ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Theoretical Computer Science 332 (2005) 313–336
Theoretical Computer Science
www.elsevier.com/locate/tcs

Greedy expansions and sets with deleted digits

Marco Pedicini*

Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche, Viale del Policlinico 137, 00161 Roma, Italy

Received 24 February 2004; received in revised form 28 October 2004; accepted 2 November 2004
Communicated by V. Pan

Abstract

We generalize a result of Daróczy and Kátai, on the characterization of univoque numbers with respect to a non-integer base (Publ. Math. Debrecen 46(3–4) (1995) 385) by relaxing the digit alphabet to a generic set of real numbers. We apply the result to derive the construction of a Büchi automaton accepting all and only the greedy sequences for a given base and digit set. In the appendix, we prove a more general version of the fact that the expansion of an element $x \in \mathbb{Q}(q)$ is ultimately periodic, if q is a Pisot number.

MSC: primary 11R06; secondary 11B83

Keywords: Non-integer bases; β-expansions; Finite automata; Pisot numbers

1. Introduction

Given an integer $m \geqslant 1$ and a real number $q > 1$, by an *expansion* of a real number x in base q with digits in $Z_m := \{0, 1, \ldots, m\}$, we mean a sequence

$$c_1, c_2, \ldots$$

satisfying

$$c_i \in Z_m \quad \text{for all } i$$

* Tel.: +39 688470274; fax: +39 64404306.
E-mail address: marco@iac.rm.cnr.it (M. Pedicini).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2004.11.002

and

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

Such expansions appear in many problems of number theory, real number computations, dynamical systems and in the theory of finite automata: [2,8–10]. There are several different algorithms for the construction of such expansions, and they have interesting and surprising properties for certain values of m and q: see, e.g., [1,5,6] the references therein.

The purpose of this paper is to extend some of these results to more general digit sets. This leads to some new difficulties, requiring new algorithms. In the first part of the paper we establish new theoretical results. They are illustrated by various examples in the second half of the work.

In order to motivate the studies of the present paper let us first recall some classical results of Parry concerning the so-called greedy expansions.

Given a non-negative real number x, let us define a sequence

$$c_1, c_2, \dots$$

by the *greedy algorithm*: if c_i is already defined for all $i < n$ then let c_n be the biggest integer in Z_m satisfying

$$\sum_{i=1}^{n} \frac{c_i}{q^i} \leqslant x. \tag{1.1}$$

One can prove that if $m \geqslant q - 1$ then

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x$$

for all $x \in [0, m/(q-1)]$. This is called the greedy expansion of x.

Let us denote by $\gamma_1, \gamma_2, \dots$ the greedy expansion of $x = 1$. If this sequence contains only a finite number of non-zero digits then let γ_k be the last non-zero element and let us denote by $\delta_1, \delta_2, \dots$ the k-periodic sequence with period $\gamma_1, \gamma_2, \dots, \gamma_{k-1}, \gamma_k - 1$.

Using this sequence, Parry obtained the following characterization of the greedy expansions [8]:

Theorem 1.1. *Assume that $m \geqslant q-1$. A sequence $c_1, c_2, \dots$ of numbers in Z_m corresponds to the greedy expansion*

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x$$

of a suitable number $x \in [0, \frac{m+1}{q})$ if and only if

$$c_{n+1}c_{n+2}\cdots < \delta_1\delta_2\dots \quad \text{in the lexicographic sense, for all } n \geqslant 1.$$

In what follows we will consider a generic set of real numbers A instead of the set Z_m; such situations arise in many problems of representation of real numbers with missing digits. This requires a more general definition of greedy expansions. We shall generalize various recent theorems obtained by different authors on this subject, and we shall illustrate our results by many examples. In the last part of the paper we will also study the close connection between greedy expansions and finite automata.

2. Quasi-greedy and greedy expansions

We study at the same time the greedy expansions and another related concept which seems to be useful to investigate: the so-called quasi-greedy expansions.

Let us fix a set $A = \{a_1, \dots, a_m\}$ of real numbers such that

$$a_1 < a_2 < \cdots < a_m.$$

Given a real number x, let us define a sequence

$$s_1, s_2, \dots$$

by the *quasi-greedy algorithm*: if s_i is already defined for all $i < n$ then let s_n be the biggest element in A satisfying

$$\sum_{i=1}^{n} \frac{s_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{q^i} < x. \tag{2.1}$$

In a similar way, let us define a sequence

$$s_1, s_2, \dots$$

by the *greedy algorithm*: if s_i is already defined for all $i < n$ then let s_n be the biggest element in A satisfying

$$\sum_{i=1}^{n} \frac{s_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{q^i} \leqslant x. \tag{2.2}$$

In the sequel, we denote by (d_i) the sequence (s_i) in (2.1) i.e., when it is defined by the quasi-greedy algorithm. We denote by (c_i) a generic sequence defined by the greedy algorithm.

The definition of the quasi-greedy expansions is meaningful if

$$x > \frac{a_1}{q-1} = \sum_{i=1}^{\infty} \frac{a_1}{q^i}.$$

Let us observe that the sequences (d_i) obtained in this way always contain infinitely many elements, different from a_1. We will say for brevity that the sequences (d_i) are *infinite*.

Note that the definition of the greedy expansions is meaningful if

$$x \geqslant \frac{a_1}{q-1} = \sum_{i=1}^{\infty} \frac{a_1}{q^i}.$$

Under some natural assumptions we obtain an expansion of x indeed:

Proposition 2.1. *If*

$$\max_{1\leqslant j\leqslant m-1}(a_{j+1}-a_j) < \frac{a_m-a_1}{q-1}, \tag{2.3}$$

then for every $x \in (a_1/(q-1), a_m/(q-1)]$ *we have*

$$\sum_{i=1}^{\infty}\frac{s_i}{q^i}=x,$$

if the sequence (s_i) *is defined by the quasi-greedy or by the greedy algorithm.*

Moreover, if $x=a_1/(q-1)$, *we have that the constant sequence* (a_1) *is the sequence given by the greedy algorithm.*

In the greedy case, we can even replace condition (2.3) with

$$\max_{1\leqslant j\leqslant m-1}(a_{j+1}-a_j) \leqslant \frac{a_m-a_1}{q-1} \tag{2.4}$$

and for all $x \in [a_1/(q-1), a_m/(q-1)]$, it is possible to obtain an expansion of x by applying the greedy algorithm.

Proof. If $x=\frac{a_m}{q-1}$, then both the quasi-greedy and the greedy algorithm provide $s_n=a_m$ for all n, and the desired equality follows.

Assume next that there are infinitely many indices n such that $s_n<a_m$. Writing $s_n=a_{j_n}$ for any such n, by the construction of the sequence (s_i) we have

$$\left(\sum_{i=1}^{n}\frac{s_i}{q^i}\right)+\left(\sum_{i=n+1}^{\infty}\frac{a_1}{q^i}\right)\leqslant x$$

(we can eventually have equality only in the case of greedy expansions) but, in both cases,

$$\left(\sum_{i=1}^{n}\frac{s_i}{q^i}\right)+\frac{a_{j_n+1}-a_{j_n}}{q^n}+\left(\sum_{i=n+1}^{\infty}\frac{a_1}{q^i}\right)\geqslant x,$$

it follows that

$$\sum_{i=n+1}^{\infty}\frac{a_1}{q^i}\leqslant x-\sum_{i=1}^{n}\frac{s_i}{q^i}\leqslant\frac{a_{j_n+1}-a_{j_n}}{q^n}+\sum_{i=n+1}^{\infty}\frac{a_1}{q^i}.$$

Letting $n\to\infty$ we obtain

$$x=\sum_{i=1}^{\infty}\frac{s_i}{q^i}$$

again.

We complete the proof by showing that only the above two cases may occur. Suppose on the contrary that there exists n such that $s_n=a_{j_n}<a_m$ and $s_i=a_m$ for all $i>n$. Then

$$\left(\sum_{i=1}^{n}\frac{s_i}{q^i}\right)+\left(\sum_{i=n+1}^{N}\frac{a_m}{q^i}\right)+\left(\sum_{i=N+1}^{\infty}\frac{a_1}{q^i}\right)\leqslant x$$

for all $N \geqslant n$, note that equality can eventually occur in the greedy case; and by the construction of (s_i),

$$\left(\sum_{i=1}^{n} \frac{s_i}{q^i}\right) + \frac{a_{j_n+1} - a_{j_n}}{q^n} + \left(\sum_{i=n+1}^{\infty} \frac{a_1}{q^i}\right) \geqslant x,$$

otherwise we could have chosen $s_n = a_k$ with some $k > j_n$; note that this time equality can eventually occur in the quasi-greedy case.

Letting $N \to \infty$ in the first inequality we obtain that

$$\left(\sum_{i=1}^{n} \frac{s_i}{q^i}\right) + \left(\sum_{i=n+1}^{\infty} \frac{a_m}{q^i}\right) \leqslant x.$$

Combining this with the second inequality we conclude that

$$\left(\sum_{i=1}^{n} \frac{s_i}{q^i}\right) + \left(\sum_{i=n+1}^{\infty} \frac{a_m}{q^i}\right) \leqslant \left(\sum_{i=1}^{n} \frac{s_i}{q^i}\right) + \frac{a_{j_n+1} - a_{j_n}}{q^n} + \left(\sum_{i=n+1}^{\infty} \frac{a_1}{q^i}\right),$$

which is equivalent to

$$\frac{a_m - a_1}{q-1} \leqslant a_{j_n+1} - a_{j_n}.$$

However this contradicts our assumption (2.3). □

In order to characterize the quasi-greedy expansions, let us introduce the quasi-greedy expansions of the differences

$$\Delta_j = a_{j+1} - a_j, \quad j = 1, \ldots, m-1$$

with respect to the translated digit set

$$A' := \{a'_j := a_j - a_1 \ : \ j = 1, \ldots, m\}$$

instead of A:

$$\Delta_j = \frac{\delta_1^j}{q} + \frac{\delta_2^j}{q^2} + \cdots \quad \text{for } j = 1, \ldots, m-1.$$

For the rest of the proof let us denote by (γ_i^j) the greedy expansion of Δ_j with respect to the set A'.

If the sequence (γ_i^j) is infinite, then it coincides with (δ_i^j). On the other hand, we have

Lemma 2.2. *If (γ_i^j) is finite for some j and its last non-zero element is $\gamma_k^j = a_p - a_1$, let us consider the sequence*

$$\sigma_i = \begin{cases} \gamma_i^j & \text{if } i < k, \\ a_{p-1} - a_1 & \text{if } i = k, \\ \delta_{i-k}^p & \text{if } i > k \end{cases}$$

then (σ_i) coincides with (δ_i^j).

Proof. Indeed, (δ_i^j) is an infinite expansion of Δ_j, and it is the largest such expansion therefore any other infinite expansion (η_i) of Δ_j must satisfy

$$(\eta_i) \leqslant (\delta_i^j),$$

so let us show that $(\eta_i) \leqslant (\sigma_i)$.

Since (γ_i^j) is finite, (η_i) and (γ_i^j) cannot be equal, so that

$$(\eta_i) < (\gamma_i^j) = \gamma_1^j \dots \gamma_h^j 00 \dots$$

as 0 is the smallest digit we have that necessarily

$$\eta_1 \dots \eta_h < \gamma_1^j \dots \gamma_h^j$$

and

$$\eta_1 \dots \eta_h \leqslant \gamma_1^j \dots \gamma_h^{j\,-} = \sigma_1 \dots \sigma_h,$$

where if $\gamma_h^j = a_{p+1}$ we denote by $\gamma_h^{j\,-}$ the digit a_p.
If we have strict inequality then

$$(\eta_i) < (\sigma_i).$$

On the other hand if we have equality, then $\eta_{h+1}\eta_{h+2}\dots$ is an infinite expansion of Δ_{p-1} and then

$$\eta_{h+1}\eta_{h+2}\dots \leqslant \delta_1^{p-1}\delta_2^{p-1}\dots$$

by definition of (δ_j^{p-1}). Hence, we have that for all infinite expansion (η_i) of Δ_j

$$(\eta_i) \leqslant \sigma_1 \dots \sigma_h \delta_1^{p-1}\delta_2^{p-1}\dots$$

so that (σ_i) is the largest infinite expansion of Δ_j and therefore coincides with (δ_i^j). □

Theorem 2.3. *Assume again that condition* (2.3) *is satisfied*:

$$\max_{1 \leqslant j \leqslant m-1} (a_{j+1} - a_j) < \frac{a_m - a_1}{q-1}.$$

Then the map $x \mapsto (s_i)$, *where* (s_i) *denotes the quasi-greedy expansion of* x (*resp. greedy expansion of* x), *is a strictly increasing bijection between the interval* $(a_1/(q-1), a_m/(q-1)]$ *and the set of infinite sequences in* A (*resp. the set of sequences in* A), *satisfying*

$$s'_{n+1}s'_{n+2}\dots \leqslant \delta_1^{j_n}\delta_2^{j_n}\dots \tag{2.5}$$

whenever $s_n = a_{j_n} < a_m$, *where we use the notation* $s'_i := s_i - a_1$ (*or where we have*

$$s'_{n+1}s'_{n+2}\dots < \delta_1^{j_n}\delta_2^{j_n}\dots, \tag{2.6}$$

respectively).

Proof. The strict monotonicity of the map $x \mapsto (s_i)$ follows from the definition of the quasi-greedy and greedy expansion.

Next, we prove that if (s_i) is the quasi-greedy or the greedy expansion of some x, then we have

$$s'_{n+1}s'_{n+2}\ldots \leqslant \delta_1^{j_n}\delta_2^{j_n}\ldots$$

for all n such that $s_n = a_{j_n} < a_m$.

Indeed, if $s_n = a_{j_n} < a_m$ for some n, then it follows from the definition of (s_n) that

$$\sum_{i=1}^{n-1}\frac{s_i}{q^i} + \frac{a_{j_n}}{q^n} + \sum_{i=n+1}^{\infty}\frac{s_i}{q^i} = x \leqslant \sum_{i=1}^{n-1}\frac{s_i}{q^i} + \frac{a_{j_n+1}}{q^n} + \sum_{i=n+1}^{\infty}\frac{a_1}{q^i}.$$

Note that equality holds only the case (s_i) quasi-greedy.

Hence, by canceling the first term on both sides, we have

$$\sum_{i=n+1}^{\infty}\frac{s_i - a_1}{q^i} \leqslant \frac{a_{j_n+1} - a_{j_n}}{q^n}$$

that is

$$\sum_{i=1}^{\infty}\frac{s'_{n+i}}{q^i} \leqslant a_{j_n+1} - a_{j_n}. \tag{2.7}$$

On the other hand, by definition, $(\gamma_i^{j_n})$ is the lexicographically largest sequence (σ_i) in A' satisfying

$$\sum_{i=1}^{\infty}\frac{\sigma_i}{q^i} = a_{j_n+1} - a_{j_n}. \tag{2.8}$$

This expansion is eventually infinite, and in case it coincides with the quasi-greedy expansion of Δ_{j_n}.

Comparing Equalities (2.7) and (2.8), we finally have

$$(s'_{n+i}) \leqslant (\delta_i^{j_n}).$$

Moreover, in the case of the greedy expansion this inequality is strict because the two sequences are different: indeed, in (2.7) we have strict inequality, while in (2.8) we have equality by the definition of $\gamma_i^{j_n}$.

The sequence $(\gamma_i^{j_n})$ can be finite or infinite where the quasi-greedy expansions $(\delta_i^{j_n})$ are infinite. In the infinite cases we are done.

Second step: It remains to establish this last inequality in the greedy case when the sequence $(\gamma_i^{j_n})$ is finite. Then $(\delta_i^{j_n}) < (\gamma_i^{j_n})$.

In order to prove the inequality

$$(s'_{n+i}) < (\delta_i^{j_n})$$

we define a strictly increasing sequence of integers $k_0 < k_1 < k_2 < \ldots$ and a corresponding sequence $r_0, r_1, r_2, \ldots$ of integers belonging to the set $\{1, \ldots, m-1\}$ as follows.

Since $(\gamma_i^{j_n})$ is finite, there exists a k such that $\gamma_k^{j_n} > 0$, and that for all $i > k$ we have $\gamma_i^{j_n} = 0$.

Then the inequality

$$(s'_{n+i}) < (\gamma_i^{j_n})$$

implies that

$$s'_{n+1}s'_{n+2}\dots s'_{n+k} < \gamma_1^{j_n}\dots\gamma_k^{j_n}.$$

Since $\delta_k^{j_n} = \gamma_k^{j_n\,-}$, it follows that

$$s'_{n+1}s'_{n+2}\dots s'_{n+k} \leqslant \delta_1^{j_n}\dots\delta_k^{j_n} \tag{2.9}$$

whenever $s_n < a_m$.

Observe that because of (2.9) we have only two possibilities: either we have a strict inequality

$$s'_{n+1}s'_{n+2}\dots s'_{n+k} < \delta_1^{j_n}\dots\delta_k^{j_n}$$

or

$$s'_{n+1}s'_{n+2}\dots s'_{n+k} = \delta_1^{j_n}\dots\delta_k^{j_n} \quad \text{and } s_{n+k} < a_m,$$

since in the second case we know that the last considered digit of the quasi-greedy expansion is strictly smaller than the last digit of the finite greedy expansion:

$$s'_{n+k} = \delta_k^{j} = \gamma_k^{j_n\,-} < \gamma_k^{j_n} \leqslant a'_m.$$

This means that $s_{n+k} = a_p < a_m$ and from the first step above, we have that

$$s'_{n+k+1}s'_{n+k+2}\dots < \gamma_1^p\gamma_2^p\dots.$$

As before we have two cases depending upon the fact the greedy expansion (γ_i^p) is infinite or not.

In the first case we have done, because we will have that

$$s'_{n+1}s'_{n+2}\dots s'_{n+k}s'_{n+k+1}s'_{n+k+2}\dots < \delta_1^{j_n}\delta_2^{j_n}\dots\delta_k^{j_n}\delta_1^p\delta_2^p\dots.$$

In the second case we have to apply the same reasoning as above: we set $r_0 = p$ and we know that

$$s'_{n+k+1}s'_{n+k+2}\dots s'_{n+k+k_1} < \gamma_1^p\gamma_2^p\dots\gamma_{k_1}^p$$

where $\gamma_{k_1}^p$ is the last non-zero digit of the (finite) greedy expansion of Δ_p.

By diminishing the last digit of the greedy expansion we obtain

$$s'_{n+k+1}s'_{n+k+2}\dots s'_{n+k+k_1} \leqslant \gamma_1^p\gamma_2^p\dots\gamma_{k_1}^{p\,-} = \delta_1^p\delta_2^p\dots\delta_{k_1}^p$$

and by applying the same argument to the last digit $a_s = \delta_{k_1}^p$, again we have $a_1 < a_s < a_m$, and by hypothesis we have that the greedy development of Δ_s, is (γ_i^s).

By iterating the previous reasoning we obtain the two sequences $k_0, k_1, \ldots$ and $r_0, r_1, \ldots$, such that

$$s'_{n+1}s'_{n+2}\ldots \leqslant \gamma_1^{j_n}\gamma_2^{j_n}\ldots\gamma_{k_0}^{j_n\,-}\gamma_1^{r_0}\gamma_2^{r_0}\ldots\gamma_{k_1}^{r_0\,-}\ldots$$

and in particular the sequence

$$\gamma_1^{j}\gamma_2^{j}\ldots\gamma_{k_0}^{j_n\,-}\gamma_1^{r_0}\gamma_2^{r_0}\ldots\gamma_{k_1}^{r_0\,-}\ldots$$

is the quasi-greedy expansion of Δ_{j_n}.

Now we prove that this inequality is strict. Indeed suppose that we have equality; this means that

$$s'_{n+1}s'_{n+2}\ldots = \delta_1^{j_n}\delta_2^{j_n}\ldots$$

from which we obtain

$$\sum_{i=1}^{\infty}\frac{s_{n+i}}{q^i} = \sum_{i=1}^{\infty}\frac{\delta_i^{j_n}+a_1}{q^i} = \sum_{i=1}^{\infty}\frac{\delta_i^{j_n}}{q^i} + \sum_{i=1}^{\infty}\frac{a_1}{q^i} = \Delta_{j_n} + \sum_{i=1}^{\infty}\frac{a_1}{q^i}.$$

In this case we would have

$$\begin{aligned} x &= \sum_{i=1}^{\infty}\frac{s_i}{q^i} = \sum_{i=1}^{n}\frac{s_i}{q^i} + \sum_{i=n+1}^{\infty}\frac{s_i}{q^i} \\ &= \sum_{i=1}^{n}\frac{s_i}{q^i} + \frac{1}{q^n}\sum_{i=1}^{\infty}\frac{s_{i+n}}{q^i} = \sum_{i=1}^{n}\frac{s_i}{q^i} + \frac{\Delta_{j_n}}{q^n} + \frac{1}{q^n}\sum_{i=1}^{\infty}\frac{a_1}{q^i} \\ &= \sum_{i=1}^{n-1}\frac{s_i}{q^i} + \frac{s_n+\Delta_{j_n}}{q^n} + \sum_{i=n+1}^{\infty}\frac{a_1}{q^i} = \sum_{i=1}^{n-1}\frac{s_i}{q^i} + \frac{s_n^+}{q^n} + \sum_{i=n+1}^{\infty}\frac{a_1}{q^i}. \end{aligned}$$

But this is an expansion of x which is lexicographically greater than (s_i), contradicting the hypothesis that (s_i) is greedy.

Third step: Now, we prove the other direction in the statement of the theorem: let (s_i) be a sequence satisfying the conditions of the theorem.

We shall prove that if (s_i) is infinite and it satisfies the condition (2.5) then it is the quasi-greedy expansion of

$$x := \sum_{i=1}^{\infty}\frac{s_i}{q^i};$$

alternatively, if (s_i) satisfies condition (2.6) then it is the greedy expansion of x.

Since $a_1 \leqslant s_i \leqslant a_m$ for all i then clearly x belongs to $[a_1/(q-1), a_m/(q-1)]$, moreover if (s_i) is infinite then not all s_i's are equal to a_1, and so x cannot be $a_1/(q-1)$. It remains to be shown that if $s_n = a_{p_n} < a_m$ for some n, then

$$\sum_{i=n+1}^{n-1}\frac{s_i}{q^i} + \frac{a_{p_n+1}}{q^n} + \sum_{i=n+1}^{\infty}\frac{a_1}{q^i} \geqslant x,$$

note that in case of greedy expansion we prove that even the strict inequality holds.

By definition,

$$x = \sum_{i=1}^{\infty} \frac{s_i}{q^i} = \sum_{i=1}^{n-1} \frac{s_i}{q^i} + \frac{a_{p_n}}{q^n} + \sum_{i=n+1}^{\infty} \frac{s_i}{q^i},$$

so that the previous inequality is equivalent to

$$\frac{a_{p_n+1}}{q^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{q^i} \geqslant \frac{a_{p_n}}{q^n} + \sum_{i=n+1}^{\infty} \frac{s_i}{q^i},$$

or to

$$\sum_{i=n+1}^{\infty} \frac{s'_i}{q^i} \leqslant \frac{a_{p_n+1} - a_{p_n}}{q^n}.$$

In order to prove this inequality, we define a strictly increasing sequence of integers $k_0 < k_1 < k_2 < \ldots$ and a corresponding sequence of integers r_0, r_1, r_2 in $\{1, \ldots, m-1\}$ as follows.

First, set $k_0 = n$ and $r_0 = p_n$. Then

$$(s'_{n+i}) \leqslant (\delta_i^{p_n})$$

by assumption. If this inequality is strict, then there exists a first integer $k_1 \geqslant n+1 = k_0+1$ such that

$$s'_{k_1} < \delta^{p_n}_{i-n} = \delta^{p_n}_{i-k_0}.$$

Then $s'_{k_1} = a_{r_1} - a_1$ for some $1 \leqslant r_1 < m$. Thus we have

$$(s'_{k_1+i}) \leqslant (\delta_i^{r_1})$$

by assumption. If this inequality is strict then there exists a first integer $k_2 \geqslant k_1+1$ satisfying

$$s'_{k_2} < \delta^{r_1}_{i-k_1};$$

then $s'_{k_2} = a_{r_2} - a_1$ for some $1 \leqslant r_2 < m$.

Iterating this argument, either we obtain two infinite sequences (k_j) and (r_j), or after a finite number of steps we obtain that $s'_{k_N+i} = \delta_i^{r_N}$ for all $i \geqslant 1$.

In the first case, we have

$$\begin{aligned}
\sum_{i=n+1}^{\infty} \frac{s'_i}{q^i} &= \sum_{j=0}^{\infty} \left(\sum_{i=k_j+1}^{k_{j+1}} \frac{s'_i}{q^i} \right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{i=k_j+1}^{k_{j+1}-1} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{s'_{k_{j+1}}}{q^{k_{j+1}}} \right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{s'_{k_{j+1}} - \delta^{r_j}_{k_{j+1}-k_j}}{q^{k_{j+1}}} \right).
\end{aligned}$$

Since

$$s'_{k_{j+1}} = a_{r_{j+1}} - a_1 < a_{r_{j+1}+1} - a_1 \leqslant \delta^{r_j}_{k_{j+1}-k_j},$$

it follows that

$$\sum_{j=0}^{\infty}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{s'_{k_{j+1}} - \delta^{r_j}_{k_{j+1}-k_j}}{q^{k_{j+1}}}\right) \leqslant \sum_{j=0}^{\infty}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right).$$

Finally, since the sequences $(\delta^1_i), \ldots, (\delta^{m-1}_i)$ are all infinite by hypothesis of the theorem, we obtain that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right)\\
&\quad < \sum_{j=0}^{\infty}\left(\sum_{i=k_j+1}^{\infty} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right)\\
&\quad = \sum_{j=0}^{\infty}\left(\frac{a_{r_j+1} - a_{r_j}}{q^{k_j}} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right)\\
&\quad = \frac{a_{p+1} - a_p}{q^n}.
\end{aligned}$$

In the second case, we obtain in a similar way that

$$\begin{aligned}
\sum_{i=n+1}^{\infty} \frac{s'_i}{q^i} &= \sum_{j=0}^{N-1}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{s'_i}{q^i}\right) + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&= \sum_{j=0}^{N-1}\left(\sum_{i=k_j+1}^{k_{j+1}-1} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{s'_{k_{j+1}}}{q^{k_{j+1}}}\right) + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&= \sum_{j=0}^{N-1}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{s'_{k_{j+1}} - \delta^{r_j}_{k_{j+1}-k_j}}{q^{k_{j+1}}}\right) + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&\leqslant \sum_{j=0}^{N-1}\left(\sum_{i=k_j+1}^{k_{j+1}} \frac{\delta^{r_j}_{i-k_j}}{q^i} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right) + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&\leqslant \sum_{j=0}^{N-1}\left(\frac{a_{r_j+1} - a_{r_j}}{q^{k_j}} + \frac{a_{r_{j+1}} - a_{r_{j+1}+1}}{q^{k_{j+1}}}\right) + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&= \frac{a_{p+1} - a_p}{q^n} + \frac{a_{r_N} - a_{r_N+1}}{q^{k_N}} + \sum_{i=k_N+1}^{\infty} \frac{\delta^{r_N}_i}{q^i}\\
&= \frac{a_{p+1} - a_p}{q^n}.
\end{aligned}$$

This completes the proof of the theorem. □

3. Unique expansions

Given an alphabet $A = \{a_1, \dots, a_m\}$ let us introduce the quasi-greedy expansion of the differences

$$\Delta_j = a_{j+1} - a_j = \frac{\delta_1^j}{q} + \frac{\delta_2^j}{q^2} + \dots$$

for $j = 1, \dots m-1$ an in Theorem 2.3.

Furthermore let us also introduce the quasi-greedy expansions $(\overline{\delta_i^j})$ of the differences

$$\overline{\Delta_j} = \overline{a_{j+1}} - \overline{a_j} = \frac{\overline{\delta_1^j}}{q} + \frac{\overline{\delta_2^j}}{q^2} + \dots$$

with respect to the *dual* alphabet $\overline{A} = \{\overline{a_1}, \dots, \overline{a_m}\}$ given by

$$\overline{a_j} = a_1 + a_m - a_{m+1-j}, \quad j = 1, \dots, m.$$

Now, let us denote by $\mathcal{A}_q$ the set of numbers x whose greedy expansion (with respect to the original alphabet A) is the unique possible expansion.

We have the following characterization of this set, which generalizes a result given in [6]:

Theorem 3.1. *Assume again that condition* (2.3) *is satisfied*:

$$\max_{1 \leqslant j \leqslant m-1} (a_{j+1} - a_j) < \frac{a_m - a_1}{q-1}.$$

Then the map $x \mapsto (c_i)$, *where* (c_i) *denotes the greedy expansion of* x, *is a strictly increasing bijection between the set* $\mathcal{A}_q$ *and the set of sequences in* A, *satisfying*

$$(c_{n+i} - a_1) < (\delta_i^j)$$

whenever $c_n = a_j < a_m$, *and*

$$(a_m - c_{n+i}) < (\overline{\delta_i^j})$$

whenever $c_n = a_{1+m-j} > a_1$.

Proof. Let (c_i) be the greedy expansion of x. If this expansion is not unique, then there exists another expansion $(d_i) < (c_i)$ of x. Then by definition we have

$$(\overline{d_i}) > (\overline{c_i})$$

and both are expansions of $(a_1+a_m)/(q-1)-x$, so that $(\overline{c_i})$ cannot be the greedy expansion of $(a_1 + a_m)/(q-1) - x$.

On the other hand, if (c_i) is the unique expansion of x, the $(\overline{c_i})$ is the unique expansion of $(a_1 + a_m)/(q-1) - x$.

Hence, $(\overline{c_i})$ is the greedy expansion of $(a_1 + a_m)/(q-1) - x$. □

4. Examples

Let us introduce the following notation for periodic expansions (c_i) with a period of length T starting at c_k:

$$(c_i) = c_1 \dots c_{k-1}(c_k \dots c_{k+T-1})^\infty.$$

Example 4.1. If we fix $q = \frac{1+\sqrt{5}}{2}$ and we consider $A = \{0, 1, 3\}$, then we have

- the alphabet A' is equal to A (as $0 \in A$), gaps in A are

$$\Delta_1 = 1 \quad \text{and} \quad \Delta_2 = 2;$$

- as stated in Proposition 2.1, for every $x \in [0, 3/(q-1)]$ (where $3/(q-1) \approx 4.8541$) there exists an expansion in base q and alphabet A;
- the greedy expansion of 1 is the sequence $11(0)^\infty$,
- the greedy expansion of 2 is the sequence $3001(0)^\infty$,
- the greedy expansion of 1.2 is the sequence

$$11(01001010100100100000)^\infty$$

which has development

$$\frac{1}{q} + \frac{1}{q^2} + \frac{q^{20}}{q^{20}-1}\left(\frac{1}{q^4} + \frac{1}{q^7} + \frac{1}{q^9} + \frac{1}{q^{11}} + \frac{1}{q^{14}} + \frac{1}{q^{17}}\right).$$

This expression is also equal to

$$1 + \frac{q^3}{q^{20}-1}\left(1 + q^3 + q^6 + q^8 + q^{10} + q^{13}\right) = 1 + \frac{(2q+1)(319q+198)}{6765q+4180},$$

the last equality has been obtained by the fact that if F_k is the Fibonacci sequence

$$F_0 = 0,\ F_1 = 1,\ F_{k+1} = F_k + F_{k-1}$$

then every power q^n can be obtained by considering the following equivalence:

$$q^n = F_n q + F_{n-1}.$$

The same argument is then used to further simplify the expression

$$1 + \frac{(2q+1)(319q+198)}{6765q+4180} = 1 + \frac{198 + 715q + 638q^2}{6765q+4180} = 1 + \frac{1353q+836}{6765q+4180}$$

and

$$6765q + 4180 = 5(1353q + 836).$$

- In order to apply Theorem 2.3, we need the quasi-greedy expansion for $\Delta_1 = 1$ and $\Delta_2 = 2$ in A' and we obtain
  - $\Delta_1 = 1 \mapsto (\eta_i^1) = 1(01)^\infty$ and
  - $\Delta_2 = 2 \mapsto (\eta_i^2) = 300(01)^\infty$,

  now, we check the content of the theorem on the greedy expansion of 1.2 as tested before it is the sequence

$$11(0100101010010010000 0)^\infty.$$

As it begins with the digit 1 we have to check that its subsequence

$$1(01001010100100100000)^\infty$$

is lexicographically smaller than the quasi-greedy expansion of $\Delta_2 = 2$, so

$$1(01001010100100100000)^\infty < 300(01)^\infty$$

and it is the case for the first digit, again we have to check that

$$(01001010100100100000)^\infty < 300(01)^\infty$$

and again it is right, the next digit is 0 so this time we have to test the subsequence starting at the 4th digit against the quasi-greedy expansion of $\Delta_1 = 1$, so

$$1001010100100100000(01001010100100100000)^\infty < 1(01)^\infty.$$

In this case we have to look until to the third digit in order to verify the theorem, and as this digit is zero the next step of the verification has to be performed again between $1010100100100000(01001010100100100000)^\infty$ and the expansion of Δ_1, i.e. $1(01)^\infty$.

The verification is then continued on the whole digit sequence; as we shall see in the next section, if the quasi-greedy expansions of the gaps in A are representable by finite automata (in the Büchi sense), we will find that any greedy expansion in A is representable by finite automata.

Example 4.2. Let $q \approx 1.3247$ be the first Pisot number (let us recall that a Pisot number is an algebraic integer $q > 1$ if all algebraic conjugates of q different of q lie in the open disc $|z| < 1$ of the complex plane). It is the only solution of modulus > 1 of the equation $x^3 = x + 1$, and consider the set $A = \{0, 1, 3\}$ as the alphabet.

The set A has deleted elements, in order to illustrate our result, we compute the greedy and quasi-greedy expansions of the gaps in A.

We have $\Delta_1 = 1$ and $\Delta_2 = 2$; their greedy expansions are:

$$1 = \frac{1}{q} + \frac{1}{q^5}$$

and

$$2 = \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q^5}.$$

The quasi-greedy expansions of the gaps are then

$$1 = \frac{1}{q} + \frac{1}{q}\sum_{i=1}^{\infty} \frac{1}{q^{5i}} = \frac{1}{q} + \frac{1}{q}\left(\frac{1}{q^5 - 1}\right).$$

In fact, by the equation $q^3 = q + 1$, we have

$$\begin{aligned}\frac{1}{q} + \frac{1}{q}\left(\frac{1}{q^5 - 1}\right) &= \frac{q^5 - 1 + 1}{q(q^5 - 1)} = \frac{q^4}{q^5 - 1}\\ &= \frac{q^2 + q}{q^3 + q^2 - 1} = \frac{q^2 + q}{(q + 1) + q^2 - 1} = \frac{q^2 + q}{q^2 + q} = 1.\end{aligned}$$

On the other hand, we have

$$\frac{1}{q^2} + \frac{1}{q^3} = \frac{q + 1}{q^3} = \frac{q^3}{q^3} = 1$$

and by adding it to the previous expansion we get

$$2 = \left(\frac{1}{q^2} + \frac{1}{q^3}\right) + \frac{1}{q} + \frac{1}{q}\left(\frac{1}{q^5 - 1}\right) = \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q}\left(\frac{1}{q^5 - 1}\right).$$

Example 4.3. We fix the alphabet $A = \{1, 3, 4, 5, 7\}$ and the base $q \approx 3.61645$, solution of the equation $q^4 - 3q^3 - 2q^2 - 3 = 0$; note that q is a Perron number (let us recall that a Perron number is an algebraic integer $q > 1$, if all algebraic conjugates of q different from q lie in the open disc $|z| < |q|$ of the complex plane). As stated in Proposition 2.1, for every $x \in [1/(q - 1), 7/(q - 1)]$ (where $1/(q - 1) \approx 0.382197$ and $7/(q - 1) \approx 2.67538$) there exists an expansion in base q and alphabet A. The first step in order to apply the theorem is to find quasi-greedy expansions for gaps in A in the alphabet $A' = \{0, 2, 3, 4, 6\}$. The gaps are $\Delta_1 = \Delta_4 = 2$ and $\Delta_2 = \Delta_3 = 1$. We have the following greedy expansions:

- $\Delta_1 = \Delta_4 = 2 \mapsto (\gamma_i^1) = (\gamma_i^4) = 6406(0)^{\infty}$,
- $\Delta_2 = \Delta_3 = 1 \mapsto (\gamma_i^2) = (\gamma_i^3) = 3203(0)^{\infty}$;

furthermore, their quasi-greedy expansions are:

- $\Delta_1 = \Delta_4 = 2 \mapsto (\eta_i^1) = (\eta_i^4) = (6404)^{\infty}$,
- $\Delta_2 = \Delta_3 = 1 \mapsto (\eta_i^2) = (\eta_i^3) = (3202)^{\infty}$.

Let us test the theorem on the greedy expansion of 1 in A. In this case it is infinite (and so it coincides with the quasi-greedy expansion):

$$1 \mapsto (c_i) = 314(1141)^{\infty}.$$

This implies that $(c_i') = 203(0030)$, and the theorem can be easily verified in this case.

5. Correspondence with automata

In this section, we apply the characterization in Theorem 2.3 in order to build out an automaton accepting all and only the greedy sequences for a given set of digits A in a given base q. The construction is based on the automata associated to the quasi-greedy expansions of gaps Δ_j in A', if they exist.

A first remark is that this construction is effective and allows us to build a finite automaton only if we can build the Büchi automata corresponding to the quasi-greedy expansions of each Δ_j's. This means that we have to suppose the regularity of quasi-greedy expansions. This is the case if the quasi-greedy expansion of every Δ_j is periodic. For example, this is satisfied if q is a Pisot number and $A = \{0, 1, \ldots, [q]\}$, by a result in the appendix to the present paper.

A second remark is the following: the automaton associated with the quasi-greedy expansion (δ_i^j) of Δ_j is defined as the Büchi automaton accepting the translated sequence $(\delta_i^{j\prime})$ where $\delta_i^{j\prime} = \delta_i^j + a_1$. This is related to the fact that condition (2.6) in Theorem 2.3 could have been equivalently stated in the following way:

$$c_{n+1}c_{n+2}\ldots < \delta_1^{j_n\prime}\delta_2^{j_n\prime}\ldots \tag{5.1}$$

whenever $c_n = a_{j_n} < a_m$.

Let us give the construction of the automaton:

Definition 1. For every periodic expansion in A'

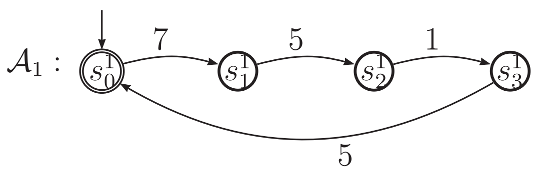
$$(d_i) = \eta_1 \ldots \eta_n(\eta_{n+1} \ldots \eta_{n+k})^\infty,$$

the Büchi automaton $\mathcal{A}$ *associated with* (d_i) is an automaton on the alphabet A with set of states

$$S = \{s_0, \ldots, s_{n+k}\}$$

and transitions $s_i \overset{\eta'_{i+1}}{\to} s_{i+1}$ for every $0 \leqslant i < n+k$ and $s_{n+k} \overset{\eta'_{n+k}}{\to} s_{n+1}$, where s_0 is the initial state and $\eta_i{}' = \eta_i + a_1$.

Example 5.1. Let us consider the quasi-greedy expansion in Example 4.3, $(\delta_i^1) = (6404)^\infty$, the associated automaton recognizing the sequence $(\delta_i^{1\prime}) = (7515)^\infty$ is



Construction 1 (*Greedy automaton*). *For every gap Δ_j in the alphabet A we consider its quasi-greedy expansion in the alphabet A' and the associated automaton $\mathcal{A}_j$, with states $S^j = \{s_i^j\}$ and set of transitions $T^j = \{s \overset{\eta'}{\to} s'\}$.*

A greedy automaton $\mathcal{A}$ recognizing every greedy sequence in base q and digit set A can be obtained by merging all the automata $\mathcal{A}_j$ in the following way:

- *we consider as set of states S of $\mathcal{A}$ the direct union of states in all $\mathcal{A}_j$'s, plus a new state s_0:*

$$S = \{s_0\} \cup \bigoplus_{j=1}^{m} S^j,$$

- *we consider the direct union of the transition sets of $\mathcal{A}_j$'s, for $1 \leqslant j < m$:*

$$T' = \bigoplus_{j=1}^{m} T^j.$$

Define

$$T_{s_0} := \{s_0 \xrightarrow{a_j} s_0^j | a_j < a_m\} \cup \{s_0 \xrightarrow{a_m} s_0\}$$

and for every state $s \neq s_0 \in S$ and transition $s \xrightarrow{a_i} s' \in T'$ with $a_i > a_1$, consider

$$T_s = \{s \xrightarrow{a_j} s_0^j | a_1 \leqslant a_j < a_i\}.$$

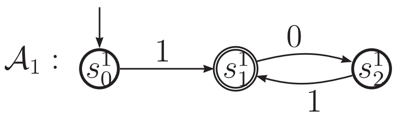
Then the set of transitions T for $\mathcal{A}$ is

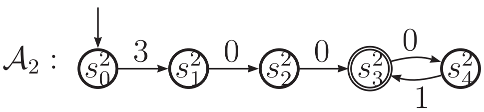
$$T = T' \cup T_{s_0} \cup \bigcup_{s \in S} T_s;$$

- *finally, the set of initial states is $I = \{s_0\}$ and the set of final states is*

$$F = \{s_0\} \cup \bigcup_{j=1}^{m-1} \{s_0^j\}.$$

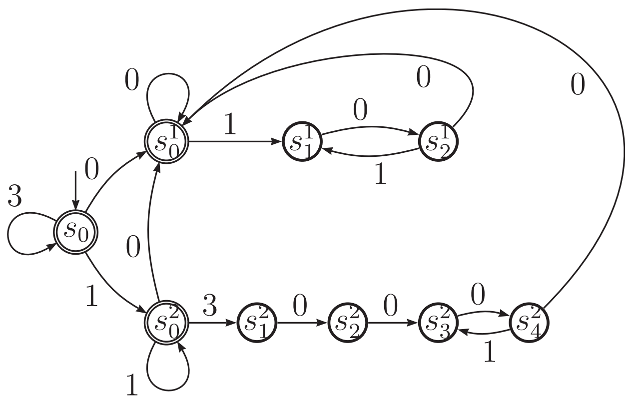
Example 5.2. Let us apply the above construction to Example 4.1. In this case it is easy to build the automaton corresponding to $\Delta_1 = 1$ and $\Delta_2 = 2$ because $A = \{0, 1, 3\}$.





Then we merge the two automata, and for any state s with a transition labeled $a_j \neq a_1$ and for every $a_i < a_j$ we add a transition with label a_i from s to the initial state of $\mathcal{A}_i$.

Here, we obtain the following automaton:



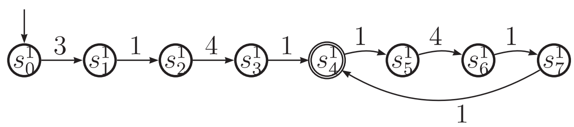
The reader may check that the greedy expansions in Example 4.1 are accepted by this automaton.

Example 5.3. Let us consider $q \approx 3.61645$ as in Example 4.3, but with the new alphabet $A = \{0, 1, 3, 4, 6\}$. We have two possible gaps $\Delta_1 = \Delta_3 = 1$ and $\Delta_2 = \Delta_4 = 2$, the admissible interval is $x \in [0, 2.29318]$ and the corresponding quasi-greedy expansions in $A' = A$ are given by

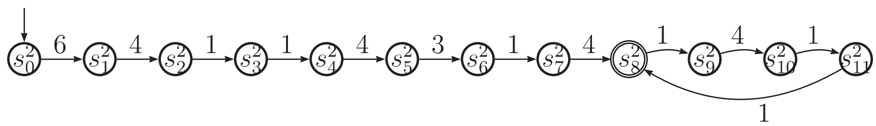
$$\Delta_1 = \Delta_3 = 1 \mapsto (\eta_i^1) = (\eta_i^3) = 3141(1411)^\infty,$$

$$\Delta_2 = \Delta_4 = 2 \mapsto (\eta_i^2) = (\eta_i^4) = 64114314(1411)^\infty.$$

The associated automata are the following:
$\mathcal{A}_1 = \mathcal{A}_3$:



$\mathcal{A}_2 = \mathcal{A}_4$:



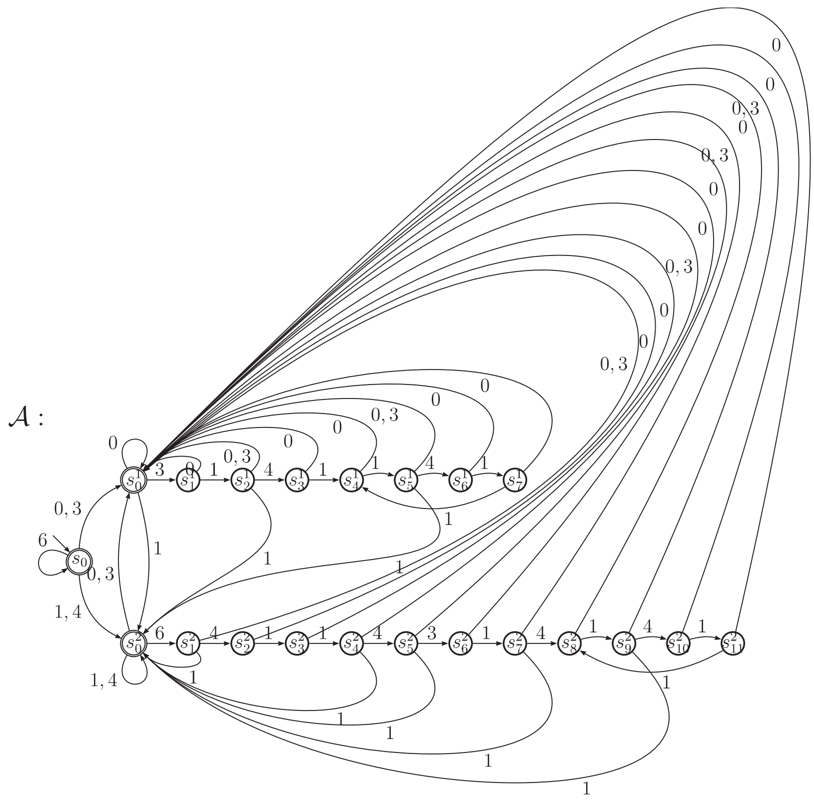


Fig. 1. The greedy automaton for $A = \{0, 1, 3, 4, 6\}$ (with repeated gaps).

The above construction is redundant in this case, because there are repeated gaps. We should merge twice the identical automaton recognizing the quasi-greedy expansions for Δ_1 and Δ_3 (and analogously for Δ_2 and Δ_4). Since the repeated automaton behave in the same way we do not repeat twice the same automaton. We consider just one copy, thereby obtaining the automaton $\mathcal{A}$ in Fig. 1.

Example 5.4. Let us consider the same q as in the previous example, but this time with the alphabet $A = \{1, 3, 4, 5, 7\}$ as in Example 4.3, which does not contain the element 0.

In this case, $A' = \{0, 2, 3, 4, 6\} \neq A$ and the quasi-greedy sequences have to be translated in order to build the greedy automata.

We have $\Delta_1 = \Delta_4 = 2 \mapsto (6404)^\infty$ and $\Delta_2 = \Delta_3 = 1 \mapsto (3202)^\infty$, the associated automata have to recognize the translated sequences in A: $(7515)^\infty$ and $(4313)^\infty$.

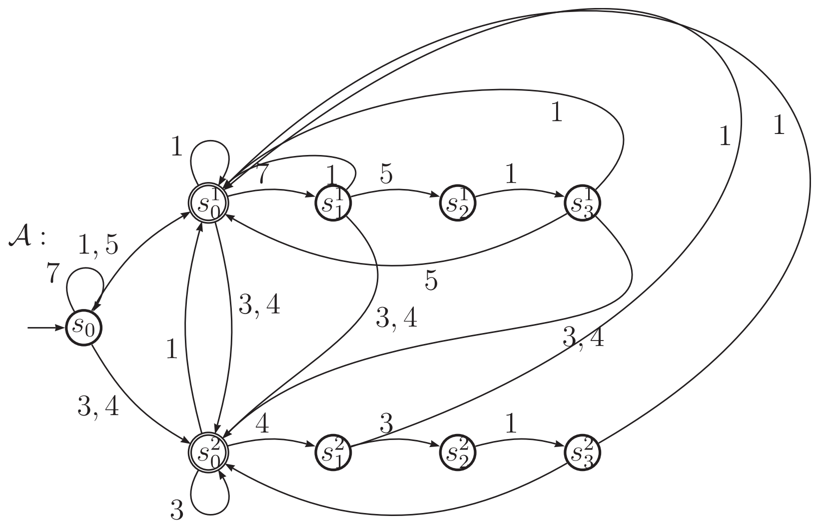
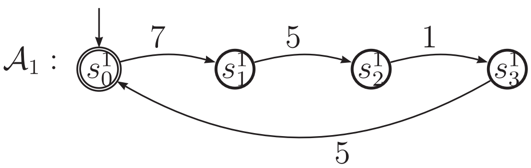
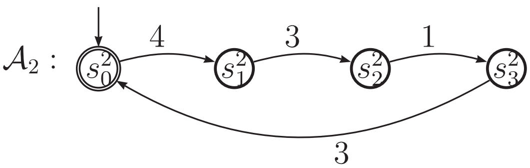


Fig. 2. The greedy automaton for $A = \{1, 3, 4, 5, 7\}$ (with repeated gaps and $0 \notin A$).





The greedy automaton recognizing all greedy expansions is then represented in Fig. 2.

Appendix A. a note on periodic expansions

It is well-known that the decimal fraction $\{x\}$ of a real number x is eventually periodic if and only if x is a rational number. This property is known to remain valid for the expansion in non-integer bases if the base is a Pisot number, see [7,10,2]. We give here a new proof of this fact by adapting an approach used by Bogmér et al. in [3]. For $x \in \mathbb{R}_+$, let $[x]$ be the integer part of x.

For the definition of Pisot numbers, see Example 4.2; for example, all rational integers $q \geqslant 2$ are Pisot numbers, and the golden section $q = \frac{1}{2}(1 + \sqrt{5})$ is also a Pisot number because its other conjugate $\frac{1}{2}(1 - \sqrt{5})$ has modulus < 1.

In order to state our result precisely, let us fix $A = \{a_1 < a_2 < \cdots < a_m\}$ and $q > 1$ as in Section 2 and let us consider the expansions (c_n) of the numbers $x \in \left[\frac{a_1}{q-1}, \frac{a_m}{q-1}\right]$ defined recursively by the following formula:

$$\begin{cases} c_1 := f(qx), \\ c_n := f\left(q^n x - \sum_{i=1}^{n-1} c_i q^{n-i}\right) \end{cases} \tag{A.1}$$

with a given function $f : \mathbb{R} \to A$.

Remark A.1. This situation describes the case of a generic sequence where the choice of the next digit is deterministically taken starting from the previous steps, and this is the case for greedy, quasi-greedy and lazy sequences:

(a) the greedy expansion is obtained if

$$f_G(y) = \max_{a \in A} \left\{ a | a \leqslant y - \frac{a_1}{q-1} \right\};$$

(b) the quasi-greedy expansion is obtained if

$$f_Q(y) = \max_{a \in A} \left\{ a | a < y - \frac{a_1}{q-1} \right\};$$

(c) the lazy expansion is obtained if

$$f_L(y) = \min_{a \in A} \left\{ a | a \geqslant y - \frac{a_m}{q-1} \right\}.$$

Another classical expansion, the β-expansion of Rényi [9] can be defined in the same way by using the function $f_\beta(y) = [y]$.

In fact, if we consider the greedy case, and we define the kth rest x_k of a given expansion (c_i) of x as

$$x_k := c_k + \frac{c_{k+1}}{q} + \frac{c_{k+2}}{q^2} + \ldots, \quad k = 1, 2, \ldots;$$

then we have

$$\begin{aligned} c_1 &:= f_G(qx_0) = \max_{a \in A} \left\{ a | a \leqslant qx - \frac{a_1}{q-1} \right\}, \\ c_n &:= f_G(x_n) = \max_{a \in A} \left\{ a | a \leqslant x_n - \frac{a_1}{q-1} \right\}. \end{aligned} \tag{A.2}$$

Note that

$$x_n = q^n \left(x - \sum_{i=1}^{n-1} \frac{c_i}{q^i} \right)$$

and so

$$c_n = f_G(x_n) = \max_{a \in A} \left\{ a | a \leqslant q^n x - q^n \sum_{i=1}^{n-1} \frac{c_i}{q^i} - \sum_{i=1}^{\infty} \frac{a_1}{q^i} \right\};$$

this shows that we choose the biggest a satisfying

$$\sum_{i=1}^{n-1} \frac{c_i}{q^i} + \frac{a}{q^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{q^i} \leqslant x,$$

that is, we apply the greedy algorithm.

Theorem A.1. *If x has an eventually periodic expansion, then x belongs to the field $\mathbb{Q}(q)$. If q is a Pisot number, then the converse also holds true.*

Remark A.2. (a) In the special case where $A = \{0, 1, \ldots, [q]\}$, the theorem reduces to an earlier result in [7,10,2]. By using the β-expansions they established the converse part if q is a Pisot number then the β-expansion of every $x \in \mathbb{Q}(q)$ is eventually periodic.

(b) Our proof will be based on a different approach of Bogmér et al. [3]. Moreover, we will show that the converse part can be established by an arbitrary type of expansion defined by some deterministic rule of similar type, for example: *quasi-greedy* expansions i.e., the lexicographically largest infinite expansion (see e.g., [4,1]) or alternatively *lazy* expansion i.e., the lexicographically smallest expansions (see e.g., [5] and again [1]).

Proof. If the expansion of $x \in \mathbb{R}$ has a period of length d, then reasoning as in the classical case where q is a rational integer, we obtain that $q^d x - x = \frac{a}{b}$. Then $x = \frac{a}{b(q^d - 1)}$ with a and $b(q^d - 1)$ belonging to $\mathbb{Z}(q)$.

Now assume that q is a Pisot number and consider a number $x = \frac{a}{b} \in \left[\frac{a_1}{q-1}, \frac{a_m}{q-1}\right]$ with $a, b \in \mathbb{Z}(q)$.

Then all numbers

$$x_k := c_k + \frac{c_{k+1}}{q} + \frac{c_{k+2}}{q^2} + \ldots, \quad k = 1, 2, \ldots$$

satisfy

$$\frac{a_1}{q-1} \leqslant x_k \leqslant \frac{a_m}{q-1}.$$

Due to the rule (A.1), it is sufficient to show that there exist two indices $n < m$ such that $x_n = x_m$: then (c_i) is possibly periodic with a period of length $m - n$.

By generalizing an argument of Bogmér [3], we will prove that the sequence (x_k) takes only finitely many different values.

Equivalently we will show that the sequence $(y_k) = (b\, x_k)$ takes only finitely many values.

Since

$$y_k = q^k a - b \sum_{i=1}^{k-1} c_i q^{k-i}, \quad k = 1, 2, \ldots,$$

the numbers y_k belongs to $\mathbb{Z}(q)$.

It is sufficient to prove that the numbers y_k and all their conjugates belong to some bounded set. Indeed, then they will be the zeros of a set of polynomials with integer coefficients whose orders and coefficients are bounded by some number independent of k (see, e.g. [11]). Since there are only a finite number of such polynomials, this will imply that the set of values y_k is finite.

As in [3], we recall from [11] that if q has the conjugates $q_1 = q$, and $q_2, \ldots, q_s$, then there exist s monomorphisms $\sigma_i : \mathbb{Q}(q) \to \mathbb{C}$, with $i = 1, \ldots, s$ such that σ_1 is the identity, and

- $\sigma_i(q) = q_i$, $i = 1, \ldots, s$,
- if $y \in \mathbb{Q}(q)$, then $\sigma_1(y), \ldots, \sigma_s(y)$ contain all conjugates of y (possibly with multiplicity).

Since q is a Pisot number, there exists a number $0 < \delta < 1$ such that $|q_i| < \delta$ for $i = 2, \ldots, s$.

Setting

$$M_k := \max\{|\sigma_1(y_k), \ldots, |\sigma_s(y_k)|\},$$

we have to show that the sequence $M_1, M_2, \ldots$ is bounded.

Let us first observe that

$$\sigma_1(y_k)| = |bx_k| \leqslant \frac{|b|\,\overline{a}}{q-1}$$

for all k with

$$\overline{a} = \max\{|a_i| : a_i \in A\}.$$

Next we note that

$$y_{k+1} = q(y_k - bc_k),$$

whence

$$\sigma_i(y_{k+1}) = q_i(\sigma_i(y_k) - \sigma_i(b)c_k),$$

so that for $i = 2, \ldots, s$ we have

$$|\sigma_i(y_{k+1})| \leqslant \delta(M_k + \overline{a}|\sigma_i(b)|).$$

Setting

$$B := \overline{a}\ \max\{|\sigma_2(b)|, \ldots, |\sigma_s(b)|\}$$

we conclude that

$$M_{k+1} \leqslant \delta(M_k + B)$$

for $k = 1, 2, \ldots$

It follows by induction that $M_{k+1} \leqslant \delta^k M_1 + (\delta^k + \delta^{k-1} + \cdots + \delta)B$. Therefore

$$M_{k+1} \leqslant M_1 + \frac{\delta}{1-\delta} B$$

for all k, so that the sequence (M_k) is bounded indeed. $\square$

References

[1] C. Baiocchi, V. Komornik, Characterizations of expansions in non-integer bases, to appear.
[2] A. Bertrand-Mathis, Développement en base θ; répartition modulo un de la suite $(x\theta^n)_{n \geqslant 0}$; langages codés et θ-shift, Bull. Soc. Math. France 114 (3) (1986) 271–323.
[3] A. Bogmér, M. Horváth, A. Sövegjártó, On some problems of I. Joó, Acta Math. Hungar. 58 (1–2) (1991) 153–155.
[4] Z. Daróczy, I. Kátai, On the structure of univoque numbers, Publ. Math. Debrecen 46 (3–4) (1995) 385–408.
[5] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansions $1=\sum_{i=1}^{\infty} q^{-n_i}$ and related problems, Bull. Soc. Math. France 118 (3) (1990) 377–390.
[6] V. Komornik, P. Loreti, On the expansions in non-integer bases, Rend. Mat. Appl. (7) 19 (4) (2000) 615–634.
[7] Y. Meyer, Nombres algébriques et analyse harmonique, Ann. Sci. École Norm. Sup. (4) 3 (1970) 75–110.
[8] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 401–416.
[9] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar. 8 (1957) 477–493.
[10] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, Bull. London Math. Soc. 12 (4) (1980) 269–278.
[11] I. Stewart, D. Tall, Algebraic number theory, second ed., Chapman and Hall Mathematics Series, Chapman & Hall, London, 1987.