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THE RELATIONSHIP BETWEEN THE JACOBI AND THE SUCCESSIVE OVERRELAXATION (SOR) MATRICES OF A k-CYCLIC MATRIX

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Abstract—Let A be a $(k-l, l)$-generalized consistently ordered matrix with T and $\mathscr{L}_\omega$ its associated Jacobi and SOR matrices whose eigenvalues μ and λ satisfy the well-known relationship $(\lambda+\omega-1)^k = \omega^k \mu^k \lambda^{k-l}$. For a subclass of the above matrices A we prove that the matrix analogue of the previous relationship holds. Exploiting the matrix relationship we show that the SOR method is equivalent to a certain monoparametric k-step iterative one when used for the solution of the fixed-point problem $x = Tx + c$.

1. INTRODUCTION

Assume that $A \in \mathbb{C}^{n,n}$ is a matrix given by

$$A = I - T, \tag{1}$$

with I being the $n \times n$ unit matrix and T being of the form

$$T := \begin{bmatrix} 0 & 0 & \dots & 0 & T_{1,k-l+1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & T_{2,k-l+2} & \dots & 0 \\ \vdots & \vdots & & & & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & T_{l,k} \\ T_{l+1,1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & T_{l+2,2} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & & & & & & \\ 0 & 0 & \dots & T_{k,k-l} & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{2}$$

where all 0s indicate block nullmatrices and all diagonal blocks are square. Obviously A in (1) is a (block) cyclic matrix of index k (*cf.* Varga [1] and belongs to the class of $(k-l, l)$-generalized consistently ordered (GCO) ones (*cf.* [2–4]). The block Jacobi matrix associated with A relative to its partitioning, is T and writing

$$T := L + U, \tag{3}$$

where L and U are strictly lower and strictly upper (block) triangular matrices, we have for the corresponding block successive overrelaxation (SOR) matrix

$$\mathscr{L}_\omega := (I - \omega L)^{-1}[(1-\omega)I + \omega U], \tag{4}$$

with $\omega \in \mathbb{C}$ being the overrelaxation parameter (*cf.* [1, 4, 5]).

Let μ be the eigenvalues of T and λ those of $\mathscr{L}_\omega$. It is well known that the two sets are connected through the relationship

$$(\lambda + \omega - 1)^k = \omega^k \mu^k \lambda^{k-l} \tag{5}$$

(*cf.* [1, pp 108–109, Exs 1, 2; 2–4]. As is known (5) for $(k, l) = (2, 1)$ is due to Young and for $(k, l) = (k, 1)$, $k \geqslant 3$ to Varga [6].

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Our main objective in this paper is to show that the matrix analogue of (5),

$$(\mathscr{L}_\omega + (\omega - 1)I)^k = \omega^k T^k \mathscr{L}_\omega^{k-l}, \tag{6}$$

holds. It should be mentioned that a slight by different version of (6) for $(k, l) = (2, 1)$ was first proved by Young and Kincaid [7], while the case $(k, l) = (k, 1)$, $k \geqslant 2$ was proved by Galanis *et al.* [8]. The result (6) with all the necessary background material, will be given in Section 2, while in Section 3 it will be shown by using (6) that the SOR method applied for the solution of the fixed-point problem

$$x = Tx + c, \tag{7}$$

$x, c \in \mathbb{C}^n$, $\det(I - T) \neq 0$, is equivalent to a monoparametric k-step iterative method of the type studied in [9]. We mention that the latter problem for $l = 1$ was completely analyzed and solved in [8].

2. THE RELATIONSHIP $(\mathscr{L}_\omega + (\omega - 1)I)^k = \omega^k T^k \mathscr{L}_\omega^{k-l}$

We begin with the statement of our main result.

Theorem

Let T, in (2)–(3), be the block Jacobi and $\mathscr{L}_\omega$, in (4), be the block SOR matrices associated with the matrix A in (1). Then for any $\omega \in \mathbb{C}$, T and $\mathscr{L}_\omega$ satisfy (6). □

First we observe that (6) is trivially satisfied for $\omega = 0$. Thus we restrict ourselves to $\omega \in \mathbb{C}\backslash\{0\}$ and set

$$\tilde{T} = \omega T, \quad \tilde{L} = \omega L, \quad \tilde{U} = \omega U. \tag{8}$$

So (3), (4) and (6) will become

$$\tilde{T} := \tilde{L} + \tilde{U}, \tag{3'}$$

$$\mathscr{L}_\omega := (I - \tilde{L})^{-1}[(1 - \omega)I + \tilde{U}] \tag{4'}$$

and

$$(\mathscr{L}_\omega + (\omega - 1)I)^k = \tilde{T}^k \mathscr{L}_\omega^{k-l}. \tag{6'}$$

In the sequel only the case $l \leqslant k - l$ will be examined. The case $l \geqslant k - l$ can be examined in the same way with the roles of L and U being interchanged. Furthermore and without loss of generality we may assume that the greatest common divisor of k and l [g.c.d. (k, l)] is one. For if g.c.d.$(k, l) = d > 1$ and $(k', l') = (k/d, l/d)$ then $\tilde{T}$ will also be a GCO $(k' - l', l')$-matrix of the form (2). So, if the theorem holds when g.c.d.$(k', l') = 1$, the validity of $(\mathscr{L}_\omega + (\omega - 1)I)^{k'} = \tilde{T}^{k'} \mathscr{L}_\omega^{k'-l}$ will imply that of (6′) because

$$\begin{aligned}
\tilde{T}^k \mathscr{L}_\omega^{k-l} &= \tilde{T}^{k'(d-1)} \tilde{T}^{k'} \mathscr{L}_\omega^{k'-l'} \mathscr{L}_\omega^{(k'-l')(d-1)} \\
&= \tilde{T}^{k'(d-1)} (\mathscr{L}_\omega + (\omega - 1)I)^{k'} \mathscr{L}_\omega^{(k'-l')(d-1)} \\
&= \tilde{T}^{k'(d-1)} \mathscr{L}_\omega^{(k'-l')(d-1)} (\mathscr{L}_\omega + (\omega - 1)I)^{k'} = \cdots \\
\cdots &= \tilde{T}^{k'(d-2)} \mathscr{L}_\omega^{(k'-l')(d-2)} (\mathscr{L}_\omega + (\omega - 1)I)^{2k'} = \cdots \\
\cdots &= (\mathscr{L}_\omega + (\omega - 1)I)^{k'd} = (\mathscr{L}_\omega + (\omega - 1)I)^k.
\end{aligned}$$

The analysis for the proof of the theorem will be based on elementary graph theory (*cf.* Varga [1] and, for more details, Harary [10]). Thus assume we are given a $k \times k$ block matrix X partitioned in accordance with T in (2). Let $K := \{1, 2, \ldots, k\}$ and let P_i, $i \in K$ be k distinct points (nodes) arranged in a row in increasing order. We shall draw the directed arc (edge) $\overrightarrow{P_iP_j}$, $i, j \in K$, joining the node i with the node j iff X_{ij} is not a nullmatrix. The graph of X, denoted by $G(X)$, is then the set of all edges associated with X. For the sake of simplicity we may write i instead of P_i and (i, j) instead of $\overrightarrow{P_iP_j}$.

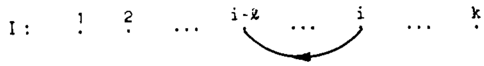

Fig. 1

From the discussion so far it is clear that in $G(\tilde{T})$ and for a given $(K\ni)i(>l)$ we will have

$$\bigcup_{j=1}^{k}(i,j)=(i,i-l), \tag{9}$$

if $i>l$ as in Fig. 1 and

$$\bigcup_{j=1}^{k}(i,j)=(i,i+k-l) \tag{10}$$

if $i\leqslant l$ as in Fig. 2. In the case of a type II edge we shall say that we have a "folding" (edge). Obviously edges of type I are associated with the matrix $\tilde{L}$, while those of type II with $\tilde{U}$. Consequently

$$G(\tilde{L})=\bigcup_{i=l+1}^{k}(i,i-l),\quad G(\tilde{U})=\bigcup_{i=1}^{l}(i,i+k-l),\quad G(\tilde{T})=G(\tilde{L})\cup G(\tilde{U}). \tag{11}$$

Let us denote the members of (6′) by B and C, that is

$$B:=(\mathscr{L}_\omega+(\omega-1)I)^k,\quad C:=\tilde{T}^k\mathscr{L}_\omega^{k-l}, \tag{12}$$

and let us consider the expansions of B and C in terms of products of $\tilde{L}$s and $\tilde{U}$s. For this (4′) must be used along with

$$\mathscr{L}_\omega=(I+\tilde{L}+\tilde{L}^2+\cdots+\tilde{L}^p)((1-\omega)I+\tilde{U}),$$

or equivalently

$$\mathscr{L}_\omega=(1-\omega)(I+\tilde{L}+\tilde{L}^2+\cdots+\tilde{L}^p)+(I+\tilde{L}+\tilde{L}^2+\cdots+\tilde{L}^p)\tilde{U}, \tag{13}$$

where $p=[(k-1)/l]$, that is the largest integer not exceeding $(k-1)/l$, and with

$$\mathscr{L}_\omega+(\omega-1)I=(1-\omega)(\tilde{L}+\tilde{L}^2+\cdots+\tilde{L}^p)+(I+\tilde{L}+\tilde{L}^2+\cdots+\tilde{L}^p)\tilde{U}. \tag{14}$$

Next, let us consider together with each nonidentically zero matrix (term) in the expansions of B and C all possible paths associated with the original graph of the factors involved in the matrix (term) in question. For example if $(k,l)=(3,1)$ then $\tilde{T}^3=(\tilde{L}+\tilde{U})^3=\tilde{L}^2\tilde{U}+\tilde{L}\tilde{U}\tilde{L}+\tilde{U}\tilde{L}^2$. So, with the first term $\tilde{L}^2\tilde{U}=\tilde{L}\tilde{L}\tilde{U}$ the path $\overrightarrow{P_3P_2}$, $\overrightarrow{P_2P_1}$, $\overrightarrow{P_1P_3}$ of length 3 will be considered and not the resulting one $\overrightarrow{P_3P_3}$ of length zero. With the second term the path $\overrightarrow{P_2P_1}$, $\overrightarrow{P_1P_3}$, $\overrightarrow{P_3P_2}$ will be considered and not $\overrightarrow{P_2P_2}$ etc. Thus, in this sense, $G(\tilde{T}^3)=(1,1)\cup(2,2)\cup(3,3)$ where each path is of length 3. Then we can prove the following statement:

Lemma 1

$G(\tilde{T}^k)$ consists of exactly one closed path from a node $i\in K$ to itself (cycle) of length k. This cycle contains precisely l foldings no two of which can be consecutive edges of it.

Proof. For any $i\in K$ there exists exactly one edge with i a starting node (see Figs 1 and 2). Let j be the ending node of a path starting from i whose length is k and which has r foldings. If to each edge (s,t) we assign the number $t-s$ then for the path in question we have $i+r(k-l)+(k-r)(l)=j$ or $i-j=(l-r)k$ showing that $j=i$ and $r=l$ because $|i-j|<k$.

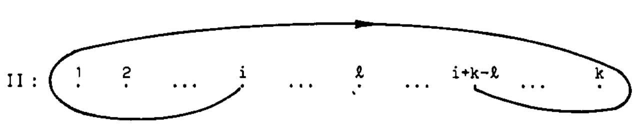

Fig. 2

That two foldings can not be consecutive edges of this path follows directly from the fact that $l \leqslant k - l$. Finally, because of the special cyclic nature of $G(\tilde{T}^k)$ and the fact that g.c.d.$(k, l) = 1$ the above cycle passes precisely once through each of the k nodes. □

From Lemma 1 (L1) many properties of the terms of the expansion of $\tilde{T}^k = (\tilde{L} + \tilde{U})^k$ and their associated graphs can be obtained. Thus we have:

Lemma 2

A nonidentically zero term of the expansion of $\tilde{T}^k = (\tilde{L} + \tilde{U})^k$ is of the general form

$$\tilde{L}^{q_1}\tilde{U}\tilde{L}^{q_2}\tilde{U}\ldots\tilde{L}^{q_l}\tilde{U}\tilde{L}^{q_{l+1}}, \tag{15}$$

where the q_js, $j = 1(1)l + 1$, can be determined uniquely for a given $i \in K$. Moreover, if the given i takes cyclically all values along its associated closed path (cycle) then all other terms of $\tilde{T}^k$ are produced in the same cyclic way by simply transferring each time in (15) only one factor, out of the k ones, from the front (left) to the back (right). □

Let us now examine how $G(\mathscr{L}_\omega)$ is derived assuming that $\omega \neq 1$ because for $\omega = 1$ the proof of the theorem is simplified. First we observe that for a given $i \in K$ the number of edges of $G(\mathscr{L}_\omega)$, with i as starting node, is $q + 2$, where $q = [i/l]$. $q + 1$ of these edges are of type I, coming from the first $q + 1$ terms $(1 - \omega)\tilde{L}^s$, $s = 0(1)q$ $(\tilde{L}^0 = I)$ of the first sum in the RHS of (13) while the $(q + 2)$nd one is a folding (edge of type II), coming from the term $\tilde{L}^q\tilde{U}$ of the second sum in (13). The set of these edges is illustrated in Fig. 3. Next we notice that, in view of (14) the set of edges of $G(\mathscr{L}_\omega + (\omega - 1)I)$, whose origin is the previous node i, is exactly the same as before except for the edge (i, i). Hence

$$G(\mathscr{L}_\omega + (\omega - 1)I) = G(\mathscr{L}_\omega) \Big\backslash \bigcup_{i=1}^{k} (i, i). \tag{16}$$

Now we are able to go on with the proof our theorem.

Proof of the theorem

For this we have to prove that if in $G(B)$ there exists a path from i to j with $l + m$ foldings then there exists an identical path in $G(C)$ and *vice versa*. Moreover the path in question is associated with terms of the expansions of B and C whose leading coefficients are the same and equal to $N(1 - \omega)^{k-l-m}$. For this we note that because of the factor $\tilde{T}^k$ in C there exists in $G(C)$ a unique path of length k from i to i with l foldings (L1). So the other m foldings of this path must come from the factor $\mathscr{L}_\omega^{k-l}$ and more specifically from the presence of m $\tilde{U}$s in the term associated with this path. Thus $0 \leqslant m \leqslant k - l$. On the other hand, it is implied that $k - l - m$ terms of the first sum in (13) are factors in the term in question, which gives as a coefficient of this term $N_C(1 - \omega)^{k-l-m}$. Furthermore, every nonidentically zero term of B will have its first k factors the same as those of a nonidentically zero term of $\tilde{T}^k$ hence its associated path will have at least l foldings. So, if in $G(B)$ i is connected with j via a path with m' foldings, $0 \leqslant m' - l \leqslant k - l$. Because of (16) this path consists of the cycle of length k with l foldings connecting i with i followed by a path with $m' - l = m$ foldings connecting i with j. The entire path is identical with that in $G(C)$ considered previously, comes from a term of the expansion of B which has $l + m$ $\tilde{U}$ factors and therefore, in view of (14), $k - l - m$ $(1 - \omega)\tilde{L}$ factors. So, the coefficient of the term in question is $N_B(1 - \omega)^{k-l-m}$. Conversely, the proof follows the same reasoning and is therefore omitted. It remains to be proved that $N_B = N_C$. Evidently N_B and N_C are equal to the number of all different paths in $G(B)$ and $G(C)$ connecting i with j with $l + m$ foldings. Let t be the number of all

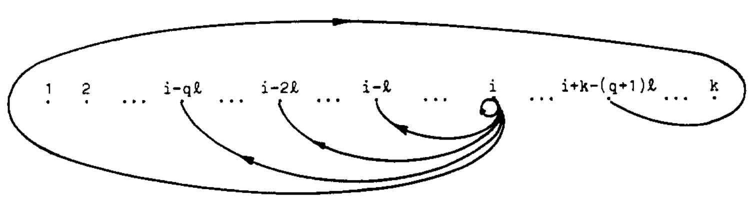

Fig. 3

intermediate nodes in $G(B)$ which are endpoints of edges of type I in one of these paths. In a path with $l+m$ foldings the endpoints of all edges of type I are $k-l-m$, so

$$N_B = \begin{cases} N_{B_{II}} = \binom{t}{k-l-m}, & \text{if the last edge is of type II;} \\ N_{B_I} = \binom{t-1}{k-l-m-1}, & \text{if the last edge is of type I.} \end{cases} \tag{17}$$

But, because of the presence of $\tilde{T}^k$ in C there exists a unique path from i to i with l foldings. Hence N_C will be the number of different paths from i to j with m foldings. The number of endpoints of edges of type I in this last part in one of these paths is $t-k+l$, that is the same t as that in $G(B)$ minus the $k-l$ edges of the cycle from i to i. We distinguish two cases:

(*a*) *The last edge of the path is of type II.* Let then p $(0 \leqslant p \leqslant k-l-m)$ be the number of edges with ending node one of the previously mentioned $t-k+l$ points corresponding to a factor $(1-\omega)^p$ of the associated term. This number is obviously $\binom{t-k+l}{p}$. But because of the factor $(1-\omega)^{k-l-m}$ in the associated product of $\tilde{L}$s and $\tilde{U}$s it is concluded, from (13), that the factor $(1-\omega)^{k-l-m-p}$ must come from the presence of $k-l-m-p$ times of the matrix $(1-\omega)I$. This implies that in the path under consideration there exist $k-l-m-p$ edges connecting a node to itself. These edges must be associated with the above p nodes, with the ending nodes of the m folding and with i itself. That is with $p+m+1$ nodes all together. But the "distribution" of $k-l-m-p$ edges to $p+m+1$ nodes is the number of combinations with repetitions of $p+m+1$ chosen $k-l-m-p$, namely

$$\binom{(p+m+1)+(k-l-m-p)-1}{k-l-m-p} = \binom{k-l}{k-l-m-p}.$$

So, we have that the total number of different paths is

$$N_C = N_{C_{II}} = \sum_{p=0}^{k-l-m} \binom{t-k+l}{p}\binom{k-l}{k-l-m-p}$$

$$= \sum_{p_1,p_2 \geqslant 0}^{p_1+p_2=k-l-m} \binom{t-k+l}{p_1}\binom{k-l}{p_2} = \binom{t-k+l+k-l}{p_1+p_2} = \binom{t}{k-l-m},$$

and $N_{C_{II}} = N_{B_{II}}$ follows from (17).

(*b*) *The last edge of the path is of type I.* The analysis is the same with the obvious differences: (i) p takes values such that $0 \leqslant p \leqslant t-k+l-1$. (ii) For a given p the number of different paths is $\binom{t-k+l-1}{p}$ and (iii) The edges which may connect a node to itself must be associated with the previous $p+m+1$ nodes and also with the node j, that is with $p+m+2$ nodes all together. Consequently

$$N_C = N_{C_I} = \sum_{p=0}^{k-l-m} \binom{t-k+l-l}{p}\binom{k-l}{k-l-m-1-p}$$

$$= \sum_{p_1,p_2 \geqslant 0}^{p_1+p_2=k-l-m} \binom{t-k+l-1}{p_1}\binom{k-l}{p_2} = \binom{t-1}{k-l-m-1}.$$

Hence, from (17), $N_{C_I} = N_{B_I}$ which concludes the proof of the theorem. □

Note. If any one of the k blocks T_{ij}, of T, in (2), is a zero matrix, (6) is valid and the proofs of the various statements so far still hold provided one assumes that there exists a *fictitious* edge connecting i with j. Of course, then T^k as well as both members of (6) are nullmatrices and a much simpler proof can be given. □

3. APPLICATIONS TO MONOPARAMETRIC k-STEP ITERATIVE METHODS

In [8] and [11] two classes of monoparametric k-step iterative methods were studied in connection with the SOR method. They were extensions and generalizations of the 2- and 4-step methods treated in [12–14] and [9]. In general the monoparametric k-step iterative methods constitute subclasses of the methods analyzed in [9], where the latter are subclasses of the k-part splitting methods introduced in [15].

For the solution of (7), (6) suggests that a monoparametric k-step iterative method equivalent to the SOR one:

$$x^{(m)} = \mathscr{L}_\omega x^{(m-1)} + \omega(I - \omega L)^{-1}c, \quad m = -k+2, -k+3, -k+4, \ldots, \tag{18}$$

can be constructed. The way of generating the new method is analogous to that in [8] for $l = 1$. More specifically let $x^{(m-k)}$ be the $(m-k)^{\text{th}}$ iteration of (18) with $m = 1, 2, 3, \ldots$ From (6) we have

$$(\mathscr{L}_\omega + (\omega - 1)I)^k x^{(m-k)} = \omega^k T^k \mathscr{L}_\omega^{k-l} x^{(m-k)},$$

or equivalently

$$\sum_{j=0}^{k} \binom{k}{j} (\omega - 1)^j \mathscr{L}_\omega^{k-j} x^{(m-k)} = \omega^k T^k \mathscr{L}_\omega^{k-l} x^{(m-k)}. \tag{19}$$

From (18) it can be obtained

$$\mathscr{L}_\omega^j x^{(m-k)} = x^{(m-k+j)} - \omega\left(\sum_{p=0}^{j-1} \mathscr{L}_\omega^p\right)(I - \omega L)c.$$

Substituting into (19) we take

$$x^{(m)} = \omega^k T^k x^{(m-l)} - \sum_{j=1}^{k} \binom{k}{j} (\omega - 1)^j x^{(m-j)}$$
$$+ \omega\left[\sum_{j=0}^{k} \binom{k}{j} (\omega - 1)^j \left(\sum_{p=0}^{k-j-1} \mathscr{L}_\omega^p\right) - \omega^k T^k \left(\sum_{p=0}^{k-l-1} \mathscr{L}_\omega^p\right)\right](I - \omega L)c.$$

Based on a statement analogous to Lemma 2 in [8], the last term in the RHS above can be simplified provided 1 is not an eigenvalue of T^k. Thus we get

$$x^{(m)} = \omega^k T^k x^{(m-l)} - \sum_{j=1}^{k} \binom{k}{j} (\omega - 1)^j x^{(m-j)} + \omega^k \sum_{j=0}^{k-1} T^j c, \quad m = 1, 2, 3, \ldots, \tag{20}$$

where $x^{(j)} \in \mathbb{C}^n$, $j = 0(-1) - k + 1$, can be taken arbitrary, and are partitioned relative to the partitioning of T. Equation (20) is a monoparametric k-step iteration method and constitutes a generalization of the one in [8]. Let us set

$$\hat{c} = [\hat{c}_1^T \hat{c}_2^T \ldots \hat{c}_k^T]^T = \omega^k \sum_{j=0}^{k-1} T^j c,$$

partitioned in the same way as the $x^{(j)}$s previously. Then (20) is split into the following k simpler and of smaller dimensions k-step iterative methods:

$$\left.\begin{aligned} x_1^{(m)} &= \omega^k \hat{T}_1 x^{(m-l)} - \sum_{j=1}^{k} \binom{k}{j} (\omega - 1)^j x_1^{(m-j)} + \hat{c}_1 \\ x_2^{(m)} &= \omega^k \hat{T}_2 x_2^{(m-l)} - \sum_{j=1}^{k} \binom{k}{j} (\omega - 1)^j x_2^{(m-j)} + \hat{c}_2 \\ &\vdots \\ x_k^{(m)} &= \omega^k \hat{T}_k x_k^{(m-l)} - \sum_{j=1}^{k} \binom{k}{j} (\omega - 1)^j x_k^{(m-j)} + \hat{c}_k \end{aligned}\right\}, \quad m = 1, 2, 3, \ldots, \tag{21}$$

where $\hat{T}_j$, $j = 1(1)k$, are cyclic products of the k nonzero submatrices of T in (2). Apart from the number zero and multiplicities, the coefficient matrices in (21) and T^k have the same eigenvalues. So, the asymptotic convergence rates of (20) and of each one of (21) are that of the SOR method (18). Therefore by applying any one of (21), say the first one, we can obtain x_1 and from (7) and (2) the other vector components of x. As was mentioned before g.c.d$(k, l) = 1$. If g.c.d.$(k, l) = d > 1$ then $k' = k/d$ and $l' = l/d$ replace k and l.

Example. In (2) let $k = 3$ and $l = 2$. Then the 3-step methods (21) will be

$$x_1^{(m)} = \omega^3 T_{1,2}T_{2,3}T_{3,1}x_1^{(m-2)} - \sum_{j=1}^{3}\binom{3}{j}(\omega - 1)^j x_1^{(m-j)} + \hat{c}_1;$$

$$x_2^{(m)} = \omega^3 T_{2,3}T_{3,1}T_{1,3}x_2^{(m-2)} - \sum_{j=1}^{3}\binom{3}{j}(\omega - 1)^j x_2^{(m-j)} + \hat{c}_2;$$

$$x_3^{(m)} = \omega^3 T_{3,1}T_{1,2}T_{2,3}x_3^{(m-2)} - \sum_{j=1}^{3}\binom{3}{j}(\omega - 1)^j x_3^{(m-j)} + \hat{c}_3.$$

Working only with the first one, provided convergence is guaranteed, x_1 is obtained. Then from (7) and (2) we have $x_3 = T_{3,1}x_1 + c_3$, $x_2 = T_{2,3}x_3 + c_2$ and $x = [x_1^T x_2^T x_3^T]^T$.

Before we close this section we point out that the study of the (optimal) convergence of (21) can be made by means of the (optimal) convergence of (18) and *vice versa*. For example in [8], $l = 1$, the optimal convergence of (18) was obtained via the (optimal) convergence of (21). Based on this observation we note that if the kth powers of the eigenvalues of T in (1) are nonnegative, with $\rho(T) < 1$ and g.c.d.$(k, l) = 1$ with $l > 1$, then the optimum SOR method (18) for the solution of (7) corresponds to $\omega = 1$ (*cf.* [3]). In such a case the optimum methods (20) and (21) are simplified to

$$x^{(m)} = T^k x^{(m-l)} + \hat{c} \tag{20'}$$

and

$$x_j^{(m)} = \hat{T}_j x^{(m-l)} + \hat{c}_j, \quad j = 1(1)k, \tag{21'}$$

respectively. The problem of (optimal) convergence in more general cases is being investigated.

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