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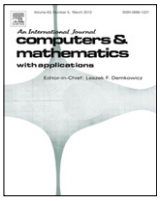

State variables and transients of fractional order differential systems

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ABSTRACT

Fractional order differentiation is generally considered as the basis of fractional calculus, but the real basis is in fact fractional order integration and particularly the fractional integrator, because definition and properties of fractional differentiation and of fractional differential systems rely essentially on fractional integration. We present the frequency distributed model of the fractional integrator and its finite dimension approximation. The simulation of FDSs, based on fractional integrators, leads to the definition of FDS internal state variables, which are the state variables of the fractional integrators, as a generalization of the integer order case.

The initial condition problem has been an open problem for a long time in fractional calculus. We demonstrate that the frequency distributed model of the fractional integrator provides a solution to this problem through the knowledge of its internal state. Beyond the solution of this fundamental problem, mastery of the integrator internal state allows the analysis and prediction of fractional differential system transients. Moreover, the finite dimension approximation of the fractional integrator provides an efficient technique for practical simulation of FDSs and analysis of their transients, with a particular insight into the interpretation of initial conditions, as illustrated by numerical simulations.

Laplace transform equations and initial conditions of the Caputo and the Riemann–Liouville derivatives are used to formulate the free responses of FDEs. Because usual equations are wrong, the corresponding free responses do not fit with real transients. We demonstrate that revised equations, including the initial state vector of the fractional integrator (used to perform differentiation) provide corrected free responses which match with real transients, as exhibited by numerical simulations.

1. Introduction

Though fractional calculus has received convincing applications in the past decades, such as robust control [1–4], fractional order PIDs [5,6], modeling and identification [7–10] of diffusive interfaces etc., the theoretical modeling of Fractional Differential Equations (FDEs) and Systems (FDSs) has not reached the same achievement as their integer order counterparts. For example, though fundamental in automatic control for its implications in controllability or observability, Lyapunov stability and optimal control, the state variable concept remains controversial and the initial condition problem is still open [11,12]. Fortunately, two main approaches have received attention. Lorenzo and Hartley have introduced the history function and the initialization function concepts which provide the framework of a global method for the analysis of the initial conditions of both fractional derivatives [13,14] and FDEs [15–18]. Recently [19], this technique has been used

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to initialize the model of a real diffusive experiment. Trigeassou has introduced the concept of the frequency distributed fractional integrator [20] which has been used for the definition of state variables [21] and to solve initial condition problems dealing with fractional derivatives [22] and FDEs [23]. Sabatier has introduced an equivalent technique [24,25], which provides a distributed model of the global system. This model emphasizes the infinite dimension of the state space variables; it has been applied to define observability [26] and to perform pseudo-state estimation [27]. The initial condition problem has also been addressed by several authors, mainly from the point of view of history and initialization functions concepts, see for example [28,29] and particularly [30].

The analysis of system transients or free responses, though essential in dynamical systems, is practically unexplored in the fractional domain. Indeed, this situation is easily explained by the difficulties encountered in the mastery of state variables and initial conditions.

On the other hand, well established results reveal to be questionable. Particularly, the popular initial conditions deduced from the Laplace transform of the Caputo derivative lead to ambiguous results, in contradiction with new state variable concepts, as it has been mentioned in some recent publications [13,14,25,22].

During the last decade, the fractional integrator technique [20] has been developed through different papers dealing with the design of the integrator and mostly its applications in simulation [31] and identification of FDEs [32–35,10]. More recently, this technique has been used to propose a Lyapunov approach to the stability of FDEs [36], and to solve initialization problems arising in FDEs [23] and fractional derivatives [37,38]. So, this paper is intended to present a synthesis of previous works on the fractional integrator and fractional differential equations and systems, mainly from the point of view of the initial condition problems and transient behaviors. The transients of the fractional integrator and of the fractional derivatives have been addressed in [39], where transients are illustrated by numerical simulations.

Though fractional order differentiation is generally considered as the basis of fractional calculus, we demonstrate that the real basis is in fact fractional order integration and particularly the fractional integrator, mainly because definition and properties of fractional differentiation and of FDSs rely essentially on fractional integration.

The initial condition problem has been an open problem for a long time in fractional calculus. We demonstrate that the frequency distributed model of the fractional integrator provides a solution to this problem through the knowledge of its infinite dimension internal state. Beyond the solution of this fundamental problem, mastery of the integrator internal state allows the analysis and prediction of fractional differential system transients. Moreover, the finite dimension approximation of the fractional integrator provides an efficient technique for practical simulation of FDSs and analysis of their transients, with a particular insight into the interpretation of initial conditions.

Laplace transform equations and initial conditions of the Caputo and the Riemann–Liouville derivatives are used to formulate the free responses of FDEs. Because usual equations are wrong [13,22,40], the corresponding free responses do not fit with real transients. We demonstrate that revised equations, including the initial state vector of the fractional integrator (used to perform differentiation) provide corrected free responses which match with real transients, as exhibited by numerical simulations.

The paper is composed of six sections. The first one is the introduction. The second section is a reminder of fractional integration and differentiation definitions. The third section is devoted to fractional integration with the frequency distributed integrator. In the fourth section, we present the simulation of FDSs with fractional integrators and their free and forced responses. Mastery of transients thanks to FDS infinite dimension state vector is illustrated by numerical simulations in the fifth section. The last section deals with free responses derived from the revised Laplace transform equations of the Caputo and Riemann–Liouville derivatives.

2. Fractional order integration and differentiation

2.1. Riemann–Liouville integration

The nth fractional order Riemann–Liouville integral (n real positive) of a function $f(t)$ is defined by the relation [41–44]:

$$I_n(f(t)) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1}$$

where $\Gamma(n)$ is the gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx. \tag{2}$$

$I_n(f(t))$ is the convolution of the function $f(t)$ with the impulse response:

$$h_n(t) = \frac{t^{n-1}}{\Gamma(n)} \tag{3}$$

of the fractional integration operator whose Laplace transform is:

$$I_n(s) = L\{h_n(t)\} = \frac{1}{s^n}. \tag{4}$$

Notice that in the integer order case ($n = 1$), the integral is characterized by $h_1(t) = H(t)$ (unit step function or Heaviside function) and

$$I_1(s) = L\{h_1(t)\} = \frac{1}{s}. \tag{5}$$

2.2. Implicit fractional differentiation

Fractional differentiation is the dual operation of the fractional integration.
Consider the fractional integration operator $I_n(s)$ whose input and output are respectively $x(t)$ and $y(t)$.
Then:

$$y(t) = I_n(x(t)) \quad \text{or} \quad Y(s) = \frac{1}{s^n} X(s). \tag{6}$$

Reciprocally, $x(t)$ is the nth order fractional derivative of $y(t)$ defined as:

$$x(t) = D_n(y(t)) \quad \text{or} \quad X(s) = s^n Y(s), \tag{7}$$

where $D_n(s) = s^n$ represents the Laplace transform of the fractional differentiation operator (for initial conditions equal to zero).

This fractional derivative definition is based on the operator $I_n(s)$, without analytical formulation of $D_n(y(t))$: it is the implicit definition of the fractional derivative. However, this implicit derivative exists only inside a closed loop, like in the simulation of fractional differential equations (Section 4).

2.3. Explicit formulations of the fractional derivative [45]

Assume that the fractional order n is included between the two integer numbers $N - 1$ and N:

$$N - 1 < n \leq N. \tag{8}$$

We can write

$$D_n(s) = s^n = \frac{1}{s^N} s^n s^N = \frac{1}{s^{N-n}} s^N, \tag{9}$$

where $\frac{1}{s^{N-n}}$ represents the fractional integration $I_{N-n}()$ and s^N the integer order differentiation $\frac{d^N()}{dt^N}$.

Then

$$L\{D_n(f)\} = D_n(s) F(s) = \frac{1}{s^{N-n}} s^N F(s) \tag{10}$$

(with zero initial conditions) and using the inverse Laplace transform, we get two expressions of $D_n(f)$:

$$D_n(f) = L^{-1}\left\{ \frac{1}{s^{N-n}} (s^N F(s)) \right\} \tag{11}$$

$$D_n(f) = L^{-1}\left\{ s^N \left(\frac{1}{s^{N-n}} F(s) \right) \right\}. \tag{12}$$

The first expression (11) corresponds to:

$$D_n(f) = h_{N-n}(t) * \frac{d^N f(t)}{dt^N} \tag{13}$$

and the second one (12) to:

$$D_n(f) = \frac{d^N}{dt^N} (h_{N-n}(t) * f(t)). \tag{14}$$

This first expression is known as the Caputo derivative [7,44]

$$D_n(f(t)) = \int_0^t \frac{(t-\tau)^{N-n-1}}{\Gamma(N-n)} \frac{d^N f(\tau)}{dt^N} d\tau \tag{15}$$

while the second one is the Riemann–Liouville derivative [44]:

$$D_n(f(t)) = \frac{d^N}{dt^N} \left\{ \int_0^t \frac{(t-\tau)^{N-n-1}}{\Gamma(N-n)} f(\tau)\, d\tau \right\}. \tag{16}$$

3. The fractional order integrator

3.1. Introduction

Because the Riemann–Liouville integral of a function $f(t)$ is the result of a convolution between $f(t)$ and the impulse response $h_n(t)$, this integral is obtained at the output of a linear system (characterized by its impulse response $h_n(t)$), excited by the input $f(t)$. This linear system is called the fractional order integrator; for convenience, it will be denoted by the Laplace transform $I_n(s)$.

This fractional integrator is the key element for FDE/FDS simulation. However, the realization of $I_n(s)$, either in analog or numerical form, is not a simple task, as in the integer order case. The reader will refer to [20,46] for a more detailed presentation.

3.2. The frequency distributed model

The fractional integrator is a linear frequency distributed system, with input $v(t)$ and output $x(t)$. Its frequency distributed state $z(\omega, t)$ verifies the differential equation (for the elementary frequency ω) (see Appendix A for a presentation of the distributed model):

$$\frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + v(t) \tag{17}$$

and the output $x(t)$ of the fractional integrator is the weighted integral (with weight $\mu_n(\omega)$) of all the contributions $z(\omega, t)$ ranging from 0 to ∞:

$$\begin{aligned} x(t) &= \int_0^{\infty} \mu_n(\omega)\, z(\omega, t)\, d\omega \\ \mu_n(\omega) &= \frac{\sin n\pi}{\pi}\, \omega^{-n} \quad 0 < n < 1. \end{aligned} \tag{18}$$

The relations (17) and (18) define the frequency distributed model of the fractional integrator; it is also known as its diffusive model [47–49]. It has also been used by several authors, without explicit reference to a frequency distributed model [50–53].

Implicitly, the state of the fractional integrator (and consequently of FDEs and FDSs) is commonly defined as its output $x(t)$. In fact, because of its definition (18), $x(t)$ is only the weighted sum of the variables $z(\omega, t)$, thus it is only a pseudo state variable and $z(\omega, t)$ is an infinite dimension distributed state variable.

Remark. Consider the integer order integrator

$$I_1(s) = \frac{1}{s} \quad \text{and} \quad \mu_1(\omega) = \delta(\omega) \tag{19}$$

$$x(t) = \int_0^{\infty} \delta(\omega)\, z(\omega, t)\, d\omega = z(0, t). \tag{20}$$

This result means that $x(t)$ and $z(0, t)$ are the same variables and that the output of the integrator characterizes completely its state in the integer order case.

3.3. Transients of the fractional integrator

Let $z(\omega, t_0)$ be the initial state of the distributed differential equation at the instant t_0.

Then, the free response of this system is

$$z(\omega, t) = z(\omega, t_0)\, e^{-\omega(t-t_0)} \tag{21}$$

and its forced response to an input $v(t)$ is the convolution of this input with the impulse response $e^{-\omega t}$.

So, the global response of the elementary system is:

$$z(\omega, t) = z(\omega, t_0)\, e^{-\omega(t-t_0)} + \int_{t_0}^{t} e^{-\omega(t-\tau)}\, v(\tau)\, d\tau. \tag{22}$$

Consequently, the response of the fractional integrator is composed of a free response term caused by the distributed initial condition $z(\omega, t_0)$, (ω ranging from $\omega = 0$ to ∞) and of a forced response term caused by the input $v(t)$, like all linear systems [54].

The distributed initial condition $z(\omega, t_0)$ can be interpreted as an initialization function summarizing all the past behavior for $t < t_0$.

The response of the fractional integrator can also expressed with the Laplace transform; this expression will be useful in the next sections.

Consider the Laplace transform of the distributed equation (17):

$$L\left\{\frac{\partial z(\omega,t)}{\partial t}\right\} = sZ(\omega,s) - z(\omega,0) = -\omega Z(\omega,s) + V(s), \tag{23}$$

where $z(\omega, 0)$ is the initial condition of the fractional integrator.

So

$$Z(\omega,s) = \frac{V(s) + z(\omega,0)}{s+\omega} \tag{24}$$

and

$$X(s) = \int_0^\infty \frac{\mu_n(\omega)\, z(\omega,0)}{s+\omega}\, d\omega + \int_0^\infty \frac{\mu_n(\omega)}{s+\omega}\, d\omega\, V(s). \tag{25}$$

Notice that:

$$\int_0^\infty \frac{\mu_n(\omega)}{s+\omega}\, d\omega = \frac{1}{s^n}. \tag{26}$$

Remark. The transients of the fractional integrator are analyzed and compared to those of the integer order integrator with the help of numerical simulations in [39].

3.4. Finite dimension approximation of the distributed model

The frequency distributed model of the fractional integrator is not directly usable. A practical model is obtained by frequency discretization of $\mu_n(\omega)$, where the function $\mu_n(\omega)$ is replaced by a multiple steps function with K steps. For an elementary step, the height is $\mu_n(\omega_k)$, and the width is $\Delta\omega_k$. Let $c_k$be the weight of the kth element:

$$c_k = \mu_n(\omega_k)\Delta\omega_k. \tag{27}$$

Then, the continuous distributed model becomes a conventional state model with dimension equal to K.

$$\begin{cases} \dfrac{d\, z_k(t)}{dt} = -\omega_k\, z_k(t) + v(t) \quad | \, k = 1 \cdots K \\ x(t) = \displaystyle\sum_{k=1}^{K} \mu_n(\omega_k)\, z_k(t)\, \Delta\omega_k \\ \quad\; = \displaystyle\sum_{k=1}^{K} c_k\, z_k(t). \end{cases} \tag{28}$$

Remark. Practically, this model of $I_n(s)$ is not satisfactory because the equivalent gain of the operator is equal to $\sum_1^K c_k$ while the static gain of the theoretical fractional integrator is infinite at $\omega = 0$. The drawback of this finite static gain is that it generates static errors in the simulation of FDEs [48].

An other approach to the discretization of the fractional integrator is provided by the frequency approach [20].

It has been demonstrated that an approximation of $I_n(s)$ is:

$$\tilde{I}_n(s) = \frac{G_n}{s} \prod_{j=1}^{J} \frac{1 + \frac{s}{\omega'_j}}{1 + \frac{s}{\omega_j}}. \tag{29}$$

The coefficient G_n is a normalizing factor, such as $I_n(s)$ and $\tilde{I}_n(s)$ are identical on a frequency interval $[\omega_b; \omega_h]$.

Oustaloup [43] has demonstrated the following relations:

$$\begin{aligned} &\omega_j = \alpha\omega'_j \quad \text{with } \alpha > 1 \\ &\omega'_{j+1} = \eta\, \omega_j \quad \text{with } \eta > 1 \end{aligned} \tag{30}$$

and

$$n' = \frac{\log(\alpha)}{\log(\alpha\, \eta)}. \tag{31}$$

It is easy to transform the model of $\tilde{I}_n(s)$ into a modal form because the ω_j are known a priori [46]. This transformation is based on the following definition:

$$\tilde{I}_n(s) = \frac{c_0}{s} + \sum_{j=1}^{J} \frac{c_j}{s+\omega_j}, \tag{32}$$

where c_0 and c_j coefficients are linked to G_n, ω_j and ω'_j by the relations:

$$\begin{aligned} c_0 &= G_n \\ c_j &= \frac{G_n(\omega_j - \omega'_j)}{\omega'_j} \prod_{\substack{i=1 \\ i\neq j}}^{J} \frac{1-\frac{\omega_j}{\omega'_i}}{1-\frac{\omega_j}{\omega_i}}. \end{aligned} \tag{33}$$

This definition of $\tilde{I}_n(s)$ corresponds to a modal state model which will be the finite dimension approximation of the frequency distributed model, used in this paper:

$$\underline{Z}^T(t) = [\, z_0 \ldots z_j \ldots z_J \,] \tag{34}$$

$$\begin{aligned} \frac{d\underline{Z}(t)}{dt} &= A\underline{Z}(t) + \underline{B}v(t) \\ x(t) &= \underline{C}^T\underline{X}(t) \end{aligned} \tag{35}$$

with:

$$A = \begin{bmatrix} 0 & & & 0 \\ & -\omega_1 & & \\ & & \ddots & \\ 0 & & & -\omega_J \end{bmatrix}; \qquad \underline{B} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{36}$$

$$\underline{C}^T = \begin{bmatrix} c_0 & c_1 & \cdots & c_J \end{bmatrix}.$$

Remark. In the direct approximation used in [48], the gain is finite, whereas with the frequency approach (Eqs. (29) and (32)), because of the integer order integrator, the gain of model (35) is infinite. So, there is no static error when the fractional integrator is used to perform simulation of FDEs (Section 4).

An other advantage of model (35) is the finite number of modes ω_j. With the direct approximation, if we use an arithmetic distribution, a quasi infinite number of modes is necessary to cover the $[\omega_b \quad \omega_h]$ frequency range. On the contrary, with the frequency approach, thanks to the geometric distribution (30) proposed by Oustaloup, only a finite number of modes is required to cover the $[\omega_b \quad \omega_h]$ interval.

So, model (35) provides an optimal compromise between precision, number of modes, and consequently reduction of computation time.

A numerical algorithm is proposed in Appendix B.

4. Simulation and state variables of fractional differential systems

4.1. FDS simulation

4.1.1. Simulation of a one derivative FDE

Consider the elementary system:

$$D_n(x(t)) + a\,x(t) = u(t). \tag{37}$$

We can write:

$$D_n(x(t)) = v(t) = u(t) - a\,x(t), \tag{38}$$

where $v(t)$ is the input of the fractional integration operator $I_n(s) = \frac{1}{s^n}$, and $x(t)$ is its output.

The simulation of this FDE can be interpreted as a closed loop system, corresponding to Fig. 1.

Consequently, $v(t)$ is the implicit derivative of $x(t)$, as defined in 2.2.

The same technique is used in the integer order case to simulate ODEs. It has been originally formulated by Lord Kelvin in 1876 [55,54] and applied by Vannever Bush [56] in the differential analyzer with a mechanical integrator, and later in analog computers [57] with electronic integrators. The same principle is used in numerical algorithms like Runge Kutta 4, but the numerical integrator is implicit in this case.

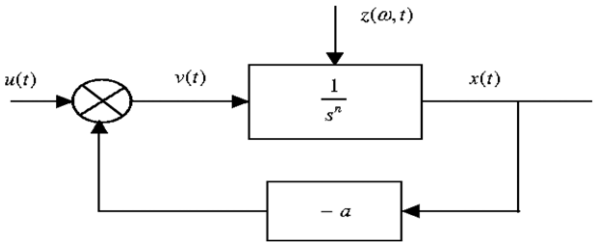

Fig. 1. Simulation of a one derivative FDE.

The conclusion is that the simulation of an ODE or of an FDE requires an integer or a fractional integrator, at the exclusion of derivatives.

Simulation is performed using the equations:

$$\begin{aligned} &\frac{\partial z(\omega, t)}{\partial t} = -\omega\, z(\omega, t) + v(t) \\ &x(t) = \int_0^{\infty} \mu_n(\omega)\, z(\omega, t)\, d\omega, \end{aligned} \tag{39}$$

where the input $v(t)$ is defined by the feedback equation:

$$v(t) = u(t) - a\, x(t). \tag{40}$$

It is fundamental to notice that the true state variable of the FDE is the internal state variable $z(\omega, t)$ of the integrator while its output $x(t)$ is the pseudo state variable of the FDE.

Practically, a numerical algorithm is used to perform simulation, where the state variables $z(\omega, t)$ have been frequency discretized into $z_j(t)$:

$$\begin{aligned} &\frac{dz_j(t)}{dt} = -\omega_j\, z_j(t) + v(t) \quad \text{for } j = 0 \text{ to } J \\ &x(t) = \sum_{j=0}^{J} c_j\, z_j(t) \\ &v(t) = u(t) - a\, x(t). \end{aligned} \tag{41}$$

For instance, using the Euler technique, the complete numerical algorithm is:

$$\begin{aligned} &v_k = u_k - a\, x_k \qquad t = k\, T_e \\ &z_{j,k+1} = \alpha_j\, z_{j,\,k} + \beta_j\, v_k \\ &x_{k+1} = \sum_{j=0}^{J} c_j\, z_{j,\,k+1}, \end{aligned} \tag{42}$$

where α_j, β_j and c_j are defined in the Appendix B.

4.1.2. Simulation of a two derivative FDE

Consider the two derivative FDE:

$$D_{m_2}(x(t)) + a_1 D_{m_1}(x(t)) + a_0\, x(t) = u(t), \tag{43}$$

where m_1 and m_2 are fractional orders verifying $m_1 < m_2$.

Let us define:

$$n_1 = m_1, \qquad n_2 = m_2 - m_1 \tag{44}$$

which are the respective orders of the two integrators $I_{n_1}(s)$ and $I_{n_2}(s)$ $(0 < n_i \leq 1)$ required for the simulation of the FDE.

Let us define

$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = D_{n_1}(x_1(t)). \end{cases} \tag{45}$$

Then:

$$D_{n_2}(x_2(t)) = -a_0\, x_1(t) - a_1\, x_2(t) + u(t) \tag{46}$$

which corresponds to the closed loop simulation system of Fig. 2:

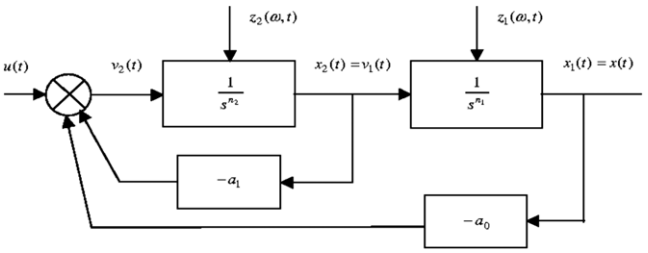

Fig. 2. Simulation of the two derivative FDE.

Simulation is performed by the following equations:

$$\begin{aligned} &\frac{\partial z_1(\omega,t)}{\partial t} = -\omega\, z_1(\omega,t) + v_1(t) \\ &x_1(t) = \int_0^\infty \mu_{n_1}(\omega)\, z_1(\omega,t)\, d\omega \\ &v_1(t) = x_2(t) \end{aligned} \tag{47}$$

$$\begin{aligned} &\frac{\partial z_2(\omega,t)}{\partial t} = -\omega\, z_2(\omega,t) + v_2(t) \\ &x_2(t) = \int_0^\infty \mu_{n_2}(\omega)\, z_2(\omega,t)\, d\omega \\ &v_2(t) = -a_0\, x_1(t) - a_1\, x_2(t) + u(t). \end{aligned} \tag{48}$$

The true state variables are $z_1(\omega,t)$ and $z_2(\omega,t)$ while $x_1(t)$ and $x_2(t)$ are the pseudo state variables.

Practically, we have to use the same numerical algorithm as previously (41) and (42).

4.1.3. Simulation of an FDS

The previous two derivative FDE corresponds to:

$$\begin{bmatrix} D_{n_1}(x_1(t)) \\ D_{n_2}(x_2(t)) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{49}$$

which is the pseudo state space model of an FDS.

In the general case, we have to define:

$$\underline{n}^T = [n_1 \dots n_i \dots n_N] \tag{50}$$

$$\underline{X}^T = [x_1 \dots x_i \dots x_N] \tag{51}$$

with $\dim \underline{X} = \dim\, \underline{n} = N,\ 0 < n_i \leq 1$.

Remark 1. If n_i is greater than one, it has to be decomposed into an integer part and a fractional one: a distributed state variable $z(\omega,t)$ is associated to the fractional part and one (or several) classical state variable $x(t)$ is associated to the integer one.

Let us define

$$D_{\underline{n}}\,(\underline{X}(t))^T = \left[D_{n_1}(x_1(t)) \ \dots\ D_{n_i}(x_i(t)) \ \dots\ D_{n_N}(x_N(t))\right]. \tag{52}$$

The corresponding pseudo state space model of the FDS is:

$$D_{\underline{n}}(\underline{X}(t)) = A\,\underline{X}(t) + \underline{B}\,u(t). \tag{53}$$

A fractional integrator $I_{n_i}(s) = \frac{1}{s^{n_i}}$ is associated to each pseudo state variable $x_i(t)$:

$$\begin{aligned} &\frac{\partial z_i(\omega,t)}{\partial t} = -\omega\, z_i(\omega,t) + v_i(t) \\ &x_i(t) = \int_0^\infty \mu_{n_i}(\omega)\, z_i(\omega,t)\, d\omega \\ &v_i(t) = D_{n_i}(x_i(t)). \end{aligned} \tag{54}$$

The true state vector is $\underline{Z}(\omega, t)$

$$\underline{Z}^T(\omega, t) = [z_1(\omega, t) \ \dots \ z_i(\omega, t) \ \dots \ z_N(\omega, t)]. \tag{55}$$

Remark 2. All FDEs can be transformed into FDSs using the generalization of the technique (44) described in [21,23]. Moreover, the output $y(t)$ of the FDE is obtained with the classical equation:

$$y(t) = \underline{C}^T \underline{X}(t) + D\,u(t). \tag{56}$$

Remark 3. Fundamentally, this simulation technique and its associated differential model are exact, because they are based on fractional integrators which are exactly modeled by infinite dimension frequency distributed systems. Consequently, the fractional integrator approach transforms the fractional differential system into an exactly equivalent integer order differential system, with infinite dimension for each fractional state variable.

This approach provides also a rigorous definition of the state of a fractional system, replacing $x_i(t)$ by $z_i(\omega, t)$: so this integrator approach allows the generalization of integer order system theory to fractional order systems.

Finally, notice that this technique applies either to commensurate or noncommensurate order fractional systems.

4.2. Response of an FDS

4.2.1. Response of a one derivative FDS

Though elementary, the response of a one derivative FDS plays an important role in the analysis of Caputo and Riemann–Liouville initial conditions (Section 6).

So we consider again:

$$D_n(x(t)) + a\,x(t) = u(t). \tag{57}$$

The Laplace transform applied to the fractional integrator $I_n(s)$ gives:

$$L\left\{\frac{\partial z(\omega, t)}{\partial t}\right\} = s\,Z(\omega, s) - z(\omega, 0) = -\omega\,Z(\omega, s) + V(s), \tag{58}$$

where $z(\omega, 0)$ is the initial condition of the fractional integrator.

So

$$Z(\omega, s) = \frac{V(s) + z(\omega, 0)}{s + \omega}. \tag{59}$$

Because

$$V(s) = U(s) - a\,X(s) \tag{60}$$

and

$$X(s) = \int_0^\infty \mu_n(\omega)\,Z(\omega, s)\,d\omega \tag{61}$$

we get

$$X(s) = \frac{s^n}{s^s + a} \int_0^\infty \frac{\mu_n(\omega)\,z(\omega, 0)}{s + \omega}\,d\omega + \frac{1}{s^n + a}\,U(s). \tag{62}$$

The first term represents the free response of the FDS, caused by the initial condition $z(\omega, 0)$, while the second term represents the forced response.

4.2.2. General case

Consider:

$$D_{\underline{n}}(\underline{X}(t)) = A\,\underline{X}(t) + \underline{B}\,u(t). \tag{63}$$

The Laplace transform applied to the fractional integrator $I_{n_i}(s)$ gives:

$$Z_i(\omega, s) = \frac{z_i(\omega, 0) + V_i(s)}{s + \omega} \tag{64}$$

and for the state vector $\underline{Z}(\omega, s)$:

$$\underline{Z}(\omega, s) = \frac{\underline{z}(\omega, 0)}{s + \omega} + \frac{\underline{V}(s)}{s + \omega}. \tag{65}$$

Let us define:

$$\underline{X}(s) = \begin{bmatrix} \vdots \\ \int_0^\infty \mu_{n_i}(\omega)\, Z_i(\omega, s) d\omega \\ \vdots \end{bmatrix} \tag{66}$$

$$\frac{1}{s^{\underline{n}}} = \begin{bmatrix} \frac{1}{s^{n_1}} & & 0 \\ & \frac{1}{s^{n_i}} & \\ 0 & & \frac{1}{s^{n_N}} \end{bmatrix} \tag{67}$$

and

$$s^{\underline{n}} = \left[\frac{1}{s^{\underline{n}}}\right]^{-1} = \begin{bmatrix} s^{n_1} & & 0 \\ & s^{n_i} & \\ 0 & & s^{n_N} \end{bmatrix} \tag{68}$$

because

$$D_{\underline{n}}(\underline{X}(s)) = \underline{V}(s) = A\,\underline{X}(s) + \underline{B}\,U(s) \tag{69}$$

we can write:

$$\underline{X}(s) = \begin{bmatrix} \vdots \\ \int_0^\infty \frac{\mu_{n_i}(\omega)\, z_i(\omega, 0)}{s + \omega} d\omega \\ \vdots \end{bmatrix} + \left[\frac{1}{s^{\underline{n}}}\right] (A\,\underline{X}(s) + \underline{B}\,U(s)\,). \tag{70}$$

Left multiplication by $s^{\underline{n}}$ gives:

$$s^{\underline{n}}\,\underline{X}(s) - A\,\underline{X}(s) = s^{\underline{n}} \begin{bmatrix} \vdots \\ \int_0^\infty \frac{\mu_{n_i}(\omega)\, z_i(\omega, 0)}{s + \omega} d\omega \\ \vdots \end{bmatrix} + \underline{B}\,U(s) \tag{71}$$

and finally:

$$\underline{X}(s) = \left[s^{\underline{n}} - A\right]^{-1} s^{\underline{n}} \begin{bmatrix} \vdots \\ \int_0^\infty \frac{\mu_{n_i}(\omega)\, z_i(\omega, 0)}{s + \omega} d\omega \\ \vdots \end{bmatrix} + \left[s^{\underline{n}} - A\right]^{-1} \underline{B}\,U(s). \tag{72}$$

Indeed, this equation generalizes the previous response of the one derivative FDS.

The global free response of the FDS is caused by the free response of each fractional integrator (25).

4.3. Initial conditions

The initial conditions correspond to $\underline{z}(\omega, 0)$. Let us consider some specific situations, with the help of the one derivative FDS.

4.3.1. System at steady state

We consider the forced response of

$$D_n(x(t)) + a\,x(t) = u(t) \tag{73}$$

to a step input $u(t) = U\,H(t)$, with no initial condition ($z(\omega, 0) = 0\,\forall\,\omega$).

Then

$$X(s) = \frac{1}{s^n + a}\,\frac{U}{s}. \tag{74}$$

Because

$$v(t) = D_n(x(t)) \quad \text{and} \quad Z(\omega, s) = \frac{1}{s + \omega} V(s). \tag{75}$$

We get:

$$Z(\omega, s) = \frac{s^n}{s^n + \omega}\,\frac{1}{s + \omega}\,\frac{U}{s}. \tag{76}$$

So we can calculate $z(\omega, \infty)$:

$$z(\omega, \infty) = \lim_{s \to 0} s Z(\omega, s) = \lim_{s \to 0} \frac{s^n}{s^n + a} \frac{U}{s + \omega}. \tag{77}$$

Thus:

$$z(\omega, \infty) = 0 \quad \forall \omega \neq 0 \tag{78}$$

and

$$z(0, \infty) = \lim_{s \to 0} \frac{s^n}{s^n + a} \frac{U}{s} = \infty \tag{79}$$

then $z(0, \infty)$ is not defined.

Nevertheless, $x(\infty)$ is perfectly defined because

$$x(\infty) = \frac{U}{a}. \tag{80}$$

So, $z(0, \infty)$ is indirectly defined by the integral equation:

$$\int_0^\infty \mu_n(\omega)\, z(0, \infty)\, d\omega = \frac{U}{a}. \tag{81}$$

Remark. With the frequency discretized model

$$x(t) = \sum_{j=0}^{J} c_j z_j(t) \tag{82}$$

and:

$$\begin{cases} z_j(\infty) = 0 \quad \forall j > 1 \\ z_0(\infty) = \dfrac{U/a}{c_0}. \end{cases} \tag{83}$$

4.3.2. System at rest

Now we consider the free response caused by $z(\omega, 0) \neq 0$.

Then

$$X(s) = \frac{s^n}{s^s + a} \int_0^\infty \frac{\mu_n(\omega)\, z(\omega, 0)}{s + \omega}\, d\omega \tag{84}$$

which represents the contribution of all the individual $z(\omega, 0)$ frequency components.

So

$$Z(\omega, s) = \frac{s^n}{s^s + a} \frac{z(\omega, 0)}{s + \omega}. \tag{85}$$

Indeed, the system is at rest for $t \to \infty$ and:

$$z(\omega, \infty) = \lim_{s \to 0} s Z(\omega, s) = \lim_{s \to 0} \frac{s\, s^n}{s^n + a} \frac{z(\omega, 0)}{s + \omega} = 0 \tag{86}$$

so we get

$$z(\omega, \infty) = 0 \quad \forall \omega. \tag{87}$$

4.3.3. Specification of initial conditions

The two previous results allow the definition of rest and steady state for an FDS.

– a system is completely at rest if

$$z_i(\omega, 0) = 0 \quad \forall \omega \text{ and } \forall i \tag{88}$$

consequently,

$$x_i(0) = 0 \quad \forall i. \tag{89}$$

Notice that reciprocally $x_i(0) = 0$ does not imply that $z_i(\omega, 0) = 0\, \forall\, \omega$!

– the component $z_i(\omega, 0)$ is in steady state if it verifies the condition:

$$\begin{cases} z_i(\omega, 0) = 0 \quad \forall \omega \neq 0 \\ \displaystyle\int_0^{\infty} \mu_{n_i}(\omega)\, z_i(0, 0)\, d\omega = x_i(0). \end{cases} \tag{90}$$

Let us consider now the ordinary initial condition of the component i:

$$\int_0^{\infty} \mu_{n_i}(\omega)\, z_i(\omega, 0)\, d\omega = x_i(0). \tag{91}$$

The specification of $x_i(0)$ is not sufficient: it is necessary to specify the frequency distribution at each frequency ω and $x_i(0)$ is only the consequence of this distribution.

5. Transients of fractional differential systems

5.1. Introduction

With the help of numerical simulations, we intend to fulfill the following objectives:

– To validate the simulation technique based on the fractional integrator
– To analyze transients (or free responses) and to validate the initialization technique based on $z(\omega, t_0)$
– To interpret the role played by the internal state variables $z(\omega, t)$.

5.2. Simulation of a one derivative FDS

Again we consider:

$$D_n(x(t)) + a\, x(t) = u(t) \tag{92}$$

with $a = 1\, n = 0.5$.

The objective is to compare the simulated output $x(t)$ to a reference one, calculated with the Mittag-Leffler function [44].

Thus, we consider the following input:

$$\begin{cases} u(t) = U \quad \text{for } 0 < t < T \\ u(t) = -U \quad \text{for } T < t < 3\,T \end{cases} \quad \text{with } U = 1\, T = 5\, s. \tag{93}$$

The reference output is:

$$\begin{aligned} x_{mit}(t) &= U\, H(t)\, \left[1 - E_{n,1}(-a\, t^n)\right] \quad 0 < t < T \\ x_{mit}(t) &= U\, H(t)\, \left[1 - E_{n,1}(-a\, t^n)\right] - 2\, U\, H(t - T)\, \left[1 - E_{n,1}(-a(t - T)^n)\right] \quad T < t < 3\,T \end{aligned} \tag{94}$$

with $E_{n,1}(-at^n)$: Mittag-Leffler function [44].

The system has been simulated with the following parameters:

$$\omega_b = 0.001 \text{ rd/s} \qquad \omega_h = 1000 \text{ rd/s} \qquad J = 20 \qquad Te = 5 \text{ ms}. \tag{95}$$

The reference and simulated outputs are displayed on Fig. 3: the two curves fit exactly.

In order to appreciate simulation accuracy, we have computed the simulation error, which is defined as:

$$\varepsilon(t) = x_{mit}(t) - x(t). \tag{96}$$

This simulation error is represented Fig. 4: the relative error is inferior to 10^{-3}, which is acceptable. Indeed, it can be reduced using lower values of T_e, enlarging the interval $\{\omega_b,\ \omega_h\}$ and increasing J, the number of cells.

Remark. Many techniques are available for the simulation of fractional order equations and systems (see for example [6,58]). Accuracy of numerical algorithms has been particularly addressed by Diethelm [50,51,40]. The Grünwald technique [59,46] is certainly the more simple approach to the simulation of FDEs. Its main drawback is the number of states ($x_{i,k-1}$ values) that have to be stored at each iteration (theoretically an infinite number) because the $x_{i,k-1}$ values correspond to an arithmetic distribution. On the contrary, the integrator approach reduces the number of states, thanks to the geometric distribution of modes (see remark of 3.4).

In [60], the authors propose a comparison of different techniques, particularly Diethelm, Grünwald and fractional integrator. Diethelm techniques are the more accurate, but they require large computation time. Grünwald approach performs a good compromise between precision and computation time. The integrator approach is the faster one, with medium precision. However, notice that the [ω_b ω_h]interval and the number of cells have not been optimized by the authors. Moreover, they have not included an integer order integrator for $1 < n < 2$ (see Remark 1 of 4.1.3).

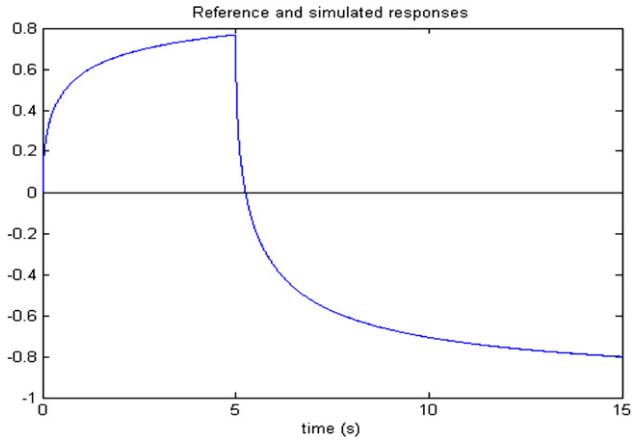

Fig. 3. Comparison of Mittag-Leffler and simulated responses.

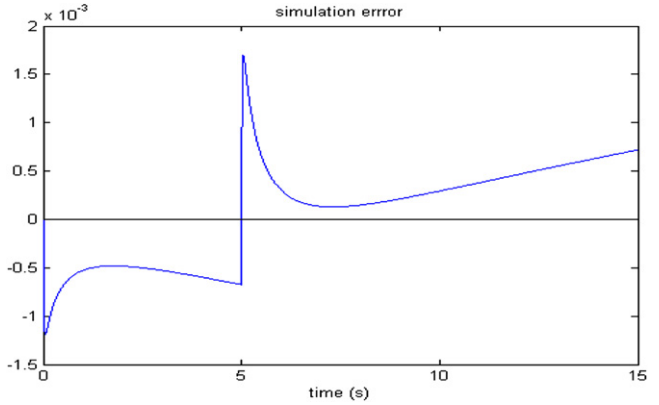

Fig. 4. Simulation error.

5.3. Initialization at $t = t_1$

On the graph of Fig. 3, we notice that $x_{mit}(t)$ (and $x(t)$) are equal to 0 at the instant $t_1 = 5.25\,s$.
So, we propose to initialize the system at $t = t_1$ by imposing $u(t) = 0$ for $t > t_1$.
So, we have modified the computation of $x_{mit}(t)$ for $t > t_1$, according to:

$$x_{mit}(t) = U\,H(t)\left[1 - E_{n,1}(-a\,t^n)\right] - 2\,U\,H(t-T)\left[1 - E_{n,1}(-a(t-T)^n)\right] + U\,H(t-t_1)\left[1 - E_{n,1}(-a(t-t_1)^n)\right] \quad t_1 < t < 3\,T \tag{97}$$

which corresponds to the equivalent input:

$$\begin{cases} u(t) = U & \text{for } 0 < t < T \\ u(t) = -U & \text{for } T < t < t_1 \\ u(t) = 0 & \text{for } t_1 < t < 3\,T \end{cases} \tag{98}$$

because $u(t) = 0$ for $t > t_1$, the reference $x_{mit}(t)$ represents the free response of the system for $t > t_1$, starting from $x(t_1) = 0$. So, we have created a reference free response which is represented Fig. 5.
On the other hand, at $t = t_1$, we have measured the state $z(\omega, t_1)$ of the fractional integrator.
So, it has been possible to initialize the system, with $z(\omega, t_1)$ and $u(t) = 0$ for $t > t_1$.
The initialized response has been plotted on Fig. 5: we can notice that the two responses fit perfectly.
The conclusions are:

- the initial state $z(\omega, t_1)$ summarizes perfectly the past behavior for $t < t_1$, so the initialized response fits perfectly the reference free response for $t > t_1$.
- this initial state $z(\omega, t_1)$ corresponds to $x(t_1) = 0$: so, we verify that the only knowledge of the pseudo state variable does not allow the prediction of the fractional system free response, it is necessary to refer to its internal state $z(\omega, t_1)$.

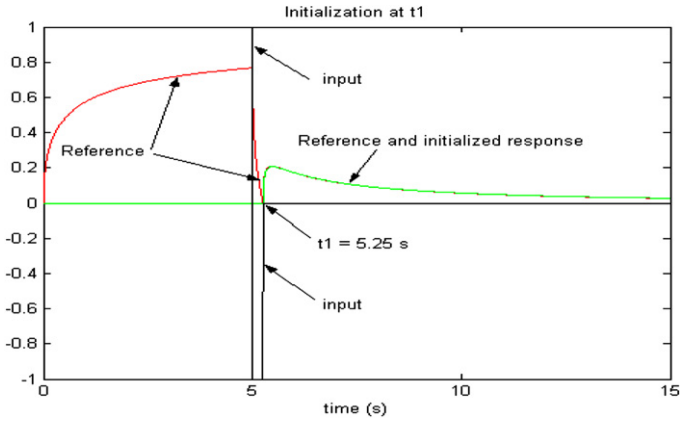

Fig. 5. Comparison of reference and initialized responses.

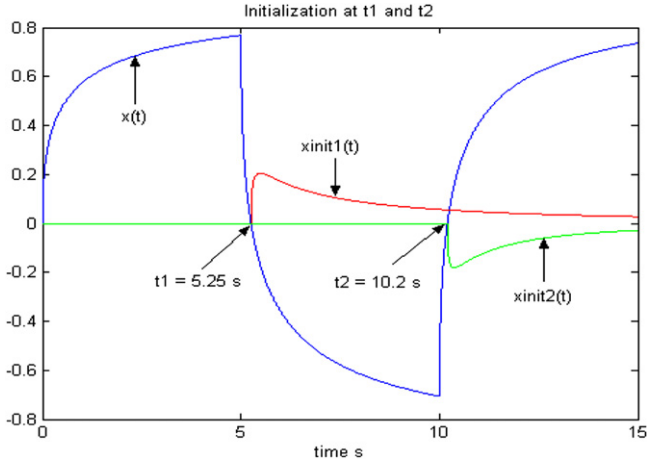

Fig. 6. Initialization at different instants.

5.4. Initialization at different instants

Finally, we have considered the following modified input:

$$\begin{cases} u(t) = U & \text{for } 0 < t < T \\ u(t) = -U & \text{for } T < t < 2T \\ u(t) = U & \text{for } 2T < t < 3T. \end{cases} \tag{99}$$

The corresponding response $x(t)$ has been plotted Fig. 6 We notice that

$$x(t) = 0 \quad \text{for } t_0 = 0,\ t_1 = 5.25\,s,\ t_2 = 10.2\,s. \tag{100}$$

Fig. 7 displays the frequency distribution of $z(\omega, t_i)$ for these three instants (the index of ω is equal to that of the discretized frequency ω_j, j varying from 0 to J).

Though the output has the same value, the three distributions are completely different.

So the system initialization at each of these instants has been represented Fig. 6: because the internal variables are different, the free responses are indeed completely different.

We can conclude that the output $x(t)$ of a fractional integrator is unable to characterize its state, the knowledge of its internal state $z(\omega, t_i)$ is necessary to predict the future behavior.

Remark 1. Indeed, these three initializations starting at $x(t_i) = 0$ are counter examples to the usual belief that the behavior of a fractional system can be characterized by $x(t_i)$, according to Caputo derivative definition.

Remark 2. $x(t)$ is not the state of the system; the true state is $z(\omega, t)$, according to:

$$x(t) = \int_0^\infty \mu_n(\omega)\, z(\omega, t)\, d\omega. \tag{101}$$

The consequence is fundamental for the definition of a Lyapunov function.

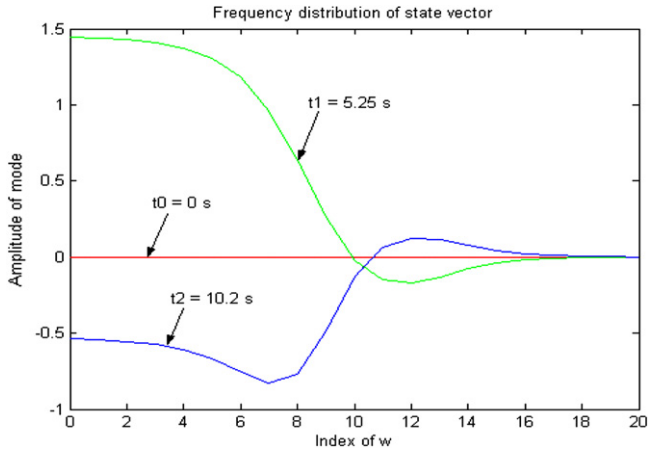

Fig. 7. Comparison of state vectors.

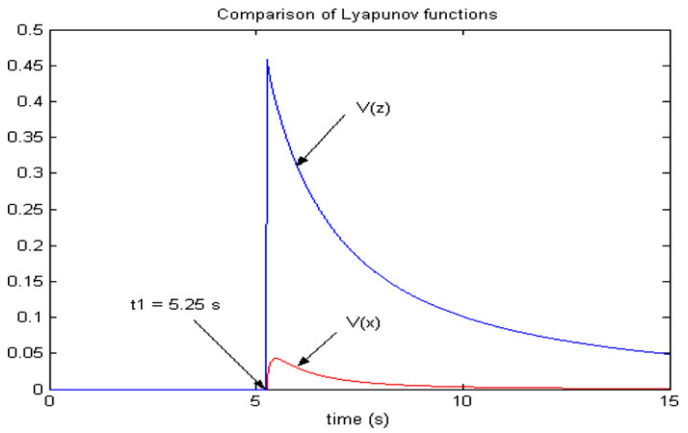

Fig. 8. Comparison of Lyapunov functions.

Consider an autonomous one derivative fractional system. As a generalization of the integer order case, it seems straightforward [61,62] to define:

$$V(x) = x^2(t). \tag{102}$$

But, because of (101),

$$V(x) = \left[\int_0^\infty \mu_n(\omega)\, z(\omega, t)\, d\omega\right]^2. \tag{103}$$

So, $V(x)$ is only a semi definite positive function.

On the contrary, if we define [36]:

$$V(z) = \int_0^\infty \mu_n(\omega)\, z^2(\omega, t)\, d\omega. \tag{104}$$

$V(z)$ is really a definite positive function.

We have represented Fig. 8 $V(x)$ and $V(z)$ with the previous example. The free response for $t \geq t_1 = 5.25\,s$ corresponds to that of an autonomous system. Because $x(t_1) = 0$, it is obvious that $V(x)$ is a semi definite positive function, whereas $V(z)$ is a positive definite function. So $V(z)$ is really a decreasing Lyapunov function, able to characterize the energy of the system.

5.5. Simulation of a two derivative FDS

We have considered the two derivative system:

$$H(s) = \frac{1}{a_0 + a_1 s^{n_1} + s^{n_1+n_2}} \tag{105}$$

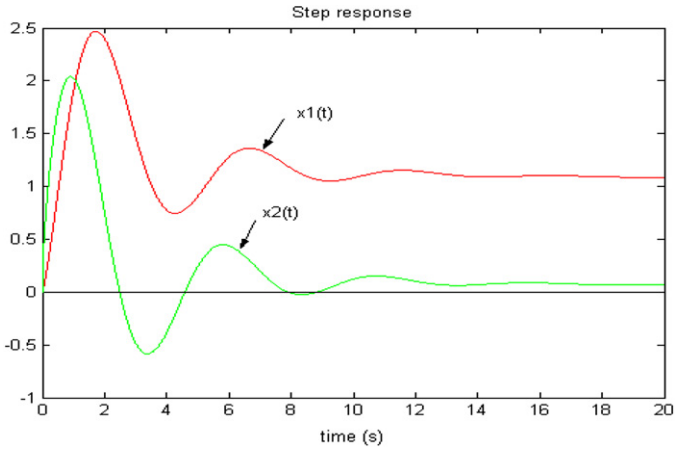

Fig. 9. Step response of the two derivative FDS.

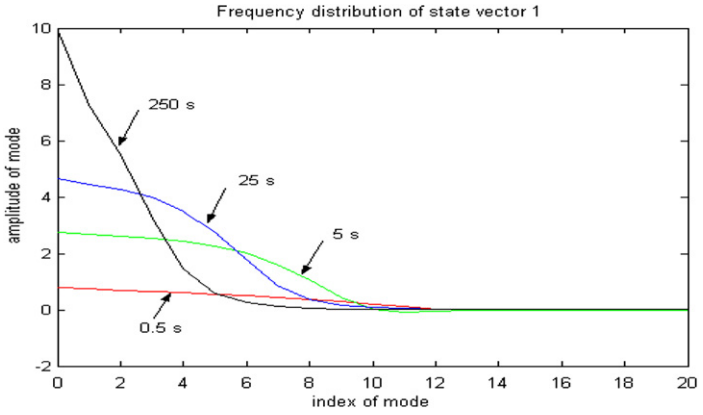

Fig. 10. Evolution of $z_1(\omega, t)$.

corresponding to the FDS:

$$\begin{bmatrix} D_{n_1}(x_1(t)) \\ D_{n_2}(x_2(t)) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{106}$$

with

$$n_1 = 0.6, \qquad n_2 = 0.4, \qquad a_0 = 1, \qquad a_1 = -1.2. \tag{107}$$

The parameters of the two fractional integrators are the same as in 5.2.

First, we have considered the step response of this system $u(t) = U\,H(t)\,U = 1$, starting from rest ($z_1(\omega, 0) = 0$, $z_2(\omega, 0) = 0$). The two variables $x_1(t)$ and $x_2(t)$ are displayed on Fig. 9.

With $a_0 = 1,\ a_1 = -1.2$, we verify that there is an oscillatory dominant mode.

Elementary analysis shows that $x_1(\infty) = 1$ and $x_2(\infty) = 0$: at $t = 20\,s$ these values are not reached.

Because $x_1(\infty)$ corresponds to steady state and $x_2(\infty)$ corresponds to rest, we have represented $z_1(\omega, t)$and $z_2(\omega, t)$ on Figs. 10 and 11, at the different instants $t = 0.5\,s,\ t = 5\,s,\ t = 25\,s,\ t = 250\,s$.

We notice that $z_1(\omega, t)$ evolves toward steady state with $z_1(0, t)$ increasing while the other modes go to zero. For $z_2(\omega, t)$, the evolution is toward rest: we notice that all the modes go to zero.

Nevertheless, it is important to notice that this evolution is very slow (though $t = 250\,s$ is a large value, it is far from infinity!) according to the long range memory phenomenon.

Moreover, we notice that the visualization of the state vectors $z_i(\omega, t)$ is a good indicator of the system evolution, better than the value of the outputs $x_i(t)$.

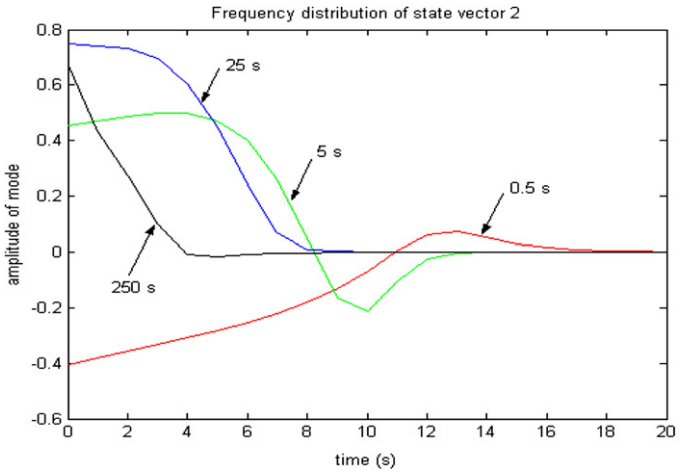

Fig. 11. Evolution of $z_2(\omega, t)$.

6. Initialization of fractional differential systems with Caputo and Riemann–Liouville derivatives

6.1. Introduction

We have demonstrated in the previous sections that the internal state variables of fractional integrators provide an efficient technique for the initialization of FDSs, which is in fact the generalization of the integer order case.

Many authors prefer to consider Caputo or Riemann–Liouville derivatives, claiming that Caputo is preferable because of its physical interpretation. In fact, the usual Laplace transform equations of these derivatives are wrong and these arguments fail.

Nevertheless, we want to demonstrate that the revised Laplace transform equations allow correct initialization, either with Caputo or Riemann–Liouville derivative.

Remark. Lorenzo and Hartley [13,18] and Trigeassou [22] have proved with different approaches that usual Laplace transform equations of fractional derivatives are wrong: they have to include either an initialization term (depending on the history function) or a free response term, corresponding to the associated $I_{1-n}(s)$ integrator.

Notice that an interpretation of fractional differentiation [63] is provided in these two complementary approaches [18,22].

6.2. Laplace transform equations

We have previously demonstrated [22,39] that the revised Laplace transform equations have to include a supplementary term, corresponding to the internal state of a specific fractional integrator.

We consider only the case $0 < n \leq 1$ corresponding to FDSs.

– Caputo derivative

Remind that

$$D_n^C(x(t)) = I_{1-n}\left(\frac{dx(t)}{dt}\right), \tag{108}$$

where $I_{1-n}(\,)$ represents the fractional integration of $\frac{dx(t)}{dt}$, with order $1-n$.

Let $z_C(\omega, t)$ be the internal state of this integrator $I_{1-n}(s)$. Then, we get:

$$L\left\{D_n^C(x(t))\right\} = s^n X(s) - s^{n-1}x(0) + \int_0^\infty \frac{\mu_{1-n}(\omega)\, z_C(\omega, 0)}{s+\omega} d\omega. \tag{109}$$

Thus, the initial conditions of the Caputo derivative are $x(0)$ and $z_C(\omega, 0)$.

– Riemann–Liouville derivative

Remind that

$$D_n^{RL}(x(t)) = \frac{d}{dt}\left[I_{1-n}(x(t))\right]. \tag{110}$$

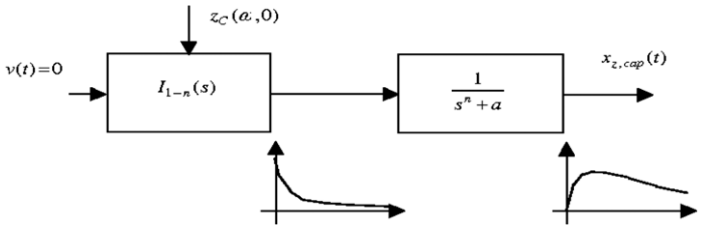

Fig. 12. Analysis of the Caputo supplementary term.

$z_{RL}(\omega, t)$ is the internal state of the fractional integrator $I_{1-n}(s)$. Then, we get:

$$L\left\{D_n^{RL}(x(t))\right\} = s^n X(s) - \{I_{1-n}(x(t))\}_0 + s\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_{RL}(\omega, 0)}{s+\omega} d\omega. \tag{111}$$

Thus, the complete initial conditions of the Riemann–Liouville derivative are $\{I_{1-n}(x(t))\}_0$ and $z_{RL}(\omega, 0)$.

Notice that the usual Laplace transform equations do not take into account the internal states of the fractional integrators.

The main interest of these equations [44,6] is to formulate the free response of an FDS, according to the considered derivative initial conditions. So, we use them to express the free response of the one derivative FDS.

6.3. Free response with Caputo derivative

Consider:

$$D_n(x(t)) + a\, x(t) = u(t) \tag{112}$$

with

$$D_n(x(t)) = D_n^C(x(t)). \tag{113}$$

Thus

$$L\left\{D_n^C(x(t))\right\} + a\, X(s) = 0, \tag{114}$$

where $L\left\{D_n^C(x(t))\right\}$ is given by (109).

So we get:

$$X(s) = \frac{s^{n-1}\, x(0)}{s^n + a} - \frac{1}{s^n + a}\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_C(\omega, 0)}{s + \omega} d\omega. \tag{115}$$

Let us define:

$$X_{Cap}(s) = \frac{s^{n-1}\, x(0)}{s^n + a} \tag{116}$$

and

$$X_{z,\,Cap}(s) = \frac{1}{s^n + a}\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_C(\omega, 0)}{s+\omega} d\omega. \tag{117}$$

Then

$$x_{Cap}(t) = L^{-1}\left\{X_{Cap}(s)\right\} = x(0)\, E_{n,\,1}\,(-a\, t^n). \tag{118}$$

This term is usually considered as the free response of the system, initialized by $x(0)$.

Indeed, this solution does not work because there is a transient, even if $x(0) = 0$!

So, it is necessary to analyze the supplementary term:

$$x_{z,\,Cap}(t) = L^{-1}\left\{X_{z,\,Cap}(s)\right\}. \tag{119}$$

$\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_C(\omega,0)}{s+\omega} d\omega$ is the free response of the $I_{1-n}(s)$ integrator (25), with the initial condition $z_C(\omega, 0)$, and $x_{z,\,Cap}(t)$ is the response of $\frac{1}{s^n+a}$ to this free response, according to Fig. 12.

Finally

$$x(t) = x_{Cap}(t) - x_{z,\,Cap}(t). \tag{120}$$

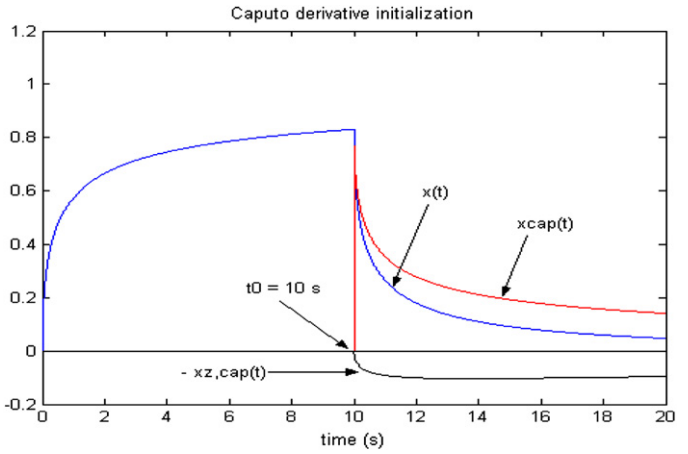

Fig. 13. Simulation of Caputo derivative initialization.

6.4. Simulation of Caputo derivative initialization

The input of the system is:

$$\begin{cases} u(t) = 1 & 0 < t < t_0 \\ u(t) = 0 & t > t_0 \end{cases} \quad \text{with } t_0 = 10\,s. \tag{121}$$

The system internal state is equal to 0 at $t = 0$.

Then, at $t = t_0$, this internal state is $z_I\,(\omega, t_0)$.

Because $u(t) = 0$ for $t > t_0$, we observe the free response of the system, initialized by $z_I\,(\omega, t_0)$: the response $x(t)$ is displayed on Fig. 13.

We compute

$$\frac{dx(t)}{dt} \quad \text{and} \quad D_n^C(x(t)) = I_{1-n}\left(\frac{dx(t)}{dt}\right) \quad \text{for } 0 < t < t_0. \tag{122}$$

At $t = t_0$, we get

$$x(t_0) \quad \text{and} \quad z_C(\omega, t_0) \quad (\text{state of } I_{1-n}(s)). \tag{123}$$

According to the procedure described in the previous section, we compute $x_{Cap}(t)$ and $x_{z,\,Cap}(t)$ for $t > t_0$.

These two signals have been plotted on Fig. 13.

Indeed, $x_{Cap}(t)$ is different from $x(t)$, though they share the same starting point $x(t_0)$.

Finally, if we add $-x_{z,\,Cap}(t)$ to $x_{Cap}(t)$, we obtain exactly $x(t)$.

So, it is possible to initialize correctly the FDS with the Caputo derivative, but the procedure is more complex than the direct one, based on $z_I\,(\omega, t_0)$!

6.5. Free response with Riemann–Liouville derivative

Consider:

$$D_n(x(t)) + a\,x(t) = u(t) \tag{124}$$

with

$$D_n(x(t)) = D_n^{RL}(x(t)). \tag{125}$$

Thus

$$L\left\{D_n^{RL}(x(t))\right\} + a\,X(s) = 0. \tag{126}$$

Using Eq. (111), we get:

$$X(s) = \frac{\{I_{1-n}(x(t))\}_0}{s^n + a} - \frac{s}{s^n + a}\int_0^\infty \frac{\mu_{1-n}(\omega)\,z_{RL}(\omega, 0)}{s + \omega}d\omega. \tag{127}$$

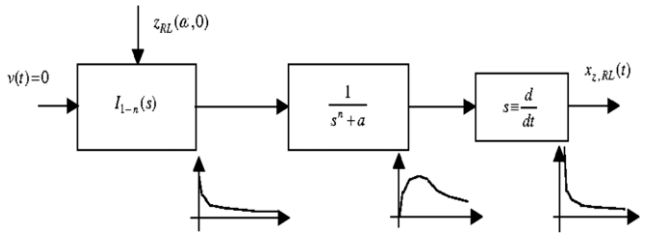

Fig. 14. Analysis of the Riemann–Liouville supplementary term.

Let us define:

$$X_{RL}(s) = \frac{\{I_{1-n}(x(t))\}_0}{s^n + a} \tag{128}$$

and

$$X_{z,\,RL}(s) = \frac{s}{s^n + a} \int_0^\infty \frac{\mu_{1-n}(\omega)\, z_{RL}(\omega, 0)}{s + \omega} d\omega. \tag{129}$$

Then

$$x_{RL}(t) = L^{-1}\ \{X_{RL}(s)\} = \ \{I_{1-n}(x(t))\}_0\ \frac{1}{t^{n-1}} E_{n,\,n}\,(-a\,t^n). \tag{130}$$

This term is called the Riemann–Liouville free response.

Indeed, it cannot fit the true free response because $\begin{cases} x_{RL}(t) \to \infty \\ t \to 0 \end{cases}$.So, it is necessary to analyze the supplementary term:

$$x_{z,\,RL}(t) = L^{-1}\ \left\{X_{z,\,RL}(s)\right\}. \tag{131}$$

$\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_{RL}(\omega,0)}{s+\omega} d\omega$ is the free response of the $I_{1-n}(s)$ integrator (25), with the initial condition $z_{RL}(\omega, 0)$, and $x_{z,\,RL}(t)$ is the integer derivative of the response of $\frac{1}{s^n+a}$ to this free response, according to Fig. 14.

Notice that $\begin{cases} x_{z,\,RL}(t) \to \infty \\ t \to 0 \end{cases}$ because of the integer derivative action.

Finally,

$$x(t) = x_{RL}(t) - x_{z,\,RL}(t). \tag{132}$$

6.6. Simulation of Riemann–Liouville derivative initialization

We use the same procedure as previously.

First, we have to compute $I_{1-n}(x(t))$ and to differentiate this integral to get $D_n^{RL}(x(t)) = \frac{d}{dt}\,[\,I_{1-n}(x(t))\,]$ for $0 < t < t_0$.

At $t = t_0$, we get

$$\{I_{1-n}(x(t))\}_0 \quad \text{and} \quad z_{RL}(\omega, t_0) \quad (\text{state of } I_{1-n}(s)). \tag{133}$$

According to the procedure described in the previous section, we compute $x_{RL}(t)$ and $x_{z,\,RL}(t)$ for $t > t_0$.

These two signals have been plotted on Fig. 15 with $x(t)$.

First, we notice an important change in the amplitude scale in comparison with Fig. 13: because of the integer derivative action, the signals $x_{RL}(t)$ and $x_{z,\,RL}(t)$ are larger than $x(t)$, particularly for $t \to t_0^+$. Nevertheless, when we add: $-x_{z,\,RL}(t)$ to $x_{RL}(t)$, we obtain exactly $x(t)$.

So it has been possible to initialize the FDS with Riemann–Liouville derivative, with a similar procedure to the Caputo derivative.

6.7. Conclusion

We have demonstrated that usual free responses predicted by Caputo or Riemann–Liouville initial conditions are wrong: they have to be corrected by a specific term including the transients of the fractional integrator.

Though it was important to verify the validity of the revised Laplace transform equations of the two derivatives, this is not a realistic approach to the initialization of an FDS.

The efficient approach is the one that generalizes the integer order technique with the appropriate state vector.

It has been demonstrated in the two previous sections that the internal states $\underline{z_I}(\omega, t_0)$ of the fractional integrators summarize the past history of the system for $t < t_0$: so this state provides the essential information which is necessary to predict the future evolution for $t > t_0$.

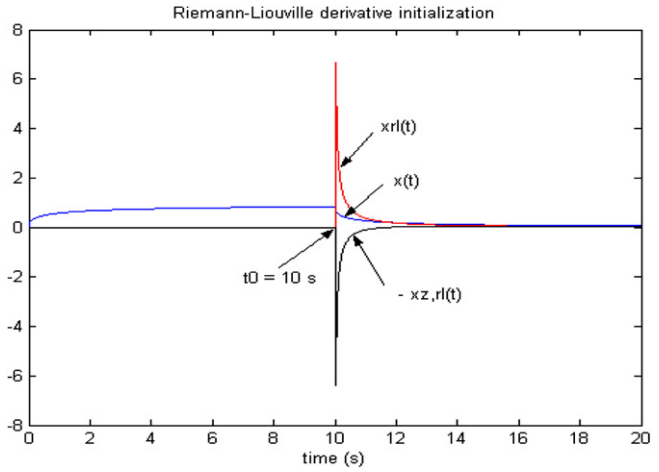

Fig. 15. Simulation of Riemann–Liouville derivative initialization.

In the fractional case, there has been a confusion between the implicit derivative (which is inherent to fractional integration) and the techniques used to compute fractional derivatives (Caputo or Riemann–Liouville): like in the integer order case, the true state of the fractional integrator is also the state of the FDS and it is the natural approach to analyze and predict transients.

On the contrary, Caputo and Riemann–Liouville derivatives are the necessary tools for the differentiation of signals operating in open loop, when the implicit derivative is not available [39].

7. Conclusion

The fractional integrator is essential to compute efficiently the Riemann–Liouville integral and particularly to perform simulation of fractional differential equations and systems. Because this integrator is a frequency distributed system, mastery of its infinite dimensional state is also essential for the mastery of FDE and FDS transients.

A fundamental result of this paper is the generalization of the integer order integrator approach for the simulation of Ordinary Differential Equations to FDEs and FDSs. Consequently, the internal state variables of fractional integrators are the components of the fractional system state vector. A specific feature of fractional systems is that each state vector component is infinite dimensional. Knowledge of this state vector allows analysis and prediction of transients. So, the initial condition problem can be solved in the same way as in the integer order case. Indeed, the main difficulty is the infinite dimension of the state vector which is an obstacle for practical applications. Some investigations with FDEs and fractional derivatives have shown that approximate finite dimension solutions can be derived using an observer based technique [23,38].

An other important result is that the usual Caputo derivative approach to FDEs is unable to derive realistic free responses. The fundamental reason of this failure is that the usual Laplace transform equations of fractional derivatives do not include the internal state of their fractional integrator. It has been shown that prediction of transients is possible with revised Laplace transform equations. Nevertheless, the Caputo and the Riemann–Liouville approach to FDSs transients is unsuited and too much complex, compared to the implicit derivative approach proposed in the paper.

Some important problems remain. Practical initialization based on state estimation by observers has to be investigated and generalized. The finite dimension approximation of the fractional integrator is an efficient technique to predict numerically FDSs transients. Nevertheless, there is a need for analytical solutions, and particularly for the transition matrix of FDSs, which is the key to derive theoretical results on controllability, observability, and optimal control of fractional systems.

Appendix A. Distributed model of the fractional integrator

The impulse response of the fractional integrator is calculated with the inverse Laplace transform

$$h_n(t) = \frac{1}{2j\pi} \int_{\gamma-j\infty}^{\gamma+j\infty} H(s)\, e^{st} ds \tag{134}$$

using a Bromwich contour [48,2,44].

Because $H(s) = \frac{1}{s^n}$ $0 < n < 1$ is a multiform function, a cut is necessary in the complex plane which leads to the contour C of Fig. 16.

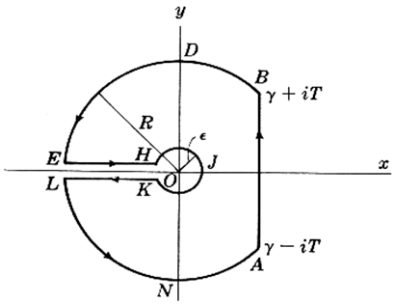

Fig. 16. Bromwich contour C.

Thus we can write:

$$\frac{1}{2j\pi}\oint_C \frac{1}{s^n}e^{st}ds = \frac{1}{2j\pi}\left[\int_{AB}+\int_{BDE}+\int_{EH}+\int_{HJK}+\int_{KL}+\int_{LNA}\right]. \tag{135}$$

Referring to Cauchy's theorem:

$$\frac{1}{2j\pi}\oint_C = 0. \tag{136}$$

Because

$$\frac{1}{2j\pi}\int_{BDE}+\frac{1}{2j\pi}\int_{LNA}=0 \quad \text{and} \quad \frac{1}{2j\pi}\int_{HJK}=0 \tag{137}$$

then:

$$\frac{1}{2j\pi}\int_{AB} = h(t) = \lim(R\to\infty, \varepsilon\to 0) - \frac{1}{2j\pi}\left[\int_{EH}+\int_{KL}\right]. \tag{138}$$

Finally, we get

$$h(t) = \int_0^\infty \frac{\sin n\pi}{\pi}\, x^{-n}\, e^{-xt}dx. \tag{139}$$

Physical considerations indicate that x corresponds to a frequency ω, so let us define $\omega = x$.

Notice that $e^{-\omega t}$ is the impulse response ($z(\omega, t) = e^{-\omega t}$) of the elementary system $\frac{1}{s+\omega}$ when its input is $v(t) = \delta(t)$.

Finally, Eq. (139) means that $h_n(t)$ is the weighted contribution of all these elementary systems, with ω ranging from $\omega = 0$ to ∞, where the frequency weight is:

$$\mu_n(\omega) = \frac{\sin n\pi}{\pi}\,\omega^{-n}. \tag{140}$$

So:

$$h_n(t) = \int_0^\infty \mu_n(\omega)\, e^{-\omega t}d\omega. \tag{141}$$

From a more general point of view, the response $z(\omega, t)$ of the elementary system to an input $v(t)$ verifies the differential equation (for the elementary frequency ω):

$$\frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + v(t) \tag{142}$$

and the output $x(t)$ of the fractional integrator is the weighted integral (with weight $\mu_n(\omega)$) of all the contributions $z(\omega, t)$ with ω ranging from 0 to ∞:

$$\begin{aligned} x(t) &= \int_0^\infty \mu_n(\omega)\, z(\omega, t)\, d\omega \\ \mu_n(\omega) &= \frac{\sin n\pi}{\pi}\,\omega^{-n} \quad 0 < n < 1. \end{aligned} \tag{143}$$

Appendix B. Fractional integrator — numerical algorithm

The user has to define the frequency range: $[\omega_b; \omega_h]$ where ω_b is the lower frequency and ω_h is the higher one. Then:

$$\begin{aligned} &\alpha = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1-n}{J}} \quad \eta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{n}{J}} \\ &\omega_1' = \omega_b \sqrt{\eta}\, \omega_J = \frac{\omega_h}{\sqrt{\eta}\, G_n} \\ &G_n = 10^{(1-n)\ \log\ \omega_b}. \end{aligned} \tag{144}$$

The intermediary frequencies verify:

$$\omega_j = \alpha\, \omega_j', \qquad \omega_{j+1}' = \eta\, \omega_j. \tag{145}$$

The coefficients c_j are provided by:

$$\begin{aligned} &c_0 = G_n \\ &c_j = \frac{G_n(\omega_j - \omega_j')}{\omega_j'} \prod_{\substack{i=1 \\ i\neq j}}^{J} \frac{1 - \frac{\omega_j}{\omega_i'}}{1 - \frac{\omega_j}{\omega_i}}. \end{aligned} \tag{146}$$

The modal differential equation

$$\frac{d\, z_j(t)}{dt} = -\omega_j\, z_j(t) + v(t) \tag{147}$$

is discretized in the time domain $\left(k = \frac{t}{Te}\right)$ using the Z transform:

$$z_{j,k} = \alpha_j\, z_{j,k-1} + \beta_j\, v_{k-1} \tag{148}$$

with:

$$\begin{aligned} &\alpha_0 = 1 \qquad \beta_0 = Te \\ &\alpha_j = e^{-\omega_j\, Te} \qquad \beta_j = \frac{1 - \alpha_j}{\omega_j} \end{aligned} \tag{149}$$

and finally:

$$x_k = \sum_{j=0}^{J} c_j\, z_{j,k}. \tag{150}$$

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