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Distributive laws and factorization ☆

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Abstract

This article shows that the distributive laws of Beck in the bicategory of sets and matrices, wherein monads are categories, determine *strict* factorization systems on their composite monads. Conversely, it is shown that strict factorization systems on categories give rise to distributive laws. Moreover, these processes are shown to be mutually inverse in a precise sense. Strict factorization systems are shown to be the strict algebras for the 2-monad $(-)^{\mathbf{2}}$ on the 2-category of categories. Further, an extension of the distributive law concept provides a correspondence with the classical factorization systems.

MSC: 18A32; 18D05

1. Introduction

In this paper we understand a *factorization system* on a category $\mathscr{K}$ to mean a pair of subcategories $(\mathscr{E}, \mathscr{M})$, each containing all the isomorphisms of $\mathscr{K}$, satisfying the diagonal fill-in condition, and further satisfying '$\mathscr{K} = \mathscr{E}\mathscr{M}$'. Of course, the equation is intended to be understood in the sense of what is called set-multiplication in elementary modern algebra texts. Part of the goal of this paper is to take that equation more

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seriously. To put it another way, factorization in the widest sense should be seen as a section for multiplication or composition. This raises the question of how categories might be multiplied or composed.

Categories are monads in a certain bicategory and after Beck [3] we know that monads in the 2-category of categories are composed with the help of distributive laws. There is much that can be said about distributive laws in any bicategory and, in particular, Beck's correspondence between distributive laws and composite monad structures holds quite generally.

We refer the reader to [14,9,11,13] for other general results about distributive laws.In this article we show the equivalence of three concepts: distributive laws in the bicategory of set-valued matrices, wherein monads correspond to categories; *strict* factorization systems on categories; and strict algebras for the 2-monad on **CAT** given by $(-)^{\mathbf{2}}$ and the structure induced by the cocommutative comonoid $\mathbf{1} \leftarrow \mathbf{2} \rightarrow \mathbf{2} \times \mathbf{2}$.

The important paper [8] showed that factorization systems on categories are equivalent to normal pseudo-algebras for the 2-monad $(-)^{\mathbf{2}}$. We extend the notion of distributive law in the bicategory of set-valued matrices to give a third concept equivalent to that of factorization system.

The next section provides a fairly detailed study of distributive laws in the bicategory **set**-**mat**. In particular, we study the composite category arising from a distributive law between categories in such a way as to subsequently reveal its factorization structures. We also identify the isomorphisms in the composite category and it is seen to be a groupoid precisely when both factors are so. This identifies 'matched pairs' of groups in the one-object case.

Strict factorization systems are defined in Section 3. See also [5]. The equivalence of these with distributive laws in **set**-**mat** follows quickly here.Strict algebras for $(-)^{\mathbf{2}}$, which we call *strict factorization algebras* are studied in detail in Section 4. Although this work does not follow directly from [8], the section is heavily influenced by that paper. We conclude with the establishment of a bijection between strict factorization systems and strict factorization algebras. After this article was written we became aware of the work of Coppey [4] which also demonstrates this bijection.

In the last section, we consider a generalization of the concept of distributive law in **set**-**mat** that allows us to extend the results of Section 3 to a correspondence between such generalized distributive laws and factorization systems. Finally, we note that 'pullback' can be seen as a distributive law in a still wider sense.

2. Distributive laws in set-mat

2.1. The objects of the bicategory **set**-**mat** (see [3]) are sets, which will be denoted by $\mathbf{X}$, $\mathbf{A}$ and so on, and in **set**-**mat** an arrow (1-cell) $\mathscr{M}:\mathbf{X}\rightarrow\mathbf{A}$ is a **set**-valued *matrix* which, to fix notation, we decree to have sets $\mathscr{M}(A,X)$ as entries, one for each pair (A,X) in $\mathbf{A}\times\mathbf{X}$. A transformation (2-cell) $t:\mathscr{M}\rightarrow\mathscr{N}:\mathbf{X}\rightarrow\mathbf{A}$ is a matrix of functions $t(A,X):\mathscr{M}(A,X)\rightarrow\mathscr{N}(A,X)$. Furthermore, we write

$$\mathbf{X}\xrightarrow{\mathscr{M}}\mathbf{A}\xrightarrow{\mathscr{E}}\mathbf{Y}=\mathbf{X}\xrightarrow{\mathscr{E}\mathscr{M}}\mathbf{Y}$$

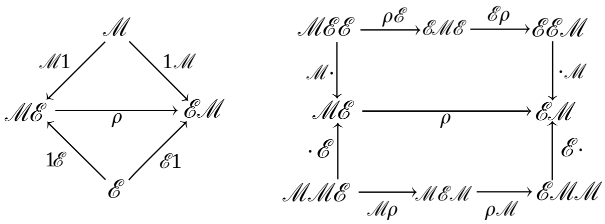
to denote composition in **set**-**mat**, with

$$\mathscr{E}\mathscr{M}(Y,X) = \sum_{A\in\mathbf{A}} \mathscr{E}(Y,A) \times \mathscr{M}(A,X).$$

It is well known that a monad $\mathscr{M}$ on an object $\mathbf{O}$ in this bicategory is precisely a category with set of objects $\mathbf{O}$.

2.2. For a suitable monoidal category $\mathscr{V}$, our remarks above and the work which follows generalize almost immediately if we replace **set**-**mat** by $\mathscr{V}$-**mat**, whose objects are sets and whose arrows are $\mathscr{V}$-valued matrices, composed with the help of $\otimes$ rather than $\times$. A monad in $\mathscr{V}$-**mat** is a category enriched in $\mathscr{V}$. On the other hand, the bicategory **set**-**mat** is biequivalent to **spn**(**set**), the bicategory of spans in the category of sets. If **spn**(**set**) is replaced by **spn**($\mathscr{E}$), where $\mathscr{E}$ is a category with pullbacks, then our work generalizes to category objects in $\mathscr{E}$.

2.3. Now if $\mathscr{M}$ and $\mathscr{E}$ are both categories with set of objects $\mathbf{O}$ then in the spirit of 2.1 we can consider distributive laws $\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ of $\mathscr{M}$ over $\mathscr{E}$ and we recall from [2] that the required equations are:



where we have denoted both transformations that provide identities by 1 and both transformations that provide composites by $\cdot$. To give merely a transformation

$$\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M} : \mathbf{O} \to \mathbf{O}$$

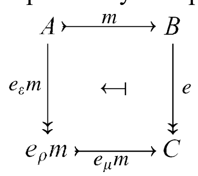
in **set**-**mat** is to give a function

$$\rho(A,C) : \mathscr{M}\mathscr{E}(A,C) \to \mathscr{E}\mathscr{M}(A,C)$$

for each pair (A,C) in $\mathbf{O}\times\mathbf{O}$. From the definition of composition of arrows it follows that to give such $\rho(A,C)$ is to give families of functions

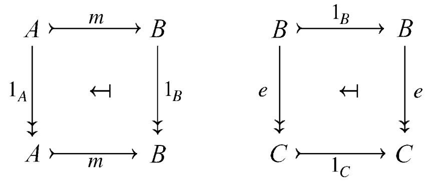
$$\Big\langle \mathscr{M}(A,B) \times \mathscr{E}(B,C) \to \sum_{I\in\mathbf{O}} \mathscr{E}(A,I) \times \mathscr{M}(I,C) \Big\rangle_{B\in\mathbf{O}}.$$

If we write $m : A \rightarrowtail B$ for an arrow in $\mathscr{M}$ and $e : B \twoheadrightarrow C$ for an arrow in $\mathscr{E}$ then it is clear that ρ provides, for each such putatively composable pair, an object $e_\rho m$ and another putatively composable pair as illustrated by



where we have also introduced an evident notation for the components of the new pair. We will call a diagram such as this a ρ-*square*.

2.4. In terms of ρ-squares the triangular distributive law equations can now be expressed as



Each of the ρ-squares expresses an equality of objects, $(1_B)_\rho m = A$ in the first case, $e_\rho(1_B) = C$ in the second case; an equality of $\mathscr{E}$ arrows; and an equality of $\mathscr{M}$ arrows:

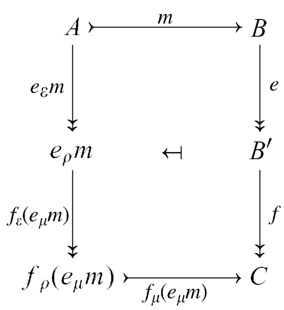
$$(1_B)_\varepsilon m = 1_A, \tag{1}$$

$$(1_B)_\mu m = m, \tag{2}$$

$$e_\varepsilon(1_B) = e, \tag{3}$$

$$e_\mu(1_B) = 1_C. \tag{4}$$

The top pentagon distributive law equation in terms of ρ-squares is given by

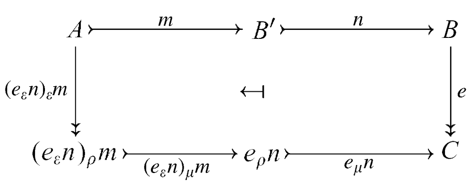


which expresses the equality of objects $(fe)_\rho m = f_\rho(e_\mu m)$ and the arrow equations

$$(fe)_\varepsilon m = f_\varepsilon(e_\mu m) \cdot e_\varepsilon m, \tag{5}$$

$$(fe)_\mu m = f_\mu(e_\mu m). \tag{6}$$

The lower pentagon of 2.3 is given similarly by



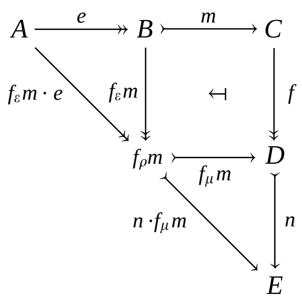
expressing the equality of objects $e_\rho(nm) = (e_\varepsilon n)_\rho m$ and the arrow equations

$$e_\varepsilon(nm) = (e_\varepsilon n)_\varepsilon m, \tag{7}$$

$$e_\mu(nm) = e_\mu n \cdot (e_\varepsilon n)_\mu m. \tag{8}$$

2.5. Inspection of the eight arrow equations that have arisen in 2.4 shows that (2) and (6) provide a left action, $-_\mu-$, of $\mathscr{E}$ on $\mathscr{M}$ while (3) and (7) provide a right action, $-_\varepsilon-$, of $\mathscr{M}$ on $\mathscr{E}$. If $\mathbf{O}$ is a one-element set, so that $\mathscr{M}$ and $\mathscr{E}$ are monoids, then the object equalities are trivial. In this case pairs of actions satisfying (1)–(8) are called *matched pairs*—at least that is the terminology in [15] when $\mathscr{M}$ and $\mathscr{E}$ are groups. In the case of monoids $\mathscr{M}$ and $\mathscr{E}$ we may as well write $\mathscr{M}$ for $\mathscr{M}(*,*)$, $\mathscr{E}$ for $\mathscr{E}(*,*)$ and $\mathscr{E}\mathscr{M}$ for the set $\mathscr{E}\mathscr{M}(*,*)$. In this case $\mathscr{E}\mathscr{M} = \mathscr{E} \times \mathscr{M}$. For groups, each matched pair is known to give rise to a group structure on the underlying set of $\mathscr{E} \times \mathscr{M}$ that is suitably compatible with the identities of $\mathscr{E}$ and $\mathscr{M}$. We will have more to say about this but it suffices here to point out that while Eqs. (5) and (8) appear somewhat bizarre when given for monoids without reference to the single object, all of the equations are entirely transparent when displayed diagrammatically with 'types' taken into account.

2.6. From the general theory of distributive laws given in [2], it follows that a distributive law $\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ in **set-mat** gives rise to a category $\mathscr{E}_\rho\mathscr{M}$, with set of objects $\mathbf{O}$, in which an arrow from A to C is given by specifying a third object, say B, and a pair $A \overset{e}{\twoheadrightarrow} B \overset{m}{\rightarrowtail} C$, with e in $\mathscr{E}$ and m in $\mathscr{M}$. Composition in $\mathscr{E}_\rho\mathscr{M}$, qua category, is given by the multiplication formula for $\mathscr{E}_\rho\mathscr{M}$, qua monad, and still following [2] we see that the composite of $A \overset{e}{\twoheadrightarrow} B \overset{m}{\rightarrowtail} C$ and $C \overset{f}{\twoheadrightarrow} D \overset{n}{\rightarrowtail} E$ is given by the following diagram:



In other words, if we denote arrows of $\mathscr{E}_\rho\mathscr{M}$ as formal composites $m \circ_B e = m \circ e : A \to C$ then

$$(n \circ f) \cdot (m \circ e) = (n \cdot f_\mu m) \circ (f_\varepsilon m \cdot e).$$

This composition satisfies the following:

(i) Identities are given by the $A \overset{1_A}{\twoheadrightarrow} A \overset{1_A}{\rightarrowtail} A$.

(ii) The assignments $e \mapsto 1 \circ e$ and $m \mapsto m \circ 1$ provide identity-on-objects functors $\mathscr{E} \to \mathscr{E}_\rho\mathscr{M}$ and $\mathscr{M} \to \mathscr{E}_\rho\mathscr{M}$, respectively.
(iii) For every $m \circ e$ in $\mathscr{E}_\rho\mathscr{M}$, $(m \circ 1)\cdot(1 \circ e) = m \circ e$.
Moreover, still appealing to the general theory of distributive laws (see [2]), compositions on $\mathscr{E}\mathscr{M}$ which satisfy (i)–(iii) are in bijective correspondence with distributive laws $\mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$. In the case of monoids $\mathscr{M}$ and $\mathscr{E}$, it follows from the general theory of distributive laws that matched pairs of monoid actions for $\mathscr{M}$ and $\mathscr{E}$, are in bijective correspondence with monoid structures satisfying (i)–(iii) on $\mathscr{E} \times \mathscr{M}$.

2.7. For monads on **set** in the 2-category of categories, the purely syntactic notion of distributive 'law' relating them is really a rewriting rule. It is when we consider the category of Eilenberg–Moore algebras for the composite monad that the connection with the 'distributive laws' of classical algebra becomes clear. The bicategory **set-mat** does not admit the construction of Eilenberg–Moore algebras as a lax limit and in 2.3, in which we introduced the ρ-square notation to display the effect of $\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$, there was no suggestion that the ρ-square 'commutes'. In the first instance, such a statement is meaningless. However, the functors in (ii) of 2.6 are faithful and (iii) of the same section suggests that we simply write e for $1 \circ e$ and m for $m \circ 1$ in $\mathscr{E}_\rho\mathscr{M}$ so that we have $m \cdot e = m \circ e$. Also, for a composable quadruple f, e, m, n, with f and e in $\mathscr{E}$ and m and n in $\mathscr{M}$, composition in $\mathscr{E}_\rho\mathscr{M}$ simplifies to $n\cdot(m \circ e)\cdot f = (n\cdot m)\circ(e\cdot f)$.

2.8. Lemma. *With the abbreviation convention of* 2.7, *ρ-squares*

$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & B \\ {\scriptstyle e_\varepsilon m}\downarrow & \leftarrow\!\shortmid & \downarrow{\scriptstyle e} \\ e_\rho m & \underset{e_\mu m}{\rightarrowtail} & C \end{array}$$

can be seen as commutative squares in $\mathscr{E}_\rho\mathscr{M}$.

Proof. This is a trivial consequence of the definition of composition in $\mathscr{E}_\rho\mathscr{M}$ given in 2.6. □

The following lemma, which we will need subsequently, suggests that there is an interesting calculus for ρ-squares.

2.9. Lemma. *Given the following configuration of ρ-squares in $\mathscr{E}_\rho\mathscr{M}$ (in which it is not assumed that the square ★ commutes):*

$$\begin{array}{ccccc} \cdot & \overset{p}{\rightarrowtail} & \cdot & & \\ {\scriptstyle h}\downarrow & \leftarrow\!\shortmid & \downarrow{\scriptstyle k} & & \\ \cdot & \overset{n}{\rightarrowtail} & \cdot & \overset{r}{\rightarrowtail} & \cdot \\ {\scriptstyle e}\downarrow & \star & \downarrow{\scriptstyle f} & \leftarrow\!\shortmid & \downarrow{\scriptstyle g} \\ I & \underset{m}{\rightarrowtail} & \cdot & \underset{g}{\rightarrowtail} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \overset{p}{\rightarrowtail} & \cdot \\ {\scriptstyle eh}\downarrow & \leftarrow\!\shortmid & \downarrow{\scriptstyle fk} \\ I & \underset{m}{\rightarrowtail} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \overset{rn}{\rightarrowtail} & \cdot \\ {\scriptstyle e}\downarrow & \leftarrow\!\shortmid & \downarrow{\scriptstyle g} \\ I & \underset{qm}{\rightarrowtail} & \cdot \end{array}$$

square ★ is a ρ-square.

Proof. From the middle diagram we see that $I=(fk)_\rho p$. From $(fk)_\rho p = f_\rho(k_\mu p)$, the object-equality preceding (5) of 2.4, and $k_\mu p = n$, in the left-most diagram, we conclude that

$$I = f_\rho n.$$

(This much follows equally from $I = g_\rho(rn)$, which we have in the right-most diagram.) Also from the middle diagram we see that $m=(fk)_\mu p$. But $(fk)_\mu p = f_\mu(k_\mu p)$ by (6) of 2.4, so

$$m = f_\mu n.$$

Starting with the right-most diagram, we observe that $e = g_\varepsilon(rn)$ which by (7) of 2.4 is $(g_\varepsilon r)_\varepsilon n$. But $g_\varepsilon r = f$, from the left-most diagram so we also have

$$e = f_\varepsilon n$$

and the three equations we have displayed show that square ★ is a ρ-square. □

In calculations it is sometimes helpful to draw ρ-squares with other orientations and suitably redirect the symbol $\longleftarrow\!\!\!\mid$ in the centre.

2.10. In anticipation of Section 4, further notation will be helpful. Writing $f:X\to A$ for a *general* arrow of $\mathscr{E}_\rho\mathscr{M}$ we can name its various components as $X \xrightarrow{e(f)} F(f) \xrightarrow{m(f)} A$. With this notation a general commutative square in $\mathscr{E}_\rho\mathscr{M}$,

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & B \end{array}$$

becomes

$$\begin{array}{ccccc} X & \xrightarrow{e(u)} & F(u) & \xrightarrow{m(u)} & Y \\ {\scriptstyle e(f)}\downarrow & & \downarrow & \longleftarrow\!\!\!\mid & \downarrow{\scriptstyle e(g)} \\ F(f) & \longrightarrow\!\!\!\!\rightarrow & I & \rightarrowtail & F(g) \\ {\scriptstyle m(f)}\downarrow & \longmapsto & \downarrow & & \downarrow{\scriptstyle m(g)} \\ A & \xrightarrow[e(v)]{} & F(v) & \xrightarrow[m(v)]{} & B \end{array}$$

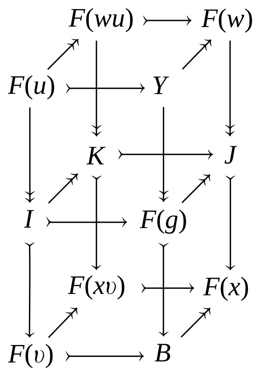
where the top-left square commutes in $\mathscr{E}$ and the bottom-right square commutes in $\mathscr{M}$. This follows immediately from the prescription for composition in $\mathscr{E}_\rho\mathscr{M}$ given in 2.6. For the moment, we will informally refer to I as the *centre object*.

Moreover, we will write $F(u,v)$ for the arrow $F(f) \twoheadrightarrow I \rightarrowtail F(g)$ in the diagram above so that we have an assignment

$$f \xrightarrow{(u,v)} g \quad \mapsto \quad F(f) \xrightarrow{F(u,v)} F(g).$$

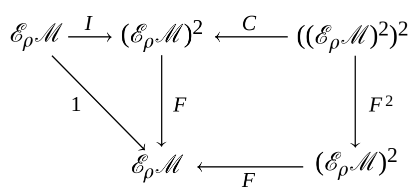
2.11. Proposition. *The assignment defines a functor* $F:(\mathscr{E}_\rho\mathscr{M})^2 \to \mathscr{E}_\rho\mathscr{M}$.

Proof. To show that F preserves identities is easy: specialize the diagrams of 2.10. to the case $g=f$, $(u,v)=(1_X,1_A)$ and apply (1) and (4) of 2.4. To show that F preserves composition, start with $f \xrightarrow{(u,v)} g \xrightarrow{(w,x)} h$ in $(\mathscr{E}_\rho\mathscr{M})^2$ and construct a 4-by-2 array of squares by pasting to the 2-by-2 array for (u,v) shown in 2.10, the corresponding 2-by-2 array for (w,x). Label the centre object of the array for (w,x) as J. Compute the composites wu and xv in primitive terms on the large diagram, using 2.6, and compute the centre object for $f \xrightarrow{(wu,xv)} h$, call it K, and supply the connecting arrows. In the middle of the resulting diagram is the following double cube:



The front face of the top cube is a ρ-square, of the square in the middle 'horizontal' plane nothing can be said initially, while the right face of the bottom cube is a ρ-square. Unfold these squares so as to configure them as in the first diagram of Lemma 2.9. The top and back faces of the top cube are ρ-squares. Unfold them so as to configure them as in the second diagram of Lemma 2.9 and observe from 2.4 that the composite square is a ρ-square. The left and bottom faces of the bottom cube are ρ-squares. Unfold them so as to configure them as in the third diagram of Lemma 2.9 and observe, again from 2.4, that the composite square is a ρ-square. The remaining faces of the cubes commute and show that the conditions of Lemma 2.9 are fulfilled. It follows that the square in the middle 'horizontal' plane is a ρ-square and this proves that $F(wu,xv)=F(w,x)F(u,v)$. □

2.12. Proposition. *The following diagrams of functors*



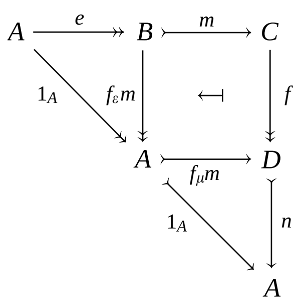
in which $I = I_{\mathscr{E}_\rho \mathscr{M}}$ *is given by identities and* $C = C_{\mathscr{E}_\rho \mathscr{M}}$ *is given by composition, commute* (*strictly*).

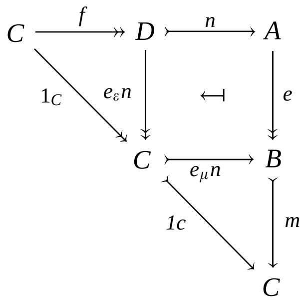
Proof. Commutativity of the triangle is easy. For the square, refer to 2.10 and write c for the common value of $vf = gu$ in $\mathscr{E}_\rho \mathscr{M}$. It follows from the meaning of composition in $\mathscr{E}_\rho \mathscr{M}$ that for the centre object we have $I = F(c)$ (while $e(c)$ is the composite arrow of the top-left square and $m(c)$ is the composite arrow of the bottom-right square). It also follows immediately that $I = F(F(u, v))$. Thus $F(c) = F(F(u, v))$ and this shows that the square above commutes on objects. Elaboration of this argument shows that the square commutes. $\square$

We close this section with an examination of the isomorphisms of $\mathscr{E}_\rho \mathscr{M}$.

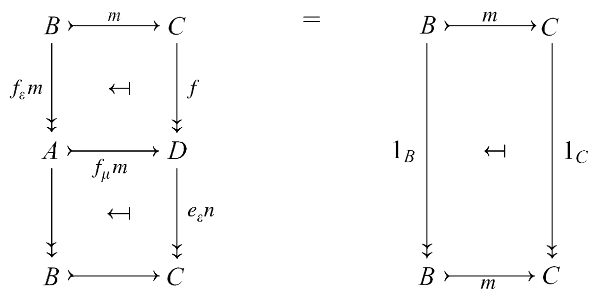
2.13. Proposition. *For* $e : A \twoheadrightarrow B$ *in* $\mathscr{E}$ *and* $m : B \rightarrowtail C$ *in* $\mathscr{M}$, $A \overset{e}{\twoheadrightarrow} B \overset{m}{\rightarrowtail} C$ *is an isomorphism in* $\mathscr{E}_\rho \mathscr{M}$ *if and only if* $e : A \twoheadrightarrow B$ *is an isomorphism in* $\mathscr{E}$ *and* $m : B \rightarrowtail C$ *is an isomorphism in* $\mathscr{M}$.

Proof. Suppose first that $A \overset{e}{\twoheadrightarrow} B \overset{m}{\rightarrowtail} C$ is an isomorphism with inverse $C \overset{f}{\twoheadrightarrow} D \overset{n}{\rightarrowtail} A$. Then from the definition of composition in 2.6 we have





From $e_\varepsilon n \cdot f = 1_C$ (in the top triangle of the large triangle on the right) we have, using both the pentagon and triangle conditions for ρ that pertain to the structure of $\mathscr{E}$,



from which it follows that $f_\varepsilon m$ is a split monomorphism in $\mathscr{E}$. But by the top triangle of the large triangle on the left we also have that $f_\varepsilon m$ is an epimorphism in $\mathscr{E}$ which is split by e. It follows that e is an isomorphism in $\mathscr{E}$ with $e^{-1} = f_\varepsilon m$. Similarly, beginning with each of the other three small triangles in the large triangles we find that m, f, and n are isomorphisms in their respective categories, with inverses again evident in the large triangles.

Conversely, assume now that $e : A \twoheadrightarrow B$ is an isomorphism in $\mathscr{E}$, with inverse $f : B \twoheadrightarrow A$, and $m : B \rightarrowtail C$ is an isomorphism in $\mathscr{M}$, with inverse $n : C \rightarrowtail B$. Consider

$$\begin{array}{ccc} C & \overset{n}{\rightarrowtail} & B \\ {\scriptstyle f_\varepsilon n}\downarrow & \leftarrow\!\mid & \downarrow{\scriptstyle f} \\ I & \underset{f_\mu n}{\rightarrowtail} & A \end{array}$$

We will show that $m \circ e$ is an isomorphism in $\mathscr{E}_\rho\mathscr{M}$ with inverse $f_\mu n \circ f_\varepsilon n$. Consider

$$\begin{array}{ccccc} A & \overset{e}{\twoheadrightarrow} & B & \overset{m}{\rightarrowtail} & C \\ & & {\scriptstyle (f_\varepsilon n)_\varepsilon m}\downarrow & \leftarrow\!\mid & \downarrow{\scriptstyle f_\varepsilon n} \\ & & J & \underset{(f_\varepsilon n)_\mu m}{\rightarrowtail} & I \\ & & & & \downarrow{\scriptstyle f_\mu n} \\ & & & & A \end{array} \qquad \begin{array}{ccccc} C & \overset{f_\varepsilon n}{\twoheadrightarrow} & I & \overset{f_\mu n}{\rightarrowtail} & A \\ & & {\scriptstyle e_\varepsilon(f_\mu n)}\downarrow & \leftarrow\!\mid & \downarrow{\scriptstyle e} \\ & & K & \underset{e_\mu(f_\mu n)}{\rightarrowtail} & B \\ & & & & \downarrow{\scriptstyle m} \\ & & & & C \end{array}$$

From $1_B = (B \overset{m}{\rightarrowtail} C \overset{n}{\rightarrowtail} B)$ in $\mathscr{M}$ we have

$$\begin{array}{ccc} B & \overset{1_B}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & \leftarrow\!\mid & \downarrow{\scriptstyle f} \\ A & \underset{1_A}{\rightarrowtail} & A \end{array} \quad = \quad \begin{array}{ccccc} B & \overset{m}{\rightarrowtail} & C & \overset{n}{\rightarrowtail} & B \\ {\scriptstyle (f_\varepsilon n)_\varepsilon m}\downarrow & \leftarrow\!\mid & {\scriptstyle f_\varepsilon n}\downarrow & \leftarrow\!\mid & \downarrow{\scriptstyle f} \\ J & \underset{(f_\varepsilon n)_\mu m}{\rightarrowtail} & I & \underset{f_\mu n}{\rightarrowtail} & A \end{array}$$

which gives $J = A$ and $(f_\varepsilon n)_\varepsilon m = f$ so that we have $(f_\varepsilon n)_\varepsilon m \cdot e = f \cdot e = 1_A$. Even more immediately we see that $f_\mu n \cdot (f_\varepsilon n)_\mu m = 1_A$ which together with the previous equation shows, by examination of the first large 'triangle' above, that $(f_\mu n \circ f_\varepsilon n) \cdot (m \circ e) = 1_A$ in $\mathscr{E}_\rho\mathscr{M}$. Starting with $1_B = (B \overset{f}{\twoheadrightarrow} A \overset{e}{\twoheadrightarrow} B)$ in $\mathscr{E}$, a similar calculation shows that $(m \circ e) \cdot (f_\mu n \circ f_\varepsilon n) = 1_C$ in $\mathscr{E}_\rho\mathscr{M}$. $\square$

2.14. Corollary. *The composite category $\mathscr{E}_\rho\mathscr{M}$ is a groupoid if and only if both $\mathscr{E}$ and $\mathscr{M}$ are groupoids.*

Taking the case $\mathbf{O} = \mathbf{1}$ we reach the speciality of ‘matched pairs’ of group actions that we learned from Mastnak [12].

2.15. Corollary. *If $\mathbf{O} = \mathbf{1}$ then the monoid structure, $\mathscr{E}_\rho\mathscr{M}$, on $\mathscr{E} \times \mathscr{M}$ is a group if and only if both $\mathscr{E}$ and $\mathscr{M}$ are groups.*

Thus, the general theory of distributive laws explains why matched pairs of group actions, for groups $\mathscr{M}$ and $\mathscr{E}$, are in bijective correspondence with group structures on $\mathscr{E} \times \mathscr{M}$ that are compatible with the identities of $\mathscr{M}$ and $\mathscr{E}$.

3. Factorization systems

3.1. A *factorization system* on a category $\mathscr{K}$ consists of a pair of subcategories $(\mathscr{E}, \mathscr{M})$, each containing all the isomorphisms of $\mathscr{K}$, satisfying the diagonal fill-in condition, with the property that, for every arrow f in $\mathscr{K}$, there is a factorization $f = m_f \cdot e_f$ with e_f in $\mathscr{E}$ and m_f in $\mathscr{M}$. An excellent reference, especially for our purposes, is [8]. As in [5], we also say that a *strict factorization system* on a category $\mathscr{K}$ consists of a pair of subcategories $(\mathscr{E}, \mathscr{M})$ of $\mathscr{K}$, each having the same set of objects as $\mathscr{K}$, with the property that, for every arrow f in $\mathscr{K}$, there is a *unique* factorization $f = m_f \cdot e_f$ with e_f in $\mathscr{E}$ and m_f in $\mathscr{M}$. The terminology is somewhat unfortunate in that a strict factorization system need not be a factorization system. However, as pointed out in [5], for each strict factorization system $(\mathscr{E}, \mathscr{M})$, there is precisely one factorization system $(\bar{\mathscr{E}}, \bar{\mathscr{M}})$ with $\mathscr{E} \subseteq \bar{\mathscr{E}}$ and $\mathscr{M} \subseteq \bar{\mathscr{M}}$ and it is given by

$$\bar{\mathscr{E}} = \{f \mid m_f \text{ is invertible}\}, \quad \bar{\mathscr{M}} = \{f \mid e_f \text{ is invertible}\}.$$

3.2. For a strict factorization system $\mathscr{S} = (\mathscr{E}, \mathscr{M})$ on a category $\mathscr{K}$ with objects $\mathbf{O}$, regard $\mathscr{E}$ and $\mathscr{M}$ as monads on $\mathbf{O}$ in **set-mat** and define

$$\rho_{\mathscr{S}} : \mathscr{M}\mathscr{E} \longrightarrow \mathscr{E}\mathscr{M} : \mathbf{O} \longrightarrow \mathbf{O}$$

as a transformation in **set-mat** by

$$A \xrightarrow{n} B \xrightarrow{f} C \quad \mapsto \quad A \xrightarrow{e_{fn}} I \xrightarrow{m_{fn}} C$$

for $n \in \mathscr{M}$ and $f \in \mathscr{E}$.

3.3. Proposition. *The transformation $\rho_{\mathscr{S}} : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ is a distributive law.*

Proof. The unitary conditions are obvious. For $A \xrightarrow{n} B$ in $\mathscr{M}$ and $B \xrightarrow{f} C \xrightarrow{g} D$ a composable pair in $\mathscr{E}$, consider first *the* $\mathscr{E}$-$\mathscr{M}$ factorization $fn = me$ and next *the* $\mathscr{E}$-$\mathscr{M}$ factorization $gm = m'e'$. Since $(gf)n = m'(e'e)$ provides an $\mathscr{E}$-$\mathscr{M}$ factorizaton it is necessarily *the* $\mathscr{E}$-$\mathscr{M}$ factorization. With these observations it is a simple matter to fill in the notation of 2.3 and get Eqs. (5) and (6) of 2.4 and the equality of objects preceding them. The set of equations for the other pentagon are derived similarly. □

3.4. For $\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ a distributive law in **set-mat**, consider the subcategories of $\mathscr{E}_\rho\mathscr{M}$, given by

$$^{\rho}\mathscr{E} = \{1 \circ e \mid e \in \mathscr{E}\} \quad \text{and} \quad \mathscr{M}^{\rho} = \{m \circ 1 \mid m \in \mathscr{M}\}.$$

Each contains all the identities of $\mathscr{E}_\rho\mathscr{M}$ and thus each has all objects of $\mathscr{E}_\rho\mathscr{M}$. In 2.7 we introduced the abbreviation convention of e for $1 \circ e$ and m for $m \circ 1$ in $\mathscr{E}_\rho\mathscr{M}$ and pointed out that $m \circ e = m \cdot e$. In fact, from the description of composition in 2.6 it is clear that $m \cdot e$ is the unique factorization of $m \circ e$ as an arrow in $^{\rho}\mathscr{E}$ followed by an arrow in $\mathscr{M}^{\rho}$ so that:

3.5. Proposition. *The pair* $(^{\rho}\mathscr{E}, \mathscr{M}^{\rho})$ *provides a strict factorization system* $\mathscr{S}_\rho$ *for* $\mathscr{E}_\rho\mathscr{M}$.

3.6. It is almost clear that $\mathscr{S}_{(-)}$ and $\rho_{(-)}$ are 'inverse constructions', relating distributive laws in **set-mat** with strict factorization systems. One could, and eventually should, pursue the relevant arrows between the concepts in question—and the relevant transformations between those—in order to exhibit $\mathscr{S}_{(-)}$ and $\rho_{(-)}$ as inverse 'biequivalences'. We stop short of doing that partly in the interests of brevity, partly because the discussion of arrows and transformations would distract the reader from the simple ideas presented here and partly because the relationship is, at the mere object level, very tight.

Still, a word or two on the matter for the reader interested in such things and familiar with [14] is warranted. In [14], Street defined for any 2-category **C** a 2-category **Mnd**(**C**) whose objects are the monads in **C**. He showed that the objects of **Mnd** (**Mnd**(**C**)) are distributive laws and thus a definition of arrow between distributive laws and of transformation between those has already been provided. The adaptation of [14] to cover bicategories as well as 2-categories is not difficult but the bicategory **set-mat** lacks the completeness property—admitting Eilenberg–Moore objects—studied in [14] and to which the definition of **Mnd** was clearly aimed. While the objects of **Mnd**(**set-mat**) are categories, the arrows of that bicategory are not functors. This situation was addressed in [16] by studying bicategories such as **set-mat** in the context of (proarrow) equipments. The forthcoming paper [S& L] will also discuss a variant of **Mnd**, namely the free completion with respect to admitting Eilenberg–Moore objects.

However, for monads $\mathscr{M}, \mathscr{M}' : \mathbf{O} \to \mathbf{O}$ in any bicategory it is clear that a transformation $\mu : \mathscr{M} \to \mathscr{M}' : \mathbf{O} \to \mathbf{O}$ can be declared to be a *homomorphism of monads* if the following diagrams commute:

$$\begin{array}{ccccc} & & \mathscr{M} & \xleftarrow{\;\cdot\;} & \mathscr{M}\mathscr{M} \\ & {}^{1}\nearrow & & & \\ 1_{\mathrm{O}} & & \downarrow \mu & & \downarrow \mu\mu \\ & {}_{1}\searrow & & & \\ & & \mathscr{M}' & \xleftarrow[\;\cdot\;]{} & \mathscr{M}'\mathscr{M}' \end{array}$$

and, further, an *isomorphism of monads* if also $\mu : \mathscr{M} \to \mathscr{M}'$ is invertible in the ambient bicategory.

3.7. Lemma. *For categories $\mathcal{M}$ and $\mathcal{M}'$ with the same set of objects, identity-on-objects functors $\mathcal{M} \to \mathcal{M}'$ are precisely homomorphisms of monads $\mathcal{M} \to \mathcal{M}'$ in* **set-mat**. *In particular, for such categories, identity-on-objects isomorphisms of categories $\mathcal{M} \xrightarrow{\simeq} \mathcal{M}'$ are precisely isomorphisms of monads in* **set-mat**.

3.8. Theorem. *For $\mathcal{S} = (\mathcal{E}, \mathcal{M})$ a strict factorization system on a category $\mathcal{K}$ there is an identity-on-objects isomorphism of categories*

$$\kappa : \mathcal{K} \xrightarrow{\simeq} \mathcal{E}_{\rho}\mathcal{M},$$

which identifies $\mathcal{S} = (\mathcal{E}, \mathcal{M})$ and $\mathcal{S}_{\rho_{\mathcal{S}}} = ({}^{\rho_{\mathcal{S}}}\mathcal{E}, \mathcal{M}^{\rho_{\mathcal{S}}})$.

For $\rho : \mathcal{M}\mathcal{E} \to \mathcal{E}\mathcal{M}$ a distributive law in **set-mat**, *there are identity-on-objects isomorphisms of categories $\mu : \mathcal{M} \xrightarrow{\simeq} \mathcal{M}^{\rho}$ and $\varepsilon : \mathcal{E} \xrightarrow{\simeq} {}^{\rho}\mathcal{E}$ which identify ρ and $\rho_{\mathcal{S}_{\rho}}$ in the sense that*

$$\begin{array}{ccc} \mathcal{M}\mathcal{E} & \xrightarrow{\rho} & \mathcal{E}\mathcal{M} \\ {\scriptstyle \mu\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon\mu} \\ \mathcal{M}^{\rho}\,{}^{\rho}\mathcal{E} & \xrightarrow[\rho_{\mathcal{S}_{\rho}}]{} & {}^{\rho}\mathcal{E}\mathcal{M}^{\rho} \end{array}$$

commutes.

Proof. For the first assertion, we define $\kappa(f) = m_f \circ e_f$. It is evidently an isomorphism of categories since, for any $m \circ e$ in $\mathcal{E}_{\rho}\mathcal{M}$, the composite me in $\mathcal{K}$ is the unique arrow with $m_{me} = m$ and $e_{me} = e$. For $m \in \mathcal{M}$, $\kappa(m) = m \circ 1$ and, for $e \in \mathcal{E}$, $\kappa(e) = 1 \circ e$.

For the second assertion we define $\mu(m) = m \circ 1$ and $\varepsilon(e) = 1 \circ e$. By the definitions of ${}^{\rho}\mathcal{E}$ and $\mathcal{M}^{\rho}$ and (ii) of 2.6, these are trivially isomorphisms. Since

$$\begin{aligned} \rho_{\mathcal{S}_{\rho}}(\mu\varepsilon(m,e)) &= \rho_{\mathcal{S}_{\rho}}(m \circ 1, 1 \circ e) \\ &= (e_{(1\circ e)\cdot(m\circ 1)}, m_{(1\circ e)\cdot(m\circ 1)}) \\ &= (e_{(e_{\mu}m)\circ(e_{\varepsilon}m)}, m_{(e_{\mu}m)\circ(e_{\varepsilon}m)}) \\ &= (1 \circ e_{\varepsilon}m, e_{\mu}m \circ 1) \\ &= (\varepsilon(e_{\varepsilon}m), \mu(e_{\mu}m)) \\ &= \varepsilon\mu(\rho(m,e)), \end{aligned}$$

the diagram commutes. □

4. Factorization algebras

4.1. In [8], Korostenski and Tholen studied the 2-monad on **CAT** whose underlying 2-functor is $(-)^{\mathbf{2}}$, whose unit $I_{(-)}$ is given by identities, and whose multiplication $C_{(-)}$ is given by composition. Evidently, Proposition 2.12 says that the functor $F : (\mathcal{E}_{\rho}\mathcal{M})^{\mathbf{2}} \to \mathcal{E}_{\rho}\mathcal{M}$ provides a strict algebra structure for this monad. It is shown in [8] that a normal pseudo-algebra for the monad $(-)^{\mathbf{2}}$ on a category $\mathcal{K}$ is precisely a factorization system on $\mathscr{K}$. To be clear, a normal pseudo-algebra structure on a category $\mathscr{K}$ consists of a functor $F : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ and an isomorphism $\alpha : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathscr{K}}$, such that $FI_{\mathscr{K}} = 1_{\mathscr{K}}$ and α satisfies the coherence conditions

$$\alpha I_{\mathscr{K}^{\mathbf{2}}} = 1_F,$$

$$\alpha (I_{\mathscr{K}})^{\mathbf{2}} = 1_F,$$

$$\alpha C_{\mathscr{K}^{\mathbf{2}}} \cdot \alpha (F^{\mathbf{2}})^{\mathbf{2}} = \alpha (C_{\mathscr{K}})^{\mathbf{2}} \cdot F\alpha^{\mathbf{2}}.$$

It follows that a strict algebra for $(-)^{\mathbf{2}}$ is also a normal pseudo-algebra for $(-)^{\mathbf{2}}$.

4.2. Mindful of the inflection terminology of [7], we call a normal pseudo-algebra for the 2-monad $(-)^{\mathbf{2}}$ on **CAT** a *factorization algebra* and we call a strict algebra for the same 2-monad a *strict factorization algebra*. In this terminology, Korostenski and Tholen [8] has shown that factorization systems and factorization algebras are equivalent concepts. We will show that strict factorization systems and strict factorization algebras are also equivalent concepts. While this is not altogether surprising, it does not immediately follow from the result in [8]. For one thing, the relating construction in the strict case is not just the restriction of that in [8] because strict factorization algebras are factorization algebras while, as noted in 3.1, strict factorization systems are not necessarily factorization systems. It is convenient in this context to call a mere functor $F : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ a *pre-factorization algebra*. In the event that $FI_{\mathscr{K}} = 1_{\mathscr{K}}$ we say that F is a *normal* pre-factorization algebra.

4.3. If F is a normal pre-factorization algebra then, as shown in [8] or [6], F provides for each commutative square in $\mathscr{K}$,

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & B \end{array}$$

considered as an arrow in $\mathscr{K}^{\mathbf{2}}$, a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle e_f}\downarrow & & \downarrow{\scriptstyle e_g} \\ F(f) & \xrightarrow{f(u,v)} & F(g) \\ {\scriptstyle m_f}\downarrow & & \downarrow{\scriptstyle m_g} \\ A & \xrightarrow[v]{} & B \end{array}$$

in $\mathscr{K}$ with $f = m_f \cdot e_f$ and $g = m_g \cdot e_g$. In other words, natural transformations $\partial_0 \xrightarrow{e} F \xrightarrow{m} \partial_1 : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ are derivable—we have $e_f = F(1_X \xrightarrow{(1_X,f)} f)$ and $m_f = F(f \xrightarrow{(f,1_A)} 1_A)$—and the factorization system arising from a normal pseudo-algebra has classes of arrows given by

$$\tilde{\mathscr{E}}_F = \{ f \mid m_f \text{ is invertible} \}, \quad \tilde{\mathscr{M}}_F = \{ f \mid e_f \text{ is invertible} \}.$$

4.4. Lemma. *The strict factorization algebra* $F:(\mathscr{E}_\rho\mathscr{M})^{\mathbf{2}} \to \mathscr{E}_\rho\mathscr{M}$ *arising from a distributive law* $\rho:\mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ *has* $e_f = e(f)$ *and* $m_f = m(f)$, *where* $e(f)$ *and* $m(f)$ *are as in* 2.10.

Proof. Let $f:X \to A$ be an arrow in $\mathscr{E}_\rho\mathscr{M}$. Then

$$e_f = F(1_X, f) = 1_{F(f)} \circ e(f) = e(f),$$

where the second equality follows by 2.10 and the third employs our convention from 2.7. Similarly, $m_f = m(f)$. □

In order to determine the classes $\tilde{\mathscr{E}}_F$ and $\tilde{\mathscr{M}}_F$ which arise from the factorization algebra coming from a distributive law $\rho:\mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$, some care is required. For $e:X \to A$ an arrow in $\mathscr{E}$, we have $e = 1_A \circ e$ in $\mathscr{E}_\rho\mathscr{M}$ and $m_e = 1_A$ is certainly an isomorphism in $\mathscr{E}_\rho\mathscr{M}$. Thus $\mathscr{E} \subseteq \mathscr{E}_F$ and similarly $\mathscr{M} \subseteq \mathscr{M}_F$.

4.5. Corollary. *For a distributive law* $\rho:\mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ *we have*

$$\tilde{\mathscr{E}}_F = \{m \circ e \mid m \text{ is invertible in } \mathscr{M}\},$$

$$\tilde{\mathscr{M}}_F = \{m \circ e \mid e \text{ is invertible in } \mathscr{E}\}.$$

Proof. For $m \circ e$ in $\mathscr{E}_\rho\mathscr{M}$, Lemma 4.4 gives $m_{m\circ e} = m \circ 1$ and by 2.13 $m \circ 1$ is an isomorphism in $\mathscr{E}_\rho\mathscr{M}$ if and only if m is an isomorphism in $\mathscr{M}$. This establishes the claim for $\tilde{\mathscr{E}}_F$ and that for $\tilde{\mathscr{M}}_F$ is of course similar. □

4.6. In studying strict factorization algebras $F:\mathscr{K}^{\mathbf{2}} \to \mathscr{K}$, it is more important to consider the classes of arrows of $\mathscr{K}$ given by

$$\mathscr{E}_F = \{f \in \mathscr{K} \mid m_f \text{ is an identity}\}, \quad \mathscr{M}_F = \{f \in \mathscr{K} \mid e_f \text{ is an identity}\}.$$

We will show at the end of this section that $\mathscr{S}_F = (\mathscr{E}_F, \mathscr{M}_F)$ is a strict factorization system on $\mathscr{K}$.

While these definitions—and for that matter the very notion of strict algebras for a monad on a 2-category—might at first seem suspect, it is important to point out that those factorization systems which arise in nature from set-theoretic image, $\mathbf{set}^{\mathbf{2}} \to \mathbf{set}$ where **set** is the category of sets, *do* come from strict factorization algebras. It is a simple matter to check the two requisite equations and to see how these are inherited by other concrete categories, their powers, and subcategories of those.

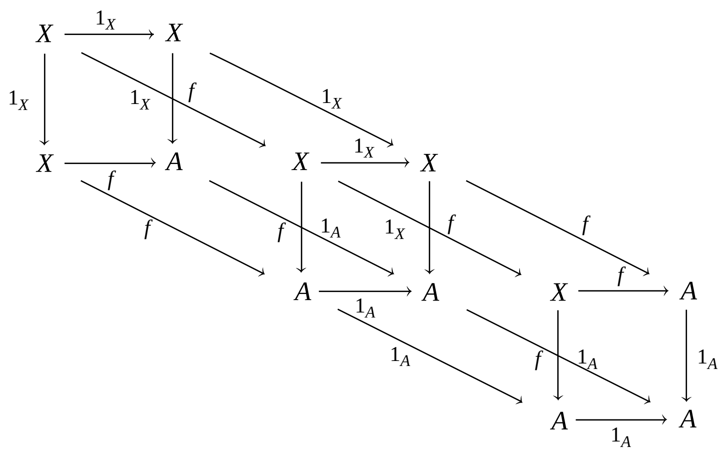
Of course, this is not a call for the abandonment of classical factorization systems. It is clear that classes (more precisely 2-categories) of categories which are described by universal properties and exactness conditions alone require the more general notion. However, as often observed by G.M. Kelly, it frequently happens that study of a more general ‘correct’ concept is facilitated by study of its strict counterpart and of the relationship between the two.

It has been said that Mac Lane's early definition of factorization system—called a 'bi-category structure' in [10]—suffered from an attempt to axiomatize too closely the notion of inclusion function, rather than injective function. Probably because inclusions are closed under composition while decomposition functions are not, it was anticipated that any attempt to capture inclusions would fail formalization because it would fail dualization. (See [1].) In this regard it is interesting to note that set-theoretic image $F:\mathbf{set}^{\mathbf{2}} \to \mathbf{set}$ gives $\mathscr{E}_F = \{\text{surjections}\}$ and $\mathscr{M}_F = \{\text{inclusions}\}$. It *happens* that one class is closed with respect to composition with isomorphisms and the other is not. Dualization of this example simply gives an '$\mathscr{M}_F$-class' which is closed with respect to composition with isomorphisms and an '$\mathscr{E}_F$-class' which is not. For a general strict factorization algebra $F:\mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ there seems to be no reason to suppose that at least one of $\mathscr{E}_F$ and $\mathscr{M}_F$ be isomorphism-closed. We turn now to a more detailed study of strict factorization algebras.

4.7. Lemma (Janelidze and Tholen see [6]).
For a normal pre-factorization algebra $F:\mathscr{K}^{\mathbf{2}} \to \mathscr{K}$, if each m_{e_f} is an epimorphism and each e_{m_f} is a monomorphism then, for each $(u,v):f \to g$ in $\mathscr{K}^{\mathbf{2}}$, $F(u,v)$ is uniquely determined by the commutativity conditions of the second diagram in 4.3.

4.8. Proposition. *If $F:\mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ is a strict factorization algebra then, for each arrow f in $\mathscr{K}$, $F(e_f)=F(f)=F(m_f)$ and $m_{e_f}:F(e_f) \to F(f)$ and $e_{m_f}:F(f) \to F(m_f)$ are identities.*

Proof. Let $f:X \to A$ be an arrow in $\mathscr{K}$ and consider the following diagram regarded as a composable pair in $(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$:



Applying the functor $FC_{\mathscr{K}}$ to this composable pair gives

$$F(f) \xrightarrow{1_{F(f)}} F(f) \xrightarrow{1_{F(f)}} F(f).$$

Applying the equal $FF^{\mathbf{2}}$ to the same pair gives

$$F(F(1_X,f))\xrightarrow{F(F(1_X,f),F(1_X,1_A))}F(F(1_X,1_A))\xrightarrow{F(F(1_X,1_A),F(f,1_A))}F(F(f,1_A)),$$

which invoking the definitions of 4.3 and $FI=1$ is

$$F(e_f)\xrightarrow{m_{e_f}}F(f)\xrightarrow{e_{m_f}}F(m_f). \qquad \square$$

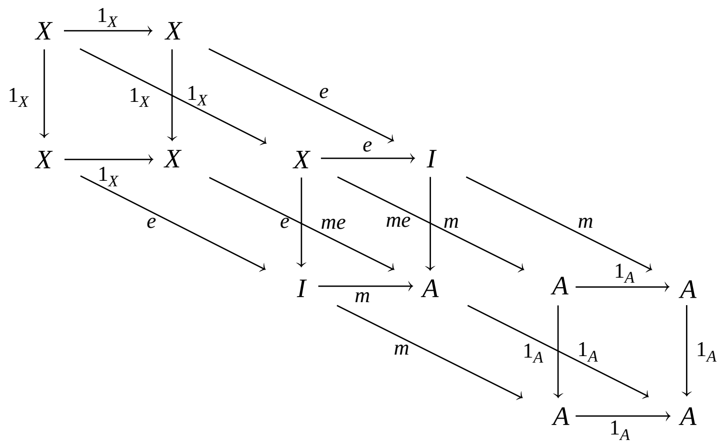
4.9. Corollary. *If $F:\mathscr{K}^{\mathbf{2}}\to\mathscr{K}$ is a strict factorization algebra then, for each arrow f in $\mathscr{K}$, $m_f\in\mathscr{M}_F$, $e_f\in\mathscr{E}_F$, and for each $(u,v):f\to g$ in $\mathscr{K}^{\mathbf{2}}$, $F(u,v)$ is uniquely determined by the commutativity conditions of the second diagram in* 4.3.

4.10. Proposition. *If $F:\mathscr{K}^{\mathbf{2}}\to\mathscr{K}$ is a strict factorization algebra then $\mathscr{E}_F\cap\mathscr{M}_F$ is precisely the class of identity arrows in $\mathscr{K}$.*

Proof. For any object $X\in\mathscr{K}$, the definitions in 4.3 give $m_{1_X}=1_X=e_{1_X}$, which shows that each identity belongs to $\mathscr{E}_F\cap\mathscr{M}_F$. Conversely, if $f=m_f\cdot e_f$ is in $\mathscr{E}_F\cap\mathscr{M}_F$ then it follows that f is an identity. $\square$

4.11. Proposition. *If $F:\mathscr{K}^{\mathbf{2}}\to\mathscr{K}$ is a strict factorization algebra then, for each arrow f in $\mathscr{K}$, the factorization $f=m_f\cdot e_f$ is the* unique *factorization of f as a composite, me, with $e\in\mathscr{E}_F$ and $m\in\mathscr{M}_F$.*

Proof. Assume that $X\xrightarrow{e}I\xrightarrow{m}A$, with $e\in\mathscr{E}_F$ and $m\in\mathscr{M}_F$. We will use the equation $FC_{\mathscr{K}}=FF^{\mathbf{2}}$ applied to the following composable pair in $(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$:



The functor $FC_{\mathscr{k}}$ applied to this composable pair gives

$$X\xrightarrow{e_{me}}F(me)\xrightarrow{m_{me}}A.$$

Now, for the moment, consider just the centre square and apply $F^{\mathbf{2}}$ to get

$$\begin{array}{ccc} X & \xrightarrow{e} & I \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle 1_I} \\ I & \xrightarrow[F(e,m)]{} & I \\ {\scriptstyle 1_I}\downarrow & & \downarrow{\scriptstyle m} \\ I & \xrightarrow[m]{} & A \end{array}$$

By the uniqueness clause of Corollary 4.9, $F(e,m) = 1_I$. So $FF^{\mathbf{2}}$ applied to the middle square gives I. Applying $FF^{\mathbf{2}}$ to the given composable pair in $(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$ gives

$$X \xrightarrow{F(F(1_X,e),F(e,me))} I \xrightarrow{F(F(me,m),F(m,1_A))} A.$$

Using uniqueness of $F(-,-)$ in the same manner as above, four times, simplifies the result to

$$X \xrightarrow{F(e,e)} I \xrightarrow{F(m,m)} A,$$

which by $FI_{\mathscr{K}} = 1_{\mathscr{K}}$ is just

$$X \xrightarrow{e} I \xrightarrow{m} A. \qquad \square$$

There is a functor $R_{\mathscr{K}} : (\mathscr{K}^{\mathbf{2}})^{\mathbf{2}} \longrightarrow (\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$ given by reflection in the diagonal, so that on objects it is described pictorially by

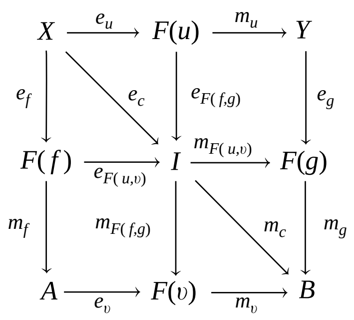
$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & B \end{array} \quad \mapsto \quad \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & B \end{array}$$

4.12. Lemma. *If $F : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ satisfies $FC_{\mathscr{K}} = FF^{\mathbf{2}}$ then it also satisfies $FC_{\mathscr{K}} = FF^{\mathbf{2}}R_{\mathscr{K}}$.*

Proof. It is clear that $C_{\mathscr{K}}R_{\mathscr{K}} = C_{\mathscr{K}}$ from which the result follows immediately. $\square$

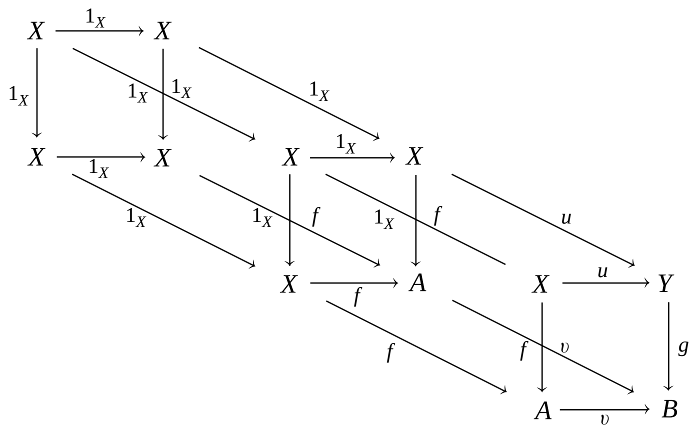
4.13. Remark. As explained in [8] the monad structure on $(-)^{\mathbf{2}}$ is completely derived from the canonical comonoid structure on $\mathbf{2}$. This derivation takes the switch functor $\mathbf{2} \times \mathbf{2} \to \mathbf{2} \times \mathbf{2}$ to $R_{\mathscr{K}}$ and $C_{\mathscr{K}}R_{\mathscr{K}} = C_{\mathscr{K}}$ follows from cocommutativity of $\mathbf{2} \to \mathbf{2} \times \mathbf{2}$.

4.14. Proposition. *If $F : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ is a strict factorization algebra then, for each commutative square $u, v : f \to g$ in $\mathscr{K}$, all regions of the following* *diagram commute*:



where we have written c *for the common value* $vf = gu$ *and* I *for the common value* $F(F(u,v)) = F(c) = F(F(f,g))$, *the last equation holding by Lemma* 4.12.

Proof. We consider first the top left-most triangular region. Application of $FC_{\mathscr{K}}$ to the $(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$ composite



gives $e_c : X \to F(c)$, since $vf = c$, while application of $FF^{\mathbf{2}}$ gives

$$X \xrightarrow{F(F(1_X,1_X),F(1_X,f))} F(f) \xrightarrow{F(F(1_X,f),F(u,v))} F(F(u,v)).$$

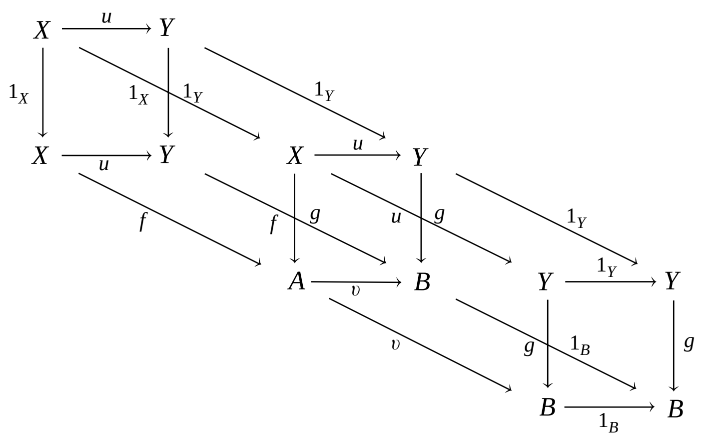
First use $F(1_X,1_X) = 1_X$ and $F(1_X,f) = e_f$ to make a preliminary simplification of this composite and then use the uniqueness clause of Corollary 4.9 to show that $F(1_X,e_f) = e_f$ and $F(e_f, F(u,v)) = e_{F(u,v)}$. Since $FC_{\mathscr{K}} = FF^{\mathbf{2}}$ we have $e_c = e_{F(u,v)} \cdot e_f$.

An entirely similar calculation shows that the bottom right-most triangle commutes. For the other two triangles apply the same idea with $FF^{\mathbf{2}}$ replaced by $FF^{\mathbf{2}}R_{\mathscr{K}}$.

For the top-right square, observe first that

$$F(u) \xrightarrow{m_u} F(1_Y) \xrightarrow{e_g} F(g) = F(u) \xrightarrow{F(u,1_Y)} F(1_Y) \xrightarrow{F(1_Y,g)} F(g),$$

which is $F(u,g)$ by functoriality of F and which in turn is $F(u \xrightarrow{(1_X,g)} c \xrightarrow{(u,1_B)} g)$. Consider the $(\mathscr{K}^2)^2$ composite



For the first factor we have

$$F(1_X, g) = F(F(1_X, 1_Y), F(f, g)) = F(1_{F(u)}, F(f, g)) = e_{F(f,g)} : F(u) \to I,$$

where the first equality uses $FC_{\mathscr{K}} = FF^2R_{\mathscr{K}}$. For the second factor we have

$$F(u, 1_B) = F(F(u, v), F(1_Y, 1_B)) = F(F(u, v), 1_{F(g)}) = m_{F(u,v)} : I \to F(g),$$

where here the first equality uses $FC_{\mathscr{K}} = FF^2$. A similar calculation shows that the bottom-left square commutes. □

4.15. Remark. It follows from Proposition 4.11 that for the square regions of Proposition 4.14 we can add

$$e_{e_g \cdot m_u} = e_{F(f,g)} \quad \text{and} \quad m_{e_g \cdot m_u} = m_{F(u,v)},$$

$$e_{e_v \cdot m_f} = e_{F(u,v)} \quad \text{and} \quad m_{e_v \cdot m_f} = m_{F(f,g)}.$$

4.16. Corollary. *If $F : \mathscr{K}^2 \to \mathscr{K}$ is a strict factorization algebra then the classes of arrows $\mathscr{E}_F$ and $\mathscr{M}_F$ are closed under composition and hence by Proposition* 4.10 *may be regarded as subcategories of $\mathscr{K}$ whose objects are those of $\mathscr{K}$.*

Proof. For $e : X \to A$ and $f : A \to Y$ in $\mathscr{E}_F$, apply Proposition 4.14 to the commutative square $e, f : e \to f$ to get

$$
\begin{array}{ccccc}
X & \xrightarrow{\ e\ } & A & \xrightarrow{\ 1_A\ } & A \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
A & \xrightarrow[\ f\ ]{} & Y & \xrightarrow{\ 1_Y\ } & Y \\
{\scriptstyle 1_A}\downarrow & & {\scriptstyle 1_Y}\downarrow & \searrow^{m_{fe}} & \downarrow{\scriptstyle 1_Y} \\
A & \xrightarrow[\ f\ ]{} & Y & \xrightarrow[\ 1_Y\ ]{} & Y
\end{array}
$$

which shows that $m_{fe} = 1_Y$. The demonstration for $\mathscr{M}_F$ is similar. □

4.17. Corollary. *If $F : \mathscr{K}^{\mathbf{2}} \to \mathscr{K}$ is a strict factorization algebra then $\mathscr{S}_F = (\mathscr{E}_F, \mathscr{M}_F)$ is a strict factorization system on $\mathscr{K}$.*

Proof. To Corollary 4.16 add Proposition 4.11. □

4.18. On the other hand, if $\mathscr{S} = (\mathscr{E}, \mathscr{M})$ is a strict factorization system on a category $\mathscr{K}$, then we can write $X \xrightarrow{e_f} F_{\mathscr{S}}(f) \xrightarrow{m_f} A$ for the unique $\mathscr{E}$-$\mathscr{M}$ factorization of an arrow $f : X \to A$ in $\mathscr{K}$. Given an arrow $u, v : f \to g$ in $\mathscr{K}^{\mathbf{2}}$ with $C_{\mathscr{K}}(u, v : f \to g) = X \xrightarrow{c} B$, consider

$$
\begin{array}{ccccc}
X & \xrightarrow{\ e_u\ } & F_{\mathscr{S}}(u) & \xrightarrow{\ m_u\ } & Y \\
{\scriptstyle e_f}\downarrow & & {\scriptstyle e_{e_g m_u}}\downarrow & & \downarrow{\scriptstyle e_g} \\
F_{\mathscr{S}}(f) & \xrightarrow[\ e_{e_v m_f}\ ]{} & I & \xrightarrow{\ m_{e_g m_u}\ } & F_{\mathscr{S}}(g) \\
{\scriptstyle m_f}\downarrow & & \downarrow & {\scriptstyle = m_{e_v m_f}} & \downarrow{\scriptstyle m_g} \\
A & \xrightarrow[\ e_v\ ]{} & F_{\mathscr{S}}(v) & \xrightarrow[\ m_v\ ]{} & B
\end{array}
$$

To see that the diagram is meaningfully labelled, examine first the upper right-hand square. We have $I = F_{\mathscr{S}}(e_g m_u)$ and since $(m_g m_{e_g m_u})(e_{e_g m_u} e_u)$ is an $\mathscr{E}$-$\mathscr{M}$ factorization for $gu = c$ it is necessarily *the* $\mathscr{E}$-$\mathscr{M}$ factorization of c and we have $I = F_{\mathscr{S}}(c)$. Of course, similar conclusions result from examining the lower left-hand square and it follows that all regions of the diagram commute. For $u, v : f \longrightarrow g$ in $\mathscr{K}^{\mathbf{2}}$ we define $F_{\mathscr{S}}(u, v) = m_{e_g m_u} e_{e_v m_f}$.

4.19. Proposition. *The definitions of* 4.18 *provide a functor* $F_{\mathscr{S}}:\mathscr{K}^{\mathbf{2}} \longrightarrow \mathscr{K}$ *and it is a factorization algebra. Moreover, the derived natural transformations* $\partial_0 \longrightarrow F_{\mathscr{S}} \longrightarrow \partial_1$ *have* f*-components* e_f *and* m_f, *respectively, as provided by* $\mathscr{E}$-$\mathscr{M}$ *factorization.*

Proof. All aspects of the statement follow immediately from uniqueness of $\mathscr{E}$-$\mathscr{M}$ factorizations. The equation $FF^{\mathbf{2}} = FC_{\mathscr{K}}$ on objects, in particular, is effectively displayed by the diagram above in 4.18 where in addition to $I = F_{\mathscr{S}}(c)$ we have also $I = F_{\mathscr{S}}(F_{\mathscr{S}}(u,v))$. □

4.20. Theorem. *For any category* $\mathscr{K}$, *the assignments*

$$F \mapsto \mathscr{S}_F \quad \text{and} \quad \mathscr{S} \mapsto F_{\mathscr{S}}$$

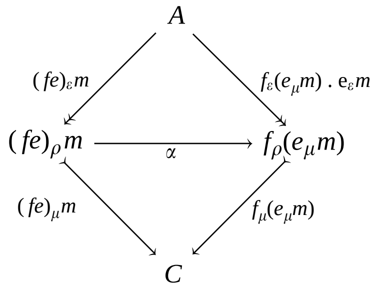
provide a bijective correspondence between strict factorization algebras on $\mathscr{K}$ *and strict factorization systems on* $\mathscr{K}$.

Proof. For any strict factorization algebra F on $\mathscr{K}$ and any $f:X \longrightarrow A$ in $\mathscr{K}$, $F_{\mathscr{S}_F}(f)$ is the object that appears in the $\mathscr{S}_F$ factorization of f and that object is $F(f)$. It follows easily that $F_{\mathscr{S}_F} = F$. For any strict factorization system $\mathscr{S} = (\mathscr{E}, \mathscr{M})$, $\mathscr{S}_{F_{\mathscr{S}}}$ is the strict factorization system $(\mathscr{E}_{F_{\mathscr{S}}}, \mathscr{M}_{F_{\mathscr{S}}})$. One has $(f:X \longrightarrow A) \in \mathscr{E}_{F_{\mathscr{S}}}$ if and only if $X \xrightarrow{f} A \xrightarrow{1_A} A$ is factorization of f as provided by $F_{\mathscr{S}}$ and the derived natural transformations $\partial_0 \longrightarrow F_{\mathscr{S}} \longrightarrow \partial_1$ which is the case if and only if $X \xrightarrow{f} A \xrightarrow{1_A} A$ is the $\mathscr{E}$-$\mathscr{M}$ factorization of f. It follows that $\mathscr{E}_{F_{\mathscr{S}}} = \mathscr{E}$, similarly $\mathscr{M}_{F_{\mathscr{S}}} = \mathscr{M}$ and $\mathscr{S}_{F_{\mathscr{S}}} = \mathscr{S}$. □

5. Relaxed distributive laws

5.1. Distributive laws in **set**-**mat**, strict factorization systems and strict factorization algebras have now been shown to be equivalent concepts. Since factorization systems and factorization algebras were shown to be equivalent concepts in [8] it is natural to look for a relaxed version of distributive law in **set**-**mat** which enables us to state and prove a counterpart for Theorem 3.8 with strict factorization systems replaced by factorization systems. With the basic correspondences at hand one suspects that the distributive law equations in 2.3 should be replaced by (coherent) specified isomorphisms—after all, if we start with a factorization system, images satisfy the object equations of 2.3 for ρ-squares merely to within isomorphism. However, a distributive law is expressed in terms of equations between *transformations* in **set**-**mat**, so that in the absence of further structure we are lacking a higher categorical dimension in which to replace equations between transformations by isomorphisms.

5.2. Consider, by way of example, Eqs. (5) and (6) of 2.3, the object equation $(fe)_\rho m = f_\rho(e_\mu m)$ and the ρ-square which immediately precedes them. In a category $\mathscr{K}$ with a factorization system and ρ understood in terms of factorization we would have instead a commutative diagram

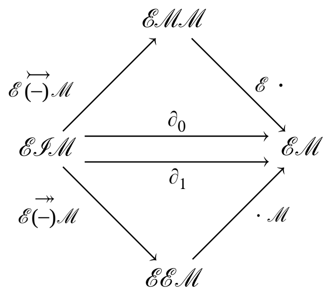


with α an isomorphism in $\mathscr{K}$. Moreover, by the diagonal fill-in property, such an α is unique. These considerations should serve to motivate the next subsection.

5.3. Given categories $\mathscr{M}$ and $\mathscr{E}$, both with set of objects $\mathbf{O}$, consider a category $\mathscr{I}$, also with set of objects $\mathbf{O}$, and identity-on-objects functors, equivalently homomorphisms of monads,

$$\overset{\twoheadrightarrow}{(-)} : \mathscr{E} \longleftarrow \mathscr{I} \longrightarrow \mathscr{M} : \overset{\rightarrowtail}{(-)}$$

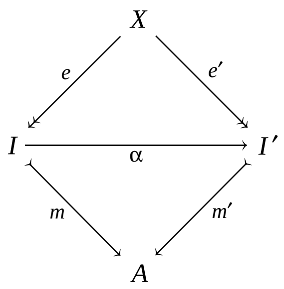
so that $\mathscr{E}$ and $\mathscr{M}$ become, in the language of ring theory, $\mathscr{I}$-algebras. Then the parallel pair $\partial_0, \partial_1 : \mathscr{E}\mathscr{I}\mathscr{M} \rightrightarrows \mathscr{E}\mathscr{M}$ defined by



extends to a category object structure on $\mathscr{E}\mathscr{M}$, for purely general reasons. (This category object structure on $\mathscr{E}\mathscr{M}$ in the category **set-mat**(**O**,**O**) is quite a different matter from the monad structure built on $\mathscr{E}\mathscr{M}$ with a distributive law.) We require of this structure that

(i) The pair (∂_0, ∂_1) is jointly monic.
(ii) The category $\mathscr{I}$ is a groupoid.
(iii) The functors $\overset{\twoheadrightarrow}{(-)}$: $\mathrm{iso}(\mathscr{E}) \leftarrow \mathscr{I} \rightarrow \mathrm{iso}(\mathscr{M}) : \overset{\rightarrowtail}{(-)}$ are isomorphisms.

From the first two of these requirements it follows that each $\mathscr{E}\mathscr{M}(X,A)$ carries the structure of an equivalence relation and that these equivalence relations are respected by the actions $\mathscr{E}\mathscr{E}\mathscr{M} \longrightarrow \mathscr{E}\mathscr{M}$ and $\mathscr{E}\mathscr{M}\mathscr{M} \longrightarrow \mathscr{E}\mathscr{M}$ provided by $\mathscr{E}$- and $\mathscr{M}$-composition, respectively. Explicitly, $X \overset{e}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} A$ is *$\mathscr{I}$-equivalent* to $X \overset{e'}{\twoheadrightarrow} I' \overset{m'}{\rightarrowtail} A$ if and only if there exists an arrow $\alpha : I \to I'$ in $\mathscr{I}$ such that $\overset{\rightarrowtail}{\alpha} \cdot e = e'$ and $m' \cdot \overset{\rightarrowtail}{\alpha} = m$. Such an α, if it exists, is necessarily unique by our first requirement above. It is helpful to draw the following diagram, considering it to be well defined and commutative without bothering to decorate α.



In particular, the diagram above makes clear our assertion that $\mathscr{I}$-equivalence is respected by pre-composition with arrows in $\mathscr{E}$ and by post-composition with arrows in $\mathscr{M}$. Observe also that e is invertible if and only if e' is invertible and m is invertible if and only if m' is invertible. By (iii) observe that if e is invertible then $X \overset{e}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} A$ is *$\mathscr{I}$-equivalent* to $X \overset{1_X}{\twoheadrightarrow} X \overset{me}{\rightarrowtail} A$, where we have made an obvious further notational simplification. Similarly, if m is invertible then $X \overset{e}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} A$ is *$\mathscr{I}$-equivalent* to $X \overset{me}{\twoheadrightarrow} A \overset{1_A}{\rightarrowtail} A$.

5.4. With the additional structure of 5.3 in place it is a simple matter to define a *distributive law of $\mathscr{M}$ over $\mathscr{E}$ with respect to $\mathscr{I}$* to mean a transformation $\rho{:}\mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ in **set-mat** with the classical Beck equations replaced by $\mathscr{I}$-equivalence. For example, the top pentagon of 2.3 now gives (all instances of) the diagram of 5.2. Moreover, still thinking of the elements of the $\mathscr{E}\mathscr{M}(A,C)$ as formal composites, as we did in 2.6, we are able to follow the prescription of 2.6 to define composites of these. This composition is readily seen to be unitary and associative to within $\mathscr{I}$-equivalence and we have a bicategory with set of objects **O** and hom categories given by the sets $\mathscr{E}\mathscr{M}(A,C)$ together with instances of $\mathscr{I}$-equivalence. We define $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$ to be the category with set of objects **O** and $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}(A,C) = \mathscr{E}\mathscr{M}(A,C)/\mathscr{I}$. We will now write $m \circ e$ for an element of $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}(A,C)$ and by the observation concluding 5.3 it is meaningful to speak of those $m \circ e$ in $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}(A,C)$ with m invertible in $\mathscr{M}$ or of those $m \circ e$ in $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}(A,C)$ with e invertible in $\mathscr{E}$. Note also that it is unambiguous to write $mi \circ e = m \circ ie$ for any isomorphism i.

5.5. For $\rho : \mathscr{M}\mathscr{E} \longrightarrow \mathscr{E}\mathscr{M}$ a distributive law with respect to $\mathscr{I}$ in **set-mat** we define

$$^{\rho}\tilde{\mathscr{E}} = \{m \circ e \mid m \text{ is invertible in } \mathscr{M}\} \quad \text{and} \quad \tilde{\mathscr{M}}^{\rho} = \{m \circ e \mid e \text{ is invertible in } \mathscr{E}\}.$$

It is not difficult to see that these classes of arrows in $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$ are closed with respect to composition and contain all isomorphisms of $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$. It is also straightforward to show that every arrow in $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$ can be factored as a composite with first factor in ${}^{\rho}\tilde{\mathscr{E}}$ and second factor in $\tilde{\mathscr{M}}^{\rho}$ and that the diagonal fill-in condition is satisfied. In short:

5.6. Proposition. *The pair* $({}^{\rho}\tilde{\mathscr{E}}, \tilde{\mathscr{M}}^{\rho})$ *provides a factorization system* $\tilde{\mathscr{S}}_{\rho}$ *for* $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$.

5.7. Of course, our definitions in the last three subsections were motivated by the consideration of starting with a classical factorization system $\mathscr{S} = (\mathscr{E}, \mathscr{M})$ on a category $\mathscr{K}$ with objects $\mathbf{O}$. Given such we can assume that, for each f in $\mathscr{K}$, a particular factorization $f = m_f \cdot e_f$ has been named. We regard $\mathscr{M}$ and $\mathscr{E}$ as monads on $\mathbf{O}$ in **set-mat** and define

$$\tilde{\rho}_{\mathscr{S}} : \mathscr{M}\mathscr{E} \longrightarrow \mathscr{E}\mathscr{M} : \mathbf{O} \longrightarrow \mathbf{O}$$

as a transformation in **set-mat** by

$$A \xrightarrow{n} B \xrightarrow{f} C \;\mapsto\; A \xrightarrow{e_{fn}} I \xrightarrow{m_{fn}} C$$

for $n \in \mathscr{M}$ and $f \in \mathscr{E}$. Of course, $\tilde{\rho}_{\mathscr{S}}$ is not a distributive law but taking $\mathscr{I}$ to be the category of all isomorphisms of $\mathscr{K}$, the structure of 5.3 is provided by the inclusions $\mathscr{I} \longrightarrow \mathscr{E}$ and $\mathscr{I} \longrightarrow \mathscr{M}$. From the well-known properties of factorization systems we have:

5.8. Proposition. *The transformation* $\tilde{\rho}_{\mathscr{S}} : \mathscr{M}\mathscr{E} \longrightarrow \mathscr{E}\mathscr{M}$ *is a distributive law of* $\mathscr{M}$ *over* $\mathscr{E}$ *with respect to* $\mathscr{I}$.

5.9. Theorem. *For* $\mathscr{S} = (\mathscr{E}, \mathscr{M})$ *a factorization system on a category* $\mathscr{K}$ *with isomorphisms* $\mathscr{I}$ *there is an identity-on-objects isomorphism of categories*

$$\kappa : \mathscr{K} \xrightarrow{\simeq} \mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$$

which identifies $\mathscr{S} = (\mathscr{E}, \mathscr{M})$ *and* $\tilde{\mathscr{S}}_{\tilde{\rho}_{\mathscr{S}}} = ({}^{\tilde{\rho}_{\mathscr{S}}}\tilde{\mathscr{E}}, \tilde{\mathscr{M}}^{\tilde{\rho}_{\mathscr{S}}})$.

For $\rho : \mathscr{M}\mathscr{E} \to \mathscr{E}\mathscr{M}$ *a distributive law with respect to* $\mathscr{I}$ *in* **set-mat**, *there are identity-on-objects isomorphisms of categories* $\mu : \mathscr{M} \xrightarrow{\simeq} \tilde{\mathscr{M}}^{\rho}$ *and* $\varepsilon : \mathscr{E} \xrightarrow{\simeq} {}^{\rho}\tilde{\mathscr{E}}$ *which identify* ρ *and* $\tilde{\rho}_{\tilde{\mathscr{S}}_{\rho}}$ *in the sense that*

$$\begin{array}{ccc} \mathscr{M}\mathscr{E} & \xrightarrow{\rho} & \mathscr{E}\mathscr{M} \\ {\scriptstyle \mu\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon\mu} \\ \tilde{\mathscr{M}}^{\rho}\,{}^{\rho}\tilde{\mathscr{E}} & \xrightarrow[\tilde{\rho}_{\tilde{\mathscr{S}}_{\rho}}]{} & {}^{\rho}\tilde{\mathscr{E}}\tilde{\mathscr{M}}^{\rho} \end{array}$$

commutes.

Proof. For the first assertion, we define $\kappa(f)$ to be the equivalence class $m_f \circ e_f$. For an arrow of $\mathscr{E}^{\mathscr{I}}_{\rho}\mathscr{M}$ represented by $m \circ e$ we define $\lambda(m \circ e) = m \cdot e$, where the composite is taken in $\mathscr{K}$. The definition of λ is seen to be sound from the definition of equivalence in 5.3. Clearly, $\lambda(\kappa(f)) = f$, while $\kappa(\lambda(m \circ e)) = m_{m\cdot e} \circ e_{m\cdot e} = m \circ e$, the last equation holding since factorization is unique up to equivalence. Thus, κ is an isomorphism which clearly identifies $(\mathscr{E}, \mathscr{M})$ and $({}^{\tilde{\rho}_{\mathscr{S}}}\tilde{\mathscr{E}}, \tilde{\mathscr{M}}^{\tilde{\rho}_{\mathscr{S}}})$.

For the second assertion we define $\mu(m) = m \circ 1$ and $\varepsilon(e) = 1 \circ e$. To show that μ is an isomorphism, consider an arrow $m \circ e$ in $\tilde{\mathscr{M}}^{\rho}$ and define $\nu(m \circ e) = me$. Clearly,

$v(\mu(m)) = m$, while $\mu(v(m \circ e)) = \mu(me) = me \circ 1 = m \circ e$ since e is an isomorphism. Thus μ and similarly ε are isomorphisms. Commutativity of the square follows from the same calculation as in the proof of Theorem 3.8. □

5.10. Remark. It should be noted that Lack and Street [9] have shown that a factorization system gives rise to a 'wreath', which provides another generalization of the notion of distributive law.

5.11. A final comment about the brief appearance of the bicategory $\mathscr{E}\mathscr{M}$ in 5.4 is in order. In related examples it will not always be desirable to pass by quotienting to a mere category. For $\mathscr{K}$ a category with pullbacks, we can regard the formation of pullback as a transformation $\rho : \mathscr{K}\mathscr{K}^{op} \longrightarrow \mathscr{K}^{op}\mathscr{K}$ in **set-mat**. Here ρ-squares are but pullback squares and in this formulation it is clear that Beck's equations are satisfied to within $\mathscr{I}$-equivalence, $\mathscr{I}$ being the isomorphisms of $\mathscr{K}$. The elements of the $\mathscr{K}^{op}\mathscr{K}(A,C)$ are arrows of the bicategory of spans in $\mathscr{K}$, and Beck composition of these, as prescribed by 2.6, is the usual composition of spans via pullback. Here we should keep the bicategorical structure and the machinery of this paper should be extended further so as provide another way of analyzing the important categories with factorization $(\mathscr{K}; \mathscr{E}, \mathscr{M})$ in which $\mathscr{E}$ is *stable* with respect to pullback. These investigations will be continued elsewhere.

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