always assume (A.1)-(A.3) and the space dimension $n \geq 3$.

We characterize the set of the eigenvalues as follows.

**Theorem 1.** The set $\{ \mu_k(\zeta) \}_{k=1}^{\infty}$ can be separated as

$$\{ \mu_k(\zeta) \}_{k=1}^{\infty} = \{ \omega_k(\zeta) \}_{k=1}^{\infty} \cup \{ \lambda_k(\zeta) \}_{k=1}^{\infty},$$

(2.4)

where $\lim_{\zeta \to 0} \omega_k(\zeta) = \omega_k$, $\lim_{\zeta \to 0} \lambda_k(\zeta) = \lambda_k \ (k \geq 1)$.

The proof of Theorem 1 (especially, the way of separation in (2.4)) is carried out with the proof of Theorem 2 in Section 5. The following theorems are the main results of this paper which concern the global behaviors of the eigenfunctions.

To state Theorems 2 and 3, we denote the corresponding eigenfunctions to $\omega_k(\zeta)$ and $\lambda_k(\zeta)$, respectively, by $\phi_{k, \zeta}$ and $\psi_{k, \zeta}$. Then, according to (2.4), we have

$$\{ \Phi_{k, \zeta} \}_{k=1}^{\infty} = \{ \phi_{k, \zeta} \}_{k=1}^{\infty} \cup \{ \psi_{k, \zeta} \}_{k=1}^{\infty} .$$

(2.5)

**Remark 2.1.** The choice of the eigenfunctions is not unique; we fix the systems of eigenfunctions $\{ \Phi_{k, \zeta} \}_{k=1}^{\infty}$, $\{ \phi_{k, \zeta} \}_{k=1}^{\infty}$, and $\{ \psi_{k, \zeta} \}_{k=1}^{\infty}$ from Definition 1 and (2.5) in the following statements in this section.

**Theorem 2.** For any sequence of positive values $\{ \zeta_m \}_{m=1}^{\infty}$ such that $\lim_{m \to \infty} \zeta_m = 0$, there exists a subsequence $\{ \sigma_m \}_{m=1}^{\infty} \subset \{ \zeta_m \}_{m=1}^{\infty}$ and the complete system of the eigenfunctions $\{ \phi_k \}_{k=1}^{\infty} \subset C^\infty(\overline{D_1 \cup D_2})$ of (2.2) corresponding to $\{ \omega_k \}_{k=1}^{\infty}$ such that $(\phi_k, \phi_m)_{k=1}^{\infty}$ and the following conditions hold for each $k \geq 1$,

$$\lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |\phi_k, \sigma_m(x) - \phi_k(x)| = 0,$$

(2.6)

$$\lim_{m \to \infty} \sup_{x = (\tau_1, \tau_2) \in Q(\sigma_m)} |\phi_k, \sigma_m(x_1, x') - V_k(x_1)| = 0,$$

(2.7)

$$\lim_{m \to \infty} \sup_{x = (\tau_1, \tau_2) \in Q(\sigma_m)} |d^{1/2}_{n-1} \sigma_m^{(n-1)/2} \psi_k, \sigma_m(x_1, x') - S_k(x_1)| = 0 \text{ or}$$

(2.8)

$$\lim_{m \to \infty} \sup_{x = (\tau_1, \tau_2) \in Q(\sigma_m)} |d^{1/2}_{n-1} \sigma_m^{(n-1)/2} \psi_k, \sigma_m(x_1, x') + S_k(x_1)| = 0.
$$

Here we denoted by $V_k$ the unique solution of the following two point boundary value problem (2.9) for each $k = 1, 2, 3, \ldots$,

$$d^2 V/dz^2 + \omega_k V = 0, \quad -1 < z < 1,$$

$$V(1) = \phi_k(p_1), \quad V(-1) = \phi_k(p_2),$$

(2.9)
and \( d_{n-1} = \pi^{(n-1)/2}/\Gamma((n+1)/2) \) which is the \((n-1)\)-dimensional Lebesgue measure of the unit ball in \( \mathbb{R}^{n-1} \).

Next we give a rather elaborate estimate of \( \psi_{k, \zeta} \) in \( D_1 \cup D_2 \).

**Theorem 3.** For any natural number \( k \geq 1 \), there exists a positive constant \( \eta_*(k) > 0 \) such that

\[
0 < \liminf_{\zeta \to 0} \inf_{x \in R_{l}(\zeta) \cup \Sigma_{l}(3\zeta)} \zeta^{(n-3)/2} |\psi_{k, \zeta}(x)| 
\leq \limsup_{\zeta \to 0} \sup_{x \in R_{l}(\zeta) \cup \Sigma_{l}(3\zeta)} \zeta^{(n-3)/2} |\psi_{k, \zeta}(x)| < +\infty, \tag{2.10}
\]

\[
0 < \liminf_{\zeta \to 0} \inf_{x \in \Sigma_{l}(\eta) \setminus \Sigma_{l}(3\zeta)} \zeta^{-(n-1)/2} |x - p_{l}|^{-2} |\psi_{k, \zeta}(x)| 
\leq \limsup_{\zeta \to 0} \sup_{x \in \Sigma_{l}(\eta) \setminus \Sigma_{l}(3\zeta)} \zeta^{-(n-1)/2} |x - p_{l}|^{-2} |\psi_{k, \zeta}(x)| < +\infty, \tag{2.11}
\]

\[
0 < \liminf_{\zeta \to 0} \sup_{x \in D_{l} \setminus \Sigma_{l}(\eta)} \zeta^{-(n-1)/2} |\psi_{k, \zeta}(x)| 
\leq \limsup_{\zeta \to 0} \sup_{x \in D_{l} \setminus \Sigma_{l}(\eta)} \zeta^{-(n-1)/2} |\psi_{k, \zeta}(x)| < +\infty, \tag{2.12}
\]

\[
0 < \liminf_{\zeta \to 0} \zeta^{-(n-1)/2} \|\psi_{k, \zeta}\|_{L^{1}(\Omega(\zeta))} 
\leq \limsup_{\zeta \to 0} \zeta^{-(n-1)/2} \|\psi_{k, \zeta}\|_{L^{1}(\Omega(\zeta))} = +\infty, \tag{2.13}
\]

\[
\lim_{\zeta \to 0} \delta_{n-1}^{1/2} \zeta^{(n-1)/2} \|\psi_{k, \zeta}\|_{L^{\infty}(\Omega(\zeta))} = 1 \tag{2.14}
\]

for any \( \eta \in (0, \eta_*(k)) \) and \( i = 1, 2 \).

We remark that \( \lim_{\zeta \to 0} \|\psi_{k, \zeta}\|_{L^{1}(\Omega(\zeta))} = 0 \) holds while \( \|\psi_{k, \zeta}\|_{L^{2}(\Omega(\zeta))} = 1 \) \((k \geq 1)\).

**Corollary.** For any natural number \( k \geq 1 \), there exist positive constants \( \zeta_0(k), \delta_{1}(k), \delta_{2}(k) \) such that

\[
\frac{\delta_{1}(k) \zeta^{(n-1)/2}}{|x - p_{l}|^{-2}} \leq |\psi_{k, \zeta}(x)| \leq \frac{\delta_{2}(k) \zeta^{(n-1)/2}}{|x - p_{l}|^{-2}} \quad (x \in \Sigma_{l}(\eta_*(k)) \setminus \Sigma_{l}(3\zeta))
\]

holds for \( \zeta \in (0, \zeta_0(k)) \) and \( i = 1, 2 \).
3. Preliminaries

We can obtain Theorems 1 and 2 by applying the methods developed in [11, 12] and further additional arguments. We prepare the revised version of [12, Theorem 2] in Proposition 3.1. But we do not give its proof, for it is almost equal to the arguments in [11, 12] except for some inessential changes. We also mention some basic a priori estimates of the solutions of the Poisson equation under our situations (Proposition 3.2) and the theorem removable singularity on the boundary (Proposition 3.3). We often use these to argue about the compactness or convergence of a family of some solutions in some portion of the domain.

We consider the following equation:

\[ Au + f_\zeta(x, v) = H_\zeta(x) \quad \text{in } \Omega(\zeta), \]
\[ \frac{\partial v}{\partial v} = 0 \quad \text{on } \partial\Omega(\zeta). \]

Here the nonlinear terms \( f_\zeta \in C^\infty(\Omega(\zeta) \times \mathbb{R}) \) and \( H_\zeta \) are given as

\[ f_\zeta(x, \zeta) = h_\zeta(x) g(\zeta) \quad (0 < \zeta < \zeta_*), \]

where \( h_\zeta, H_\zeta, \) and \( g \) satisfy the conditions,

(i) \( g \in C^\infty(\mathbb{R}), \limsup_{\zeta \to -\infty} g(\zeta) < 0, \liminf_{\zeta \to -\infty} g(\zeta) > 0, \)

(ii) \( h_\zeta, H_\zeta \in C^\infty(\Omega(\zeta)) \) and there exist a sequence of positive values \( \{\zeta_m\}_{m=1}^\infty \) and \( h, H \in C^\infty([1, 1]) \) such that
\[ h_\zeta(x) > 0 \text{ in } \Omega(\zeta), \quad \lim_{m \to \infty} \zeta_m = 0, \]

\[ \lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |h_{\zeta_m}(x) - h(x)| = 0 \quad (i = 1, 2), \]

\[ \lim_{m \to \infty} \sup_{x \in Q(\zeta_m)} |h_{\zeta_m}(x_1, x') - \hat{h}(x_1)| = 0, \]

\[ \lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |H_{\zeta_m}(x) - H(x)| = 0 \quad (i = 1, 2), \]

\[ \lim_{m \to \infty} \sup_{x \in Q(\zeta_m)} |H_{\zeta_m}(x_1, x') - \hat{H}(x_1)| = 0. \]

Remark 3.1. \( h(1) = h(p_1), \quad \hat{H}(1) = H(p_1), \quad \hat{h}(-1) = h(p_2), \quad \hat{H}(-1) = H(p_1) \) are automatically satisfied by (ii).

In the above situation, we have the following.

**Proposition 3.1.** Assume \( n \geq 3 \). For each \( \zeta \in (0, \zeta_\ast) \), let \( v_\zeta \) be any solution of (3.1). Then there exist a subsequence \( \{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty \) and functions \( w \in C^\infty(D_1 \cup D_2) \) and \( V \in C^\infty([-1, 1]) \) such that the following conditions are satisfied:

\[ \Delta w + h(x) g(w) = H(x) \quad \text{in } D_1 \cup D_2, \quad \text{(3.2)} \]

\[ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D_1 \cup \partial D_2, \]

\[ \frac{d^2 V}{dz^2} + h(z) g(V) = \hat{H}(z) \quad \text{for } z \in (-1, 1), \quad \text{(3.3)} \]

\[ V(1) = w(p_1), \quad V(-1) = w(p_2), \]

\[ \lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |v_{\sigma_m}(x) - w(x)| = 0, \quad \text{(3.4)} \]

\[ \lim_{m \to \infty} \sup_{x \in Q(\sigma_m)} |v_{\sigma_m}(x_1, x') - V(x_1)| = 0. \quad \text{(3.5)} \]

Remark 3.2. In the case that \( h_\zeta \equiv 1, \quad H_\zeta \equiv 0 \), Proposition 3.1 is equal to [12, Theorem 2].

Remark 3.3. If there is a priori bound to \( v_\zeta \) in the sense of sup-norm, i.e., there is a constant \( c > 0 \) such that

\[ (i)' \quad \|v_\zeta\|_{L^\infty(\Omega(\zeta))} \leq c \quad (0 < \zeta < \zeta_\ast), \]

we do not need the assumption (i), because we can modify \( g(\zeta) \) for large values of \( \zeta \) so that (i) holds.

Let \( D \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial D \) and \( G \) be a subset of \( \partial D \). We consider

\[ \Delta u = f \text{ in } D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \setminus G. \quad \text{(3.6)} \]
PROPOSITION 3.2. Let $\alpha \in (0, 1)$ and $p \in (1, \infty)$ be constants. For any positive constants $\eta_1$ and $\eta_2$ such that $0 < \eta_2 < \eta_1$ and for any integer $k \geq 0$, there exist three positive constants $c_1 = c_1(\eta_1, \eta_2)$, $c_2 = c_2(\eta_1, \eta_2, k)$, and $c_3 = c_3(\eta_1, \eta_2, k)$ such that

\[ \|u\|_{C^{k+1}(\mathcal{D}\setminus \Sigma(\eta_1))} \leq c_1 \left\{ \|u\|_{C^0(\mathcal{D}\setminus \Sigma(\eta_2))} + \|f\|_{C^0(\mathcal{D}\setminus \Sigma(\eta_2))} \right\}, \]

\[ \|u\|_{C^{k+2}(\mathcal{D}\setminus \Sigma(\eta_1))} \leq c_2 \left\{ \|u\|_{C^0(\mathcal{D}\setminus \Sigma(\eta_2))} + \|f\|_{C^0(\mathcal{D}\setminus \Sigma(\eta_2))} \right\}, \]

\[ \|u\|_{W^{k+2,p}(\mathcal{D}\setminus \Sigma(\eta_1))} \leq c_3 \left\{ \|u\|_{L^p(\mathcal{D}\setminus \Sigma(\eta_2))} + \|f\|_{W^{k,p}(\mathcal{D}\setminus \Sigma(\eta_2))} \right\}, \]

for any $u$ and $f$ which satisfy (3.6). Here we have defined the set $\Sigma(\eta) = \{x \in D \mid \text{dis}(x, G) < \eta\}$ for $\eta > 0$.

For the definitions of the norms, see [6]. These inequalities can be proved in the same way as the classical Schauder estimates in D. Gilbarg and N. S. Trudinger [6; Chaps. 4, 6, 8, 9].

PROPOSITION 3.3. Let $u$ belong to $H^1(\Sigma) \cup (L^\infty(\Sigma) \cap C^2(\Sigma \setminus \{0\}))$ which satisfies the following equation (3.10), where the set $\Sigma = \{(x_1, x') \in \mathbb{R}^n \mid x_1 > 0, |x| < c\}$ ($c > 0$) and $\xi$ is an arbitrary constant

\[ \Delta u + \xi u = 0 \text{ in } \Sigma, \quad \partial u / \partial x_1 = 0 \text{ on } \partial \Sigma \cap \{x_1 = 0\} \setminus \{0\}. \]  

Then $u \in C^\infty(\Sigma \cup \{(0, x') \in \mathbb{R}^n \mid |x'| < c\})$. In particular, the boundary condition in (3.10) is satisfied at $x = 0$.

Reflecting with respect to the hyperplane $x_1 = 0$, we extend $u$ as a solution of the same equation on the domain $\{0 < |x| < c\}$ and the problem is reduced to the removability of an interior isolated singularity.

4. CONSTRUCTION OF AUXILIARY FUNCTIONS

In this section we prepare some notations and important auxiliary functions which will be used in the proof of the theorems.

We define the following sets for a positive constant $l$ such that $0 < l < 3\zeta*$:

\[ \Gamma'(\zeta) = \{(x_1, x') \in \Gamma(\zeta) \mid 1 - l < x_1 \leq 1 - 2\zeta'\}, \]

\[ \gamma_1(\zeta) = \{(x_1, x') \in \Gamma(\zeta) \mid x_1 = 1 - l\}, \]

\[ \gamma_3(\zeta) = \partial \Sigma_1(3\zeta) \setminus \partial D_1, \]

\[ \gamma_4 = \partial \Sigma_1(l) \setminus \partial D_1, \]

\[ \mathcal{O}(\zeta) = \Sigma_1(l), \]

$n^+$ and $n^-$ are the unit normal vectors on $\gamma_3(\zeta) \cup \gamma_3(\zeta)$ which are, respectively, outward and inward about the set $R_1(\zeta) \cup \Sigma_1(3\zeta)$. 

In the proof of Theorem 3, we need some comparison functions in the moving portion \( \mathcal{Q}(\zeta) \) which is the neighborhood of the point \( p_1 \). Hereafter in this section we construct some auxiliary functions. We use two radially symmetric solutions \( \phi_1, \phi_2 \) of the following equation which are explicitly expressed as follows in (i) and (ii):

\[
A\phi + M\phi = 0.
\] (4.1)

(i) In the case \( n \geq 3 \), odd,

\[
\phi_1(r) = (M^{1/2}r)^{-(n-2)/2} J_{(n-2)/2}(M^{1/2}r),
\]

\[
\phi_2(r) = (-1)^{[n/2]} (M^{1/2}r)^{-(n-2)/2} J_{-(n-2)/2}(M^{1/2}r),
\]

where \([m]\) is the largest integer which does not exceed \( m \).

(ii) In the case \( n \geq 2 \), even,

\[
\phi_1(r) = (M^{1/2}r)^{-(n-2)/2} J_{(n-2)/2}(M^{1/2}r),
\]

\[
\phi_2(r) = - (M^{1/2}r)^{-(n-2)/2} Y_{(n-2)/2}(M^{1/2}r).
\]

Here \( J_\nu \) and \( Y_\nu \) are the Bessel function and the Neumann function,

\[
J_\nu(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(v+m+1) \Gamma(m+1)} (r/2)^{\nu+2m},
\]
\[
Y_N(r) = \frac{2}{\pi} J_N(r) \left( c + \log \frac{r}{2} \right) - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (N+m)!} \left( \sum_{p=1}^{m} \frac{1}{p} + \sum_{p=1}^{N} \frac{1}{p} \right) (r/2)^{N+2m} - \frac{1}{\pi} \sum_{m=0}^{N-1} \frac{(N-1-m)!}{m!} (r/2)^{-N+2m},
\]

where \( c \) is the Euler constant (cf. [5]).

We use the following properties of \( \phi_1 \) and \( \phi_2 \).

**Lemma 4.1.** Assume \( n \geq 3 \). For any positive constant \( M \), there exist positive constants \( c_0(M) \), \( c_1(M) \), and \( c_2(M) \) such that \( \phi_1 \) and \( \phi_2 \) are expressed as

\[
\phi_1(r) = A_1(r), \quad \phi_2(r) = \frac{A_2(r)}{r^{n-2}},
\]

where \( A_1 \in C^\infty([0, \infty)) \) and \( A_2 \in C^\infty((0, \infty)) \) satisfy the following properties,

\[
0 < c_1(M) \leq A_i(r) \leq c_2(M) \quad \text{for} \quad r \in (0, c_0(M)],
\]

\[
A_i'(r) < 0 \quad \text{for} \quad r \in (0, c_0(M)], \quad \lim_{r \to 0} A_i'(r) = 0 \quad \text{for} \quad i = 1, 2.
\]

**Definition 4.** Let \( q_\zeta^{(M)} \) and \( s_\zeta^{(M)} \) be the solutions of the following boundary value problems in \( R_1(\zeta) \cup \Sigma_1(3\zeta) \), respectively,

\[
\begin{align*}
\Delta q + Mq &= 0 \quad \text{in} \quad R_1(\zeta) \cup \Sigma_1(3\zeta), \\
\partial q/\partial n &= 0 \quad \text{on} \quad \partial(R_1(\zeta) \cup \Sigma_1(3\zeta)) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta)), \\
q &= \zeta \quad \text{on} \quad \gamma_2(\zeta), \\
A_i s + M s &= 0 \quad \text{in} \quad R_1(\zeta) \cup \Sigma_1(3\zeta), \\
\partial s/\partial n &= 0 \quad \text{on} \quad \partial(R_1(\zeta) \cup \Sigma_1(3\zeta)) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta)), \\
s &= \zeta \quad \text{on} \quad \gamma_2(\zeta) \cup \gamma_3(\zeta).
\end{align*}
\]

**Lemma 4.2.** There exist positive constants \( \alpha, \beta, \zeta_0 \) such that

\[
0 \leq q_\zeta^{(M)}(x) \leq \zeta, \quad \zeta \leq s_\zeta^{(M)}(x) \leq 2\zeta \quad \text{in} \quad R_1(\zeta) \cup \Sigma_1(3\zeta),
\]

\[
\alpha \leq \frac{\partial q_\zeta^{(M)}}{\partial n} \leq \beta \quad \text{on} \quad \gamma_2(\zeta), \quad \alpha \leq -\frac{\partial q_\zeta^{(M)}}{\partial n} \leq \beta \quad \text{on} \quad \gamma_3(\zeta) \quad \text{for} \quad \zeta \in (0, \zeta_0),
\]

\[
\limsup_{\zeta \to 0} \frac{1}{\zeta^2} \sup_{x \in \gamma_2(\zeta) \cup \gamma_3(\zeta)} \left| \frac{\partial s_\zeta^{(M)}}{\partial n}(x) \right| < \infty.
\]
Sketch of proof of Lemma 4.2. We change the scale of the variable and the functions $q_\zeta^{(M)}$ and $s_\zeta^{(M)}$ around the point $p_1$ as

$$x - p_1 = \zeta(y - p_1),$$

$$\bar{q}_\zeta(y) = \frac{1}{\zeta} q_\zeta^{(M)}(p_1 + \zeta(y - p_1)), \quad (4.4)$$

$$\bar{s}_\zeta(y) = \frac{1}{\zeta} (s_\zeta^{(M)}(p_1 + \zeta(y - p_1)) - \zeta),$$

where $\bar{q}_\zeta$ and $\bar{s}_\zeta$ are defined on the fixed domain $E = R_1(1) \cup \Sigma_1(3)$. By letting $\zeta \to 0$ in the transported equations defined on $E$ from (4.2) and (4.3), we can obtain the asymptotic behaviors of $\bar{q}_\zeta$ and $\bar{s}_\zeta$ elaborately. The conclusion of Lemma 4.2 can be obtained through this procedure.

Now we construct some comparison functions, $\varphi_\zeta^+, \tilde{\varphi}_\zeta^+, \varphi_\zeta^-, \tilde{\varphi}_\zeta^-$. 

\textbf{Definition 5.}

$$\varphi_\zeta^+(x) = \begin{cases} 
\frac{\zeta^n - 1}{|x - p_1|^{n-2}} & \text{for } x \in \Sigma_1(1) \setminus \Sigma_1(3\zeta), \\
\frac{1}{2 \cdot 3^{n-1}} \left( (n - 2) A_2(3\zeta) - 3\zeta A_2'(3\zeta) \right) q_\zeta^{(M)}(x) \\
+ \frac{A_2(3\zeta)}{3^{n-2}} s_\zeta^{(M)}(x) & \text{for } x \in \Sigma_1(3\zeta) \cup R_1(\zeta), \\
\frac{\alpha}{4 \cdot 3^{n-1} \beta M^{1/2}} \left( (n - 2) A_2(3\zeta) - 3\zeta A_2'(3\zeta) \right) \\
\times \sin M^{1/2}(-x_1 + (1 - 2\zeta)) \\
+ \frac{\zeta}{3^{n-1}} \left\{ \frac{1}{2\beta} \left( (n - 2) A_2(3\zeta) - 3\zeta A_2'(3\zeta) \right) + 3A_2(3\zeta) \right\} \\
\times \cos M^{1/2}(-x_1 + (1 - 2\zeta)) & \text{for } x \in \Gamma'(\zeta). \\
\end{cases}$$

$$\tilde{\varphi}_\zeta^+(x) = \begin{cases} 
A_1(|x - p_1|) & \text{for } x \in \Sigma_1(1) \setminus \Sigma_1(3\zeta), \\
- \frac{1}{2\beta} A_1(3\zeta) q_\zeta(x) + \frac{A_1(3\zeta)}{\zeta} s_\zeta^{(M)}(x) & \text{for } x \in R_1(\zeta) \cup \Sigma_1(3\zeta) \\
- \frac{\alpha A_1'(3\zeta)}{4\beta M^{1/2}} \sin M^{1/2}(-x_1 + (1 - 2\zeta)) + \left( A_1(3\zeta) - \frac{\zeta A_1'(3\zeta)}{2\beta} \right) \\
\times \cos M^{1/2}(-x_1 + (1 - 2\zeta)) & \text{for } x \in \Gamma'(\zeta). \\
\end{cases}$$
\begin{align*}
\varphi_\zeta^-(x) &= \begin{cases} 
\frac{\zeta^{n-1}}{|x-p_1|^{n-2}} \frac{A_2(|x-p_1|)}{|x-p_1|^{n-2}} & \text{for } x \in \Sigma_1(I) \setminus \Sigma_1(3\zeta), \\
\frac{2}{3^{n-1}} (n-2) A_2(3\zeta) - 3\xi A'_2(3\zeta) & q_\xi^{(M)}(x) \\
+ \frac{A_2(3\zeta)}{3^{n-2}} s_\xi^{(M)}(x) & \text{for } x \in \Sigma_1(3\zeta) \cup R_1(\zeta),
\end{cases} \\
\phi_\zeta^-(x) &= \begin{cases} 
\frac{4\beta}{3^{n-1} \alpha M^{1/2}} ((n-2) A_2(3\zeta) - 3\xi A'_2(3\zeta)) \\
\times \sin M^{1/2}(-x_1 + (1 - 2\zeta)) & \text{for } x \in \Gamma'(\zeta),
\end{cases} \\
\phi_\zeta^+(x) &= \begin{cases} 
\frac{A_1(|x-p_1|)}{\zeta} & \text{for } x \in \Sigma_1(I) \setminus \Sigma_1(3\zeta), \\
-\frac{2}{\alpha} A_1(3\zeta) q_\xi^{(M)}(x) - \frac{A_1(3\zeta)}{\zeta} s_\xi^{(M)}(x) & \text{for } x \in R_1(\zeta) \cup \Sigma_1(3\zeta),
\end{cases} \\
\phi_\zeta^+(x) &= \begin{cases} 
-\frac{4\beta A_1'(3\zeta)}{\alpha M^{1/2}} \sin M^{1/2}(-x_1 + (1 - 2\zeta)) + \left( A_1(3\zeta) - \frac{2\xi A_1'(3\zeta)}{\alpha} \right) \\
\times \cos M^{1/2}(-x_1 + (1 - 2\zeta)) & \text{for } x \in \Gamma'(\zeta).
\end{cases}
\end{align*}

The following properties can be easily checked from the construction of \( \varphi_\zeta^+, \phi_\zeta^+, \phi_\zeta^-, \phi_\zeta^- \) and Lemmas 4.1 and 4.2.

**Lemma 4.3.** The functions \( \varphi_\zeta^+, \phi_\zeta^+, \phi_\zeta^-, \phi_\zeta^- \) are continuous in \( \partial(\zeta) \) and smooth in \( \partial(\zeta) \setminus (\gamma_2(\zeta) \cup \gamma_3(\gamma)) \) and satisfy
\[
\begin{align*}
\Delta \varphi_\zeta^+ + M \varphi_\zeta^+ &= 0, & \Delta \phi_\zeta^+ + M \phi_\zeta^+ &= 0, & \text{in } \partial(\zeta) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta)), \\
\Delta \varphi_\zeta^- + M \varphi_\zeta^- &= 0 & \Delta \phi_\zeta^- + M \phi_\zeta^- &= 0, & \text{in } \partial(\zeta) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta)),
\end{align*}
\]
\[
\begin{align*}
\partial \varphi_\zeta^+/\partial \nu &= 0, & \partial \phi_\zeta^+/\partial \nu &= 0, & \partial \varphi_\zeta^-/\partial \nu &= 0, \\
\partial \phi_\zeta^-/\partial \nu &= 0 & \text{on } \partial(\zeta) \setminus (\gamma_1(\zeta) \cup \gamma_4)
\end{align*}
\]

**Lemma 4.4.**
\[
\begin{align*}
\frac{\partial \varphi_\zeta^+}{\partial n^+} + \frac{\partial \varphi_\zeta^-}{\partial n^-} < 0, & \quad \frac{\partial \phi_\zeta^+}{\partial n^+} + \frac{\partial \phi_\zeta^-}{\partial n^-} < 0 & \text{for } x \in \gamma_2(\zeta) \cup \gamma_3(\zeta), \\
\frac{\partial \varphi_\zeta^+}{\partial n^+} + \frac{\partial \varphi_\zeta^-}{\partial n^-} > 0, & \quad \frac{\partial \phi_\zeta^+}{\partial n^+} + \frac{\partial \phi_\zeta^-}{\partial n^-} > 0 & \text{for } x \in \gamma_2(\zeta) \cup \gamma_3(\zeta),
\end{align*}
\]
where $\varphi^+_\xi$ and $\tilde{\varphi}^+_\xi$ (resp. $\varphi^-_\xi$ and $\tilde{\varphi}^-_\xi$) play the roles of upper solutions (resp. lower solutions). The above auxiliary functions depend on $M$ and their domain of definition $\Omega(\xi)$ depends on $l$. In the proof of Theorem 3, we use $\varphi^+_\xi$, $\tilde{\varphi}^+_\xi$, $\varphi^-_\xi$, $\tilde{\varphi}^-_\xi$ for adequately large $M > 0$ and small $l > 0$, respectively.

5. PROOF OF THEOREMS 1 AND 2

In this section we prove Theorems 1 and 2 together. By the comparison theorems in Courant and Hilbert [5], $\mu_k(\xi)$ is smaller than the $k$th eigenvalue of $-\Delta$ in $\Omega(\xi)$ for the Dirichlet Boundary condition on $\partial \Omega(\xi)$, which is smaller than the $k$th eigenvalue $\omega'_k$ of $-\Delta$ in $D_1 \cup D_2$ for the Dirichlet boundary condition on $\partial D_1 \cup \partial D_2$. Thus we have $0 \leq \mu_k(\xi) \leq \omega'_k$ for $k \geq 1$ and $\xi \in (0, \zeta^*)$, and so we have $0 \leq \liminf_{\xi \to 0} \mu_k(\xi) \leq \limsup_{\xi \to 0} \mu_k(\xi) \leq \omega'_k$ for any $k \geq 1$.

For $\{\Phi_{k, \xi}\}_{k=1}^{\infty}$ in Definition 1, we define two subsets $K_1$ and $K_2$ of the set of natural numbers $\mathbb{N}$ by the conditions

$$\lim_{n \to 0} \|\Phi_{k, \xi}\|_{L^\infty(\Omega(\xi))} = \infty \quad \text{for } k \in K_1,$$  

(5.1)

$$\limsup_{n \to 0} \|\Phi_{k, \xi}\|_{L^\infty(\Omega(\xi))} < \infty \quad \text{for } k \in K_2.$$  

(5.2)

Hereafter we put $m_i = \#(K_i) \ (0 \leq m_i \leq \infty)$ and denote by $n_i(k)$ the $k$th element of $K_i$, i.e., $K_i = \{n_i(k)\}_{k=1}^{m_i}$ where $n_i(k) < n_i(k+1)$ for $i = 1, 2$. In the latter part of this section (Lemma 5.4) we will prove that $m_1 = m_2 = \infty$ and $K_1 \cup K_2 = \mathbb{N}$.

Put $\Phi'_{k, \xi}(x) = \Phi_{k, \xi}(x)/\|\Phi_{k, \xi}\|_{L^\infty(\Omega(\xi))}$ and then we have the following.

**LEMMA 5.1.** For any sequence of positive values $\{\xi_m\}_{m=1}^{\infty}$ such that $\lim_{m \to \infty} \xi_m = 0$, there exists a subsequence $\{\sigma_m\}_{m=1}^{\infty} \subset \{\xi_m\}_{m=1}^{\infty}$ such that $\lim_{m \to \infty} \mu_k(\sigma_m)$ exists for $k \in K_1$ and belongs to $\{\lambda_k\}_{k=1}^{\infty}$ and $\lambda_j(k) < \lambda_{j+1}(k)$ hold for $1 \leq k \leq m_1 - 1$ where the number $j(k)$ is defined by the condition $\lambda_j(k) = \lim_{m \to \infty} \mu_n(k)(\sigma_m)$ $(1 \leq k \leq m_1)$ and then $\liminf_{\xi \to 0} \mu_n(k)(\xi) \geq \lambda_k$ $(1 \leq k \leq m_1)$ follows. The corresponding eigenfunctions satisfy the following conditions:

$$\lim_{\xi \to 0} \|\Phi_{n(k), \xi}\|_{L^2(D_1 \cup D_2)} = 0 \quad (1 \leq k \leq m_1),$$

(5.3)

$$\lim_{\xi \to 0} \|\Phi'_{n(k), \xi}\|_{L^\infty(D_1 \cup D_2)} = 0 \quad (1 \leq k \leq m_1).$$  

(5.4)
\[
\lim_{m \to \infty} \sup_{x \in \Omega(\sigma_m)} |\Phi'_{n_1(k), \sigma_m}(x_1, x') - S_{j(k)}(x_1)| = 0 \quad (1 \leq k \leq m), \tag{5.5}
\]

\[
\lim_{m \to \infty} \sup_{x \in \Omega(\sigma_m)} |\Phi'_{n_1(k), \sigma_m}(x_1, x') + S_{j(k)}(x_1)| = 0
\]

where \( S_j(z) = \sin(j\pi/2)(1 - z) \) \((j \geq 1)\).

**Proof of Lemma 5.1.** Since, for \( k \geq 1 \) and \( \zeta \in (0, \zeta_*), \)

\[
\int_{\Omega(\zeta)} |\nabla \Phi_{k, \zeta}|^2 \, dx = \mu_k(\zeta) \int_{\Omega(\zeta)} |\Phi_{k, \zeta}|^2 \, dx = \mu_k(\zeta) \leq \omega_k, \tag{5.6}
\]

we have

\[
\lim_{m \to \infty} \| \Phi'_{n_1(k), \zeta_m} \|_{H^1(\Omega(\zeta_m))} = 0 \quad \text{for} \quad 1 \leq k \leq m_1, \tag{5.7}
\]

while \( \| \Phi'_{k, \zeta} \|_{L^2(\Omega(\zeta))} = 1 \) for \( k \geq 1. \)

First we choose a subsequence \( \{\zeta''_m\}_{m=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty} \) such that \( \lim_{m \to \infty} \mu_{n_1(k)}(\zeta'_m) \) exists \((= \xi_k)\). Applying Proposition 3.1 to the following equation, for \( \zeta \in \{\zeta''_m\} \) with (5.7),

\[
\Delta \Phi'_{n_1(k), \zeta} + \mu_{n_1(k)}(\zeta) \Phi'_{n_1(k), \zeta} = 0 \quad \text{in} \quad \Omega(\zeta),
\]

\[
\partial \Phi'_{n_1(k), \zeta}/\partial n = 0 \quad \text{on} \quad \partial \Omega(\zeta),
\]

\[
|\Phi'_{n_1(k), \zeta}(x)| \leq 1 \quad \text{in} \quad \Omega(\zeta), \tag{5.8}
\]

we conclude that

\[
\lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |\Phi'_{n_1(k), \zeta_m}(x)| = 0, \tag{5.9}
\]

and that there exists a subsequence \( \{\sigma_m\}_{m=1}^{\infty} \) and a function \( T_k \) in \( C^\infty([-1, 1]) \) which satisfies

\[
d^2T_k/dz^2 + \xi_k T_k = 0, \quad -1 < z < 1,
\]

\[
T_k(\pm 1) = 0, \quad \sup_{-1 < z < 1} |T_k(z)| = 1, \tag{5.10}
\]

From (5.10), \( T_k \) turns out to be some eigenfunction of (2.3) and we conclude \( \xi_k \in \{\lambda_k\}_{k=1}^{\infty} \). We repeat the above argument for \( k = 1, 2, 3, \ldots \) and apply the method of the diagonal argument and we obtain the subsequence \( \{\sigma_m\}_{m=1}^{\infty} \) common to all \( k \geq 1. \) Equation (5.4) can be obtained in the above argument from the arbitrariness of \( \{\zeta_m\} \) and (5.8). Next we will
prove (5.3). Assume the contrary to (5.3), that is, there exists a subsequence \( \{ \kappa_m \}_{m=1}^{\infty} \) such that

\[
\lim_{m \to \infty} \inf \| \Phi_{\kappa_m} \|_{L^2(D_1 \cup D_2)} > 0 \quad \text{for some } 1 \leq k \leq m_1.
\]

By (5.6), we obtain a convergent subsequence of \( \{ \Phi_{\kappa_m} \}_{m=1}^{\infty} \) in \( L^2(D_1 \cup D_2) \) by the Rellich theorem; i.e., there exist a subsequence \( \{ \kappa_m' \}_{m=1}^{\infty} \subset \{ \kappa_m \}_{m=1}^{\infty} \) and \( \Phi \in H^1(D_1 \cup D_2) \) such that

\[
\lim_{m \to \infty} \mu_{\kappa_m}(\xi) = \xi \quad \text{(exists and belongs to } \{ \lambda_k \}_{k=1}^{\infty})
\]

and

\[
\lim_{m \to \infty} \| \Phi_{\kappa_m} - \Phi \|_{L^2(D_1 \cup D_2)} = 0.
\]  

(5.11)

Applying (3.9) in Proposition 3.2 to (5.8) for \( \zeta \in \{ \kappa_m' \} \) with \( G = \{ p_i \}, D = D_1, p = 2 \), repeatedly, we obtain the compactness of \( \{ \Phi_{\kappa_m} \}_{m=1}^{\infty} \) in \( C^{\infty}(\bigcup_{j=1}^{m} (D_j \setminus \Sigma_j(\eta))) \) for any \( \eta > 0 \) and we see

\[
\Phi \in C^{\infty}(D_1 \cup D_2), \quad \Phi \equiv 0,
\]

\[
A \Phi + \zeta \Phi = 0 \text{ in } D_1 \cup D_2, \quad \partial\Phi / \partial v = 0 \text{ on } (\partial D_1 \setminus \{ p_1 \}) \cup (\partial D_2 \setminus \{ p_2 \}).
\]  

(5.12)

From \( \Phi \in H^1(D_1 \cup D_2) \) and the property of the removable singularity Proposition 3.3, the Neumann boundary condition in (5.12) can be extended up to \( p_1 \) and \( p_2 \). Thus \( \Phi \) turns out to be an eigenfunction of (2.2) and we conclude that \( \xi \in \{ \omega_k \}_{k=1}^{\infty} \) and this contradicts the assumption (A.3) and (5.11). This concludes (5.3). Next we prove \( \xi_{\kappa_m}(k_1) < \xi_{\kappa_m}(k_2) \) for \( 1 \leq k_1 < k_2 \leq m_1 \). By the orthonormality and (5.3), we have, for \( j_1, j_2 \in K_1 \),

\[
\int_{Q(\sigma_m)} \Phi_{j_1, \sigma_m} \Phi_{j_2, \sigma_m} dx - \delta_{j_1, j_2} \int_{D_1 \cup D_2} \Phi_{j_1, \sigma_m} \Phi_{j_2, \sigma_m} dx \leq \| \Phi_{j_1, \sigma_m} \|_{L^2(D_1 \cup D_2)} \| \Phi_{j_2, \sigma_m} \|_{L^2(D_1 \cup D_2)} \to 0 \quad \text{as } m \to \infty \) (5.13)
\]

and we have, from (5.10),

\[
d_{n-1}^{1/2} \sigma_m^{(n-1)/2} \| \Phi_{j_1, \sigma_m} \|_{L^2(Q(\sigma_m))} \delta_{n-1}^{1/2} \sigma_m^{(n-1)/2} \| \Phi_{j_2, \sigma_m} \|_{L^2(Q(\sigma_m))} \times
\]

\[
\int_{Q(\sigma_m)} (\Phi'_{j_1, \sigma_m} / d_{n-1}^{1/2} \sigma_m^{(n-1)/2}) (\Phi'_{j_2, \sigma_m} / d_{n-1}^{1/2} \sigma_m^{(n-1)/2}) dx \to \delta_{n, j_1} \quad \text{as } m \to \infty .
\]  

(5.14)

By using (5.10), we easily see

\[
\lim_{m \to \infty} \int_{Q(\sigma_m)} \Phi'_{j, \sigma_m}(x)^2 / d_{n-1} \sigma_m^{(n-1)} dx = 1 \quad (j \in K_1)
\]
and by (5.14) for \( j_1 = j_2 = j \in K_1 \), we have
\[
\lim_{m \to \infty} d_{n-j}^{1/2} \sigma_{m-1}^{(m-1)/2} \| \Phi_{j, \sigma_{m}} \|_{L^2(\Omega(\sigma_{m}))} = 1 \quad (j \in K_1).
\]

Letting \( m \to \infty \) in (5.14), we obtain for \( j = n_1(k) \), \( 1 \leq k \leq m_1 \),
\[
\int_{-1}^{1} T_{k_1}(z) T_{k_2}(z) \, dz = \delta_{k_1, k_2}, \quad 1 \leq k_1, k_2 \leq m_1.
\]

Therefore we conclude \( \tilde{\xi}_{k_1} < \tilde{\xi}_{k_2} \) for \( 1 \leq k_1 < k_2 \leq m_1 \), because each eigenvalue \( \lambda_k \) in (2.3) is simple. Define \( j(k) \) by \( \lambda_{j(k)} = \tilde{\xi}_k (1 \leq k \leq m_1) \). From (5.8) and \( T_k = S_{j(k)} \) or \( -S_{j(k)} \), (5.5) holds. Thus we have completed the proof of Lemma 5.1.

**Lemma 5.2.** For any sequence of positive values \( \{ \xi_m \}_{m=1}^{\infty} \) such that \( \lim_{m \to \infty} \xi_m = 0 \), there exists a subsequence \( \{ \sigma_m \}_{m=1}^{\infty} \subset \{ \xi_m \}_{m=1}^{\infty} \) such that \( \lim_{m \to \infty} \mu_k(\sigma_m) \) exists for \( k \in K_2 \) and belongs to \( \{ \omega_k \}_{k=1}^{\infty} \) and there exists a map \( j' : \{ 1, 2, 3, \ldots, m_2 \} \to \mathbb{N} \) such that \( \lim_{m \to \infty} \mu_{\omega_{j'_{\tilde{k}}}j_{\tilde{k}}} \sigma_m = \omega_{j'_{k}j+k+1} \) \( (1 \leq k \leq m_2 - 1) \). Thus \( \lim_{m \to \infty} \mu_{\omega_{j'_{k}}}j_{\tilde{k}} \sigma_m \geq \omega_{k} \) \( (1 \leq k \leq m_2) \) follows. There exists a system of orthonormalized eigenfunctions \( \{ \Phi_k \}_{k=1}^{m_2} \) in (2.2) corresponding to \( \{ \omega_{j'_{k}} \}_{k=1}^{m_2} \) such that

\[
\lim_{m \to \infty} \sup_{x \in \Omega(\sigma_m)} |\Phi_{n_2(k), \sigma_m}(x) - \Phi_k(x)| = 0 \quad (1 \leq k \leq m_2) \quad (5.15)
\]

\[
(\Phi_{k_1}, \Phi_{k_2})_{L^2(\Omega(\sigma_m))} = \delta_{k_1, k_2} \quad (1 \leq k_1, k_2 \leq m_2). \quad (5.16)
\]

**Proof of Lemma 5.2.** For each \( k \geq 1 \), there exists \( M_k \geq 0 \) from (5.2) such that \( \| \Phi_{n_2(k), \xi} \|_{L^2(\Omega(\xi))} \leq M_k \) \( (0 < \xi < \xi_*) \). First we choose a subsequence \( \{ \sigma_m \}_{m=1}^{\infty} \subset \{ \xi_m \}_{m=1}^{\infty} \) such that \( \lim_{m \to \infty} \mu_{\omega_{j'_{k}}}j_{\tilde{k}} \sigma_m \geq \omega_{k} \) \( (1 \leq k \leq m_2) \) follows. Then we can apply Proposition 3.1 to the boundary value problem

\[
\Delta \Phi_{n_2(k), \zeta} + \mu_{n_2(k)}(\zeta) \Phi_{n_2(k), \zeta} = 0 \quad \text{in} \ \Omega(\zeta),
\]

\[
\partial \Phi_{n_2(k), \zeta} / \partial v = 0 \quad \text{on} \ \partial \Omega(\zeta),
\]

\[
|\Phi_{n_2(k), \zeta}(x)| \leq M_k \quad \text{in} \ \Omega(\zeta). \quad (5.17)
\]

We obtain a subsequence \( \{ \sigma_m \}_{m=1}^{\infty} \) and functions \( \Phi_k \in C^\infty(D_1 \cup \overline{D_2}) \) and \( V_k \in C^\infty([-1, 1]) \) such that

\[
\lim_{m \to \infty} \sup_{x \in \Omega(\sigma_m)} |\Phi_{n_2(k), \sigma_m}(x) - \Phi_k(x)| = 0, \quad (5.18)
\]

\[
\lim_{m \to \infty} \sup_{x \in \Omega(\sigma_m)} |\Phi_{n_2(k), \sigma_m}(x, x') - V_k(x_1)| = 0,
\]

\[
\Delta \Phi_k + \xi_k \Phi_k = 0 \quad \text{in} \ D_1 \cup D_2, \quad \partial \Phi_k / \partial v = 0 \quad \text{on} \ \partial D_1 \cup \partial D_2,
\]

\[
\frac{d^2 V_k}{dz^2} + V_k = 0, \quad -1 < z < 1, \quad V_k(1) = \Phi_k(p_1), \quad V_k(-1) = \Phi_k(p_2). \quad (5.19)
\]
From \( \| \Phi_k \|_{L^2(D_1 \cup D_2)} = 1 \), \( \zeta_k \in \{ \omega_k \}_{k=1}^{\infty} \). Repeating the above argument for \( k = 1, 2, 3, \ldots \) and applying the method of the diagonal argument, we obtain a subsequence \( \{ \sigma_m \}_{m=1}^{\infty} \) common to all \( k \geq 1 \). By (5.18) and the orthonormality of \( \{ \Phi_k, \zeta \}_{k=1}^{\infty} \), we obtain (5.16) and \( j'(k) < j'(k + 1) \) \( (1 \leq k \leq m_1 - 1) \). We have completed the proof of Lemma 5.2.

**Remark 5.1.** In the argument of Lemma 5.1, by the arbitrariness of \( \{ \xi_m \}_{m=1}^{\infty} \), we have from (5.14)

\[
\lim_{\xi \to 0} d_{n-1}^{1/2} \zeta^{(n-1)/2} \| \Phi_k, \zeta \|_{L^n(\Omega(\zeta))} = 1 \quad \text{for any } k \in K_1.
\]

**Lemma 5.3.**

\[
\lim_{\zeta \to 0} \mu_{n_1(k)}(\zeta) = \lambda_k \quad (1 \leq k \leq m_1),
\]

\[
\lim_{\zeta \to 0} \mu_{n_2(k)}(\zeta) = \omega_k \quad (1 \leq k \leq m_2).
\]

Equivalently, \( j(k) = k \) \( (1 \leq k \leq m_1) \) and \( j'(k) = k \) \( (1 \leq k \leq m_2) \).

**Proof of Lemma 5.3.** Assume that (5.21) does not hold. We define \( j_1 \) and \( j_2 \) to be the smallest numbers such that \( 0 \leq j_1 \leq m_1, 0 \leq j_2 \leq m_2 \) and

\[
\lim_{\zeta \to 0} \mu_{n_1(k)}(\zeta) = \lambda_k \quad (1 \leq k \leq j_1 - 1),
\]

\[
\lim_{\zeta \to 0} \mu_{n_1(j_1)}(\zeta) > \lambda_{j_1},
\]

\[
\lim_{\zeta \to 0} \mu_{n_2(k)}(\zeta) = \omega_k \quad (1 \leq k \leq j_2 - 1),
\]

\[
\lim_{\zeta \to 0} \mu_{n_2(j_2)}(\zeta) > \omega_{j_2}.
\]

By Lemmas 5.1 and 5.2 and (A.3), we easily see that \( \lim_{\zeta \to 0} \mu_k(\zeta) \in \{ \lambda_k \}_{k=1}^{\infty} \) for \( k \in K_1 \) and \( \lim_{\zeta \to 0} \mu_k(\zeta) \in \{ \omega_k \}_{k=1}^{\infty} \) for any \( k \in K_2 \). So we see that \( \lim_{\zeta \to 0} \mu_{n_1(j_1)}(\zeta) \neq \lim_{\zeta \to 0} \mu_{n_2(j_2)}(\zeta) \) from (A.3). To deduce a contradiction, we divide the argument into two cases:

(i) the case \( \lim_{\zeta \to 0} \mu_{n_1(j_1)}(\zeta) < \lim_{\zeta \to 0} \mu_{n_2(j_2)}(\zeta) \),

(ii) the case \( \lim_{\zeta \to 0} \mu_{n_1(j_1)}(\zeta) > \lim_{\zeta \to 0} \mu_{n_2(j_2)}(\zeta) \).

**Case (i).** We define a test function as

\[
\phi_{n_1}(x) = \begin{cases} 
\frac{1}{d_{n-1}^{1/2} \zeta^{(n-1)/2}} \sin \left( j_1 \pi / 2 \right) \left( \frac{x_1}{1 - 2 \zeta} + 1 \right) & \text{for } x \in \Gamma(\zeta), \\
0 & \text{for } x \in \Omega(\zeta) \setminus \Gamma(\zeta),
\end{cases}
\]

\[
\phi_{n_1}(x) = \sum_{k=1}^{n_1(j_1) - 1} (\phi_{n_1} \cdot \Phi_k, \zeta)_{L^2(\Omega(\zeta))} \Phi_k, \zeta(x).
\]
Concerning this test function, we can easily calculate \( \lim_{\xi \to 0} \| \Phi_\xi \|_{L^2(\Omega(\zeta))} = 1 \), \( \lim_{\xi \to 0} \int_{\Omega(\zeta)} |\nabla \Phi_\xi |^2 \, dx = \lambda_{j_1} \), \( (\Phi_\xi \cdot \Phi_k, \zeta)_{L^2(\Omega(\zeta))} = 0 \) (\( 1 \leq k \leq n_1(j_1) - 1 \)).

By using the min–max principle, we obtain \( \lim \sup_{\xi \to 0} \mu_{n_1(j_1)}(\zeta) \leq \lambda_{j_1} \). This contradicts to the assumption (5.22).

**Case (ii).** Let \( \{ \zeta_m \}_{m=1}^\infty \) be a sequence of positive values which converges to 0 such that

\[
\lim_{m \to \infty} \mu_{n_2(j_2)}(\zeta_m) = \lim_{\zeta \to 0} \mu_{n_2(j_2)}(\zeta).
\]

Let \( \{ \phi_j \}_{j=1}^\infty \) be an arbitrary complete system of the eigenfunctions to \( \{ \omega_j \}_{j=1}^\infty \) of the eigenvalue problem (2.2) such that

\[
(\phi_{k_1}, \phi_{k_2})_{L^2(D_1 \cup D_2)} = \delta_{k_1, k_2} \quad (k_1, k_2 \geq 1).
\]

Applying Lemma 5.2 for this \( \{ \zeta_m \}_{m=1}^\infty \), we obtain \( \{ \sigma_m \}_{m=1}^\infty \) and \( \phi_1, \phi_2, \ldots, \phi_{j_2-1} \in C^\infty (\overline{D_1 \cup D_2}) \) where each \( \phi_j \) is the eigenfunction in (2.2) corresponding to the eigenvalue \( \omega_j = \lim_{\xi \to 0} \mu_{n_2(j_2)}(\zeta) \) for \( j \) such that \( 1 \leq j \leq j_2 - 1 \), and \( (\phi_j, \phi_k)_{L^2(D_1 \cup D_2)} = \delta_{j, k} \) \( (1 \leq j, k \leq j_2 - 1) \). By dim L.h. \( \{ \phi_1, \phi_2, \ldots, \phi_{j_2-1} \} = \dim \text{L.h.} \{ \phi_1, \phi_2, \ldots, \phi_{j_2-1} \} + 1 \), we choose \( \delta \in \text{L.h.} \{ \phi_1, \phi_2, \ldots, \phi_{j_2-1} \} \) such that \( (\delta, \phi_j)_{L^2(D_1 \cup D_2)} = 0 \) for \( j = 1, 2, \ldots, j_2 - 1 \) and \( \| \delta \|_{L^2(D_1 \cup D_2)} = 1 \) where L.h. \( U \) is the linear subspace of \( L^2(D_1 \cup D_2) \) generated by the subset \( U \).

To construct a test function, we prepare a cut-off function \( e_\xi \in C^\infty (\overline{D_1 \cup D_2}) \) such that

\[
0 \leq e_\xi (x) \leq 1 \quad \text{for } x \in D_1 \cup D_2, \quad \text{supp } e_\xi \subset \bigcup_{i=1}^{2} (\overline{D_i \setminus \Sigma_i (2\zeta)}).
\]

\[
e_\xi (x) = 1 \quad \text{for any } x \in \bigcup_{i=1}^{2} (D_i \setminus \Sigma_i (t_\zeta)), \quad \sum_{\zeta \to 0} \| e_\xi \|_{H^1(D_1 \cup D_2)} = 0,
\]

where \( t_\zeta > 0 \) and \( \lim_{\xi \to 0} t_\zeta = 0 \). We can construct such a family of functions \( \{ e_\xi \}_{0 < \zeta < \zeta_*} \) by applying the method of [4]. We define a test function as

\[
\Phi_\xi (x) = \begin{cases} 
    e_\xi (x) \phi(x) & \text{for } x \in D_1 \cup D_2, \\
    0 & \text{for } x \in \Omega(\zeta),
\end{cases}
\]

\[
= \Phi_\xi (x) = \Phi_\xi (x) - \sum_{k=1}^{n_2(j_2)-1} (\Phi_\xi \cdot \Phi_k, \zeta)_{L^2(\Omega(\zeta))} \Phi_k, \zeta (x).
\]

By an easy calculation, we have

\[
\lim_{m \to \infty} \| \Phi_{\sigma_m} \|_{L^2(\Omega(\sigma_m))} = 1, \quad \lim_{m \to \infty} \sup_{\Omega(\sigma_m)} |\nabla \Phi_{\sigma_m} |^2 \, dx \leq \omega_{j_2}.
\]

By the min–max principle, we conclude \( \lim \sup_{m \to \infty} \mu_{n_2(j_2)}(\sigma_m) \leq \omega_{j_2} \). This
contradicts assumptions (5.23) and (5.24). Through both cases (i) and (ii), we have deduced a contradiction and we have completed the proof of (5.21).

We rearrange \( \{ \lambda_k \}_{k=1}^{\infty} \cup \{ \omega_k \}_{k=1}^{\infty} \) in increasing order and denote it by \( \{ \mu_k \}_{k=1}^{\infty} \).

**Lemma 5.4.** \( m_1 = m_2 = \infty, \ K_1 \cup K_2 = \mathbb{N}, \lim_{\zeta \to 0} \mu_k(\zeta) = \mu_k \ (k \ge 1) \).

**Proof of Lemma 5.4.** We assume \( \min(m_1, m_2) < \infty \) and deduce a contradiction. We divide the argument into two cases:

\[
(1) \lambda_{m_1 + 1} < \omega_{m_2 + 1} \quad \quad (2) \lambda_{m_1 + 1} > \omega_{m_2 + 1}.
\]

For convenience we put \( \lambda_\infty = \infty \) and \( \omega_\infty = \infty \).

**Case (1).** In this case, \( m_1 < \infty \). Put \( m'_2 = \max \{ k \mid \omega_k < \lambda_{m_1 + 1} \} \) and \( s = m_1 + m'_2 \) and we have \( 1, 2, ..., s \in K_1 \cup K_2 \) and \( s + 1 \notin K_1 \) from Lemma 5.3. Then, by the similar argument as in Lemma 5.2, there exist a sequence \( \{ \zeta_m \}_{m=1}^{\infty} \), a value \( \xi \), and a function \( \phi \neq 0 \) such that

\[
\lim_{m \to \infty} \mu_{s+1}(\zeta_m) = \xi, \quad \lim_{m \to \infty} ||\Phi_{s+1, \zeta_m} - \phi||_{L^\infty(D_1 \cup D_2)} = 0, \quad \text{and} \quad \Delta \phi + \xi \phi = 0 \text{ in } D_1 \cup D_2, \quad \frac{\partial \phi}{\partial v} = 0 \text{ on } \partial D_1 \cup \partial D_2.
\]

Next, applying Lemmas 5.2 and 5.3, we have a subsequence \( \{ \sigma_m \}_{m=1}^{\infty} \subset \{ \zeta_m \}_{m=1}^{\infty} \) and an orthonormal of eigenfunctions \( \{ \phi_k \}_{k=1}^{m_2} \) such that \( \lim_{m \to \infty} \mu_{s+1}(\zeta_m) = \xi, \) and \( \lim_{m \to \infty} \| \Phi_{s+1, \zeta_m} - \phi_k \|_{L^\infty(D_1 \cup D_2)} = 0 \) for \( 1 \le k \le m'_2 \). From this, \( \xi \ge \omega_{m_2 + 1} > \lambda_{m_1 + 1} = \mu_{s+1} \) and then \( \limsup_{\zeta \to 0} \mu_{s+1}(\zeta) > \mu_{s+1} \) holds. On the other hand, by the method similar to that Case (i) in the proof of Lemma 5.3 we can prove \( \limsup_{\zeta \to 0} \mu_{s+1}(\zeta) \leq \lambda_{m_1 + 1} = \mu_{s+1} \) by using the test function made from \( \sin(m_1 + 1) \pi(x_1/(1 - 2\zeta) + 1) \) and Lemmas 5.1 and 5.2. This is a contradiction.

**Case (2).** In this case, \( m_2 < \infty \). Put \( m'_1 = \max \{ k \mid \lambda_k < \omega_{m_2 + 1} \} \) and \( s = m_1 + m'_1 \) and then \( 1, 2, ..., s \in K_1 \cup K_2 \) and \( s + 1 \notin K_1 \) follow from Lemma 5.3. Then there exists a sequence \( \{ \zeta_m \}_{m=1}^{\infty} \) such that \( \lim_{m \to \infty} \xi_m = 0 \) and \( \lim_{m \to \infty} \| \Phi_{s+1, \zeta_m} \|_{L^\infty(\Omega(\zeta_m))} = \infty. \) Applying an argument similar to that in Lemma 5.1 and choosing a convergent subsequence of \( \{ \Phi_{s+1, \zeta_m} \}_{m=1}^{\infty} \) (cf. \( \Phi_{k, \zeta_m} = \Phi_{k, \zeta_m} \|_{L^\infty(\Omega(\zeta_m))} \)), we obtain some eigenvalue \( \xi \) and its eigenfunction \( T \) of (2.3). But by applying Lemma 5.1 and the same argument there, we can prove that \( (T, S_1) \xi_{l-1}, l_1) = 0 \) \((1 \leq k \leq m'_1) \) and we have \( \xi \geq \lambda_{m_1 + 1} \) and then we have \( \limsup_{\zeta \to 0} \mu_{s+1}(\zeta) \leq \omega_{m_2 + 1} = \mu_{s+1} \). This is a contradiction and we complete both Cases (1) and (2). We conclude \( m_1 = m_2 = \infty \). Together with
this and Lemma 5.3, we conclude \( K_1 \cup K_2 = \mathbb{N} \). This completes the proof of Lemma 5.4.

**Proof Theorem 1.** Define \( \lambda_k(\zeta) \equiv \mu_{n_k(k)}(\zeta), \omega_k(\zeta) \equiv \mu_{n_2(k)}(\zeta) \) for \( k \geq 1, \zeta \in (0, \zeta_*) \). We conclude Theorem 1 by Lemmas 5.3 and 5.4.

**Proof of Theorem 2.** From Lemmas 5.1 to 5.4, we obtain a complete orthonormal system of eigenfunctions \( \{ \phi_k \}_{k=1}^\infty \) in \( L^2(D_1 \cup D_2) \) and their associated functions \( \{ V_k \}_{k=1}^\infty \subset C^\infty([-1, 1]) \) from (5.19). The properties (2.6)-(2.9) are deduced from Lemmas 5.1 to 5.4 and Theorem 1.

### 6. PROOF OF THEOREM 3

We take each \( k \geq 1 \) and fix it and we confine ourselves to investigate the behavior of \( \psi_{k, \zeta} \) in \( \tilde{\Omega}(\zeta) \). Multiplying \( \psi_{k, \zeta} \) by \(-1\) if necessary, we may assume, by (2.8) in Theorem 2,

\[
\lim_{\zeta \to 0} \sup_{x \in \tilde{\Omega}(\zeta)} |d_{n-1}^{1/2} r^{(n-1)/2} \psi_{k, \zeta}(x_1, x') - \frac{k \pi}{2} (1 - x_1)| = 0.
\]

Recall that \( \lim_{\zeta \to 0} d_{n-1}^{1/2} r^{(n-1)/2} \| \psi_{k, \zeta} \|_{L^2(\Omega(\zeta))} = 1 \).

We put the parameter \( M = \omega_k + 1 \) and \( l = \min(c_0(M), 1/2k) \) where \( c_0(M) \) is the constant in Lemma 4.1 for the above \( M \). We use the auxiliary functions in Section 4 defined on the region \( \tilde{\Omega}(\zeta) \) for the above constants \( M \) and \( l \). We abbreviate the subscript \( k \) and denote \( \lambda_k(\zeta), \psi_{k, \zeta} \) by \( \lambda(\zeta), \psi_\zeta \), respectively. We prepare some notations:

\[
\psi_\zeta(x) = \psi_\zeta(x)/\| \psi_\zeta \|_{L^2(\Omega(\zeta))}, \quad \bar{\psi}_\zeta(x) = \psi_\zeta(x)/\| \psi_\zeta \|_{L^2(D_1 \cup \Sigma_1(\zeta))},
\]

\[
\alpha_\zeta = \sup_{x \in D_1 \setminus \Sigma_1(l)} |\psi_\zeta(x)|, \quad \bar{a}_\zeta = \sup_{x \in D_1 \setminus \Sigma_1(l)} |\bar{\psi}_\zeta(x)|.
\]

Remark that \( \lim_{\zeta \to 0} \alpha_\zeta = 0 \) by Theorem 2. \( \psi_\zeta \) satisfies

\[
\Delta \psi_\zeta + \lambda(\zeta) \psi_\zeta = 0 \quad \text{in} \ \tilde{\Omega}(\zeta),
\]

\[
\partial \psi_\zeta/\partial v = 0 \quad \text{on} \ \partial \tilde{\Omega}(\zeta) \setminus \gamma_1(\zeta),
\]

\[
\lim_{\zeta \to 0} \sup_{x \in \gamma_1(\zeta)} |\psi_\zeta(x) - a| = 0.
\]

where \( 0 < a = \sin(k \pi l/2) < 2^{-1/2} \).

**Lemma 6.1.** \( \lim_{\zeta \to 0} \psi_\zeta = 0 \) in \( C^\infty(D_1 \setminus \Sigma_1(\eta)) \) for any \( \eta > 0 \).

**Proof of Lemma 6.1.** Applying (3.7), and (3.8) of Proposition 3.2 to \( \{ \psi_\zeta \}_{0 < \zeta < \zeta_*} \) successively, we see that the family \( \{ \psi_\zeta \}_{0 < \zeta < \zeta_*} \) is compact in
$C^\infty(D_1\setminus\Sigma_1(\eta))$ for any $\eta > 0$. By a diagonal argument, for any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that $\lim_{m \to \infty} \zeta_m = 0$, there exists a subsequence $\{\sigma_m\}_{m=1}^\infty$ and $\psi \in C^\infty(D_1\setminus\{p_1\})$ such that

$$\lim_{m \to \infty} \psi_{\sigma_m} = \psi \quad \text{in } C^\infty(D_1\setminus\Sigma_1(\eta)) \text{ for } \eta > 0,$$

$$\Delta \psi + \lambda \psi = 0 \quad \text{in } D_1,$$

$$\sigma \psi \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial D_1\setminus\{p_1\},$$

$$|\psi(x)| \leq 1 \quad \text{in } D_1,$$  \hspace{1cm} (6.3)

where $\lambda = \lim_{\zeta \to 0} \lambda(\zeta) \in \{\lambda_k\}_{k=1}^\infty$. By the argument of the removable singularity (Proposition 3.3), the Neumann boundary condition in (6.3) can be extended up to $p_1$. If $\psi \not\equiv 0$, this means that $\lambda \in \{\lambda_k\}_{k=1}^\infty$, but this contradicts the assumption (A.3). We conclude that $\psi \equiv 0$ in $D_1$ and Lemma 6.1.

We remark that $\lim_{\zeta \to 0} \tilde{a}_\zeta = 0$ follows from Lemma 6.1.

**Lemma 6.2.** There exists a constant $c_1 > 0$, such that

$$|\psi_\zeta(x)| \leq c_1 \left( \frac{1}{\zeta} \phi_\zeta^+(x) + \tilde{a}_\zeta \tilde{\phi}_\zeta^+(x) \right) \quad \text{for } x \in \gamma_1(\zeta) \cup \Sigma_1(l).$$  \hspace{1cm} (6.4)

**Proof of Lemma 6.2.** Denote by $\varphi_\zeta$ the right side of (6.4). It is easy to see by Lemma 4.1 that there exist positive constants $c'$ and $c''$ such that

$$\varphi_\zeta(x)/\zeta \geq c' \quad \text{for any } x \in \gamma_2(\zeta) \ (0 < \zeta < \zeta_*),$$

$$\tilde{\phi}_\zeta^+(x) \geq c'' \quad \text{for any } x \in \gamma_4(0 < \zeta < \zeta_*).$$

We put $c_1 = 2 \max(1/c', 1/c'')$ and prove (6.4) for this $c_1$. It is easy to see $\psi_\zeta(x) < \varphi_\zeta(x)$ for $x \in \gamma_3(\zeta) \cup \gamma_4$. We define a constant

$$\kappa_* \equiv \sup\{ \kappa \in [0, 1) \mid \varphi_\zeta(x) - \kappa \psi_\zeta(x) \geq 0 \text{ in } \gamma_3(\zeta) \cup \Sigma_1(l) \}.$$

We assume that $0 \leq \kappa_* < 1$ and will deduce a contradiction. If $0 \leq \kappa_* < 1$, then $K_* = \{ x \in \gamma_2(\zeta) \cup \Sigma_1(l) \mid \varphi_\zeta(x) - \kappa \psi_\zeta(x) = 0 \} \neq \emptyset$. By the definition of $\kappa_*$ and $K_*$ and the smoothness of $\psi_\zeta$ and the nonsmoothness of $\varphi_\zeta$ on $\gamma_3(\zeta)$ by Lemma 4.4, we see

$$\varphi_\zeta(x) - \kappa_\zeta \psi_\zeta(x) \geq 0 \quad \text{in } R_1(\zeta) \cup \Sigma_1(l)$$

and $K_* = (\gamma_2(\zeta) \cup \Sigma_1(l)) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta) \cup \gamma_4)$. If there exists a point $x_* \in K_* \setminus \partial(R_1(\zeta) \cup \Sigma_1(l) \cup \gamma_3(\zeta))$, we have the differential inequality by the equations satisfied by $\varphi_\zeta$ and $\psi_\zeta$,

$$\Delta(\varphi_\zeta - \kappa_\zeta \psi_\zeta) + \lambda(\zeta)(\varphi_\zeta - \kappa_\zeta \psi_\zeta) = (-M + \lambda(\zeta)) \varphi_\zeta < 0,$$  \hspace{1cm} (6.5)
in some neighborhood of $x_*$. But from $\phi_\zeta(x_*) - \kappa_* \tilde{\psi}_\zeta(x_*) = 0$ and the strong maximum principle, $\phi_\zeta - \kappa_* \tilde{\psi}_\zeta \equiv 0$ (near $x_*$) follows and again from (6.4), $\phi_\zeta \equiv 0$ follows. This is a contradiction and we conclude that $K_* = \delta(R_1(\zeta) \cup \Sigma_1(l)) \setminus \gamma_3(\zeta)$. We take any $x_* \in K_*$ and we also have the differential inequality (6.4) near the boundary point $x_*$. By $\phi_\zeta(x_*) - \kappa_* \tilde{\psi}_\zeta(x_*) = 0$ and the inequality $\phi_\zeta(x) - \kappa_* \tilde{\psi}_\zeta(x) > 0$ for $x \in (R_1(\zeta) \cup \Sigma_1(l)) \setminus (\gamma_2(\zeta) \cup \gamma_3(\zeta) \cup \gamma_4)$ and the Hopf Lemma, we obtain
$$\partial(\phi_\zeta - \kappa_* \tilde{\psi}_\zeta)/\partial v < 0.$$ On the other hand, $\phi_\zeta$ and $\tilde{\psi}_\zeta$ satisfy the Neumann boundary condition at $x_*$ and this contradicts the above inequality. Therefore we conclude $K_* = 1$ and obtain $\phi_\zeta(x) - \tilde{\psi}_\zeta(x) \geq 0$. Applying the same argument to $\phi_\zeta$ and $-\tilde{\psi}_\zeta$, we have $\phi_\zeta(x) + \tilde{\psi}_\zeta(x) \geq 0$ in $R_1(\zeta) \cup \Sigma_1(l)$ and we have completed the proof of Lemma 6.2.

**Lemma 6.3.** $\limsup_{\zeta \to \infty} \frac{\tilde{\alpha}_\zeta}{\zeta^{n-2}} < \infty$.

**Proof of Lemma 6.3.** We assume that there exists a sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ which converges to 0 such that
$$\lim_{m \to \infty} \frac{\tilde{\alpha}_{\zeta_m}}{\zeta_m^{n-2}} = \infty. \tag{6.6}$$
We consider the function
$$\tilde{\psi}_\zeta^*(x) = \frac{\tilde{\psi}_\zeta(x)}{\tilde{\alpha}_\zeta} \quad \text{in} \quad D_1 \setminus \Sigma_1(3\zeta), \tag{6.7}$$
which satisfies
$$\sup_{x \in D_1 \setminus \Sigma_1(l)} |\tilde{\psi}_\zeta^*(x)| = 1. \tag{6.8}$$
$\tilde{\psi}_\zeta^*$ satisfies the equation
$$A\tilde{\psi}_\zeta^* + \lambda(\zeta_m) \tilde{\psi}_\zeta^* = 0 \quad \text{in} \quad D_1, \tag{6.9}$$
$$\partial \tilde{\psi}_\zeta^*/\partial v = 0 \quad \text{on} \quad \partial D_1 \setminus \partial \Sigma_1(3\zeta_m).$$
By Lemma 6.2, we have
$$|\tilde{\psi}_\zeta^*(x)| \leq \frac{c_1}{\tilde{\alpha}_\zeta} \left( \frac{1}{\zeta_m} \phi_\zeta(x) + \tilde{\alpha}_\zeta \phi_\zeta^*(x) \right) \quad \text{for} \quad x \in R_1(\zeta_m) \cup \Sigma_1(l) \tag{6.10}$$
$$= c_1 \left( \frac{\zeta_m^{n-2}}{\tilde{\alpha}_\zeta} A_2(|x - p_1|) + A_1(|x - p_1|) \right) \quad \text{for} \quad x \in \Sigma_1(l) \setminus \Sigma_1(3\zeta_m)$$
\[505/77:2-10\]
and we also have
\begin{align}
|\tilde{\psi}_{\sigma_m}^*(x)| &\leq c_1 \left( \frac{cA_2(|x-p_1|)}{|x-p_1|^{n-2}} + \tilde{\psi}_{\sigma_m}^+(x) \right) \quad \text{for } x \in \Sigma_1(l) \setminus \Sigma_1(3\zeta_m), \\
|\tilde{\psi}_{\sigma_m}^*(x)| &\leq 1 \quad \text{for } D_1 \setminus \Sigma_1(l),
\end{align}
(6.11)
by putting \( c = \sup_{m \geq 1} \beta_m^{-2}/\alpha_m \) which is finite by (6.6). By applying (3.7) and (3.8) in Proposition 3.2 successively to (6.9) and (6.11) with a diagonal argument, we can choose a subsequence \( \{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty \) and a function \( \psi^* \) such that
\begin{align}
\lim_{m \to \infty} \tilde{\psi}_{\sigma_m}^* = \psi^* \quad \text{in } C^\infty(D_1 \setminus \Sigma_1(\eta)) \text{ for any } \eta > 0, \\
A\psi^* + \lambda\psi^* = 0 \quad \text{in } D_1, \\
\frac{\partial \psi^*}{\partial \nu} = 0 \quad \text{on } \partial D_1 \setminus \{p_1\},
\end{align}
(6.12)
(6.13)
\begin{align}
|\psi^*(x)| &\leq \max(1, c_1 c_1(M)) \quad \text{in } D_1.
\end{align}
By the removability of singularity (Proposition 3.3), the boundary condition in (6.13) is extended up to \( p_1 \). On the other hand, (6.13) and (6.14) imply \( \lambda \in \{\omega_k\}_{k=1}^\infty \). But this contradicts the assumption (A.3). Therefore we complete the proof of Lemma 6.3.

**Lemma 6.4.** There exists a constant \( c_2 > 0 \) such that
\begin{align}
|\psi^*(x)| &\leq c_2 (a \varphi^+(x) + \alpha \tilde{\psi}^+(x)) \quad \text{for } x \in \Gamma^*(\zeta) \cup R_1(\zeta) \cup \Sigma_1(l), \\
|\psi^*(x)| &\leq \alpha \zeta, \quad \text{for } x \in D_1 \setminus \Sigma_1(l).
\end{align}

**Proof of Lemma 6.4.** The first inequality can be proved by the same method as Lemma 6.2 and the second one is trivial from the definition of \( \psi^* \).

**Lemma 6.5.** \( \lim \sup \zeta \to 0 \alpha \zeta/\zeta^{n-1} < \infty \).

**Proof of Lemma 6.5.** From Lemma 6.4, we have
\begin{align}
|\psi^*(x)| &\leq c_2 (a \zeta + \alpha \zeta) \quad \text{for } x \in D_1 \cup R_1(\zeta).
\end{align}
(6.14)
On the other hand, by the definition of \( \psi^* \), we have
\begin{align}
\alpha \zeta &= \sup_{x \in D_1 \setminus \Sigma_1(l)} |\psi^*(x)| = (\|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))})^{-1} \sup_{x \in D_1 \setminus \Sigma_1(l)} |\psi^*(x)| \\
&= (\|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))})^{-1} \|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))} \sup_{x \in D_1 \setminus \Sigma_1(l)} |\tilde{\psi}_\zeta(x)| \\
&= \|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))} \sup_{x \in D_1 \setminus \Sigma_1(l)} |\tilde{\psi}_\zeta(x)| = \|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))} \tilde{\alpha} \zeta \\
\alpha \zeta &= \tilde{\alpha} \zeta \|\psi^*\|_{L^\infty(D_1 \cup R_1(\zeta))}. 
\end{align}
(6.15)
By using (6.16), (6.17), and the property of \( \varphi_\zeta^- \), there exists a constant \( c_4 > 0 \) such that
\[
\psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) \geq c_4 \varphi_\zeta^-(x) \quad \text{for } x \in \tilde{\mathcal{O}}(\zeta). \]

Put \( \tilde{\kappa}_* = \sup\{\kappa \in [0, 1] | \psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) - \kappa c_4 \varphi_\zeta^-(x) \geq 0 \text{ in } \tilde{\mathcal{O}}(\zeta)\}. \)

By a method similar to that in Lemma 6.2, we can prove \( \tilde{\kappa}_* = 1 \). This completes the proof of Lemma 6.6.

By Lemma 6.6, we get the estimates from below in (2.10)-(2.13). Equation (2.14) follows from Remark 5.1 and Lemmas 5.3 and 5.4. By (6.14) and (6.15),
\[
\alpha_\zeta / \tilde{\alpha}_\zeta = \| \psi_\zeta' \|_{L^\infty(D_1 \cup \mathcal{R}_1(\zeta))} \leq c_2 (a_\zeta' + a_\zeta). \]

We have
\[
(1 - c_2 \tilde{\alpha}_\zeta) \alpha_\zeta \leq c_2 a_\zeta \tilde{\alpha}_\zeta. \]

Using \( \lim_{\zeta \to 0} \tilde{\alpha}_\zeta = 0 \), and Lemma 6.3, we conclude
\[
\limsup_{\zeta \to 0} \alpha_\zeta / \zeta^n - 1 \leq c_2 a \limsup_{\zeta \to 0} \tilde{\alpha}_\zeta / \zeta^{n-2} < \infty. \]

This concludes Lemma 6.5.

With the aid of the estimate of Lemma 6.5, in the inequality of Lemma 6.4, we obtain the estimates from above in (2.10)-(2.13) in Theorem 3.

Next we prove the estimates from below.

**Lemma 6.6.** There exists a constant \( c_4 > 0 \) such that
\[
\psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) \geq c_4 \varphi_\zeta^-(x) \quad \text{for } x \in \tilde{\mathcal{O}}(\zeta). \]

**Proof of Lemma 6.6.** By \( \tilde{\phi}_\zeta^+(x) \geq c_1(M) \) (for \( x \in \gamma_4 \)) and Lemma 6.5, there exists a constant \( c_3, c_3' > 0 \) such that
\[
\psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) \geq c_3' \zeta^n - 1 \quad \text{for } x \in \gamma_4. \quad (6.16)\]

By \( \lim_{\zeta \to 0} \sup_{x \in \gamma_4(\zeta)} |\psi_\zeta'(x) - a| = 0 \) and \( \lim_{\zeta \to 0} \alpha_\zeta = 0 \), we have
\[
\psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) \geq a/2 \quad \text{on } \gamma_4(\zeta) \text{ for small } \zeta > 0. \quad (6.17)\]

First we prove
\[
\psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) > 0 \quad \text{in } \tilde{\mathcal{O}}(\zeta). \quad (6.18)\]

Put \( \kappa_* = \sup\{\kappa \in [0, 1] | \kappa \psi_\zeta'(x) + c_3 \zeta^n - 1 \tilde{\phi}_\zeta^+(x) > 0 \text{ in } \tilde{\mathcal{O}}(\zeta)\}. \) If \( \kappa_* < 1 \), by applying the same method as Lemma 6.2, we can deduce a contradiction and for \( \kappa_* = 1 \), repeating the same argument, we obtain (6.18).
7. Generalizations

Theorems 1, 2, and 3 can be simply generalized to the case where the potential term is added in the equation.

We consider the eigenvalue problem

\[ \Delta \Phi^* + h_\zeta(x) \Phi^* + \mu^* \Phi^* = 0 \quad \text{in } \Omega(\zeta), \]
\[ \frac{\partial \Phi^*}{\partial \nu} = 0 \quad \text{on } \partial \Omega(\zeta), \]
(7.1)

where there exist \( h \in C^\infty(D_1 \cup D_2), \tilde{h} \in C^\infty([-1, 1]) \) such that

\[ \lim_{\zeta \to 0} \sup_{x \in D_1 \cup D_2} |h_\zeta(x) - h(x)| = 0, \]
\[ \lim_{\zeta \to 0} \sup_{x \in \partial \Omega(\zeta)} |h_\zeta(x, x') - \tilde{h}(x_1)| = 0. \]

**Definition 6.** Let \( \{\mu^*_k(\zeta)\}_{k=1}^\infty \) and \( \{\Phi^*_k(\zeta)\}_{k=1}^\infty \) be the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of the orthonormalized eigenfunctions of the eigenvalue problem (7.1).

**Definition 7.** Let \( \{\omega_k^*\}_{k=1}^\infty \) and \( \{\lambda_k^*\}_{k=1}^\infty \) be respectively the sequence of the eigenvalues arranged in increasing order (counting multiplicity) of (7.2) and (7.3),

\[ \Delta \phi^* + h(x) \phi^* + \omega^* \phi^* = 0 \quad \text{in } D_1 \cup D_2, \]
\[ \frac{\partial \phi^*}{\partial \nu} = 0 \quad \text{on } \partial D_1 \cup D_2, \]
\[ d^2 S^*/dz^2 + \tilde{h}(z) S^* + \lambda^* S^* = 0, \quad -1 < z < 1, \]
\[ S^*(1) = S^*(-1) = 0. \]
(7.2) (7.3)

We assume the condition

(A.4) \( \{\omega_k^*\}_{k=1}^\infty \cap \{\lambda_k^*\}_{k=1}^\infty = \emptyset. \)

**Theorem 4.** Assume (A.1), (A.2), (A.4), and \( n \geq 3 \). The sets \( \{\mu_k^*(\zeta)\}_{k=1}^\infty \) and \( \{\Phi_k^*(\zeta)\}_{k=1}^\infty \) are separated as

\[ \{\mu_k^*(\zeta)\}_{k=1}^\infty = \{\omega_k^*(\zeta)\}_{k=1}^\infty \cup \{\lambda_k^*(\zeta)\}_{k=1}^\infty, \]
\[ \{\Phi_k^*(\zeta)\}_{k=1}^\infty = \{\phi_k^*(\zeta)\}_{k=1}^\infty \cup \{\psi_k^*(\zeta)\}_{k=1}^\infty. \]
such that
\[
\lim_{\zeta \to 0} \omega^*_\zeta(z) = \omega^*_0, \quad \lim_{\zeta \to 0} \lambda^*_\zeta(z) = \lambda^*_0.
\]

for any sequence of positive values \(\{\zeta_m\}_{m=1}^\infty\) such that \(\lim_{m \to \infty} \zeta_m = 0\), there exists a subsequence \(\{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty\) and the complete system of the eigenfunctions \(\{\phi^*_k\}_{k=1}^\infty \subset C^\infty(D_1 \cup D_2)\) of (7.2) and \(\{S^*_m\}_{k=1}^\infty \) of (7.3), respectively, such that \((\phi^*_k \cdot \phi^*_m)_{\zeta \in (\rho_1 \cup \rho_2)} = \delta_{k,m}, (S^*_k \cdot S^*_m)_{L^2((-1,1))} = \delta_{k,m}\) for \(k, m \geq 1\), and the following conditions for each \(k \geq 1\),

\[
\lim_{m \to \infty} \sup_{x \in D_1 \cup D_2} |\phi^*_k, \sigma_m(x) - \phi^*_k(x)| = 0,
\]

\[
\lim_{m \to \infty} \sup_{x = (x_1, x') \in Q(\sigma_m)} |\phi^*_k, x_1, x') - \phi^*_k(x_1)| = 0,
\]

or

\[
\lim_{m \to \infty} \sup_{x = (x_1, x') \in Q(\sigma_m)} \left| d^{1/2} \sigma_m (n-1/2) \psi^*_k, \sigma_m(x_1, x') - S^*_k(x_1) \right| = 0,
\]

Here we denoted by \(V^*_k\) the unique solution of the two point boundary value problem (7.4) for each \(k = 1, 2, 3, \ldots\),

\[
d^2V^*/dz^2 + h(z) V^* + \omega^* V^* = 0, \quad -1 < z < 1,
\]

\[
V^*(1) = \phi^*_k(p_1), \quad V^*(-1) = \phi^*_k(p_2).
\]

(7.4)

For any \(k \geq 1\), there exists a positive constant \(\eta_*(k) > 0\) such that

\[
0 < \lim_{\zeta \to 0} \inf_{x \in R(\zeta) \cup \Sigma(3\zeta)} \zeta^{(n-3)/2} |\psi^*_k, \zeta(x)|
\]

\[
\leq \lim_{\zeta \to 0} \sup_{x \in R(\zeta) \cup \Sigma(3\zeta)} \zeta^{(n-3)/2} |\psi^*_k, \zeta(x)| < +\infty,
\]

\[
0 < \lim_{\zeta \to 0} \inf_{x \in \Sigma(\eta) \cup \Sigma(3\zeta)} \zeta^{-(n-1)/2} |x - p_1|^{n-2} \psi^*_k, \zeta(x)|
\]

\[
\leq \lim_{\zeta \to 0} \sup_{x \in \Sigma(\eta) \cup \Sigma(3\zeta)} \zeta^{-(n-1)/2} |x - p_1|^{n-2} \psi^*_k, \zeta(x)| < +\infty,
\]

\[
0 < \lim_{\zeta \to 0} \sup_{x \in D(\eta) \cup \Sigma(3\zeta)} \zeta^{-(n-1)/2} |\psi^*_k, \zeta(x)|
\]

\[
\leq \lim_{\zeta \to 0} \sup_{x \in D(\eta) \cup \Sigma(3\zeta)} \zeta^{-(n-1)/2} |\psi^*_k, \zeta(x)| < +\infty,
\]

\[
0 < \lim_{\zeta \to 0} \inf_{\zeta \in (0, \eta_*(k))} \zeta^{-(n-1)/2} \left\| \psi^*_k, \zeta \right\|_{L^2(D(\zeta))}
\]

\[
\leq \lim_{\zeta \to 0} \sup_{\zeta \in (0, \eta_*(k))} \zeta^{-(n-1)/2} \left\| \psi^*_k, \zeta \right\|_{L^2(D(\zeta))} < +\infty,
\]

for \(\eta \in (0, \eta_*(k))\) and \(i = 1, 2\).
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REFERENCES